

# Nilpotent normal form for divergence-free vector fields and volume-preserving maps

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Received 11 June 2007; received in revised form 14 August 2007; accepted 25 August 2007

Available online 5 September 2007

Communicated by M. Silber

## Abstract

We study the normal forms for incompressible flows and maps in the neighborhood of an equilibrium or fixed point with a triple eigenvalue. We prove that when a divergence-free vector field in  $\mathbb{R}^3$  has nilpotent linearization with maximal Jordan block then, to arbitrary degree, coordinates can be chosen so that the nonlinear terms occur as a single function of two variables in the third component. The analogue for volume-preserving diffeomorphisms gives an optimal normal form in which the truncation of the normal form at any degree gives an exactly volume-preserving map whose inverse is also polynomial with the same degree.

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**Keywords:** Nilpotent normal form; Divergence-free vector fields; Volume-preserving maps

## 1. Introduction

A system of ODEs  $\dot{\xi} = v(\xi, t)$  gives rise to a volume-preserving flow when the vector field  $v$  has zero divergence,

$$\nabla \cdot v = 0. \quad (1)$$

Similarly, a diffeomorphism  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is volume preserving when its Jacobian has unit determinant,  $\det Df(\xi) = 1$ . Volume-preserving dynamics arises, for example, in the flow of a Lagrangian element in an incompressible fluid [17,10,12,5,29,30,23,24], or the tracing of lines in a magnetic field, [33,19,16]. Thus such flows are of interest in the study of the motion of passive tracers in time-dependent, incompressible fluids and for the dynamics of charged particles in strong magnetic fields. Volume-preserving maps are a natural generalization of area-preserving maps to higher dimensions. They also arise as

the normal form for certain homoclinic bifurcations for three-dimensional systems [15,14], and as integrators for incompressible flows [27,26,18,32] and thus have intrinsic mathematical interest [6,3,34,28,20,13,21,22].

An equilibrium of a divergence-free vector field can undergo a number of codimension-one bifurcations. For example when one of the eigenvalues vanishes and the remaining eigenvalues do not lie on the imaginary axis, then the resulting bifurcation is generically of standard saddle-node type. Similarly, when a single pair of eigenvalues lie on the imaginary axis, the bifurcation is of Hopf type. Since the eigenvalues of the equilibrium satisfy  $\sum_{i=1}^n \lambda_i = 0$ , there are two more exotic cases that are codimension-one only for low dimensions. For three dimensions, the codimension-one configuration  $\{0, i\omega, -i\omega\}$  can give rise, through a Hopf-saddle node bifurcation to the creation of a periodic orbit surrounded by a family of invariant tori [4]. The final codimension-one case occurs in four dimensions when the eigenvalues are  $\{i\omega_1, -i\omega_2, i\omega_2, -i\omega_1\}$ ; this bifurcation is analogous to the Hamiltonian–Hopf bifurcation [4].

The linearization of a volume-preserving map at a fixed-point  $x^* = f(x^*)$  gives a matrix  $M = Df(x^*)$  with determinant one, so that the multipliers satisfy  $\prod_{i=1}^d \lambda_i = 1$ .

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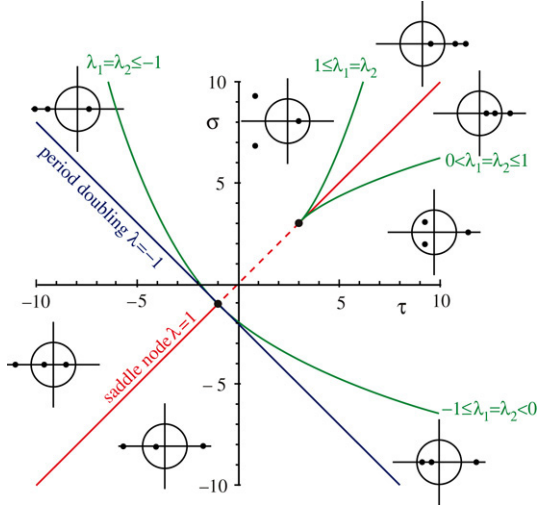


Fig. 1. Classification of the eigenvalues for a three-dimensional, volume-preserving map as a function of the trace  $\tau$  and second trace  $\sigma$ .

For the three-dimensional case, the characteristic polynomial has the form

$$p(\lambda) = \det(\lambda \mathbb{I} - M) = \lambda^3 - \tau \lambda^2 + \sigma \lambda - 1, \quad (2)$$

where  $\tau$  denotes the trace,  $\tau = \text{tr } M$ , and  $\sigma$  the second trace,  $\sigma = \frac{1}{2}((\text{tr } M)^2 - \text{tr } M^2)$ . Consequently,  $(\tau, \sigma)$  parametrize the space of volume-preserving matrices that are linearly conjugate to  $M$ , at least when the eigenvalues are distinct. The eigenvalue configurations in this space, see Fig. 1 [20,21], show that there are two codimension-two points. The first, at  $(\tau, \sigma) = (-1, -1)$  where the eigenvalues are  $(1, -1, -1)$  corresponds to simultaneous period-doubling and saddle-node bifurcations, and the second, at  $(\tau, \sigma) = (3, 3)$ , to the triple eigenvalue  $(1, 1, 1)$ . These are connected by a line segment  $\{\tau = \sigma : -1 < \tau < 3\}$  on which two eigenvalues are on the unit circle, and the third is equal to one, corresponding to simultaneous saddle-node and Neimark–Sacker bifurcations.

In this paper we study the normal forms for the codimension-two cases of a three-dimensional incompressible flow with eigenvalues  $\{0, 0, 0\}$  and of a volume-preserving map with multipliers  $\{1, 1, 1\}$ . We first recall some of the standard results on normal forms.

Near an equilibrium, a set of ODEs with a smooth vector field takes the form

$$\dot{\xi} = v(\xi) = J\xi + b(\xi), \quad (3)$$

for  $\xi \in \mathbb{R}^n$ , with a vector field  $v$  whose linear part is  $J\xi$  and nonlinear part is  $b(\xi) = O(2)$ . A local normal form is a conjugate system

$$\dot{\eta} = w(\eta) = J\eta + c(\eta), \quad (4)$$

that is “simpler” in some sense. For example, it is usually desirable to eliminate as many of the nonlinear terms in  $b$  as possible since then the vector field  $w$  will have fewer parameters. A first step in this process is to choose coordinates in which  $J$  itself is simple, and it is typical to begin by normalizing the linear part by choosing  $J$  to be in Jordan normal form.

Since the linear parts of (3) and (4) are the same, the transformation  $\xi \rightarrow \eta$  can be assumed – to lowest order – to be the identity,

$$\eta = \psi(\xi) = \xi + h(\xi). \quad (5)$$

Though the “simplest” form of a flow may only be topologically conjugate to the original system (for example, this is what the Hartman–Grobman theorem provides when  $J$  is hyperbolic), it is not possible to explicitly construct  $w$  unless we assume that both the vector fields and the transformation are smooth. When  $\psi$  is a diffeomorphism then (5) implies that the vector fields of (3) and (4) are related by

$$\begin{aligned} D\psi(\xi)v(\xi) &= w(\psi(\xi)) \Rightarrow \\ b(\xi) + Dh(\xi)J\xi + Dh(\xi)b(\xi) &= Jh(\xi) + c(\xi + h(\xi)). \end{aligned}$$

