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Correlation functions of the KdV hierarchy and applications to intersection numbers over $\overline{\mathcal{M}}_{g,n}$

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ABSTRACT

We derive an explicit generating function of correlation functions of an arbitrary tau-function of the KdV hierarchy. In particular applications, our formulation gives closed formulæ of a new type for the generating series of intersection numbers of ψ-classes as well as of mixed ψ- and κ-classes in full genera.

1. Introduction and results

The famed Korteweg–de Vries (KdV) equation

$$u_t = u u_x + \frac{1}{12} u_{xxx}, \tag{1.1}$$

has long been known to be *integrable*. It belongs to an infinite family of pairwise commuting nonlinear evolutionary PDEs called *the KdV hierarchy*. The hierarchy can be described in terms of isospectral deformations of the Lax operator

$$L = \partial_x^2 + 2\,u(x). \tag{1.2}$$

Namely, the k-th equation of the KdV hierarchy, in the normalization of the present paper, reads

$$L_{t_k} = [A_k, L], \tag{1.3}$$

$$A_k := \frac{1}{(2k+1)!!}\left(L^{\frac{2k+1}{2}}\right)_+, \quad k \geq 0. \tag{1.4}$$

Here, the independent variables $t_0, t_1, t_2, \ldots$ are called *times*. The symbol $\left(L^{\frac{2k+1}{2}}\right)_+$ stands for the differential part of the pseudo-differential operator $L^{\frac{2k+1}{2}}$, see e.g. the book [1] for details. The $k = 1$ equation of (1.3) coincides with (1.1). As customary in the literature, we shall identify t_0 with the spatial variable x.

The notion of *tau-function* for the KdV hierarchy was introduced by the Kyoto school [2–4] during the 1970s–1980s. In 1991, E. Witten, in his study of two-dimensional quantum gravity [5], conjectured that the generating function of the intersection numbers of ψ-classes on the Deligne–Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable algebraic curves is a tau-function of the KdV hierarchy. Witten's conjecture was

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later proved by M. Kontsevich [6]; see [7–9] for several alternative proofs. Moreover the so-called "tau structures" of KdV-like hierarchies became one of the central subjects in the study of the deep relation between integrable hierarchies and Gromov–Witten invariants [10–12].

The original motivations of the Witten's conjecture identify the tau-function $\tau = \tau(t_0, t_1, t_2, \dots)$ of a particular solution to the KdV hierarchy with the partition function of 2D quantum gravity. The time variables $t_0, t_1, t_2, \dots$ are identified with coupling constants associated with observables $\tau_0 = 1, \tau_1, \tau_2, \dots$ of the quantum theory. Thus, the quantum correlators of 2D quantum gravity are just logarithmic derivatives of the tau-function

$$\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_n} \rangle = \left. \frac{\partial^n \log \tau(t_0, t_1, t_2, \dots)}{\partial t_{k_1} \partial t_{k_2} \dots \partial t_{k_n}} \right|_{t_0 = t_1 = t_2 = \dots = 0}. \tag{1.5}$$

Tau-functions of some other solutions to the KdV hierarchy along with the corresponding correlators of the form (1.5) proved to be of interest for other applications, in particular to the study of topology of Deligne–Mumford moduli spaces (see below). The main goal of this paper is to provide a simple algorithm, in the framework of the theory of Lax operator and its eigenfunctions, for computation of n-point correlators of an arbitrary solution to the KdV hierarchy.

Let us begin with basics of the theory of KdV tau-functions in the version of M. Sato et al. Denote by $\mathbf{t} = (t_0, t_1, t_2, \dots)$ the infinite vector of time variables. Let $\mathcal{B} = \mathbb{C}[[t_k,\ k = 0, 1, 2, \dots]]$ be the Bosonic Fock space. A *Sato tau-function* $\tau(\mathbf{t})$ of the KdV hierarchy is an element in $\mathcal{B}$ satisfying the Hirota bilinear identities

$$\operatorname*{res}_{z=\infty} \tau(\mathbf{t} - [z^{-1}])\, \tau(\tilde{\mathbf{t}} + [z^{-1}]) \exp\left(\sum_{j \ge 1} \frac{t_j - \tilde{t}_j}{(2j+1)!!} z^{2j+1} \right) z^{2p}\, dz = 0, \quad \forall\, \mathbf{t},\ \tilde{\mathbf{t}},\ p = 0, 1, 2, \dots. \tag{1.6}$$

Here $\mathbf{t} - [z^{-1}] := \left(t_0 - z^{-1}, \dots, t_k - \frac{(2k-1)!!}{z^{2k+1}}, \dots \right)$. The residue is understood in a formal way, namely, as (minus) the coefficient of z^{-1} in the formal expansion at $z = \infty$. Given an arbitrary tau-function $\tau(\mathbf{t})$, then $u(\mathbf{t}) = \partial_x^2 \log \tau(\mathbf{t})$ is a solution of the KdV hierarchy (1.3).

Conversely, let $u(\mathbf{t})$ be an arbitrary (formal) solution of the KdV hierarchy (1.3); then there exists [1] a tau-function $\tau(\mathbf{t})$ such that $\partial_x^2 \log \tau(\mathbf{t}) = u(\mathbf{t})$. The tau-function of $u(\mathbf{t})$ is uniquely determined up to a *gauge freedom*

$$\tau(\mathbf{t}) \mapsto \exp\Big(\alpha_{-1} + \sum_{j \ge 0} \alpha_j t_j \Big) \tau(\mathbf{t}) \tag{1.7}$$

where the coefficients $\alpha_j,\ j \ge -1$ are arbitrary constants.

Let $\tau(\mathbf{t})$ be any tau-function of $u(\mathbf{t})$. Define the *wave* and *dual wave functions* by

$$\psi(z; \mathbf{t}) = \frac{\tau(\mathbf{t} - [z^{-1}])}{\tau(\mathbf{t})} e^{\vartheta(z;\mathbf{t})}, \qquad \psi^*(z; \mathbf{t}) = \psi(-z; \mathbf{t}) = \frac{\tau(\mathbf{t} + [z^{-1}])}{\tau(\mathbf{t})} e^{-\vartheta(z;\mathbf{t})}, \tag{1.8}$$

where the phase ϑ is given by

$$\vartheta(z; \mathbf{t}) := \sum_{j=0}^{\infty} t_j \frac{z^{2j+1}}{(2j+1)!!}. \tag{1.9}$$

The gauge freedom (1.7) affects $\psi(z; \mathbf{t})$ by a multiplicative factor of the form

$$g(z) = \exp\left(- \sum_{k=0}^{\infty} \frac{\alpha_k \cdot (2k-1)!!}{z^{2k+1}} \right) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{1.10}$$

The wave functions are (formal) eigenfunctions of the Lax operator

$$L\, \psi = z^2\, \psi, \qquad L\, \psi^* = z^2\, \psi^*. \tag{1.11}$$

Their dependence on time variables is specified by the following compatible system

$$\psi_{t_k} = A_k \psi, \qquad \psi^*_{t_k} = -A_k \psi^*, \qquad k \ge 0. \tag{1.12}$$

where the differential operators A_k are defined in (1.4). As $z \to \infty$ their (formal) asymptotic behaviours are

$$\psi(z; \mathbf{t}) = \left(1 + \mathcal{O}(z^{-1})\right) e^{\vartheta(z;\mathbf{t})}, \qquad \psi^*(z; \mathbf{t}) = \left(1 + \mathcal{O}(z^{-1})\right) e^{-\vartheta(z;\mathbf{t})}. \tag{1.13}$$

Also they satisfy an infinite system of bilinear relations

$$\operatorname*{res}_{z=\infty} \psi(z; \mathbf{t})\, \psi^*(z; \tilde{\mathbf{t}})\, z^{2p}\, dz = 0, \quad \forall\, \mathbf{t},\ \tilde{\mathbf{t}},\ p = 0, 1, \dots. \tag{1.14}$$

See e.g. [1, Thm. 6.3.8] for proofs of these statements. Depending on the context, the wave and dual wave functions (as well as the τ-function) can be defined analytically following the approach of G. Segal and G. Wilson [13]; also see Remark 4.2.

Definition 1.1. For any tau-function $\tau(\mathbf{t})$ of the KdV hierarchy, we call the functions

$$\langle\langle \tau_{k_1} \tau_{k_2} \dots \tau_{k_n} \rangle\rangle(\mathbf{t}) := \frac{\partial^n \log \tau}{\partial t_{k_1} \dots \partial t_{k_n}}(\mathbf{t}), \quad n \ge 1 \tag{1.15}$$

and the numbers

$$\langle \tau_{k_1}\tau_{k_2}\dots\tau_{k_n}\rangle := \langle\langle \tau_{k_1}\tau_{k_2}\dots\tau_{k_n}\rangle\rangle(\mathbf{t}=0) \tag{1.16}$$

the n-**point correlation functions** and the n-**point correlators** of the tau-function respectively. For every $n \geq 1$ we define the generating function of n-point correlation functions by

$$F_n(z_1,\dots,z_n;\mathbf{t}) := \sum_{k_1,\dots,k_n=0}^{\infty} \langle\langle \tau_{k_1}\dots\tau_{k_n}\rangle\rangle(\mathbf{t})\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n+1)!!}{z_n^{2k_n+2}}. \tag{1.17}$$

Evaluating it at $\mathbf{t}=0$

$$F_n(z_1,\dots,z_n) := F_n(z_1,\dots,z_n;\mathbf{0}) \tag{1.18}$$

one obtains generating functions of n-point correlators of $\tau(\mathbf{t})$.

The notation (1.15) is borrowed from the literature in quantum gravity [5].

We are now in a position to formulate the first main result of the paper.

Theorem 1.2. *Let τ be any tau-function of the KdV hierarchy and let ψ, ψ^* be defined by (1.8). The generating function of one-point correlation functions has the following expression*

$$F_1(z;\mathbf{t}) = \frac{1}{2}\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right) - \vartheta_z(z) \tag{1.19}$$

where

$$\Psi(z;\mathbf{t}) = \begin{pmatrix} \psi(z;\mathbf{t}) & \psi^*(z;\mathbf{t}) \\ -\psi_x(z;\mathbf{t}) & -\psi^*_x(z;\mathbf{t})\end{pmatrix} \tag{1.20}$$

and $\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the Pauli matrix.

The proof is in Section 3. The identity (1.19) gives a unique reconstruction, up to a multiplicative constant, of tau-function from the knowledge of the corresponding wave functions and thus provides a converse to Sato's result.

Remark 1.3. The reader conversant with the theory of isomonodromic deformation will observe similarity with the Jimbo–Miwa–Ueno definition of isomonodromic tau-function [14,15].

Remark 1.4. The theory developed by Sato et al. gives a formal expansion of τ by using the infinite Grassmannian approach (see for example [16–18] for details)

$$\tau = \sum_{\mu\in\mathbb{Y}} \pi_\mu\, \mathfrak{s}_\mu \tag{1.21}$$

which provides derivatives of τ at $\mathbf{t}=0$ in terms of Plücker coordinates. Here, $\mathbb{Y}$ denotes the set of Young diagrams, π_μ are Plücker coordinates of a point in the Sato Grassmannian, $\mathfrak{s}_\mu = \mathfrak{s}_\mu(T)$ are Schur polynomials in $T=(T_1,T_2,\dots)$ with $t_k = -(2k+1)!!\,T_{2k+1}$. However, the representation (1.21) does not help much with computation of logarithmic derivatives of tau-function.

A well-known formula (see e.g. [19, pag. 389]), expressing the generating function of $\langle\langle\tau_0\tau_j\rangle\rangle$, can be easily obtained as an immediate corollary of Theorem 1.2:

Corollary 1.5. *The following equality holds true:*

$$1+\sum_{j=0}^{\infty}\frac{(2j+1)!!}{z^{2j+2}}\langle\langle\tau_0\tau_j\rangle\rangle = \psi(z)\,\psi^*(z). \tag{1.22}$$

It has been shown in [20,19] that any pair of wave and dual wave functions ψ, ψ^* satisfy

$$\psi(z)\,\psi^*(z) = 1+\sum_{j=0}^{\infty}\frac{Res_\partial L^{\frac{2j+1}{2}}}{z^{2j+2}}. \tag{1.23}$$

Hence from (1.23) and (1.22) we have $\langle\langle\tau_0\tau_j\rangle\rangle = \frac{1}{(2j+1)!!}Res_\partial L^{\frac{2j+1}{2}}$, $j\geq 0$.

Our next result expresses the generating function of two-point correlation functions in terms of wave and dual wave functions.