Under the assumption that  $v, w, h \in C^\infty$ , the normal form can be computed by power series expansion. One way to accomplish this is to transform the terms of each degree successively. When this normalization has been carried out for all terms through degree  $d - 1$ , i.e.,  $v = w + O(d)$ , we let  $h(\xi) = O(d)$  in (5); these degree  $d$  terms must then solve the “homological equation”

$$\mathcal{L}_J(h)(\xi) = c(\xi) - b(\xi) + O(d + 1). \quad (6)$$

Here  $\mathcal{L}_J$  is the homological operator defined by

$$\mathcal{L}_J = \text{ad}_J \equiv [J\xi, \cdot], \quad (7)$$

where  $[\cdot, \cdot]$  is the Lie bracket of vector fields:

$$\begin{aligned} \text{ad}_J h(\xi) &= (J\xi \cdot \nabla)h(\xi) - (h(\xi) \cdot \nabla)J\xi \\ &= Dh(\xi)J\xi - Jh(\xi) \\ &= \mathcal{D}_J h(\xi) - Jh(\xi), \end{aligned} \quad (8)$$

where we introduce the linear operator  $\mathcal{D}_J \equiv J\xi \cdot \nabla$ . The problem of constructing  $h$  at degree  $d$  then reduces to linear algebra on the finite-dimensional space,  $F_d^n$ , of  $n$ -dimensional vectors of polynomials in  $n$  variables of homogeneous degree  $d$ . In this case, a solution  $h \in F_d^n$  to (7) exists providing  $c - b \in \text{rng } \mathcal{L}_J \cap F_d^n$ . If the operator  $\mathcal{L}_J$  were surjective on  $F_d^n$ , then we could set  $c = 0$  to eliminate all nonlinear terms. However in cases of interest, the homological operator typically has a nontrivial kernel, and  $c$  must be chosen to eliminate any terms in  $b$  that are not in  $\text{rng } \mathcal{L}_J$ .

Recall that any matrix  $J = S + N$  has a unique splitting into commuting matrices such that  $S$  is semisimple and  $N$  is nilpotent. Moreover,  $\ker J = \ker S \cap \ker N$ . It can be similarly shown that  $\mathcal{L}_J = \mathcal{L}_S + \mathcal{L}_N$  is a semisimple–nilpotent splitting of the homological operator on  $F_d^n$ . The construction of a normal form requires the selection of a complement to  $\text{rng } \mathcal{L}_J$ ; this can be chosen to be the intersection of complements to the ranges of the semisimple and nilpotent parts.

For the semisimple case, it is easy to find a basis for  $F_d^n$  in which  $\mathcal{L}_S$  is also diagonal (the vector monomials, see Section 2). In this case  $\ker \mathcal{L}_S$  is a complement to  $\text{rng } \mathcal{L}_S$ , so that  $c$  can be chosen to be the terms in  $b$  that correspond to eigenfunctions of the homological operator with

zero eigenvalue. In this way the construction for the semisimple case reduces to characterizing  $\ker \mathcal{L}_S$ .

The construction of a complement to the range of the nilpotent homological operator for a nilpotent matrix  $N$  is not as easy. There are two commonly used methods:

- Given any scalar product  $\langle \cdot, \cdot \rangle$  on the space of polynomial vector fields, then an orthogonal complement to the range is  $\text{coker } \mathcal{L}_N = \ker(\mathcal{L}_N)^*$ , the kernel of the adjoint. A nice choice of scalar product, generalizing the Frobenius inner product of matrices, leads to  $(\mathcal{L}_N)^* = \mathcal{L}_{N^*}$  [2,11]. This is called inner-product style in [25].
- From the representation theory of  $\mathfrak{sl}(2)$  and the Jacobson–Morozov embedding theorem, for any nilpotent  $N$  there are matrices  $M$  and  $T$  that together give a representation of  $\mathfrak{sl}(2)$ . It then follows that  $\ker \mathcal{L}_M$  is a complement to the range of  $\mathcal{L}_N$  [7,9]. This is called  $\mathfrak{sl}(2)$  style in [25].

In general these approaches lead to different complements. Unfortunately, neither of these complements typically has the “simplest” form. For example, one could declare the form with the minimal number of nonlinear terms at each degree to be simplest. Murdock has discussed a separate procedure that can be appended to either style to simplify the normal form in this way [25]. Below we will use the term *simplified normal form* to refer to the specific result of Murdock’s procedure.

In this paper we study an incompressible vector field in three dimensions with a multiplicity-three eigenvalue  $\lambda = 0$  so that  $J = 0 + N$ . Generically  $\lambda$  has geometric multiplicity one, so  $N$  has a single Jordan block. We will show that the complement to  $\text{rng } \mathcal{L}$  can be selected so that the normal form is particularly simple:

**Theorem 1.** Consider a smooth vector field (3) on  $\mathbb{R}^3$ ,  $\xi = (x, y, z)^T \in \mathbb{R}^3$ , with vanishing divergence, (1) and linear part

$$Dv(0) = N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (9)$$

The vector field  $v$  can be transformed by a volume-preserving, near-identity transformation (5) into the form

$$N\xi + \hat{e}_3 \rho(x, y), \quad \hat{e}_3 = (0, 0, 1)^T \quad (10)$$

up to terms of arbitrary degree. Since the divergence-free condition is linear, the truncation of the normal form at any degree is also divergence free.

The simple complement proposed here extends Murdock’s results to the case of divergence-free vector fields. For the particular  $N$  we are considering, the result for general vector fields was obtained in [11]. Their results state that the nonlinear part of the transformed vector field has the form  $\hat{e}_3(z\varphi_1 + y\varphi_2 + \varphi_3)$  where  $\varphi_i = \varphi_i(x, y^2 - 2xz)$  are arbitrary polynomials. We will reconstruct this result in Section 3, since we need an intermediate form from this construction to obtain Theorem 1. Indeed, as we explain in Section 4, it is not possible to simply impose zero divergence on this simplified form. In other words, the operations of imposing zero divergence and computing the simplified normal form do not commute.

A similar theorem holds for dimension two, i.e., the divergence-free case of the Takens–Bogdanov bifurcation. Generalization to four or more dimensions is not as straightforward, as we discuss in Section 7.

Normal forms for maps can be found by analogous means. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a smooth diffeomorphism

$$f(\xi) = J(\xi + b(\xi)), \quad (11)$$

where  $b = O(2)$  represents the nonlinear terms. A formal conjugacy (5)  $\psi \circ f = g \circ \psi$  to a normal form  $g(\eta) = J(\eta + c(\eta))$  can be found if we can solve the homological equation  $L_J h(\xi) = c(\xi) - b(\xi)$  at each degree for  $h$ , where

$$L_J \equiv \text{Ad}_{J^{-1}} - \text{id} \quad (12)$$

and  $\text{Ad}_J h(\xi) \equiv Jh(J^{-1}\xi)$ . The terms  $c$  in the normal form should again be selected to be in a complement to the range of the mapping homological operator  $L$ . Construction of this complement depends upon the properties of  $J$ .

Here we consider the case of volume-preserving maps that have a fixed point with multipliers  $\lambda = 1$  of multiplicity three. The mapping analogue of Theorem 1 is

**Theorem 2.** Consider a smooth, volume-preserving diffeomorphism (11) on  $\mathbb{R}^3$  with the linear part

$$Df(0) = J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I} + N. \quad (13)$$

By a volume-preserving, near-identity transformation the map  $f$  can be put into the form

$$g(\xi) = J(\xi + \hat{e}_3 \rho(x, y)), \quad \hat{e}_3 = (0, 0, 1)^T \quad (14)$$

up to terms of arbitrary high degree. The truncation of the normal form at any degree is exactly volume preserving with a polynomial inverse of the same degree as the polynomial  $\rho$ .

Note that the last statement, that the truncation of the map is volume preserving, is nontrivial since this is a nonlinear condition, while the analogous statement in Theorem 1 is trivial because the divergence-free condition is linear.

The essential step for the transition from vector fields to maps is to use the exponential of the vector field and the exponential of  $\text{ad}$  instead of  $\text{Ad}$ . However, since the simplified normal form style is basis-dependent, additional coordinate transformations are needed to transfer the result. We are motivated by the approach of [1].