Proposition 1.6. *Let τ be a tau-function of the KdV hierarchy and let ψ, ψ^* be defined by (1.8). The generating function of two-point correlation functions has the following expression*

$$F_2(z,w;\mathbf{t}) = \frac{\frac{1}{2}\mathcal{R}_x(w)\mathcal{R}_x(z) - \mathcal{R}(w)\mathcal{R}(z)\chi(z)\chi(-z) - \mathcal{R}(z)\mathcal{R}(w)\chi(w)\chi(-w) - (z^2+w^2)}{(z^2-w^2)^2} \tag{1.24}$$

where $\mathcal{R}=\mathcal{R}(z;\mathbf{t}) = \psi(z;\mathbf{t})\,\psi^*(z;\mathbf{t})$, $\chi=\chi(z;\mathbf{t}) = \partial_x\log\psi(z;\mathbf{t})$; the explicit dependence on $\mathbf{t}$ of the functions $\mathcal{R}(z;\mathbf{t})$, $\mathcal{R}(w;\mathbf{t})$, etc. is omitted from (1.24).

The proof is in Section 4.1. Note that both $\mathcal{R}$ and χ are intended as formal Laurent series in z^{-1} whose coefficients are differential polynomials in u. The formula (1.24) was also derived in [10] through the Hadamard–Seeley expansion of $e^{\varepsilon L}$, $\varepsilon \to 0$.

Now we formulate the result for n-point correlation functions ($n \geq 2$); it includes the case $n = 2$ but we have preferred to highlight it separately in Proposition 1.6 for clarity's sake.

Theorem 1.7 (*Main Theorem*)**.** *Let τ be any tau-function of the KdV hierarchy and let $\psi = \psi(z; \mathbf{t})$, $\psi^* = \psi^*(z; \mathbf{t})$ be defined by (1.8). Let $\mathcal{R} = \mathcal{R}(z; \mathbf{t})$ be as above. Denote Θ the matrix given by one of the following equivalent expressions*

$$\Theta(z; \mathbf{t}) = \frac{1}{2}\begin{bmatrix} -\mathcal{R}_x & -2\mathcal{R} \\ \mathcal{R}_{xx} - 2(z^2 - 2\,u)\mathcal{R} & \mathcal{R}_x \end{bmatrix} \tag{1.25}$$

$$= \frac{1}{2}\begin{bmatrix} -(\psi\,\psi^*)_x & -2\,\psi\,\psi^* \\ 2\,\psi_x\,\psi_x^* & (\psi\,\psi^*)_x \end{bmatrix} \tag{1.26}$$

$$= z\,\Psi(z; \mathbf{t})\,\sigma_3\,\Psi^{-1}(z; \mathbf{t}) \tag{1.27}$$

where the matrix $\Psi(z; \mathbf{t})$ is defined in (1.20). Then the generating function of n-point correlation functions with $n \geq 2$ has the following expression

$$F_n(z_1, \ldots, z_n; \mathbf{t}) = -\frac{1}{n}\sum_{r \in S_n} \frac{\operatorname{Tr}\left(\Theta(z_{r_1}) \cdots \Theta(z_{r_n})\right)}{\prod_{j=1}^{n}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{n,2}\frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2}. \tag{1.28}$$

Here $\delta_{n,2}$ is the Kronecker delta, S_n denotes the group of permutations of $\{1, \ldots, n\}$, and for a permutation $r = [r_1, \ldots, r_n]$, r_{n+1} is understood as r_1.

The proof is in Section 4.

Remark 1.8. The reader can verify that the right side of (1.28) is regular on the diagonals $z_i = z_j$.

Example 1.9. The next example is $n = 3$ for which the theorem above yields (on account of the cyclicity of the trace)

$$F_3(z_1, z_2, z_3; \mathbf{t}) = -\frac{\operatorname{Tr}\left(\Theta(z_1)\Theta(z_2)\Theta(z_3)\right) - \operatorname{Tr}\left(\Theta(z_2)\Theta(z_1)\Theta(z_3)\right)}{(z_1^2 - z_2^2)(z_2^2 - z_3^2)(z_3^2 - z_1^2)}. \tag{1.29}$$

Remark 1.10. We have been recently made aware of an interesting preprint from Si-Qi Liu [21], where the $n = 2, 3$ cases of Theorem 1.7 have also been obtained independently. Jian Zhou also privately communicated a different nice proof of Theorem 1.7 [22].

Application to intersection numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$. Recall that $\overline{\mathcal{M}}_{g,n}$ denotes the Deligne–Mumford moduli space of stable curves of genus g with n marked points; let us also recall

Witten's conjecture (*Kontsevich's Theorem*)**.** *The partition function of 2D quantum gravity*

$$Z(\mathbf{t}) = \exp\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\ldots,k_n \geq 0}\langle \tau_{k_1}\cdots\tau_{k_n}\rangle\, t_{k_1}\cdots t_{k_n}\right) \tag{1.30}$$

is a tau-function of the KdV hierarchy (1.3). Here $\langle \tau_{k_1}\ldots\tau_{k_n}\rangle$ are intersection numbers of ψ-classes over the Deligne–Mumford moduli spaces

$$\langle \tau_{k_1}\ldots\tau_{k_n}\rangle := \begin{cases} \displaystyle\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\ldots\psi_n^{k_n}, & \text{if } g := \dfrac{k_1 + \cdots + k_n - n + 3}{3} \text{ is a nonnegative integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.31}$$

In the above integrals, ψ_i is the first Chern class of the i-th tautological line bundle over $\overline{\mathcal{M}}_{g,n}$.

The partition function $Z(\mathbf{t})$ is now generally known as the *Witten–Kontsevich tau-function*. Witten's conjecture implies that $u^{WK}(\mathbf{t}) := \partial_x^2 \log Z(\mathbf{t})$ is a particular solution of (1.3), for which reason we call it the *Witten–Kontsevich solution*, also known as *the topological solution*. It can be specified by the initial data

$$u^{WK}(\mathbf{t})|_{t_0 = x,\ t_{\geq 1} = 0} = x. \tag{1.32}$$

Applying Theorems 1.2 and 1.7 to the Witten–Kontsevich tau-function, we obtain

Theorem 1.11. *Let $M(z)$ denote the following matrix-valued formal series*

$$M(z) = \frac{1}{2}\begin{pmatrix} -\displaystyle\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+4} & -2\displaystyle\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g} \\ 2\displaystyle\sum_{g=0}^{\infty}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g+2} & \displaystyle\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+4} \end{pmatrix}. \tag{1.33}$$

The generating functions of n-point intersection numbers (1.31)

$$F_n^{WK}(z_1,\dots,z_n) := \sum_{k_1,\dots,k_n=0}^{\infty} \langle \tau_{k_1}\dots\tau_{k_n}\rangle \frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n+1)!!}{z_n^{2k_n+2}}, \quad n\geq 1 \tag{1.34}$$

are given by the following formulæ:

$$F_1^{WK}(z) = \sum_{g=1}^{\infty}\frac{(6g-3)!!}{24^g\cdot g!} z^{-(6g-2)}, \tag{1.35}$$

$$F_n^{WK}(z_1,\dots,z_n) = -\frac{1}{n}\sum_{r\in S_n}\frac{\mathrm{Tr}\left(M(z_{r_1})\cdots M(z_{r_n})\right)}{\prod_{j=1}^{n}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{n,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}, \quad n\geq 2. \tag{1.36}$$

The expression (1.35) is equivalent to the well-known formula (for example see in [23,24]):

$$\langle \tau_{3g-2}\rangle = \frac{1}{24^g\cdot g!}, \quad g\geq 1. \tag{1.37}$$

The formula (1.36) seems to be new.

Remark 1.12. In [24], A. Okounkov provided an expression for *different* generating functions of the same intersection numbers. The relationship between the two approaches is the following

Our approach	Okounkov's approach
$\sum_{k_1,\dots,k_n=0}^{\infty}\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n+1)!!}{z_n^{2k_n+2}}\langle\tau_{k_1}\dots\tau_{k_n}\rangle$	$\sum_{k_1,\dots,k_n=0}^{\infty} x_1^{k_1}\cdots x_n^{k_n}\langle\tau_{k_1}\dots\tau_{k_n}\rangle$ (1.38)

It should be apparent that the two generating functions are related by a (formal) multiple Laplace transform. Okounkov's approach produced integral expressions (but reducing to explicit formulæ for $n\leq 3$). On the other hand the formula (1.36) is absolutely explicit and allows an efficient computation of the n-point numbers also for very high genera. See Section 8 for more details.

Remark 1.13. In [25] M. Bergère and B. Eynard introduced correlators associated with the Christoffel–Darboux kernel obtained from solutions to a general 2×2 system of linear ODEs with rational coefficients (see Definition 2.3 in [25]). In this case one can use a suitably modified Jimbo–Miwa–Ueno formula [14] as the definition of the tau-function. The construction of [25] was extended in [26] to Christoffel–Darboux kernels associated with larger systems of ODEs. Applications of this approach to computation of intersection numbers on the Deligne–Mumford moduli spaces have not been discussed in Refs. [25,26]. We are grateful to B. Eynard who, after the first version of the present paper appeared on arXiv, kindly communicated us that, in the particular case of Airy kernel it can be shown that the Bergère–Eynard generating functions of the correlators, after a suitable normalization, reproduce intersection numbers of ψ-classes. It is an interesting observation that certainly deserves a further study.

Application to higher Weil–Petersson volumes. The *Main Theorem* also allows us to compute higher Weil–Petersson volumes, by which we mean integrals of mixed ψ- and κ-classes of the form

$$\langle\kappa_1^{d_1}\dots\kappa_\ell^{d_\ell}\tau_{k_1}\dots\tau_{k_n}\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\dots\psi_n^{k_n}\kappa_1^{d_1}\dots\kappa_\ell^{d_\ell}. \tag{1.39}$$

Recall [27–29] that κ-classes as elements of the Chow ring $A^*\left(\overline{\mathcal{M}}_{g,n}\right)$ are defined by

$$\kappa_i = f_*\left(\psi_{n+1}^{i+1}\right)\in A^i\left(\overline{\mathcal{M}}_{g,n}\right), \quad i\geq 0 \tag{1.40}$$

where $f:\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ is the universal curve (the forgetful map). The class $\kappa_0 = 2g-2+n$ is just a constant, so it will not appear below. The integrals (1.39) take zero values unless

$$\sum_{j=1}^{n}k_j + \sum_{j=1}^{l} j\,d_j = 3g-3+n. \tag{1.41}$$

We denote by $Z^\kappa(\mathbf{t};\mathbf{s})$ the partition function of higher Weil–Petersson volumes

$$Z^\kappa(\mathbf{t};\mathbf{s}) = \exp\left(\sum_{g,n,\ell\geq 0}\frac{1}{n!}\sum_{d_1,\dots,d_\ell,k_1,\dots,k_n\geq 0}\langle\kappa_1^{d_1}\dots\kappa_\ell^{d_\ell}\tau_{k_1}\dots\tau_{k_n}\rangle_{g,n}\, t_{k_1}\cdots t_{k_n}\frac{s_1^{d_1}\dots s_\ell^{d_\ell}}{d_1!\cdots d_\ell!}\right), \tag{1.42}$$

where $\mathbf{s}$ denotes the infinite vector of independent variables $(s_1,s_2,\dots)$. It is a KdV tau-function of a family of solutions depending on the parameters $\mathbf{s}$ [30]. The $\mathbf{s}=\mathbf{0}$ evaluation of this function gives the Witten–Kontsevich tau-function: $Z(\mathbf{t}) = Z^\kappa(\mathbf{t};\mathbf{0})$. By $\psi^{WK}(z;\mathbf{t})$, $\psi^\kappa(z;\mathbf{t};\mathbf{s})$ denote the corresponding wave functions

$$\psi^{WK}(z;\mathbf{t}) := \frac{Z(\mathbf{t}-[z^{-1}])}{Z(\mathbf{t})}\mathrm{e}^{\vartheta(z;\mathbf{t})}, \qquad \psi^\kappa(z;\mathbf{t};\mathbf{s}) := \frac{Z^\kappa(\mathbf{t}-[z^{-1}];\mathbf{s})}{Z^\kappa(\mathbf{t};\mathbf{s})}\mathrm{e}^{\vartheta(z;\mathbf{t})}. \tag{1.43}$$

Notations. (i) $\mathbb{Y}$ will denote the set of all partitions. (ii) For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, we denote by $\ell(\lambda) = card\{i = 1, 2, \ldots \mid \lambda_i \neq 0\}$ the *length* of λ, by $|\lambda| = \sum_{i\geq 1} \lambda_i$ the *weight* of λ, by $m_i(\lambda)$ the *multiplicity* of i in λ, $i = 1, \ldots, \infty$. Let

$$m(\lambda)! := \prod_{i=1}^{\infty} m_i(\lambda)!. \tag{1.44}$$

The partition of 0 is denoted by (0); $\ell((0)) = |(0)| = 0$. For an arbitrary sequence of indeterminates $q_1, q_2, \ldots$ denote $q_\lambda := q_{\lambda_1} \cdots q_{\lambda_{\ell(\lambda)}}$; we use $\prec$ for the reverse lexicographic ordering on $\mathbb{Y}$, e.g. $\mathbb{Y}_3 = \{(3) \prec (2, 1) \prec (1^4)\}$, $\mathbb{Y}_4 = \{(4) \prec (3, 1) \prec (2^2) \prec (2, 1^2) \prec (1^4)\}$.

We give two ways of representing the generating function of the intersection numbers $\langle \kappa_1^{d_1} \ldots \kappa_\ell^{d_\ell} \tau_{k_1} \ldots \tau_{k_n}\rangle$. The first one represents the generating function of intersections of ψ- and κ-classes in terms of the generating functions of intersections of ψ-classes obtained in Theorem 1.11. The second approach is based on an explicit representation of the wave function $\psi^\kappa(z; \mathbf{t}; s)$ in terms of the wave function of the Witten–Kontsevich solution. These two approaches are formulated as, respectively, the first and the second part of the following theorem.

Theorem 1.14. *Part I. For a given $\lambda \in \mathbb{Y}$ and for any $n \geq 0$*

$$\sum_{k_1,\ldots,k_n\geq 0} \langle \kappa_{\lambda_1} \ldots \kappa_{\lambda_{\ell(\lambda)}} \tau_{k_1} \ldots \tau_{k_n}\rangle \frac{(2k_1 + 1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_n + 1)!!}{z_n^{2k_n+2}} = (-1)^{\ell(\lambda)} \sum_{|\mu|=|\lambda|} \frac{L_{\lambda\mu}}{m(\mu)!} \operatorname*{res}_{w_1=\infty} \cdots \operatorname*{res}_{w_{\ell(\mu)}=\infty}$$
$$\times \frac{w_1^{2\mu_1+3}}{(2\mu_1 + 3)!!} \cdots \frac{w_{\ell(\mu)}^{2\mu_{\ell(\mu)}+3}}{(2\mu_{\ell(\mu)} + 3)!!} F^{WK}_{\ell(\mu)+n}(w_1, \ldots, w_{\ell(\mu)}, z_1, \ldots, z_n)\, dw_1 \cdots dw_{\ell(\mu)}. \tag{1.45}$$

Here $L_{\lambda\mu}$ are transition matrices from the monomial basis of symmetric functions to the power sum basis [31], see Lemma 7.3 for an explicit expression for the entries of this matrix.
Part II. The wave function corresponding to higher Weil–Petersson volume satisfies

$$\psi^\kappa(z; \mathbf{t}; \mathbf{s}) = \exp\left(\sum_{k=1}^{\infty} \frac{h_k(-\mathbf{s})z^{2k+3}}{(2k + 3)!!}\right) \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)} s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \partial_{t_{\mu_1+1}} \ldots \partial_{t_{\mu_{\ell(\mu)}+1}} \psi^{WK}(z; \mathbf{t}). \tag{1.46}$$

Denote by $F^\kappa_n(z_1, \ldots, z_n; \mathbf{s})$ the generating function of the higher Weil–Petersson volumes, i.e.

$$F^\kappa_n(z_1, \ldots, z_n; \mathbf{s}) := \sum_{l\geq 0} \sum_{\substack{k_1,\ldots,k_n\geq 0\\ d_1,\ldots,d_l\geq 0}} \langle \kappa_1^{d_1} \ldots \kappa_\ell^{d_\ell} \tau_{k_1} \ldots \tau_{k_n}\rangle \frac{(2k_1 + 1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_n + 1)!!}{z_n^{2k_n+2}} \frac{s_1^{d_1} \ldots s_\ell^{d_\ell}}{d_1! \cdots d_\ell!}, \quad n \geq 1. \tag{1.47}$$

Then

$$F^\kappa_1(z; \mathbf{s}) = \frac{1}{4z} \left(-A(z; \mathbf{s})\, B_z(-z; \mathbf{s}) + B_z(z; \mathbf{s})\, A(-z; \mathbf{s}) + B(z; \mathbf{s})\, A_z(-z; \mathbf{s}) - A_z(z; \mathbf{s})\, B(-z; \mathbf{s})\right), \tag{1.48}$$

$$F^\kappa_n(z_1, \ldots, z_n; \mathbf{s}) = -\frac{1}{n} \sum_{r\in S_n} \frac{\operatorname{Tr}\left(M^\kappa(z_{r_1}; \mathbf{s}) \cdots M^\kappa(z_{r_n}; \mathbf{s})\right)}{\prod_{j=1}^{n}(z_{r_j}^2 - z_{r_{j+1}}^2)} - \delta_{n,2} \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2}, \quad n \geq 2. \tag{1.49}$$

Here $A(z; \mathbf{s}) := \psi^\kappa(z; \mathbf{0}; \mathbf{s})$, $B(z; \mathbf{s}) := \psi^\kappa_x(z; \mathbf{0}; \mathbf{s})$,

$$M^\kappa(z; \mathbf{s}) := \frac{1}{2} \begin{pmatrix} -[A(z; \mathbf{s})B(-z; \mathbf{s}) + A(-z; \mathbf{s})B(z; \mathbf{s})] & -2A(z; \mathbf{s})A(-z; \mathbf{s}) \\ 2B(z; \mathbf{s})B(-z; \mathbf{s}) & A(z; \mathbf{s})B(-z; \mathbf{s}) + A(-z; \mathbf{s})B(z; \mathbf{s}) \end{pmatrix} \tag{1.50}$$

(cf. definition (1.26) of the matrix Θ).

The polynomials $h_k = h_k(x_1, \ldots, x_k)$, $k \geq 1$ in the formula (1.46) are defined by the well known generating function

$$1 + \sum_{k\geq 1} h_k(x)z^k = e^{\sum_{i\geq 1} x_i z^i}.$$

Example 1.15. Using Theorem 1.14 we obtain in particular

$$\langle \kappa_{3g-3}\rangle_{g,0} = \frac{1}{24^g \cdot g!}, \quad g \geq 1, \tag{1.51}$$

$$\langle \kappa_1 \tau_{3g-3}\rangle_{g,1} = 3\frac{12g^2 - 12g + 5}{5!! \cdot 24^g \cdot g!}, \quad g \geq 1, \tag{1.52}$$

$$\langle \kappa_2 \tau_{3g-4}\rangle_{g,1} = 3\frac{72g^3 - 132g^2 + 95g - 35}{7!! \cdot 24^g \cdot g!}, \quad g \geq 2, \tag{1.53}$$

$$\langle \kappa_3 \tau_{3g-5}\rangle_{g,1} = \frac{1296g^4 - 3888g^3 + 4482g^2 - 2835g + 945}{9!! \cdot 24^g \cdot g!}, \quad g \geq 2. \tag{1.54}$$

Moreover, we find

$$A^{(1)}(z) = -\frac{z^5}{5!!}c + \frac{z^5}{5!!}q - \frac{z^2}{2\cdot 5!!}c, \tag{1.55}$$

$$A^{(2)}(z) = -\frac{z^7}{7!!}c + \frac{8z^7+5z}{8\cdot 7!!}q - \frac{z^4}{2\cdot 7!!}c, \tag{1.56}$$

$$A^{(1^2)}(z) = \left(\frac{z^{10}}{225} + \frac{11z^7}{1575} - \frac{z^4}{2520}\right)c + \left(-\frac{z^{10}}{225} - \frac{z^7}{210} + \frac{3z}{560}\right)q, \tag{1.57}$$

$$B^{(1)}(z) = -\frac{z^6}{5!!}q + \frac{z^3}{2\cdot 5!!}q + \frac{4z^6-6}{4\cdot 5!!}c, \tag{1.58}$$

$$B^{(2)}(z) = -\frac{z^8}{7!!}q + \frac{z^5}{2\cdot 7!!}q + \frac{8z^8-7z^2}{8\cdot 7!!}c, \tag{1.59}$$

$$B^{(1^2)}(z) = \left(-\frac{z^{11}}{225} - \frac{z^8}{210} + \frac{z^5}{150} - \frac{z^2}{240}\right)c + \left(\frac{z^{11}}{225} + \frac{4z^8}{1575} - \frac{13z^5}{2520}\right)q. \tag{1.60}$$

Here $c = c(z)$, $q = q(z)$ are Faber–Zagier series (6.10), (6.11); for a partition λ, $A^\lambda(z)$ is the coefficient of $s_\lambda := \prod_{j=1}^{\ell(\lambda)} s_{\lambda_j}$ in the $\mathbf{s}$-expansion of $A(z;\mathbf{s})$, similarly for definition of $B^\lambda(z)$.

Organization of the paper In Section 2, we briefly review the KdV hierarchy. In Section 3, we prove Theorem 1.2 and derive some useful lemmas. In Section 4, we prove Proposition 1.6 and Theorem 1.7. Application to the Witten–Kontsevich solution is presented in Section 6. Application to higher Weil–Petersson volumes is presented in Section 7. Further remarks are given in Section 8. We list useful formulæ and tables of intersection numbers, and apply a generalization of the Kac–Schwarz operator to higher Weil–Petersson volumes in appendices.

2. A brief reminder of the KdV hierarchy

In this preliminary section we review some useful facts about the KdV hierarchy (1.3), and derive generalized Kac–Schwarz operators associated to string equations.

Let $\mathcal{A}_u$ be the ring of differential polynomials in u. Define a family of differential polynomials $\Omega_{p;0}(u; u_x, \dots, u_p)$ with zero constant term for $p \geq 0$ through the Lenard–Magri recursion

$$\partial_x \Omega_{p;0} = \frac{1}{2p+1}\left(2\,u\,\partial_x + u_x + \frac{1}{4}\partial_x^3\right)\Omega_{p-1;0}, \quad p \geq 1, \tag{2.1}$$

$$\Omega_{-1;0} = 1. \tag{2.2}$$

Here, $u_k := \partial_x^k u$, $k \geq 1$ are jet variables. Then the KdV hierarchy (1.3) can be equivalently written as

$$u_{t_p} = \partial_x \Omega_{p;0}, \quad p \geq 0. \tag{2.3}$$

The first few $\Omega_{p;0}$ are given by

$$\Omega_{0;0} = u, \tag{2.4}$$

$$\Omega_{1;0} = \frac{u^2}{2} + \frac{1}{12}u_{xx}, \tag{2.5}$$

$$\Omega_{2;0} = \frac{u^3}{6} + \left(\frac{1}{12}u\,u_{xx} + \frac{1}{24}u_x^2\right) + \frac{u_4}{240}, \tag{2.6}$$

$$\Omega_{3;0} = \frac{u^4}{24} + \left(\frac{1}{24}u^2 u_{xx} + \frac{1}{24}u\,u_x^2\right) + \left(\frac{1}{240}u\,u_4 + \frac{1}{120}u_x u_3 + \frac{1}{160}u_{xx}^2\right) + \frac{u_6}{6720}. \tag{2.7}$$

In general, we have [10,19,20]

$$\Omega_{p;0} = \frac{1}{(2p+1)!!}Res_\partial L^{\frac{2p+1}{2}}, \quad \forall p \geq 0. \tag{2.8}$$

Let $\psi(z;\mathbf{t})$ and $\psi^*(z;\mathbf{t})$ be a pair of wave and dual wave functions of the KdV hierarchy. As in the Introduction, we define the following two formal series in z^{-1} by

$$\mathcal{R}(z;\mathbf{t}) := \psi(z;\mathbf{t})\,\psi^*(z;\mathbf{t}), \quad \chi(z;\mathbf{t}) := \partial_x \log\psi(z;\mathbf{t}). \tag{2.9}$$

We remind [1] that the function $\mathcal{R}$ coincides with the diagonal value of the *resolvent* of the Lax operator L. One can easily see from (1.23) and (2.8) that it gives the generating series of $\Omega_{p;0}$, i.e.

$$\mathcal{R}(z;\mathbf{t}) = 1 + \sum_{k=0}^{\infty}\frac{(2k+1)!!}{z^{2k+2}}\Omega_{k;0}(u(\mathbf{t}); u_x(\mathbf{t}), \dots, u_k(\mathbf{t})). \tag{2.10}$$

It is well known that the following two lemmas hold true for $\mathcal{R}(z)$ and $\chi(z)$, for which we omit proofs.

Lemma 2.1. *The function $\mathcal{R} = \mathcal{R}(z; u; u_x, \dots)$ is the unique solution in $\mathcal{A}_u[[z^{-1}]]$ of the following ODE*

$$\mathcal{R}\,\mathcal{R}_{xx} - \frac{1}{2}\,\mathcal{R}_x^2 + (4\,u - 2\,z^2)\,\mathcal{R}^2 = -2\,z^2, \tag{2.11}$$

$$\mathcal{R} = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{2.12}$$

Furthermore, the following formula holds true for $\mathcal{R}(z)$

$$\mathcal{R}(z; u, u_x, \dots)|_{u_j=0,\, j\geq 1} = \left(1 - \frac{2\,u}{z^2}\right)^{-\frac{1}{2}}. \tag{2.13}$$

In the above formulæ it is understood that $\mathcal{R}(z; \mathbf{t}) = \mathcal{R}(z; u(\mathbf{t}); u_x(\mathbf{t}), \dots)$.

One can recognize that $\mathcal{R}(z; u; u_x, \dots)$ is a full genus correction of gradient of the period of the one-dimensional Frobenius manifold [32,10].