These results can be extended to the (formal) unfolding of these bifurcations, as stated in

**Theorem 3.** The unfoldings of the vector field of Theorem 1 and the map of Theorem 2 are obtained by replacing the polynomial  $\rho(x, y)$  in (10) and (14) by

$$\epsilon + \mu_1 x + \mu_2 y + \rho(x, y; \epsilon, \mu_1, \mu_2),$$

where the lowest-order terms of  $\rho$  in  $x$  and  $y$  are quadratic.

## 2. Polynomial vector fields

Normal form theory uses formal power series of vector fields. In this section we introduce our notation for these vector fields and give some of the results about the polynomial subspaces and their bases that we will use.

For each point  $\xi \in \mathbb{R}^n$  and each  $m \in \mathbb{N}^n$ , let  $\xi^m = \xi_1^{m_1} \xi_2^{m_2} \dots \xi_n^{m_n}$  denote a scalar monomial with degree  $|m| = \sum_{i=1}^n m_i$ . Since the homological operators  $\mathcal{L}$  (7) and  $\mathcal{L}$  (12) preserve the degree, one can restrict to the finite-dimensional spaces of polynomial vector fields with fixed degree.

Let  $P^n$  be the space of formal (without regard to convergence) power series in  $n$  variables and  $P_d^n \subset P^n$  be the polynomials with homogeneous degree  $d$ , i.e., if  $\theta \in P_d^n$ , then  $\theta(s\xi) = s^d \theta(\xi)$  for all  $s \in \mathbb{R}$ . One basis for  $P_d^n$  is the set of all monomials  $\{\xi^m : |m| = d\}$ . Consequently the dimension of  $P_d^n$  is

$$\dim P_d^n = \binom{n+d-1}{d} = \frac{(n+d-1)!}{(n-1)!d!}.$$

The space of vector fields in  $\mathbb{R}^n$  with components in  $P^n$  is denoted  $F^n$  and its degree  $d$  subspace is denoted

$$F_d^n = \{h(\xi) : h_i(\xi) \in P_d^n, i = 1, \dots, n\}. \quad (15)$$

The space  $F_d^n$  is a vector space with basis given by all “vector monomials,”

$$p_{i,m} = \xi^m \hat{e}_i = (0, \dots, 0, \xi^m, 0, \dots, 0)^T, \quad |m| = d, \quad 1 \leq i \leq n, \quad (16)$$

consequently,

$$\dim F_d^n = n \dim P_d^n.$$

For example,  $\dim P_d^2 = d+1$  and  $\dim F_d^3 = 3(d+2)(d+1)/2$ . Many of the operators we consider that act on  $F^n$  have  $F_d^n$  as invariant subspaces, i.e., they preserve the degree.

The subspace of divergence-free vector fields of degree  $d$  is denoted by  $V_d^n$ :

$$V_d^n = \{v \in F_d^n : \nabla \cdot v = 0\}. \quad (17)$$

Since  $\nabla \cdot$  is a map from  $F_d^n$  onto  $P_{d-1}^n$ , the condition of vanishing divergence lowers the dimension by  $\dim P_{d-1}^n$ , and hence

$$\dim V_d^n = \dim F_d^n - \dim P_{d-1}^n. \quad (18)$$

A basis for  $V_d^n$  can be constructed as follows:

**Lemma 4.** *A basis of  $V_d^n$  consists of  $n \dim P_{d-1}^{n-1}$  basis vectors  $p_{i,\tilde{m}}$  where the  $\tilde{m}$  indicates that  $\tilde{m}_i = 0$ ,  $|\tilde{m}| = d$ , and  $(n-1) \dim P_{d-1}^n$  basis vectors of the form*

$$v_{i,m} = \xi^m [(1+m_{i+1})\xi_i \hat{e}_i - (1+m_i)\xi_{i+1} \hat{e}_{i+1}], \quad |m| = d-1, \quad (19)$$

with  $1 \leq i \leq n-1$ .

**Proof.** The divergence of the basis vector  $p_{i,m} \in F_d^n$  is

$$\nabla \cdot p_{i,m} = \partial_i \xi^m = m_i \xi_1^{m_1} \dots \xi_i^{m_i-1} \dots \xi_n^{m_n},$$

and this vanishes only if  $m_i = 0$ . Consequently, letting  $\check{F}_d^n = \text{span}\{p_{i,\tilde{m}} : \tilde{m}_i = 0, |m| = d\}$ , and  $\tilde{F}_d^n = \text{span}\{p_{i,\tilde{m}} : \tilde{m}_i \neq 0, |m| = d\}$ , we can write  $F_d^n = \check{F}_d^n \oplus \tilde{F}_d^n$ . Note that  $\check{F}_d^n \subset V_d^n$ , and that its dimension is  $n \dim P_{d-1}^{n-1}$  since the  $i$ th component of a vector in this space is an arbitrary polynomial depending on all  $n-1$  variables except  $\xi_i$ .

To find the remaining basis vectors for  $V_d^n$ , we need to find the divergence-free subset of  $\tilde{F}_d^n$ . For any vector in the latter space there must be some  $i$  such that its  $i$ th component depends upon  $\xi_i$ ; consequently, an alternative basis is the set of vectors of the form

$$\tilde{v}_{a,m} = \xi^m \begin{pmatrix} a_1 \xi_1 \\ \vdots \\ a_n \xi_n \end{pmatrix},$$

where  $|m| = d-1$  and the vectors  $a$  are chosen to give any basis for  $\mathbb{R}^n$ . Such a vector has zero divergence if

$$\begin{aligned} \nabla \cdot \tilde{v}_{a,m} &= \sum_{i=1}^n \partial_i (\xi^m a_i \xi_i) = \sum_{i=1}^n a_i (\xi_i \partial_i \xi^m + \xi^m) \\ &= \xi^m \sum_{i=1}^n a_i (m_i + 1) = 0. \end{aligned}$$

This equation simply states that the vector  $a$  must be orthogonal to the vector  $(1+m_i)_{i=1,\dots,n}$ . For a fixed  $m$ , there are  $n-1$  such vectors, hence there are  $(n-1) \dim P_{d-1}^n$  such vector fields. The independent solutions can be chosen such that only two  $a_i$ s are nonzero, as in (19). These vectors are independent of each other because the monomials  $\xi^m$  are distinct.

Finally, note that the dimensions of the two subsets add up:

$$\begin{aligned} \dim V_d^n &= \dim F_d^n - \dim P_{d-1}^n \\ &= n \dim P_{d-1}^{n-1} + (n-1) \dim P_{d-1}^n. \quad \square \end{aligned}$$

There is a natural inner product on the space of polynomial vector fields that generalizes the Frobenius inner product [2,11]. For each  $p, q \in F_d^n$ , define

$$\langle p, q \rangle \equiv p(\partial_\xi) \cdot q(\xi)|_{\xi=0}, \quad (20)$$

where “ $\cdot$ ” denotes the Euclidean scalar product and  $p(\partial_\xi)$  is the differential operator with each occurrence of  $\xi_i$  in  $p(\xi)$  replaced by the derivative  $\frac{\partial}{\partial \xi_i}$ . For example, for two vector monomials (16),

$$\langle p_{i,m}, p_{j,\tilde{m}} \rangle = m! \delta_{m,\tilde{m}} \delta_{i,j},$$

where  $m! \equiv m_1! m_2! \dots m_n!$ . It is easy to see that (20) satisfies the requirements for an inner product:  $\langle p, q \rangle = \langle q, p \rangle$ , and  $\langle p, p \rangle > 0$  when  $p \neq 0$ . For linear vector fields,  $d=1$ , this inner product reduces to

$$\langle A\xi, B\xi \rangle = \sum_{i,j=1}^n A_{ij} B_{ij},$$

the inner product that defines the Frobenius norm for matrices.

Using the inner product, the two subsets of basis vectors for  $V_d^n$  are orthogonal:

**Lemma 5.** *The basis vectors  $p_{i,\tilde{m}}, v_{j,\tilde{m}}$  of  $V_d^n$  are orthogonal for any  $i, j \in [1, n]$  and  $m, \tilde{m} \in \mathbb{N}^d$ .*

**Proof.** This follows because  $p_{i,\tilde{m}}$  is missing  $\xi_i$  in the  $i$ th component, but  $v_{j,\tilde{m}}$  always has  $\xi_i$  in component  $i$  if that component is nonvanishing.  $\square$

We define one additional subspace of  $F_d^n$ :

$$U_d^n = \{\theta(\xi)\xi : \theta(\xi) \in P_{d-1}^n\}. \quad (21)$$

A basis for  $U_d^n$  is given by the set vector fields

$$u_m = \xi^m \sum_{i=1}^n \xi_i \hat{e}_i = \xi^m \xi, \quad |m| = d-1.$$

For example,

$$U_2^3 = \text{span} \left\{ \begin{pmatrix} x^2 \\ xy \\ xz \end{pmatrix}, \begin{pmatrix} xy \\ y^2 \\ yz \end{pmatrix}, \begin{pmatrix} xz \\ yz \\ z^2 \end{pmatrix} \right\}. \quad (22)$$

Note that each basis vector in  $U_d^n$  has nonvanishing divergence  $\nabla \cdot u_m = (n+d-1)\xi^m$ ; indeed, this space is the complement of the divergence-free space.

**Lemma 6.** *For the inner product (20), the spaces  $U_d^n$  and  $V_d^n$  are orthogonal complements in  $F_d^n$ .*

**Proof.** For any  $v \in V_d^n$  and  $u = \theta(\xi)\xi \in U_d^n$  we have

$$\langle \theta(\xi)\xi, v \rangle = \langle \theta, \nabla \cdot v \rangle = 0.$$

Thus  $U_d^n$  and  $V_d^n$  are orthogonal. That they are complementary spaces simply follows from the observation that their dimensions add to that of  $F_d^n$ . By (21), the dimension of  $U_d^n$  is the same as that of degree  $d-1$  polynomials,

$$\dim U_d^n = \dim P_{d-1}^n.$$

Therefore, (18) gives  $\dim F_d^n = \dim V_d^n + \dim U_d^n$ .  $\square$

**Remark 1.** The orthogonal decomposition of  $F_d^n$  seems reminiscent of the Hodge–de Rham decomposition of vector fields—when  $n=3$ , this is the Helmholtz decomposition,  $h = \nabla \times \psi + \nabla \phi$ . However, the decomposition  $F_d^3 = V_d^3 \oplus U_d^3$  is different—in particular the curl of a vector  $\theta\xi$  in  $U_d^3$  does not vanish in general. The difference is that for the Helmholtz decomposition, the scalar product between vector fields is defined by the integral of the Euclidean scalar product (in  $\mathbb{R}^3$ ). Nevertheless, the dimension of the space of gradient vector fields that are not harmonic vector fields is  $\dim P_{d+1}^3 - (2(d+1) + 1) = \dim P_{d-1}^3$ , the same dimension as  $U_d^3$ .

### 3. Normal form for vector fields

In this section we will use the inner-product style to compute the complement to  $\text{rng } \mathcal{L}$ .<sup>2</sup> This will give the formal normal

form for a vector field with a triple-zero eigenvalue. The results here reproduce those of [11]. We will use an intermediate form of this result in the next section to project onto the divergence-free case and complete the proof of Theorem 1.

The main advantage of the inner product (20) is that it allows a simple construction of the adjoint of  $\mathcal{L}_N = \text{ad}_N$  on the space of polynomial vector fields [2,11]. Along the way, we prove the analogous result for the operator  $\text{Ad}_J$  that will be used for the map case.

**Lemma 7.** *Using the inner product (20),  $(\text{Ad}_J)^* = \text{Ad}_{J^*}$  and  $(\text{ad}_N)^* = \text{ad}_{N^*}$ .*

**Proof.** From the definition (20)

$$\langle p, \text{Ad}_J q \rangle = p(\partial_\xi) \cdot J q(J^{-1}\xi)|_{\xi=0}.$$

Change coordinates to  $\eta = J^{-1}\xi$ , noting that  $\partial_\xi = J^{-1*}\partial_\eta$  to obtain

$$\begin{aligned} \langle p, \text{Ad}_J q \rangle &= (J^* p(J^{-1*}\partial_\eta)) \cdot q(\eta)|_{\eta=0} \\ &= q(\partial_\eta) \cdot (J^* p(J^{-1*}\eta))|_{\eta=0} = \langle q, \text{Ad}_{J^*} p \rangle. \end{aligned} \quad (23)$$

The analogous statement for  $\text{ad}_N$  follows from  $\text{Ad}_{e^{tN}} = e^{t\text{ad}_N}$  and  $(e^{tN})^* = e^{tN^*}$ . Setting  $J = e^{tN}$  in (23) and differentiating with respect to  $t$  at  $t=0$  gives the result.  $\square$

This lemma implies that  $\ker \text{ad}_{N^*} = \text{coker } \text{ad}_N$  is the orthogonal complement to the range of  $\mathcal{L}_N$ . Using (8), the cokernel of  $\text{ad}_N$  is therefore determined by the solutions of

$$\text{ad}_{N^*} h = \mathcal{D}_{N^*} h - N^* h = 0. \quad (24)$$

Here we solve (24) for the  $n$ -dimensional generalization of (9)

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots \end{pmatrix}, \quad (25)$$

that is,  $N_{i,i+1} = 1$ , and  $N_{ij} = 0$  otherwise. For the adjoint of  $N$ , the linear operator  $\mathcal{D}$  becomes

$$\mathcal{D}_{N^*} = \sum_{i=1}^{n-1} \xi^i \frac{\partial}{\partial \xi^{i+1}},$$

and (24) is equivalent to the system of linear PDEs

$$\mathcal{D}_{N^*} h_1 = 0, \quad \mathcal{D}_{N^*} h_j = h_{j-1}, \quad j = 2, 3, \dots, n. \quad (26)$$

This system is easily solved by the method of characteristics. The solution of the characteristic equations  $\dot{\xi} = N^* \xi$  and  $\dot{h} = N^* h$  is formally easy:

$$\begin{aligned} \xi(t) &= e^{tN^*} \xi^0, \\ h(t) &= e^{tN^*} h^0. \end{aligned} \quad (27)$$

A solution of the PDE is obtained by inverting the equations for  $\xi(t)$  to solve for the invariants  $\xi^0 = e^{-tN^*} \xi$ . However, we must use one of the equations to eliminate time. The easiest way to do this is to use the equation for the second variable since it is

<sup>2</sup> In our case this gives the same result as the  $\mathfrak{sl}(2)$  style.



Table 1

Dimensions of the various (co)kernels for the homological operator  $\text{ad}_N$  with  $N$  in (25) in dimension  $n$  at degree  $d$ 

$\dim(\ker \text{ad}_N)$						$\dim(\ker \text{ad}_N \cap V_d^n)$						$\dim P_d^{n-1}$					
$n \setminus d$	2	3	4	5	6	$n \setminus d$	2	3	4	5	6	$n \setminus d$	2	3	4	5	6
2	2	2	2	2	2	2	1	1	1	1	1	2	1	1	1	1	1
3	4	6	7	9	10	3	3	4	5	6	7	3	3	4	5	6	7
4	7	12	17	24	31	4	6	10	14	19	25	4	6	10	15	21	28
5	11	21	36			5	10	18	31			5	10	20	35		
6	15					6	14					6	15				