Lemma 2.2. *The function $\chi = \chi(z; \mathbf{t}) = \partial_x \log \psi(z; \mathbf{t})$ is the unique formal solution in $\mathcal{A}_u[[z^{-1}]]$ of the Riccati equation*

$$\chi_x + \chi^2 + 2\,u - z^2 = 0, \tag{2.14}$$

$$\chi(z; \mathbf{t}) = z + \sum_{k=1}^{\infty} \frac{\chi_k(\mathbf{t})}{z^k}, \quad z \to \infty. \tag{2.15}$$

Lemma 2.3 *([19]). Let $\psi = \psi(z; \mathbf{t})$ and $\psi^* = \psi^*(z; \mathbf{t})$ be a pair of wave and dual wave functions of the KdV hierarchy. The following formula holds true:*

$$\psi_x\,\psi^* - \psi\,\psi_x^* = 2\,z, \tag{2.16}$$

$$\chi(z; \mathbf{t}) = \frac{1}{2}\,\partial_x \log \mathcal{R}(z; \mathbf{t}) + \frac{z}{\mathcal{R}(z; \mathbf{t})}. \tag{2.17}$$

Proof. The identity (2.16) is well-known. The proof of (2.17) is straightforward by using (2.16) and (2.9). □

3. One-point correlation functions of the KdV hierarchy

In this section we prove Theorem 1.2 by using the bilinear identities (1.6). Following [19,33,10], we will frequently use the following differential operator

$$\nabla(z) := \sum_{k=0}^{\infty} \frac{(2k+1)!!}{z^{2k+2}} \frac{\partial}{\partial t_k}. \tag{3.1}$$

Lemma 3.1. *For any tau-function $\tau(\mathbf{t})$ of the KdV hierarchy and any $z \in \mathbb{C}$, $z \neq 0$ we have*

$$\frac{\tau_x(\mathbf{t} - [z^{-1}])\,\tau(\mathbf{t} + [z^{-1}])}{2\,z\,\tau(\mathbf{t})^2} - \frac{\tau_x(\mathbf{t} + [z^{-1}])\,\tau(\mathbf{t} - [z^{-1}])}{2\,z\,\tau(\mathbf{t})^2} + \frac{\tau(\mathbf{t} - [z^{-1}])\,\tau(\mathbf{t} + [z^{-1}])}{\tau(\mathbf{t})^2} = 1. \tag{3.2}$$

The proof of this lemma is a straightforward application of Lemma 2.3 and hence omitted.

Proof of Theorem 1.2. Let $\tau(\mathbf{t})$ be any tau-function of the KdV hierarchy. Let $\psi(z; \mathbf{t})$ and $\psi^*(z; \mathbf{t})$ be the corresponding wave and dual wave functions (1.8). The bilinear identities (1.6), (1.14) imply that $\forall\, \mathbf{t}, \widetilde{\mathbf{t}}, w$

$$\oint_{R<|z|<|w|} \tau(\mathbf{t} - [w^{-1}] - [z^{-1}])\tau(\widetilde{\mathbf{t}} + [w^{-1}] + [z^{-1}]) e^{\sum_{j\geq 0} (t_{2j+1} - \tilde{t}_{2j+1} - 2\,(2j-1)!!\,w^{-2j-1})\,\frac{z^{2j+1}}{(2j+1)!!}}\, dz = 0. \tag{3.3}$$

Here R is a sufficiently large number. Note that

$$e^{-2\sum_{j\geq 0} \frac{1}{2j+1} \left(\frac{z}{w}\right)^{2j+1}} = \frac{w - z}{w + z}. \tag{3.4}$$

Applying the operator $\nabla(w)$ to the above identity (3.3) and then setting $\widetilde{\mathbf{t}} = \mathbf{t}$, we obtain that $\forall\, \mathbf{t}, w$,

$$\sum_{k=0}^{\infty} \frac{(2k+1)!!}{w^{2k+2}} \left[\oint_{R<|z|<|w|} \left(\tau(\mathbf{t} - [w^{-1}] - [z^{-1}])\, \tau_{t_k}(\mathbf{t} + [w^{-1}] + [z^{-1}])\, \frac{w - z}{w + z} \right.\right.$$
$$\left.\left. - \frac{z^{2k+1}}{(2k+1)!!}\, \tau(\mathbf{t} - [w^{-1}] - [z^{-1}])\, \tau(\mathbf{t} + [w^{-1}] + [z^{-1}])\, \frac{w - z}{w + z} \right) dz \right] = 0. \tag{3.5}$$

Noticing now that

$$\frac{w-z}{w+z}=\frac{2\,w}{z+w}-1,\quad z\to -w, \tag{3.6}$$

$$\frac{w-z}{w+z}=-1+2\,\frac{w}{z}+\mathcal{O}(z^{-2}),\quad z\to\infty \tag{3.7}$$

we have

$$\begin{aligned}
&\oint_{R<|z|<|w|}\tau(\mathbf{t}-[w^{-1}]-[z^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}]+[z^{-1}])\,\frac{w-z}{w+z}\,\frac{dz}{2i\pi}\\
&=-2\,w\,\tau(\mathbf{t})\,\tau_{t_k}(\mathbf{t})+\oint_{R<|w|<|z|}\tau(\mathbf{t}-[w^{-1}]-[z^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}]+[z^{-1}])\,\frac{w-z}{w+z}\,\frac{dz}{2i\pi}\\
&=-2\,w\,\tau(\mathbf{t})\,\tau_{t_k}(\mathbf{t})-\operatorname*{res}_{z=\infty}\left(\tau(\mathbf{t}-[w^{-1}])-\tau_{t_0}(\mathbf{t}-[w^{-1}])\,z^{-1}+\mathcal{O}(z^{-2})\right)\\
&\quad\times\left(\tau_{t_k}(\mathbf{t}+[w^{-1}])+\tau_{t_0t_k}(\mathbf{t}+[w^{-1}])\,z^{-1}+\mathcal{O}(z^{-2})\right)\left(-1+2\,w\,z^{-1}+\mathcal{O}(z^{-2})\right)dz\\
&=-2\,w\,\tau(\mathbf{t})\,\tau_{t_k}(\mathbf{t})+2\,w\,\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}])+\tau_{t_0}(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}])-\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_0t_k}(\mathbf{t}+[w^{-1}]).
\end{aligned} \tag{3.8}$$

Similarly we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\oint_{R<|z|<|w|}\frac{z^{2k+1}}{(2k+1)!!}\,\tau(\mathbf{t}-[w^{-1}]-[z^{-1}])\,\tau(\mathbf{t}+[w^{-1}]+[z^{-1}])\,\frac{w-z}{w+z}\,\frac{dz}{2i\pi}\\
&=\oint_{R<|z|<|w|}\tau(\mathbf{t}-[w^{-1}]-[z^{-1}])\,\tau(\mathbf{t}+[w^{-1}]+[z^{-1}])\,\frac{z}{(w+z)^2}\,\frac{dz}{2i\pi}\\
&=-\tau(\mathbf{t})\,\tau(\mathbf{t})+\tau(\mathbf{t})\,\tau(\mathbf{t})=0.
\end{aligned} \tag{3.9}$$

Here we have used the following expansions

$$\frac{z}{(w+z)^2}=\frac{-w}{(z+w)^2}+\frac{1}{z+w},\quad z\to -w, \tag{3.10}$$

$$\frac{z}{(w+z)^2}=\frac{1}{z}+\mathcal{O}(z^{-2}),\quad z\to\infty. \tag{3.11}$$

Substituting (3.8) and (3.9) into (3.5) we arrive at

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}\,\tau(\mathbf{t})^2}\left(\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}])+\frac{1}{2\,w}\,\tau_{t_0}(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}])\right.\\
&\quad\left.-\frac{1}{2\,w}\,\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_0t_k}(\mathbf{t}+[w^{-1}])\right)=\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\,\frac{\tau_{t_k}(\mathbf{t})}{\tau(\mathbf{t})}.
\end{aligned} \tag{3.12}$$

Replacing $w\to -w$ in (3.12) we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}\,\tau(\mathbf{t})^2}\left(\tau(\mathbf{t}+[w^{-1}])\,\tau_{t_k}(\mathbf{t}-[w^{-1}])-\frac{1}{2\,w}\tau_{t_0}(\mathbf{t}+[w^{-1}])\,\tau_{t_k}(\mathbf{t}-[w^{-1}])\right.\\
&\quad\left.+\frac{1}{2\,w}\tau(\mathbf{t}+[w^{-1}])\,\tau_{t_0t_k}(\mathbf{t}-[w^{-1}])\right)=\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\,\frac{\tau_{t_k}(\mathbf{t})}{\tau(\mathbf{t})}.
\end{aligned} \tag{3.13}$$

Now let us look at the r.h.s. of (1.19). It is straightforward to compute the following products

$$\begin{aligned}
\psi_x\psi_z^* &= -\frac{\tau_{t_0}(\mathbf{t}-[z^{-1}])}{\tau(\mathbf{t})^2}\sum_{k=0}^{\infty}\tau_{t_k}(\mathbf{t}+[z^{-1}])\,\frac{(2k+1)!!}{z^{2k+2}}-\frac{\tau_{t_0}(\mathbf{t}-[z^{-1}])\tau(\mathbf{t}+[z^{-1}])}{\tau(\mathbf{t})^2}\vartheta_z\\
&\quad -z\,\frac{\psi(z;\mathbf{t})}{\tau(\mathbf{t})}\sum_{k=0}^{\infty}\tau_{t_k}(\mathbf{t}+[z^{-1}])\frac{(2k+1)!!}{z^{2k+2}}\mathrm{e}^{-\vartheta}-z\,\vartheta_z\,\psi(z;\mathbf{t})\psi^*(z;\mathbf{t})\\
&\quad +\frac{\psi(z;\mathbf{t})\tau_{t_0}(\mathbf{t})}{\tau(\mathbf{t})^2}\sum_{k=0}^{\infty}\tau_{t_k}(\mathbf{t}+[z^{-1}])\frac{(2k+1)!!}{z^{2k+2}}\mathrm{e}^{-\vartheta}-\frac{\tau_{t_0}(\mathbf{t})}{\tau(\mathbf{t})^2}\,\vartheta_z\,\psi(z;\mathbf{t})\psi^*(z;\mathbf{t}),
\end{aligned} \tag{3.14}$$

$$\psi_z\psi_x^*=-\psi_x\psi_z^*|_{z\to -z}, \tag{3.15}$$

$$\begin{aligned}
\psi\,\psi_{xz}^* &= \sum_{k=0}^{\infty}\frac{(2k+1)!!}{z^{2k+2}\,\tau(\mathbf{t})^2}\,\tau(\mathbf{t}-[z^{-1}])\tau_{t_0t_k}(\mathbf{t}+[z^{-1}])-\frac{\tau(\mathbf{t}-[z^{-1}])\,\tau_{t_0}(\mathbf{t}+[z^{-1}])}{\tau(\mathbf{t})^2}\vartheta_z\\
&\quad -\psi(z;\mathbf{t})\,\psi^*(z;\mathbf{t})-\left(z\,\psi(z;\mathbf{t})+\frac{\tau_{t_0}(\mathbf{t})}{\tau(\mathbf{t})}\,\psi(z;\mathbf{t})\right)\psi_z^*(z;\mathbf{t}),
\end{aligned} \tag{3.16}$$

$$\psi_{xz}\psi^*=-\psi\,\psi_{xz}^*|_{z\to -z}. \tag{3.17}$$

As a result we have

$$
\begin{aligned}
&\frac{\psi_x\psi_w^* + \psi_w\psi_x^* - \psi\psi_{xw}^* - \psi_{xw}\psi^*}{4\,w}\\
&= \frac{1}{4\,w}\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}\tau(\mathbf{t})^2}\Big[\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_0t_k}(\mathbf{t}+[w^{-1}]) - \tau_{t_0}(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}])\\
&\quad - \tau(\mathbf{t}+[w^{-1}])\,\tau_{t_0t_k}(\mathbf{t}-[w^{-1}]) + \tau_{t_0}(\mathbf{t}+[w^{-1}])\,\tau_{t_k}(\mathbf{t}-[w^{-1}])\Big]\\
&\quad - \frac{1}{2}\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}\tau(\mathbf{t})^2}\Big[\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_k}(\mathbf{t}+[w^{-1}]) + \tau(\mathbf{t}+[w^{-1}])\,\tau_{t_k}(\mathbf{t}-[w^{-1}])\Big]\\
&\quad + \frac{\vartheta_w}{2\,\tau(\mathbf{t})^2}\left[\frac{1}{w}\,\tau(\mathbf{t}-[w^{-1}])\,\tau_{t_0}(\mathbf{t}+[w^{-1}]) - \frac{1}{w}\,\tau(\mathbf{t}+[w^{-1}])\,\tau_{t_0}(\mathbf{t}-[w^{-1}]) - 2\,\tau(\mathbf{t}-[w^{-1}])\,\tau(\mathbf{t}+[w^{-1}])\right]\\
&= -\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\frac{\tau_{t_k}(\mathbf{t})}{\tau(\mathbf{t})} - \vartheta_w.
\end{aligned}
\tag{3.18}
$$

The last equality uses (3.12), (3.13) and Lemma 3.1. The theorem is proved. □

4. Multi-point correlation functions of the KdV hierarchy

In this section we apply Theorem 1.2 to derive generating functions of multi-point correlation functions of the KdV hierarchy. We will need the following formulae for the time derivatives of wave functions.

Lemma 4.1. *Let $\psi(z;\mathbf{t})$ and $\psi^*(z;\mathbf{t})$ be any pair of wave and dual wave functions and let $\mathcal{R}(z;\mathbf{t}) = \psi(z;\mathbf{t})\,\psi^*(z;\mathbf{t})$. It follows that*

$$\nabla(z)\,\psi(w) = \frac{2\,\mathcal{R}(z)\psi_x(w) - \mathcal{R}_x(z)\psi(w)}{2\,(z^2-w^2)}, \tag{4.1}$$

$$\nabla(z)\,\psi_x(w) = \frac{\mathcal{R}_x(z)\psi_x(w) - \left[\mathcal{R}_{xx}(z) - 2\,\mathcal{R}(z)(w^2-2\,u)\right]\psi(w)}{2\,(z^2-w^2)}, \tag{4.2}$$

$$\nabla(z)\,\psi_w(w) = \frac{2\,\mathcal{R}(z)\psi_{xw}(w) - \mathcal{R}_x(z)\psi_w(w)}{2\,(z^2-w^2)} - w\,\frac{2\,\mathcal{R}(z)\psi_x(w) - \mathcal{R}_x(z)\psi(w)}{(z^2-w^2)^2}, \tag{4.3}$$

$$\nabla(z)\,\Psi(w) = \frac{1}{2(z^2-w^2)}\begin{pmatrix} -\mathcal{R}_x(z) & -2\,\mathcal{R}(z)\\ \mathcal{R}_{xx}(z) - 2\,(w^2-2\,u)\mathcal{R}(z) & \mathcal{R}_x(z)\end{pmatrix}\Psi(w), \tag{4.4}$$

$$\nabla(z)\,\Psi(w) = \frac{1}{z^2-w^2}\Theta(z)\Psi(w) + \begin{pmatrix} 0 & 0\\ \mathcal{R}(z) & 0\end{pmatrix}\Psi(w). \tag{4.5}$$

Proof. The identity (4.1) is a standard result; e.g. see the formula (3.12) in [20] or (11.24) in [19]. The others follow from (4.1). □

Remark 4.2. The definition of the matrices Ψ and Θ is valid not just in a formal sense. In many cases the matrix Ψ solves a Riemann–Hilbert problem; for example if the potential $u(x)$ is in an appropriate Schwartz class or it belongs to the class of quasi-periodic functions (as in the finite gap integration problems). In these cases the identities for generating functions in the next section have also an analytic meaning and allow to tackle the problem of studying large time behaviours. This is interesting e.g. in the case of the particular solution of Witten–Kontsevich (see below), because the matrix Ψ could be written not just as a formal series but analytically in terms of Airy functions. We postpone this type of analysis to a forthcoming publication [34].

Remark 4.3. The matrix Θ is a resolvent of the matrix-valued Lax operator; see e.g. pages 158–159 in [1].

4.1. Generating function of two-point correlation functions

In this subsection we prove Proposition 1.6.

Proof of Proposition 1.6. Applying the operator $\nabla(w)$ on both sides of (1.19) we obtain

$$\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\sum_{j=0}^{\infty}\frac{(2j+1)!!}{z^{2j+2}}\langle\langle\tau_k\tau_j\rangle\rangle = \frac{1}{2}\,\nabla(w)\,\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right) - \nabla(w)\vartheta_z(z). \tag{4.6}$$

Clearly

$$\nabla(w)\vartheta_z(z) = \frac{z^2+w^2}{(z^2-w^2)^2}. \tag{4.7}$$

It is easy to see that $\Psi^{-1}(z) = \frac{1}{2\,z}\sigma_2\Psi(z)^T\sigma_2$. Here and below we will use Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{4.8}$$

We have

$$\begin{aligned}
&\nabla(w)\,\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right)\\
&\quad=\frac{1}{2z}\mathrm{Tr}\left[\sigma_2\nabla(w)\left(\Psi(z)^T\right)\sigma_2\Psi_z(z)\sigma_3+\sigma_2\Psi(z)^T\sigma_2\nabla(w)\left(\Psi_z(z)\right)\sigma_3\right]\\
&\quad=\frac{2}{(w^2-z^2)^2}\mathrm{Tr}\left(\Theta(w)\Theta(z)\right)-\frac{i\,\mathrm{Tr}\left[\Psi(z)^T(\Theta(w)^T\sigma_2+\sigma_2\,\Theta(w))\Psi_z(z)\sigma_1\right]}{2z\,(w^2-z^2)}\\
&\qquad-\frac{i}{2z}\,\mathrm{Tr}\left[\Psi(z)^T(Q(w)^T\sigma_2+\sigma_2\,Q(w))\Psi_z(z)\sigma_1\right].
\end{aligned}\tag{4.9}$$

Here

$$Q(w):=\begin{pmatrix}0&0\\ \mathcal{R}(w)&0\end{pmatrix}.\tag{4.10}$$

Noticing that $Q(w)^T\sigma_2+\sigma_2\,Q(w)=0,\ \Theta(w)^T\sigma_2+\sigma_2\,\Theta(w)=0$, we have

$$\nabla(w)\left[\mathrm{Tr}\left(\Psi^{-1}(z)\Psi_z(z)\sigma_3\right)\right]=\frac{2}{(w^2-z^2)^2}\mathrm{Tr}\left(\Theta(w)\Theta(z)\right).\tag{4.11}$$

Hence

$$\sum_{k=0}^{\infty}\frac{(2k+1)!!}{w^{2k+2}}\sum_{j=0}^{\infty}\frac{(2j+1)!!}{z^{2j+2}}\langle\langle\tau_k\tau_j\rangle\rangle=\frac{\mathrm{Tr}\left(\Theta(w)\Theta(z)\right)-z^2-w^2}{(w^2-z^2)^2}\tag{4.12}$$

$$=\frac{\frac12\mathcal{R}_x(w)\mathcal{R}_x(z)-\frac12(\mathcal{R}_{xx}(w)\mathcal{R}(z)+\mathcal{R}_{xx}(z)\mathcal{R}(w))+(z^2+w^2-4u)\mathcal{R}(w)\mathcal{R}(z)-z^2-w^2}{(w^2-z^2)^2}.\tag{4.13}$$

Substituting (2.11) into the formula we obtain (1.24). The theorem is proved. □

Example 4.4. The first few two-point correlation functions are given by

$$\langle\langle\tau_0^2\rangle\rangle=u,\qquad\langle\langle\tau_0\tau_1\rangle\rangle=\frac{u^2}{2}+\frac{1}{12}u_{xx},\tag{4.14}$$

$$\langle\langle\tau_1^2\rangle\rangle=\frac{u^3}{3}+\left(\frac{u_x^2}{24}+\frac{u\,u_{xx}}{6}\right)+\frac{1}{144}u_{xxxx},\tag{4.15}$$

$$\langle\langle\tau_1\tau_2\rangle\rangle=\frac{u^4}{8}+\left(\frac{u\,u_x^2}{12}+\frac18u^2u_{xx}\right)+\left(\frac{1}{90}u\,u_4+\frac{23\,u_{xx}^2}{1440}+\frac{1}{60}\,u_xu_3\right)+\frac{1}{2880}u_6.\tag{4.16}$$

To end this subsection we point out that, for an arbitrary solution $u(\mathbf{t})$ of the KdV hierarchy, the Sato tau-function τ and the axiomatic tau-function τ^{KdV} defined in [10] coincide up to a gauge factor of the form $\mathrm{e}^{\alpha_{-1}+\sum_{j\ge0}t_j\alpha_j}$. Indeed, by definition we know that $\forall\,p,q,r$,

$$\langle\langle\tau_p\tau_q\rangle\rangle=\langle\langle\tau_q\tau_p\rangle\rangle,\qquad\langle\langle\tau_p\tau_q\tau_r\rangle\rangle=\langle\langle\tau_q\tau_r\tau_p\rangle\rangle=\langle\langle\tau_r\tau_p\tau_q\rangle\rangle.\tag{4.17}$$

From Proposition 1.6 and Lemma 2.1 we know that $\langle\langle\tau_p\tau_q\rangle\rangle\in\mathcal{A}_u$ and that

$$\langle\langle\tau_p\tau_q\rangle\rangle|_{u_j=0,\,j\ge1}=\frac{u^{p+q+1}}{p!q!(p+q+1)}.\tag{4.18}$$

So τ satisfies the defining properties of τ^{KdV}.

4.2. Generating functions of multi-point correlation functions

Notation. For a permutation $r\in S_n,\ n\ge2$ denote

$$P(r)=-\prod_{j=1}^{n}\frac{1}{z_{r_j}^2-z_{r_{j+1}}^2}\tag{4.19}$$

where r_{n+1} is understood as r_1.

Lemma 4.5. *The following formula holds true:*

$$\nabla(z)\Theta(w)=\frac{1}{z^2-w^2}[\Theta(z),\Theta(w)]+[Q(z),\Theta(w)]\tag{4.20}$$

with Q defined in (4.10).

Proof. We have, using the Leibniz rule,

$$\begin{aligned}\nabla(z)\,\Theta(w) &= \nabla(z)\left[\Psi(w)\sigma_3\Psi(w)^{-1}\right] = \nabla(z)\left(\Psi(w)\right)\sigma_3\Psi(w)^{-1} + \Psi(w)\sigma_3\nabla(z)\left(\Psi(w)^{-1}\right)\\ &= \left[\frac{1}{z^2-w^2}\Theta(z)\Psi(w) + Q(z)\Psi(w)\right]\sigma_3\Psi(w)^{-1}\\ &\quad + \Psi(w)\sigma_3\left[\frac{1}{z^2-w^2}\Psi(w)^{-1}\sigma_2\Theta(z)^T\sigma_2 + \Psi(w)^{-1}\sigma_2 Q(z)^T\sigma_2\right]\\ &= \frac{1}{z^2-w^2}[\Theta(z),\Theta(w)] + [Q(z),\Theta(w)]. \quad \square\end{aligned} \tag{4.21}$$

Proof of Theorem 1.7. We use mathematical induction on n. For $n = 2$, the theorem has been already verified in Proposition 1.6. Suppose it is true for $n = p$ $(p \geq 2)$. For $n = p + 1$,

$$\begin{aligned}&\sum_{k_1,\ldots,k_{p+1}=0}^{\infty} \frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_{p+1}+1)!!}{z_{p+1}^{2k_{p+1}+2}}\langle\langle\tau_{k_1}\ldots\tau_{k_{p+1}}\rangle\rangle\\ &= \nabla(z_{p+1})\sum_{k_1,\ldots,k_p=0}^{\infty}\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_p+1)!!}{z_{p+1}^{2k_p+2}}\langle\langle\tau_{k_1}\ldots\tau_{k_p}\rangle\rangle\\ &= \nabla(z_{p+1})\left(\frac{1}{p}\sum_{r\in S_p}P(r)\,\mathrm{Tr}\left(\Theta(z_{r_1})\cdots\Theta(z_{r_p})\right) - \delta_{p,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}\right)\\ &= \frac{1}{p}\sum_{r\in S_p}P(r)\sum_{j=1}^{p}\mathrm{Tr}\left[\Theta(z_{r_1})\cdots\nabla(z_{p+1})\left(\Theta(z_{r_j})\right)\cdots\Theta(z_{r_p})\right]\\ &= \frac{1}{p}\sum_{r\in S_p}P(r)\sum_{j=1}^{p}\mathrm{Tr}\left[\Theta(z_{r_1})\cdots\left(\frac{1}{z_{p+1}^2-z_{r_j}^2}[\Theta(z_{p+1}),\Theta(z_{r_j})] + [Q(z_{p+1}),\Theta(z_{r_j})]\right)\cdots\Theta(z_{r_p})\right]\\ &\overset{\star}{=} \frac{1}{p}\sum_{r\in S_p}P(r)\sum_{j=1}^{p}\left(\frac{1}{z_{p+1}^2-z_{r_j}^2} - \frac{1}{z_{p+1}^2-z_{r_{j-1}}^2}\right)\mathrm{Tr}\left(\Theta(z_{r_1})\cdots\Theta(z_{r_{j-1}})\Theta(z_{p+1})\Theta(z_{r_j})\cdots\Theta(z_{r_p})\right)\\ &= \frac{1}{p}\sum_{j=1}^{p}\sum_{r\in S_p}P(r)\frac{z_{r_j}^2-z_{r_{j-1}}^2}{(z_{p+1}^2-z_{r_j}^2)(z_{p+1}^2-z_{r_{j-1}}^2)}\mathrm{Tr}\left(\Theta(z_{p+1})\Theta(z_{r_j})\cdots\Theta(z_{r_p})\Theta(z_{r_1})\cdots\Theta(z_{r_{j-1}})\right)\\ &= \frac{1}{p}\sum_{j=1}^{p}\sum_{r\in S_p}P\left([p+1,r_j,\ldots,r_p,r_1,\ldots,r_{j-1}]\right)\mathrm{Tr}\left(\Theta(z_{p+1})\Theta(z_{r_j})\cdots\Theta(z_{r_p})\Theta(z_{r_1})\cdots\Theta(z_{r_{j-1}})\right)\\ &\overset{\dagger}{=} \sum_{r\in S_p}P\left([p+1,r]\right)\,\mathrm{Tr}\left(\Theta(z_{p+1})\Theta(z_{r_1})\cdots\Theta(z_{r_p})\right)\\ &= \frac{1}{p+1}\sum_{r\in S_{p+1}}P(r)\,\mathrm{Tr}\left(\Theta(z_{r_1})\cdots\Theta(z_{r_{p+1}})\right).\end{aligned} \tag{4.22}$$