**Theorem 1** for  $n = 2, 3$  implies that the first two rows of the middle and right tables coincide for any  $d$ .

linear in  $t$ . Denote the first three coordinates by  $(x, y, z)$  so that  $\xi = (x, y, z, \dots, \xi_n)$ . Then (27) gives  $y(t) = y^0 + tx^0$ . When  $x(t) = x^0 \neq 0$  the characteristics all pass through the surface  $y^0 = 0$ . So up to this singularity we can choose this surface to define the initial conditions. To do this, set  $t = y/x$ , to obtain the invariants

$$\xi^0(x, y, \dots, \xi_n) = e^{-\frac{y}{x} N^*} \xi. \quad (28)$$

These invariants are rational functions with denominator  $x^{k-1}$ . The first invariant is simply  $x$  itself, and the solution of the PDE will depend on arbitrary functions of the invariants. Thus we can clear the denominators in the remaining equations and define  $n - 1$  polynomial invariants

$$\begin{aligned} \alpha &= \xi_1^0 = x, \\ \beta &= 2x\xi_3^0 = 2zx - y^2, \\ \gamma &= 3x^2\xi_4^0 = 3x^2w - 3xyz + y^3, \end{aligned} \quad (29)$$

and so forth. Therefore, a formal solution to (26) is

$$h(\xi) = e^{(y/x)N^*} \phi(x, \beta, \gamma, \dots), \quad (30)$$

for an arbitrary vector valued function  $\phi$  of the  $n - 1$  invariants. For example, for  $n = 2$  the solution is

$$h = \begin{pmatrix} \phi_1(x) \\ \frac{y}{x} \phi_1(x) + \phi_2(x) \end{pmatrix}, \quad (31)$$

and when  $n = 3$  we obtain

$$h = \begin{pmatrix} \phi_1(x, \beta) \\ \frac{y}{x} \phi_1(x, \beta) + \phi_2(x, \beta) \\ \frac{y^2}{2x^2} \phi_1(x, \beta) + \frac{y}{x} \phi_2(x, \beta) + \phi_3(x, \beta) \end{pmatrix}. \quad (32)$$

The desired solution for  $h$  is in  $F_d^n$ . It is, however, a nontrivial problem to obtain the most general polynomial solution. Even if we assume that the functions  $\phi_i$  are polynomials, the solution (30) is generally rational because it contains powers of  $t = y/x$ . For  $n = 2$  it is clear that  $\phi_1(x)$  must be  $x$  times a polynomial and the solution is simply

$$h = \begin{pmatrix} x\varphi_1 \\ y\varphi_1 + \varphi_2 \end{pmatrix}, \quad (33)$$

where  $\varphi_i$  are polynomials that depend upon  $x$  only.

For  $n = 3$  one might first think that  $\phi_1 = x^2\varphi_1$  is required, since it appears with denominator  $x^2$  in the third component of (32). However, this is not the most general polynomial solution, since we can replace  $y^2$  by  $2zx - \beta$  to eliminate one power of  $x$  in the denominator and then remove the second by setting  $\phi_1 = x\varphi_1$  and  $\phi_3 = \frac{\beta}{2x}\varphi_1 + \varphi_3$ . This changes the third term to  $z\varphi_1 + \frac{y}{x}\phi_2 + \varphi_3$ . Finally setting  $\phi_2 = x\varphi_2$ , we obtain the polynomial solution for  $n = 3$

$$h = \begin{pmatrix} x\varphi_1 \\ y\varphi_1 + x\varphi_2 \\ z\varphi_1 + y\varphi_2 + \varphi_3 \end{pmatrix}, \quad (34)$$

where  $\varphi_i = \varphi_i(x, \beta)$  are polynomials of appropriate degree. This solution was shown to be the general polynomial solution in [8,11].

With these forms it is easy to obtain the dimensions of the kernel of  $\text{ad}_{N^*}$  for  $n = 2$  and 3 giving the entries in the first two rows of the leftmost pane of Table 1.

**Lemma 8.** The dimension of  $\ker \text{ad}_{N^*}$  in  $P_d^2$  is 2 and in  $P_d^3$  is  $\lceil 3d/2 + 1 \rceil$ .

**Proof.** This is a simple counting argument, based on the polynomial forms. For  $n = 2$  at degree  $d$ ,  $h$  contains two monomials,  $\varphi_1 = ax^{d-1}$  and  $\varphi_2 = bx^d$  so the dimension is always two.

For  $n = 3$ , the polynomials  $\varphi_i$  in (34) whose arguments are of degree one and two, respectively, must be chosen appropriately. The even and odd cases can be treated separately: a degree  $2m$  polynomial has the form

$$\sum_{l=0}^m a_l x^{2l} \beta^{m-l} \in P_{2m}^3, \quad (35)$$

and a degree  $2m + 1$  polynomial has the form

$$\sum_{l=0}^m b_l x^{2l+1} \beta^{m-l} \in P_{2m+1}^3. \quad (36)$$

Each of these sums has  $m + 1$  arbitrary coefficients, so the degree  $d$  case has  $\lceil \frac{d+1}{2} \rceil$  coefficients.

To apply this to the form (34), note that when  $h$  is degree  $d$ , then  $\varphi_1$  and  $\varphi_2$  have degree  $d - 1$ . Thus each has  $\lceil \frac{d}{2} \rceil$  coefficients. The function  $\varphi_3$  is degree  $d$ , so it has  $\lceil \frac{d+1}{2} \rceil$  coefficients. Thus total number of coefficients is

$$2 \left\lceil \frac{d}{2} \right\rceil + \left\lceil \frac{d+1}{2} \right\rceil = \left\lceil \frac{3d}{2} + 1 \right\rceil.$$

Since the each monomial in each function represents an independent vector in  $h$ , this is the same as the dimension.  $\square$

**Remark 2.** The construction of a general polynomial solution to (26) when  $n > 3$  is complicated by the fact that there are polynomial combinations of the invariants that have  $\alpha = x$  as a factor. For example the combination  $\beta^3 + \gamma^2 = \alpha^2 \delta$  where  $\delta$  is a quartic polynomial in  $(x, y, z, w)$ . Consequently, the power of  $x$  in the denominator of terms involving these combinations is different than its nominal value. It can be shown that it is the only nontrivial relation in the case  $n = 4$ , and this leads to the polynomial form for this case [8,25]. The case  $n = 5$  was given in [9]. The dimension of the degree  $d$  subspaces is complicated by these relations when  $n > 3$  (see Table 1, for  $n \geq 4$ ). Since the higher-dimensional systems correspond to bifurcations with codimension larger than two, they are also more rare as dynamical systems.

#### 4. Divergence-free vector fields

In this section we complete the proof of Theorem 1. The remaining task is to compute a complement to the range of  $\text{ad}_N$  for divergence-free vector fields. We start by showing that for any matrix  $N$ , the splitting  $F_d^n = V_d^n \oplus U_d^n$  block diagonalizes  $\text{ad}_N$ , that is

$$\text{ad}_N V_d^n \subset V_d^n, \text{ and } \text{ad}_N U_d^n \subset U_d^n. \quad (37)$$

Moreover, since  $(\text{ad}_N)^* = \text{ad}_{N^*}$  this also applies to  $\text{ad}_{N^*}$ .

This block diagonalization allows us to compute  $\text{coker ad}_N$  for divergence-free vector fields by simply projecting the full  $\text{coker}$  onto  $V_d^n$ . This projection can be accomplished by solving an ODE for one of the components of  $h$ . However, the result is not the simplified normal form of Theorem 1. The last step is to show that the simplified form is also a complement to  $\text{rng ad}_N$  by showing that its projection onto  $\text{coker ad}_N$  is of full rank.

We begin by verifying (37) in the next two lemmas.