Here, for a permutation $r = [r_1,\ldots,r_\ell] \in S_\ell$, $r_0 := r_\ell$, and we have used the facts that both $P(r)$ and matrix trace of products are invariant under the cyclic permutation. In the step marked with an asterisk, we have used that the terms of the form

$$\sum_{j=1}^{p}\mathrm{Tr}\left[\Theta(z_{r_1})\cdots\overbrace{[Q(z_{p+1}),\Theta(z_{r_j})]}^{j\text{-th place}}\cdots\Theta(z_{r_p})\right] \tag{4.23}$$

are zero; to see this the reader can notice that the above is the infinitesimal action of the conjugation action by $e^{sQ(z_{p+1})}$ and the trace $\mathrm{Tr}(\Theta(z_{r_1})\cdots\Theta(z_{r_p}))$ is invariant under simultaneous conjugation of the matrices. In the step marked with † we have relabelled the summation over permutations. In the last step we have used that the summand is invariant under cyclic permutations of the indices. The theorem is proved. $\square$

5. On string equation and Kac–Schwarz operator

In the remaining part of this paper we will consider a subset of solutions of the KdV hierarchy [10] whose tau-functions are specified by string equations:

$$L_{-1}\tau = 0, \quad L_{-1} := \sum_{k\geq 0}\tilde{t}_{k+1}\frac{\partial}{\partial t_k} + \frac{1}{2}\tilde{t}_0^2 \tag{5.1}$$

where $\tilde{t}_k = t_k - c_k,\ k \geq 0$, with arbitrary constants c_k. Note that the string equation (5.1) uniquely determines a tau-function up to a multiplicative constant independent of $\mathbf{t}$. The Witten–Kontsevich tau-function and the generating function of higher Weil–Petersson volumes are particular examples in this class of solutions; e.g., for the Witten–Kontsevich solution $c_1 = 1$, $c_k = 0$ for $k \neq 1$.

It was observed by V. Kac and A.S. Schwarz [35] that the string equation for the Witten–Kontsevich tau-function implies a differential equation in the spectral parameter z for the associated wave function. The equation can be written in terms of the linear differential operator $S_z = \frac{1}{z}\partial_z - \frac{1}{2z^2} - z$ that will be called *Kac–Schwarz operator*. In this section, following A. Buryak [36], we give a generalization of the Kac–Schwarz operator for the general class of solutions satisfying (5.1).

Definition 5.1. The generalized Kac–Schwarz operator S_z associated to the string equation (5.1) is defined as the following differential operator of z:

$$S_z = \frac{1}{z}\partial_z - \frac{1}{2z^2} - \sum_{k=0}^{\infty} \frac{c_k}{(2k-1)!!} z^{2k-1} \tag{5.2}$$

where the constants c_k are the shifts in the times t_k in (5.1).

The above definition also can been generalized to the Gelfand–Dickey hierarchy as in [35,37].

Proposition 5.2. *Let $K(z;\mathbf{t}) := \psi(z;\mathbf{t}) \cdot \tau(\mathbf{t}) = \tau(\mathbf{t} - [z^{-1}])\, \mathrm{e}^{\vartheta(z)}$. Then the following formula holds true:*

$$L_{-1}K = S_z K. \tag{5.3}$$

Proof. On one hand we have

$$\begin{aligned}
L_{-1}K &= L_{-1}\left(\tau(\mathbf{t} - [z^{-1}]) \cdot \mathrm{e}^{\vartheta}\right) \\
&= \mathrm{e}^{\vartheta} \cdot \sum_{k=0}^{\infty} \tilde{t}_{k+1} \frac{\partial \tau(\mathbf{t} - [z^{-1}])}{\partial t_k} + \sum_{k\geq 0} \frac{\tilde{t}_{k+1}\, z^{2k+1}}{(2k+1)!!} K + \frac{(\tilde{t}_0)^2}{2} K \\
&= \mathrm{e}^{\vartheta} \cdot \sum_{k=0}^{\infty} \left(\tilde{t}_{k+1} - \frac{(2k+1)!!}{z^{2k+3}}\right) \frac{\partial \tau(\mathbf{t} - [z^{-1}])}{\partial t_k} + \sum_{k\geq 0} \frac{\tilde{t}_{k+1}\, z^{2k+1}}{(2k+1)!!} K + \frac{(\tilde{t}_0 - z^{-1})^2}{2} K \\
&\quad + \mathrm{e}^{\vartheta} \cdot \sum_{k=0}^{\infty} \frac{(2k+1)!!}{z^{2k+3}} \frac{\partial \tau(\mathbf{t} - [z^{-1}])}{\partial t_k} - \frac{1}{2z^2} K + \frac{1}{z}\tilde{t}_0 K \\
&= \mathrm{e}^{\vartheta} \cdot \sum_{k=0}^{\infty} \frac{(2k+1)!!}{z^{2k+3}} \frac{\partial \tau(\mathbf{t} - [z^{-1}])}{\partial t_k} - \frac{1}{2z^2} K + \sum_{k\geq -1} \frac{\tilde{t}_{k+1}\, z^{2k+1}}{(2k+1)!!} K.
\end{aligned} \tag{5.4}$$

On another hand we have

$$\frac{1}{z}\partial_z K = \mathrm{e}^{\vartheta} \sum_{k=0}^{\infty} \frac{(2k+1)!!}{z^{2k+3}} \frac{\partial \tau(\mathbf{t} - [z^{-1}])}{\partial t_k} + \sum_{k\geq 0} \frac{t_k z^{2k-1}}{(2k-1)!!} K. \tag{5.5}$$

Since $\widetilde{t_k} = t_k - c_k$, we obtain $L_{-1}K = \frac{1}{z}\partial_z K - \frac{1}{2z^2}K - \sum_{k=0}^{\infty} \frac{c_k z^{2k-1}}{(2k-1)!!} K$, which was our statement. □

Proposition 5.3. *Let $u_0(x) := \partial_x^2 \log \tau(x, \mathbf{0})$ and let $f(z;x) := \psi(z;x,\mathbf{0})$; then*

$$S_z f(z;x) = -c_1 f_x(z;x) - \sum_{k\geq 1} c_{k+1} \psi_{t_k}(z;x,\mathbf{0}), \tag{5.6}$$

$$S_z f_x(z;x) = -c_1\left(z^2 - 2u_0(x)\right) f(z;x) - \sum_{k\geq 1} c_{k+1} \psi_{t_0 t_k}(z;x,\mathbf{0}), \tag{5.7}$$

$$\psi_{t_k}(z;x,\mathbf{0}) = \frac{1}{(2k+1)!!} \sum_{i=0}^{k} (2i-1)!!\, z^{2k-2i} \left(\Omega_{i-1;0}|_{u\to u_0} f_x(z;x) - \frac{1}{2} f(z;x)\partial_x \Omega_{i-1;0}|_{u\to u_0} \right), \tag{5.8}$$

$$\begin{aligned}
\psi_{t_0 t_k}(z;x,\mathbf{0}) \;=\; & \frac{1}{(2k+1)!!} \sum_{i=0}^{k} (2i-1)!!\, z^{2k-2i} \Bigg(\partial_x \Omega_{i-1;0}|_{u\to u_0} f_x(z;x) + \Omega_{i-1;0}|_{u\to u_0} \left(z^2 - 2u_0(x)\right) f(z;x) \\
& - \frac{1}{2} f_x(z;x)\partial_x \Omega_{i-1;0}|_{u\to u_0} - \frac{1}{2} f(z;x)\partial_x^2 \Omega_{i-1;0}|_{u\to u_0} \Bigg).
\end{aligned} \tag{5.9}$$

Proof. Taking $t_k = 0,\ k \geq 1$ in (5.3) gives

$$S_z f(z;x) = \frac{c_0^2}{2} f(z;x) - \sum_{k\geq 0} c_{k+1} \frac{\partial \psi}{\partial t_k}(z;x,\mathbf{0}) - \sum_{k\geq 0} c_{k+1} \frac{\partial \log \tau}{\partial t_k}(x,\mathbf{0}) f(z;x). \tag{5.10}$$

Taking $t_k = 0,\ k \geq 1$ in (5.1) and substituting it into the above equality one obtains (5.6). Eq. (5.7) is obtained by taking x-derivative of (5.6). Eq. (5.8) can be derived from the defining Eqs. (1.11) and (1.12), which is a standard expression hence omitted. Taking the x-derivative in (5.8) yields (5.9). □

We will apply in Appendix B results of Propositions 5.2, 5.3 to computation of higher Weil–Petersson volumes.

6. Application to the Witten–Kontsevich tau-function

In this section we consider a particular solution of the KdV hierarchy (1.3). Namely, we are going to apply Theorems 1.2, 1.7 and the Witten–Kontsevich theorem to derive n-point functions of the intersection numbers (1.31).

To do so we first derive the *initial datum* of the wave function $\psi^{WK}(z;\mathbf{t})$. Denote $f^{WK}(z;x) = \psi^{WK}(z;\mathbf{t})|_{\mathbf{t}_{\geq 1}=\mathbf{0}}$. Noting that $u_0^{WK}(x) = x$, we have

$$f_{xx}^{WK} + 2\,x f^{WK} = z^2 f^{WK}. \tag{6.1}$$

Solutions of this ODE can be expressed as linear combinations of Airy functions. Imposing the asymptotic condition $f^{WK}(z;x) = \left(1+\mathcal{O}(z^{-1})\right)\exp(xz)$, it is easy to obtain

$$f^{WK}(z;x) = G(z)\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{1}{3}}\mathrm{Ai}\left(2^{-\frac{2}{3}}(z^2-2\,x)\right) \tag{6.2}$$

where $G(z) = 1+\mathcal{O}(z^{-1})$ is yet to be determined and is a function independent of x. The asymptotic expansion of (6.2) is of the indicated form only in a suitable sector; the dual wave function admitting the required asymptotic in the same sector is found in Appendix A. To fix the gauge freedom we need to enforce the *string equation*

$$\sum_{k=0}^{\infty} t_{k+1}\frac{\partial Z}{\partial t_k} + \frac{t_1^2}{2}Z = \frac{\partial Z}{\partial t_1} \tag{6.3}$$

which is derived by Witten [5]. One immediately reads off from (6.3) that

$$c_0 = 0, \qquad c_1 = 1, \qquad c_k = 0 \quad (k\geq 2). \tag{6.4}$$

So, due to Definition 5.1, the Kac–Schwarz operator associated to $Z(\mathbf{t})$ reads

$$S_z^{WK} = \frac{1}{z}\partial_z - \frac{1}{2z^2} - z. \tag{6.5}$$

Proposition 5.3 with the values (6.4) yields immediately that

$$S_z^{WK} f^{WK}(z;x) = -\partial_x f^{WK}(z;x) = \sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{2}{3}}G(z)\mathrm{Ai}'\left(2^{-\frac{2}{3}}(z^2-2\,x)\right). \tag{6.6}$$

On the other hand from (6.2) we have

$$S_z^{WK} f^{WK}(z;x) = \sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{1}{3}}\left(2^{\frac{1}{3}}G(z)\mathrm{Ai}'\left(2^{-\frac{2}{3}}(z^2-2\,x)\right) + \frac{G'(z)}{z}\mathrm{Ai}\left(2^{-\frac{2}{3}}(z^2-2\,x)\right)\right). \tag{6.7}$$

Comparing (6.6), (6.7) we conclude that $G(z)$ is a constant and thus $G(z)\equiv 1$.

Definition 6.1. We define $A^{WK}(z) := \psi^{WK}(z;\mathbf{0}) = f^{WK}(z;0)$, $B^{WK}(z) := \psi_x^{WK}(z;\mathbf{0}) = f_x^{WK}(z;0)$.

Setting $x=0$ in (6.7) yields

$$\begin{aligned} A^{WK}(z) &= \sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{1}{3}}\mathrm{Ai}\left(2^{-\frac{2}{3}}z^2\right) \simeq c(z),\\ B^{WK}(z) &= -\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}\,2^{\frac{2}{3}}\,\mathrm{Ai}'\left(2^{-\frac{2}{3}}z^2\right) \simeq z\,q(z) \end{aligned} \tag{6.8}$$

where the symbol $\simeq$ stands for asymptotic equivalence, and $c(z)$ and $q(z)$ are known as the Faber–Zagier series [16,36] which can be computed from the known expansions of Airy functions

$$\mathrm{Ai}(z) \sim \frac{\mathrm{e}^{-\zeta}}{2\sqrt{\pi}z^{1/4}}\sum_{k=0}^{\infty}\frac{(6k-1)!!}{(2k-1)!!}\frac{(-216\,\zeta)^{-k}}{k!}, \quad \zeta = \frac{2}{3}z^{3/2},\ z\to\infty,\ |\arg z|<\pi. \tag{6.9}$$

So

$$c(z) = \sum_{k=0}^{\infty} C_k z^{-3k}, \qquad C_k = \frac{(-1)^k}{288^k}\frac{(6k)!}{(3k)!(2k)!}, \tag{6.10}$$

$$q(z) = \sum_{k=0}^{\infty} q_k z^{-3k}, \qquad q_k = \frac{1+6k}{1-6k}C_k. \tag{6.11}$$

For future reference, we also have, from Proposition 5.3

Lemma 6.2. *$A^{WK}(z)$, $B^{WK}(z)$ satisfy the following system of ODEs*

$$\begin{cases} S_z^{WK}A^{WK}(z) = -B^{WK}(z),\\ S_z^{WK}B^{WK}(z) = -z^2A^{WK}(z). \end{cases} \tag{6.12}$$

The Faber–Zagier series satisfy the well-known relation

$$c(z)\,q(-z)+c(-z)\,q(z)=2, \tag{6.13}$$

which expresses (2.16) of Lemma 2.3. Furthermore they possess the following properties.

Lemma 6.3. *The following equalities hold true:*

$$c(z)\cdot c(-z)=\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g}, \tag{6.14}$$

$$q(z)\cdot q(-z)=-\sum_{g=0}^{\infty}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g}, \tag{6.15}$$

$$c(z)\cdot q(-z)=1-\frac{1}{2}\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+3}, \tag{6.16}$$

$$q(z)\cdot c(-z)=1+\frac{1}{2}\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+3}. \tag{6.17}$$

Proof. By straightforward computations and by using summation formulæ for hypergeometric series (i.e. Dixon's identities and contiguous identities [38]). □

Due to the definition of $Z(\mathbf{t})$, the relation between the intersection numbers and the n-point correlation functions of $Z(\mathbf{t})$ consists simply in an evaluation at $\mathbf{t}=0$:

$$\langle\tau_{k_1}\dots\tau_{k_n}\rangle=\langle\!\langle\tau_{k_1}\dots\tau_{k_n}\rangle\!\rangle|_{\mathbf{t}=0}. \tag{6.18}$$

Proof of Theorem 1.11. Let us first consider one-point intersection numbers. According to Theorem 1.2 and the Witten–Kontsevich theorem, we have

$$\begin{aligned}F_1^{WK}(z)&=\frac{1}{4}\left[q(z)\,c'(-z)+c'(z)\,q(-z)+\frac{c(-z)}{z}\left(q(z)+z\,q'(z)\right)+\frac{c(z)}{z}\left(-q(-z)+z\,q'(-z)\right)\right]\\&=z^2\left(1-\frac{c(z)c(-z)+q(z)q(-z)}{2}\right)=\sum_{g=1}^{\infty}\frac{(6g-3)!!}{24^g\cdot g!}z^{-(6g-2)}.\end{aligned} \tag{6.19}$$

Thus we obtain

$$\langle\tau_j\rangle=\begin{cases}\dfrac{1}{24^g\cdot g!} & j=3g-2,\\ 0 & \text{otherwise},\end{cases} \tag{6.20}$$

i.e. $\langle\tau_{3g-2}\rangle=\langle\tau_{3g-2}\rangle_g=\frac{1}{24^g\cdot g!}$.

Now we consider the n-point function for intersection numbers of ψ-classes. Substituting (6.8) into (1.27) and using Lemma 6.3 we obtain

$$\begin{aligned}\Theta(z;0)&=\frac{1}{2}\begin{pmatrix}z\left(c(z)q(-z)-c(-z)q(z)\right) & -2\,c(z)c(-z)\\ -2z^2q(z)q(-z) & -z\left(c(z)q(-z)-c(-z)q(z)\right)\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}-\displaystyle\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+4} & -2\displaystyle\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g}\\ 2\displaystyle\sum_{g=0}^{\infty}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g+2} & \displaystyle\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+4}\end{pmatrix}=:M(z).\end{aligned} \tag{6.21}$$

So we have

$$F_n^{WK}(z_1,\dots,z_n)=\frac{1}{n}\sum_{r\in S_n}P\left(r\right)\,\mathrm{Tr}\left(M(z_{r_1})\cdots M(z_{r_n})\right)-\delta_{n,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}. \tag{6.22}$$

The theorem is proved. □

Theorem 1.11 allows an efficient way of computing the intersection numbers of ψ-classes. E.g., for the two point function, write

$$\mathrm{Tr}\left(M(z_1)M(z_2)\right)=\sum_{k_1,k_2=-1}^{\infty}a_{k_1,k_2}\,z_1^{-2k_1}z_2^{-2k_2}, \tag{6.23}$$

where a_{k_1,k_2} are rational numbers defined by

$$a_{k_1,k_2} = \begin{cases} \dfrac{(6g_1-5)!!\cdot(6g_2-5)!!}{2\cdot 24^{g_1+g_2-2}\cdot(g_1-1)!\cdot(g_2-1)!}, & \text{if } k_1 = 3g_1-2,\ k_2 = 3g_2-2,\ g_1,g_2\geq 1,\\ -\dfrac{(6g_1-1)!!\cdot(6g_2-1)!!}{24^{g_1+g_2}\cdot g_1!\cdot g_2!}\dfrac{6g_2+1}{6g_2-1}, & \text{if } k_1 = 3g_1,\ k_2 = 3g_2-1,\ g_1,g_2\geq 0,\\ -\dfrac{(6g_1-1)!!\cdot(6g_2-1)!!}{24^{g_1+g_2}\cdot g_1!\cdot g_2!}\dfrac{6g_1+1}{6g_1-1}, & \text{if } k_1 = 3g_1-1,\ k_2 = 3g_2,\ g_1,g_2\geq 0,\\ 0, & \text{otherwise.} \end{cases} \tag{6.24}$$

Then formula (1.36) yields the following explicit expressions for two-point correlators

$$\langle\tau_{3g_1+1}\tau_{3g_2+1}\rangle = \int_{\overline{\mathcal{M}}_{g_1+g_2+1,2}} \psi_1^{3g_1+1}\psi_2^{3g_2+1} = \frac{\sum_{\ell=0}^{3g_1+1}(3g_1+2-\ell)\cdot a_{\ell-1,3g-\ell}}{(6g_1+3)!!\cdot(6g_2+3)!!},\quad g_1,g_2\geq 0, \tag{6.25}$$

$$\langle\tau_{3g_1+2}\,\tau_{3g_2}\rangle = \int_{\overline{\mathcal{M}}_{g_1+g_2+1,2}} \psi_1^{3g_1+2}\psi_2^{3g_2} = \frac{\sum_{\ell=0}^{3g_1+2}(3g_1+3-\ell)\cdot a_{\ell-1,3g-\ell}}{(6g_1+5)!!\cdot(6g_2+1)!!},\quad g_1,g_2\geq 0. \tag{6.26}$$

For example, $\int_{\overline{\mathcal{M}}_{51,2}}\psi_1^{61}\psi_2^{91}$ is equal to

$$\frac{9386050172836412587500989359024403743277403220016343379}{1295911182815633150530109902582474075123568533734585207009074641418782551472383393793315566660413635620549920713937621921151313421430423552000000000000}.$$

The above algorithm can also be easily applied to computation of high genus multipoint intersection numbers,[1] e.g.

$$\langle\tau_{20}\tau_{21}\tau_{22}\rangle = \frac{59907930252114536543946157271}{34410236643719662106010647646034081605299946240000}\quad(\text{genus } 21), \tag{6.27}$$

$$\langle{\tau_8}^2{\tau_9}^2\rangle = \frac{15779395279487}{15064643317373337600}\quad(\text{genus } 11). \tag{6.28}$$

Relationship with topological recursion. Let us briefly comment on the relationship between the formulæ (1.36) and functions of Eynard–Orantin type, sometimes referred to as solutions of the *topological recursions*. Recall that $Z(\mathbf{t})$ admits the genus expansion

$$\log Z(\mathbf{t}) = \sum_{g=0}^{\infty}\mathcal{F}_g(\mathbf{t}), \tag{6.29}$$

where $\mathcal{F}_g$ are genus g free energies corresponding to the Witten–Kontsevich solution

$$\mathcal{F}_g(\mathbf{t}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1+\cdots+k_n=3g-3+n}\langle\tau_{k_1}\cdots\tau_{k_n}\rangle_{g,n}\,t_{k_1}\cdots t_{k_n}. \tag{6.30}$$

Define

$$W_{g,n}^{WK}(z_1,\ldots,z_n) = \nabla(z_1)\cdots\nabla(z_n)\mathcal{F}_g(\mathbf{t})\,|_{\mathbf{t}=\mathbf{0}} \tag{6.31}$$

where we remind the reader that

$$\nabla(z) = \sum_{k=0}^{\infty}\frac{(2k+1)!!}{z^{2k+2}}\frac{\partial}{\partial t_k}. \tag{6.32}$$

Hence we have

$$F_n^{WK}(z_1,\ldots,z_n) = \sum_{g=0}^{\infty}W_{g,n}^{WK}(z_1,\ldots,z_n). \tag{6.33}$$

The functions $W_{g,n}^{WK}(z_1,\ldots,z_n)$ are introduced by Eynard and Orantin [39]; see also [40,41]. It is proved for example in [41] that Eynard–Orantin's topological recursions for $W_{g,n}^{WK}$ are equivalent to the Virasoro constraints [33,10].

[1] A table of first few intersection numbers has been given in the Appendix to the preprint version arXiv:1504.06452 of the present paper.

7. Application to higher Weil–Petersson volumes

In this section we apply Theorem 1.7 to the solution of higher Weil–Petersson volumes, for which we mean intersection numbers of mixed κ- and ψ-classes:

$$\langle \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell} \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell}. \tag{7.1}$$

These numbers are zero unless

$$\sum_{j=1}^{\ell} j\, d_j + \sum_{j=1}^{n} k_j = 3g - 3 + n. \tag{7.2}$$

Theorem 7.1 *([42,30,10,43,44]). The partition function $Z^\kappa(\mathbf{t};\mathbf{s})$ defined by*

$$Z^\kappa(\mathbf{t};\mathbf{s}) = \exp\left(\sum_{g=0}^{\infty} \sum_{n,\ell \geq 0} \frac{1}{n!} \sum_{\sum j d_j + \sum k_j = 3g-3+n} \langle \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell} \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n}\, t_{k_1} \cdots t_{k_n} \frac{s_1^{d_1} \dots s_\ell^{d_\ell}}{d_1! \cdots d_\ell!} \right) \tag{7.3}$$

is a particular tau-function of the KdV hierarchy (1.3). Moreover,[2]

$$Z^\kappa(\mathbf{t};\mathbf{s}) = Z(t_0, t_1, t_2 - h_1(-\mathbf{s}), t_3 - h_2(-\mathbf{s}), \dots, t_{k+1} - h_k(-\mathbf{s}), \dots) \tag{7.4}$$

where $h_k(\mathbf{s})$ are polynomials in $s_1, s_2, \dots$ defined through

$$\sum_{k=0}^{\infty} h_k(\mathbf{s})\, x^k = \exp\left(\sum_{j=1}^{\infty} s_j x^j \right). \tag{7.5}$$

Observe that, equivalently $Z^\kappa(\mathbf{t};\mathbf{s})$ has the form

$$Z^\kappa(\mathbf{t};\mathbf{s}) = \exp\left(\sum_{g=0}^{\infty} \sum_{n,q,\ell \geq 0} \sum_{\substack{\sum m_j = n \\ \sum j d_j + \sum j m_j = 3g-3+n}} \langle \kappa_1^{d_1} \dots \kappa_\ell^{d_\ell} \tau_0^{m_0} \dots \tau_q^{m_q} \rangle_{g,n} \frac{t_0^{m_0} \cdots t_q^{m_q}}{m_0! \cdots m_q!} \frac{s_1^{d_1} \dots s_\ell^{d_\ell}}{d_1! \cdots d_\ell!} \right). \tag{7.6}$$

It is easy to see that $Z(\mathbf{t}) = Z^\kappa(\mathbf{t};\mathbf{0})$.

There exist several methods for computing the integrals (7.1), including application of the Virasoro constraints [46,9,47,43,41,44], the *quasi-triviality* approach [10,48,16], as well as an interesting method in the original paper of [27]. We propose a yet different approach, based on Theorems 1.7 and 7.1.