**Lemma 9.** *The volume-preserving subspace  $V_d^n$  is an invariant subspace of  $\text{ad}_N$  for any matrix  $N$  ( $N$  need not be traceless or nilpotent); in other words,  $\text{ad}_N V_d^n \subset V_d^n$ .*

**Proof.** We will show that  $\nabla \cdot (\mathcal{L}_N v) = \mathcal{D}_N(\nabla \cdot v)$ , so that  $\nabla \cdot v = 0$  implies that  $\nabla \cdot (\text{ad}_N v) = 0$ . Writing the latter in components gives

$$\begin{aligned} \partial_i (\text{ad}_N v)_i &= \partial_i (N_{jk} \xi_k \partial_j v_i - N_{ij} v_j) \\ &= [Dv, N]_{ii} + (N\xi \cdot \nabla) \partial_i v_i, \end{aligned}$$

where  $[\cdot, \cdot]$  is the matrix commutator and we use the summation convention. The first term is simply the trace of the commutator of  $Dv$  and  $N$ , but  $\text{tr}[A, B] = 0$  for any two matrices. The last term is  $\mathcal{D}_N(\nabla \cdot v)$ , which vanishes since  $v \in V_d^n$ .  $\square$

**Remark 3.** Notice that it was *not* necessary to assume  $\text{tr } N = 0$  in this Lemma because the trace of the commutator of any two matrices vanishes. For other Lie algebras, e.g.  $\mathfrak{sp}$  or  $\mathfrak{so}$ , it is necessary to assume that  $N$  is in the algebra to get a block

diagonalization since otherwise the commutator  $[Dv, N]$  is not in the algebra.<sup>3</sup>

This lemma implies that the operator  $\text{ad}_N$  acting on a vector in components  $v + u$  is block upper-triangular. To show that it is block diagonal, we must also show that  $\text{ad}_N$  leaves the orthogonal complement  $U_d^n$  invariant.

**Lemma 10.** *The nonvolume-preserving subspace  $U_d^n$  is an invariant subspace of  $\text{ad}_N$  for any matrix  $N$ . Thus  $\text{ad}_N U_d^n \subset U_d^n$ .*

**Proof.** From (21), the general element of  $U_d^n$  has the form  $\theta(\xi)\xi$  where  $\theta \in P_d^n$  is a scalar polynomial. To apply  $\text{ad}_N$  to such a vector, we must compute the Jacobian  $D(\theta\xi)$ , which has components  $D(\theta\xi)_{ij} = \partial_j(\theta\xi_i) = \xi_i \partial_j \theta + \theta \delta_{ij}$ . Therefore,

$$\begin{aligned} \text{ad}_N \theta \xi &= D(\theta\xi)N\xi - N\theta\xi = \xi(\nabla \theta \cdot N\xi) + \theta N\xi - N\theta\xi \\ &= (\nabla \theta \cdot N\xi)\xi. \end{aligned}$$

The result is again of the form of scalar, degree  $d$  polynomial times  $\xi$ , and therefore is still in  $U_d^n$ .  $\square$

Thus  $\text{coker ad}_N$  can be split into two subspaces

$$\text{coker ad}_N = (\text{coker ad}_N \cap V_d^n) \oplus (\text{coker ad}_N \cap U_d^n),$$

that are invariant under  $\text{ad}_N$ . Consequently for a divergence-free vector field (3), the solution  $h$  to the homological equation (6) can be taken to be itself divergence free, and the normal form can be selected to be in  $\text{coker ad}_N \cap V_d^n$ .

We will only construct these subspaces for the two- and three-dimensional cases.

The two-dimensional case is simple. The general solution of the homological equation in this case was given in (31). Imposing  $\nabla \cdot h = 0$  implies that  $\phi_1(x) = 0$ , so that  $h = \hat{e}_2 \phi_2(x)$ . This implies that  $\dim \text{coker ad}_N \cap V_d^2 = 1$ , as shown in Table 1. Note that the resulting vector field already has the “simplified” form of Theorem 1. In this case, we could also have imposed the volume-preserving condition on the polynomial solution (33) and obtained the same result. Neither of these two statements are true for the three-dimensional case

**Lemma 11.** *The intersection  $(\text{coker ad}_N) \cap V_d^3$  is the set of vector fields of the form*

$$h = \begin{pmatrix} x^2 \psi_1 \\ xy \psi_1 + x \psi_2 \\ \frac{1}{2} y^2 \psi_1 + y \psi_2 + \psi_3 \end{pmatrix}, \quad (38)$$

where  $\psi_i = \psi_i(x, \beta) \in P_d^3$  and  $\psi_3$  satisfies

$$-2\partial_\beta \psi_3 = x \partial_x \psi_1 + 3\psi_1 + \beta \partial_\beta \psi_1. \quad (39)$$

The dimension of the cokernel of  $\text{ad}_N$  restricted to  $V_d^3$  is  $d + 1$  as in Table 1.

<sup>3</sup> Preserving symmetry in the normal form is different. There a matrix group  $G$  exists so that  $[G, N] = 0$ . This is replacing the condition on the differential of  $K$  and on  $N$ . When  $G$  is unitary this implies  $[G, N^*] = 0$ . However, there is no associated invariant subspace, instead we find  $[\text{Ad}_G, \text{ad}_N] = 0$  and  $[\text{Ad}_G, \text{ad}_{N^*}] = 0$ , which means that the cokernel is invariant under the action of  $G$ . This implies that the normal form has the same symmetry group. However, this does not mean that there is a subspace invariant under  $\text{ad}_N$ .

**Proof.** For arbitrary functions  $\phi_i$ , setting the divergence of (32) to zero yields

$$-2\partial_\beta\phi_3 = \frac{1}{x}\partial_x\phi_1 + \frac{1}{x^2}(\phi_1 + \beta\partial_\beta\phi_1).$$

For any given  $\phi_1$  and  $\phi_2$ , this ODE for the dependence  $\phi_3$  on  $\beta$  can be integrated with respect to  $\beta$ . In order that  $\phi_3$  be a polynomial we require  $\phi_1 = x^2\psi_1$ . Similarly we need to set  $\phi_2 = x\psi_2$  in order to make the 2nd component of  $h$  polynomial.

The polynomial  $\psi_1$  has degree  $d-2$ , so by (35) and (36) it has  $\lceil \frac{d-1}{2} \rceil$  coefficients. Similarly  $\psi_2$  is of degree  $d-1$  and has  $\lceil \frac{d}{2} \rceil$  coefficients. For given  $\psi_1$  and  $\psi_2$ ,  $\psi_3$  is determined up to an arbitrary function of  $x$ , which introduces a single term  $cx^d$  at fixed degree  $d$ . Thus the total number of coefficients is

$$\left\lceil \frac{d-1}{2} \right\rceil + \left\lceil \frac{d}{2} \right\rceil + 1 = d+1. \quad \square$$

For completeness, we also can obtain a representation for the nonvolume-preserving block.

**Lemma 12.** *The intersection  $(\text{coker ad}_N) \cap U_d^3$  is the set of vector fields  $\{\theta\xi : \theta(x, \beta) \in P_{d-1}^3, \beta = 2zx - y^2\}$ . This space has dimension  $\lceil \frac{d}{2} \rceil$ .*

**Proof.** From Lemma 10 we see that we need to solve the equation

$$\text{ad}_{N^*}\theta\xi = (\nabla\theta \cdot N^*\xi) = \mathcal{D}_{N^*}\theta = 0;$$

which is the same as the first PDE in the system (26). The method of characteristics, as in Section 3 implies that the general solution is an arbitrary function of the invariants (28) as before. When  $n=3$  the invariants  $x$  and  $\beta$  have no combinations that have  $x^k$  as a factor, so the general polynomial solution is

$$u = \theta(x, \beta)\xi,$$

for a polynomial function  $\theta$ . Since  $\theta$  has degree  $d-1$ , (35) and (36), imply that it has  $\lceil \frac{d}{2} \rceil$  coefficients.  $\square$

Lemma 11 shows that the dimension of  $\ker \text{ad}_N^* \cap V_d^3$  is the same as that of the space of a single polynomial in two variables (given in the right pane of Table 1). This makes it plausible that the divergence-free vector fields of the form  $h = (0, 0, \rho(x, y))^T$  form a possible complement of the range of  $\text{ad}_N$ . This is proved in the next lemma.

The *simplified normal form* was introduced by Murdock for the general case [25]. He argued that a simplified complement to the range of  $\text{ad}_N$  has the form of (34) but with  $h_1 = h_2 = 0$ . We show next that a similar projection can be done in the divergence-free case.

**Lemma 13.** *The set  $\{\hat{e}_3\rho(x, y) : \rho \in P_d^2\}$  is a complement to  $\text{rng ad}_N \cap V_d^3$ .*

**Proof.** We need to demonstrate that the orthogonal projection of the new set to the volume-preserving subspace  $\text{coker ad}_N \cap V_d^n$  is one-to-one. In other words, the matrix of inner products of bases for the two spaces is nonsingular.

Consider bases for the two subspaces. A basis for the simplified complement are the  $d+1$  vector monomials  $\{\hat{e}_3x^ky^{d-k}\}$  where  $k=0, 1, \dots, d$ .

The basis for  $\text{coker ad}_N \cap V_d^3$  given in (38) can be constructed by using monomials for the functions  $\psi_1$  and  $\psi_2$ . For example when  $d=2m$  is even, the monomial basis elements correspond to  $\psi_1 = x^{2j-2}\beta^{m-j}$  and  $\psi_2 = x^{2j-1}\beta^{m-j}$ , for  $j=1, 2, \dots, m$ . The final basis element corresponds to the free function obtained from integrating (39),  $\psi_3 = x^{2m}$ . The  $d+1$  basis vectors are then

$$h_{1,j} = \begin{pmatrix} x^{2j}\beta^{m-j} \\ x^{2j-1}y\beta^{m-j} \\ \frac{1}{2}\left(y^2 - \frac{m+j+1}{m-j+1}\beta\right)x^{2j-2}\beta^{m-j} \end{pmatrix},$$

$$h_{2,j} = \begin{pmatrix} 0 \\ x^{2j}\beta^{m-j} \\ x^{2j-1}y\beta^{m-j} \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 \\ 0 \\ x^{2m} \end{pmatrix}.$$

The inner product of a general basis vector with  $h_{1,j}$  gives

$$\begin{aligned} \langle \hat{e}_3x^ky^{2m-k}, h_{1,j} \rangle &= \left\langle x^ky^{2m-k}, \frac{1}{2}\left(y^2 - \frac{m+j+1}{m-j+1}\beta\right)x^{2j-2}\beta^{m-j} \right\rangle \\ &= -\frac{m+1}{m-j+1} \langle x^ky^{2m-k}, x^{2j-2}(-y^2)^{m-j+1} \rangle. \end{aligned}$$

This is clearly nonzero only for  $k=2j-2$ , that is when  $k=0, 2, \dots, d-2$ . The second set of functions  $h_{2,j}$  have inner products with the simplified basis given by

$$\begin{aligned} \langle \hat{e}_3x^ky^{2m-k}, h_{2,j} \rangle &= \langle x^ky^{2m-k}, x^{2j-1}\beta^{m-j} \rangle \\ &= \langle x^ky^{2m-k}, x^{2j-1}(-y^2)^{m-j} \rangle. \end{aligned}$$

which is nonzero only for  $k=2j-1$ , or  $k=1, 3, \dots, d-1$ . The final inner product  $\langle \hat{e}_3x^ky^{2m-k}, h_3 \rangle$  is nonzero when  $k=2m$ . Hence up to a permutation the matrix of scalar products between the basis vectors is diagonal and thus has full rank. Thus the orthogonal projection of the set  $\hat{e}_3\rho(x, y)$  to  $\text{coker ad}_N \cap V_d^3$  is one-to-one. A similar calculation pertains for the case that  $d=2m+1$  is odd; in this case the monomials in  $\psi_1$  cover the odd powers of  $x$  and the monomials in  $\psi_2$  cover the even powers.  $\square$

We have now proved Theorem 1.

**Remark 4.** Note that while Murdock's simplified normal form (obtained by setting  $h_1 = h_2 = 0$  in (34))

$$h(x, y, z) = \hat{e}_3(z\varphi_1(x, \beta) + y\varphi_2(x, \beta) + \varphi_3(x, \beta)), \quad (40)$$

does form a complement to  $\text{rng ad}_N$  for the general, nonvolume-preserving case, we cannot simply project the simplified normal form onto the volume-preserving subspace. Indeed, the only degree  $d$ , volume-preserving elements for the simplified normal form are

$$h(x, y, z) = \hat{e}_3(ay^2x^{d-1} + byx^{d-1} + cx^d),$$

giving a dimension of three, which is too small (except for  $d=2$ ). The point is that the transformation from inner product



form to simplified normal form is equivalent to changing the scalar product. The new scalar product is defined by declaring the range of  $\text{ad}_N$  and the proposed complement to be orthogonal. Assuming that the dimensions are right this is always possible. However, the new scalar product need not respect the divergence-free conditions. More precisely, when the adjoint of  $\text{ad}_N$  is defined with respect to this scalar product, then  $V_d^n$  and  $U_d^n$  need not be invariant subspaces.

**Remark 5.** We have shown that  $\text{ad}_N$  can be block diagonalized into the divergence-free subspace and its orthogonal complement. The basis given in Section 2 achieves this block diagonalization by Lemma 6. Thus it is interesting to compare the different representations of the complement of the kernel. We have just proved that another “simplified” complement is

$$\tilde{h} = \hat{e}_3 \rho(x, y) + \theta(x, \beta) \xi,$$

which is different from (40). Nevertheless, the projection argument works in this case as well.

## 5. Volume-preserving maps

The normal form that we have just obtained for divergence-free vector fields can essentially be transferred to volume-preserving mappings by exponentiating  $\text{ad}_N$  to obtain the homological operator for (12). In this section we adapt the results of Bridges and Cushman [1] for Hamiltonian vector fields to divergence-free vector fields. The main difference is that in the volume-preserving case there is no scalar generating function.

There are three basic steps to transferring Theorem 1 to the mapping result Theorem 2. First we note that the mapping homological operator  $L$  has the same adjoint properties as the flow operator  $\mathcal{L}$  with respect to the inner product (20). Then we use the fact that  $\text{Ad}$  is the exponential of  $\text{ad}$  to relate the fundamental spaces of these operators. Finally we show that it is possible to choose coordinates in which the linear form of the map is the exponential of the nilpotent block  $N$  for the flow.

As for the vector-field case, we will use the inner-product style. Recall that Lemma 7 implies that  $(\text{Ad}_J)^* = \text{Ad}_{J^*}$  using the inner product (20). Thus the mapping homological operator (12) satisfies

$$(L_J)^* = (\text{Ad}_{J^{-1}} - \mathbb{I})^* = \text{Ad}_{J^{-1*}} - \mathbb{I} = L_{J^*}.$$

Consequently, the construction of the normal form for the mapping (11) reduces to finding a representation for  $\ker L_{J^*}$ . Recall that  $\text{Ad}_{e^N} = e^{\text{ad}_N}$ , thus there is a relation between the homological operators:

$$L_J = \text{Ad}_{J^{-1}} - \mathbb{I} = \exp(\text{ad}_{-\log J}) - \mathbb{I},$$

providing  $J$  has a (real) logarithm. This is true for (14) since none of the eigenvalues of  $J$  are negative. We will thus consider  $\text{ad}_{-\log J}$  as the appropriate homological operator instead of the more difficult homological operator for maps.

To use the relation between  $\text{Ad}$  and  $\text{ad}$ , we need to relate their fundamental spaces. The basic lemma relates the spaces for any nilpotent operator. This crucial step was inspired by Bridges and Cushman [1].

**Lemma 14.** *If  $N$  is a nilpotent matrix then  $M = \exp N - \mathbb{I}$  and  $N$  have the same kernel and range.*

**Proof.** Note that  $M = NB = BN$  where  $B = (\mathbb{I} + \frac{1}{2}N + \dots + \frac{1}{m!}N^{m-1})$  is a polynomial in  $N$  because  $N$  is nilpotent. We claim that  $B$  is nonsingular. Indeed when  $N$  is in Jordan form, then  $B$  is upper triangular and has a diagonal of all ones. Now, if  $w \in \text{rng } M$  then there is a  $v$  such that  $w = Mv = NBv$ , so  $w \in \text{rng } N$ . Moreover, if  $w \in \text{rng } N$  then  $w = Nv = (NB)B^{-1}v = MB^{-1}v$  so  $w \in \text{rng } M$ . Thus  $\text{rng } M = \text{rng } N$ .

Similarly if  $z \in \ker M$  then  $0 = B^{-1}Mz = Nz$  and if  $z \in \ker N$  then  $0 = BNz = Mz$ . So  $\ker M = \ker N$ .  $\square$

In order to apply the previous lemma to  $\text{ad}_N$  we need the following well-known fact about adjoint representation of nilpotent matrices.

**Lemma 15.** *If  $N$  is nilpotent, then  $\text{ad}_N$  is nilpotent.*

**Proof.** This follows from the fact that there exists a triad  $N, M, K$  that form a representation for  $\mathfrak{sl}(2)$ , i.e., that  $[N, M] = K$ ,  $[K, N] = 2N$ , and  $[K, M] = -2M$  and for any such representation, the matrices  $N$  and  $M$  are nilpotent and  $K$  is semisimple. Finally the operators  $\text{ad}_N$ ,  $\text{ad}_M$  and  $\text{ad}_K$  also form such a representation. This proof is along the lines of Murdock [25] Thm 2.5.2, but it is a general theorem, see e.g. [31].  $\square$

The last step fixes the problem that Theorem 1 is stated only for nilpotent Jordan blocks  $N$ , but  $\log J = \log(\mathbb{I} + N)$  is not of this form. The remedy is to first do a linear change of coordinates that puts the volume-preserving map into a form where its linear part is  $\exp N$  instead of  $\mathbb{I} + N$ . Then Theorem 2 connects to Theorem 1. The transformation must be chosen so that it does not destroy the simplified form of the map, in which all the nonlinearity is concentrated in the last component.

**Lemma 16.** *Suppose  $N$  is a nilpotent matrix, then there is an invertible matrix  $T$  such that  $T \exp NT^{-1} = \mathbb{I} + N$ . Moreover the conjugacy can be chosen such that  $T$  leaves  $\hat{e}_n$  invariant.*

**Proof.** While this lemma can be proved by induction for arbitrary dimensions, since we will apply it only for  $n = 2$  and  $n = 3$ , we simply give the matrices for these cases. For  $n = 2$  there is nothing to do since  $N_2^2 = 0$ , so that  $\exp N_2 = \mathbb{I} + N_2$ . For the  $3 \times 3$  case we can see that

$$\exp N_3 = \mathbb{I} + N_3 + \frac{1}{2}N_3^2 = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have  $T_3 \exp N_3 = (\mathbb{I} + N_3)T_3$  using

$$T_3 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad \square$$

Thus, using the new coordinate system, we can apply Theorem 1 to the map case, giving the promised normal form (14).

To finish the proof of [Theorem 2](#) we only need to verify the claims about the properties of the normal form. Firstly every truncation of the map is *exactly* volume preserving. The determinant of the Jacobian of  $g(\xi) = J(\xi + \hat{e}_3\rho(x, y))$  is the product of  $\det J = 1$  and  $\det(\mathbb{I} + D\hat{e}_3\rho(x, y)) = 1$ , since  $D\hat{e}_3\rho(x, y)$  is lower triangular. Another way of viewing this result is to say that time-one map of the vector field  $\hat{e}_3\rho(x, y)$  is the map  $\xi + \hat{e}_3\rho(x, y)$ . Secondly the inverse of the map can be explicitly computed. Let  $\xi' = g(\xi)$  using (14). Then

$$\begin{aligned}\xi &= g^{-1}(\xi') = J^{-1}\xi' - \hat{e}_3\rho(x, y) \\ &= J^{-1}\xi' - \hat{e}_3\rho(x' - y' + z', y' - z').\end{aligned}$$

This completes the proof of [Theorem 2](#).

## 6. Unfolding

The unfolding of the triple eigenvalue collisions treated above is obtained by considering a family depending on  $\epsilon$  and additional unfolding parameters  $\mu_i$ . The parameter  $\epsilon$  will be used to unfold the saddle-node bifurcation, while the parameters  $\mu_i$  will ensure that an arbitrary spectrum of the linearization can be obtained near the bifurcation  $\epsilon = \mu_i = 0$ . The original map is now considered to depend upon  $\epsilon$ , and is expanded in a power series in  $(\epsilon, \mu)$ . The zero-order terms have been treated in the previous section. Normalization now proceeds order by order in  $(\epsilon, \mu)$ . The main observation (see, e.g. [11]) is that at every order in the parameters the same homological operator is obtained. Thus all the results obtained above hold at each order, and hence instead of the polynomial  $\rho(x, y)$  with constant coefficients now each coefficient becomes a power series in  $(\epsilon, \mu)$ .

The essential new feature is that now we also get a homological equation for terms of degree 0 and 1 (previously these terms were absent by assumption). At degree 0 the homological equation states that the constant vector must be in the kernel of the adjoint of  $N$ . The kernel of  $N^*$  is  $\hat{e}_3$ , which gives the constant term in the unfolding. At degree 1 the homological equation states that the linear terms must commute with the adjoint of  $N$ . The general matrix that commutes with  $N^*$  is lower diagonal banded:

$$\begin{pmatrix} \mu_3 & 0 & 0 \\ \mu_2 & \mu_3 & 0 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}.$$

Since we are dealing with divergence-free vector fields we require  $\mu_3 = 0$ . Now the second component of the vector field would have an entry  $x\mu_2$ . To bring this into our simple form in which only the third component contains nonlinear and unfolding terms we can simply add an element from the range of the homological operator (at degree 1). Thus we can modify this by adding any matrix (times  $\xi$ ) that does not commute with  $N^*$ . Hence we can simply remove  $\hat{e}_2x\mu_2$ , since it is in the range. In this way we find that the third component of the unfolded vector field becomes

$$\epsilon + \mu_1x + \mu_2y + \rho(x, y; \epsilon, \mu_1, \mu_2).$$

Under generic conditions on  $p$  one of the two parameters  $\mu_i$  can be removed. Crossing over from divergence-free vector fields to volume-preserving maps works as before. This proves [Theorem 3](#).

Interestingly the map thus obtained when  $\rho$  is restricted to be a quadratic polynomial is exactly the map that was studied earlier in [22]. In that paper this family of maps was studied because of its algebraic property that it is polynomial with polynomial inverse. We now see that this family of maps is also interesting because it is the unfolding of the volume-preserving saddle-node bifurcation.

## 7. Conclusion

While we have obtained a normal form for three dimensions, our results do not apply to the higher-dimensional case. Indeed, [Table 1](#), which compares the dimension of the kernel of the homological operator in the volume-preserving subspace and the dimension of the set spanned by  $\hat{e}_n p(\xi_1, \dots, \xi_{n-1})$ , shows that these two dimensions are no longer the same when  $n > 3$ . It appears to be true that the dimension of the kernel is no larger and is strictly smaller for all degrees when  $n > 5$ . Thus we would conjecture that the span of  $\hat{e}_n p(\xi_1, \dots, \xi_{n-1})$  still contains a complement to the range so that our normal form can still be used (though it would not be the simplest normal form since it could contain more terms than strictly necessary). To prove this result the main problem is to obtain a description of the *polynomial* kernel of  $\text{ad}_N^*$ ; this becomes very complicated when  $n > 3$ , see [9,25]. In higher dimensions most of the proof works, however, there is no simple general *polynomial* solution to the homological equation available as it was given in (34).

In a future paper we plan to study the dynamics of the unfolding of (14).

## Acknowledgements

HRD was supported in part by a Leverhulme Research Fellowship, and he would like to thank MSRI Berkeley and UC Boulder for their hospitality. JDM was supported in part by NSF grant DMS-0202032 and the Mathematical Sciences Research Institute. The authors would like to thank Jeroen Lamb and James Murdock for helpful suggestions.

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