7.1. Proof of Theorem 1.14

Before entering into the proof proper, we need a few preparations.

Lemma 7.2. *The following formula holds true*

$$\exp\left(\sum_{m \geq 1} h_m(\mathbf{s}) q_m \right) = \sum_{\lambda \in \mathbb{Y}} \frac{s_\lambda}{m(\lambda)!} \sum_{|\mu| = |\lambda|} L_{\lambda\mu} \frac{q_\mu}{m(\mu)!}, \tag{7.7}$$

where $L_{\lambda\mu}$ is the transition matrix from the monomial basis to the power sum basis. We have used the shorthands

$$s_\lambda := \prod_{j=1}^{\ell(\lambda)} s_{\lambda_j}, \qquad q_\mu := \prod_{j=1}^{\ell(\mu)} q_{\mu_j}. \tag{7.8}$$

Proof. By Taylor expansion of the exponential we have

$$\begin{aligned} \exp\left(\sum_{m \geq 1} h_m(\mathbf{s}) q_m \right) &= \sum_{n=0}^{\infty} \sum_{\sum k_j = n} \prod_{j=1}^{\infty} \frac{q_j^{k_j}}{k_j!} \prod_{j=1}^{\infty} h_j(\mathbf{s})^{k_j} \\ &= \sum_{\mu \in \mathbb{Y}} \frac{q_\mu}{m(\mu)!} h_\mu(\mathbf{s}) = \sum_{\mu \in \mathbb{Y}} \frac{q_\mu}{m(\mu)!} \sum_{|\lambda| = |\mu|} L_{\lambda\mu} s_\lambda, \end{aligned} \tag{7.9}$$

where the last equality uses the definition of the transition matrix between homogeneous basis and power sum basis. □

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2 Recently Mattia Cafasso brought our attention to an interesting paper [45] where shifts of arguments of tau-functions have been represented in the Grassmannian approach. It would be interesting to apply the methods of [45] in order to get new insight to our computation of the $A(z;\mathbf{s})$, $B(z;\mathbf{s})$ series (see Eqs. (7.28), (7.29)). We plan to do it in a subsequent publication.

It is known that the matrix $(L_{\lambda\mu})$ equals the product of the character table and the Kostka matrix. Now we are going to give two explicit formulæ for $L_{\lambda\mu}$ for given λ, μ. Write $\mathbb{Y} = \{\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \ldots\}$, where $\sigma_0 = (0),\ \sigma_1 = (1),\ \sigma_2 = (2),\ \sigma_3 = (1^2)$, etc. Let $p(n)$ be the number of partitions of weight n, i.e. $p(0) = p(1) = 1,\ p(2) = 2,\ p(3) = 3, \ldots$; and let $r_n := \sum_{k=1}^{n} p(k),\ n \ge 1,\ r_0 := 0$.

According to the Macdonald's book [31] $L_{\lambda\mu}$ counts the number of maps f such that $f(\lambda) = \mu$. The following lemma provides an efficient way for computing the matrices $L_{\lambda\mu}$.

Lemma 7.3. *For any λ, μ satisfying $|\lambda| = |\mu|$, denote $v(\lambda) = \ell(\lambda) - m_1(\lambda)$. The following formula holds true:*

$$L_{\lambda\mu} = \sum_{1 \le k_1, \ldots, k_{v(\lambda)} \le \ell(\mu)} \begin{pmatrix} m_1(\lambda) \\ \mu_1 - \sum_{j=1}^{v(\lambda)} \delta_{1,k_j}\lambda_j, \ldots, \mu_{\ell(\mu)} - \sum_{j=1}^{v(\lambda)} \delta_{\ell(\mu),k_j}\lambda_j \end{pmatrix}. \tag{7.10}$$

Proof. By noticing that the combinatorial meaning of r.h.s. of (7.10) is the same as that of $L_{\lambda\mu}$. $\square$

Remark 7.4. Let $\kappa_{\lambda\mu} = \frac{L_{\lambda\mu}}{m(\mu)!}$. For any fixed weight $|\lambda| = |\mu|$, $(\kappa_{\lambda\mu})$ is a lower triangular matrix with integer entries which we call κ-matrix. Note that we have used the reverse lexicographic ordering for partitions. The first several κ-matrices are given by

$$(\kappa_{\lambda\mu}) = (1), \quad |\lambda| = |\mu| = 1; \qquad (\kappa_{\lambda\mu}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 2; \tag{7.11}$$

$$(\kappa_{\lambda\mu}) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 3; \tag{7.12}$$

$$(\kappa_{\lambda\mu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 4 & 3 & 6 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 4; \tag{7.13}$$

$$(\kappa_{\lambda\mu}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 4 & 3 & 3 & 1 & 0 \\ 1 & 5 & 10 & 10 & 15 & 10 & 1 \end{pmatrix}, \quad |\lambda| = |\mu| = 5. \tag{7.14}$$

We have the following observations:

(i) The sum of the last row of $(\kappa_{\lambda\mu}),\ |\lambda| = |\mu|$ is a Bell number. Indeed taking in

$$\exp\left[\sum_{k\ge1} h_k(\mathbf{s})q_k\right] = \sum_{\lambda,\mu\in\mathbb{Y}} \frac{\kappa_{\lambda\mu}}{m(\lambda)!} s_\lambda q_\mu \tag{7.15}$$

$q_1 = q_2 = \cdots = 1$ we obtain

$$\exp\left[\exp\left[\sum_{j\ge1} s_k\right] - 1\right] = \sum_{\lambda\in\mathbb{Y}} \frac{s_\lambda}{m(\lambda)!} \sum_{\mu\in\mathbb{Y}_{|\lambda|}} \kappa_{\lambda\mu}. \tag{7.16}$$

Then taking $s_2 = s_3 = \cdots = 0$ we obtain

$$\exp\left(\exp(s_1) - 1\right) = \sum_{k=0} \frac{s_1^k}{k!} \sum_{\mu\in\mathbb{Y}_k} \kappa_{(1^k)\mu}, \tag{7.17}$$

i.e. $\sum_{\mu\in\mathbb{Y}_k} \kappa_{(1^k)\mu} = B_k$ which is the k-th Bell number.

(ii) The sum of any row of $(\kappa_{\lambda\mu}),\ |\lambda| = |\mu|$ is a Bell number. More precisely,

$$\sum_{\mu\in\mathbb{Y}_{|\lambda|}} \kappa_{\lambda\mu} = B_{\ell(\lambda)}. \tag{7.18}$$

Indeed, due to the combinatorial meaning, for a fixed weight the sum only depends on the number of rows of λ. Since in (ii) we have obtained the sum for $\lambda = (1^k)$, the formula (7.18) follows for any λ.

We are now in a position to prove Theorem 1.14.

Proof of Part I. Note that for any partition $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots) \in \mathbb{Y}$, we have

$$\langle \kappa_{\lambda_1} \ldots \kappa_{\lambda_{\ell(\lambda)}} \tau_{k_1} \ldots \tau_{k_n} \rangle = \frac{\partial^{n+\ell(\lambda)} \log Z^\kappa}{\partial s_{\lambda_1} \ldots \partial s_{\lambda_{\ell(\lambda)}} \partial t_{k_1} \ldots \partial t_{k_n}} (\mathbf{0}; \mathbf{0}). \tag{7.19}$$

According to Theorem 7.1,

$$Z^\kappa(\mathbf{t};\mathbf{s}) = e^{-\sum_{k\geq 1} h_k(-\mathbf{s})\,\partial_{t_{k+1}}} Z(\mathbf{t}). \tag{7.20}$$

Therefore we can write the following expansion

$$\log Z^\kappa(\mathbf{t};\mathbf{s}) = \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)} s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \partial_{t_{\mu_1+1}} \dots \partial_{t_{\mu_{\ell(\mu)}+1}} \log Z(\mathbf{t}). \tag{7.21}$$

From (7.21) we collect the coefficient of s_λ in the generating function $F_n^\kappa(z_1,\dots,z_n;\mathbf{s})$. For this purpose we can use the generating formulæ in Theorems 1.2 and 1.7 to arrive at

$$\sum_{k_1,\dots,k_n\geq 0} \langle \kappa_{\lambda_1}\dots\kappa_{\lambda_{\ell(\lambda)}} \tau_{k_1}\dots\tau_{k_n}\rangle \frac{(2k_1+1)!!}{z_1^{2k_1+2}} \cdots \frac{(2k_n+1)!!}{z_n^{2k_n+2}}$$
$$= (-1)^{\ell(\lambda)} \sum_{|\mu|=|\lambda|} \frac{L_{\lambda\mu}}{m(\mu)!} \operatorname*{res}_{w_1=\infty} \dots \operatorname*{res}_{w_{\ell(\mu)}=\infty} w^\mu F_{\ell(\mu)+n}(w_1,\dots,w_{\ell(\mu)},z_1,\dots,z_n)\, dw_1\cdots dw_{\ell(\mu)} \tag{7.22}$$

where for a partition μ, $w^\mu := \frac{w_1^{2\mu_1+3}}{(2\mu_1+3)!!}\cdots\frac{w_{\ell(\mu)}^{2\mu_{\ell(\mu)}+3}}{(2\mu_{\ell(\mu)}+3)!!}$. This concludes the proof of Part I.

Remark 7.5. Note that

$$\partial_{t_{k_1}}\dots\partial_{t_{k_n}} \log Z^\kappa(\mathbf{t};\mathbf{s}) = e^{-\sum_{k\geq 1} h_k(-\mathbf{s})\,\partial_{t_{k+1}}} \partial_{t_{k_1}}\dots\partial_{t_{k_n}} \log Z(\mathbf{t}). \tag{7.23}$$

Let $F_n^{WK}(z_1,\dots,z_n;\mathbf{t})$ and $F_n^\kappa(z_1,\dots,z_n;\mathbf{t};\mathbf{s})$ denote the generating functions of n-point correlation functions corresponding to $Z(\mathbf{t})$ and $Z^\kappa(\mathbf{t};\mathbf{s})$, respectively. Then we have

$$F_n^\kappa(z_1,\dots,z_n;\mathbf{t};\mathbf{s}) = e^{-\sum_{k\geq 1} h_k(-\mathbf{s})\,\partial_{t_{k+1}}} F_n^{WK}(z_1,\dots,z_n;\mathbf{t})$$
$$= \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)} s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \partial_{t_{\mu_1+1}} \dots \partial_{t_{\mu_{\ell(\mu)}+1}} F_n^{WK}(z_1,\dots,z_n;\mathbf{t}). \tag{7.24}$$

Proof of Part II. By definition of the wave function we have

$$\psi^\kappa(z;\mathbf{t};\mathbf{s}) = \frac{Z^\kappa(\mathbf{t}-[z^{-1}];\mathbf{s})}{Z^\kappa(\mathbf{t};\mathbf{s})} \exp(\vartheta(z;\mathbf{t}))$$
$$= \frac{Z^\kappa(\mathbf{t}-[z^{-1}];\mathbf{s})}{Z^\kappa(\mathbf{t};\mathbf{s})} \exp(\vartheta(z;\mathbf{t})) \exp\left(-\sum_{k=2}^\infty \frac{h_{k-1}(-\mathbf{s}) z^{2k+1}}{(2k+1)!!}\right) \exp\left(\sum_{k=2}^\infty \frac{h_{k-1}(-\mathbf{s}) z^{2k+1}}{(2k+1)!!}\right)$$
$$= \exp\left(\sum_{k=2}^\infty \frac{h_{k-1}(-\mathbf{s}) z^{2k+1}}{(2k+1)!!}\right) \exp\left(-\sum_{k\geq 1} h_k(-\mathbf{s})\,\partial_{t_{k+1}}\right) \psi^{WK}(z;\mathbf{t})$$
$$= \exp\left(\sum_{k=1}^\infty \frac{h_k(-\mathbf{s}) z^{2k+3}}{(2k+3)!!}\right) \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)} s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \partial_{t_{\mu_1+1}} \dots \partial_{t_{\mu_{\ell(\mu)}+1}} \psi^{WK}(z;\mathbf{t}). \tag{7.25}$$

Similarly we have

$$\psi_x^\kappa(z;\mathbf{t};\mathbf{s}) = \exp\left(\sum_{k=1}^\infty \frac{h_k(-\mathbf{s}) z^{2k+3}}{(2k+3)!!}\right) \sum_{\lambda\in\mathbb{Y}} \frac{(-1)^{\ell(\lambda)} s_\lambda}{m(\lambda)!} \sum_{|\mu|=|\lambda|} L_{\lambda\mu} \frac{(-1)^{\ell(\mu)}}{m(\mu)!} \partial_{t_{\mu_1+1}} \dots \partial_{t_{\mu_{\ell(\mu)}+1}} \psi_x^{WK}(z;\mathbf{t}). \tag{7.26}$$

Let

$$A(z;\mathbf{s}) := \psi^\kappa(z;\mathbf{0};\mathbf{s}), \qquad B(z;\mathbf{s}) := \psi_x^\kappa(z;\mathbf{0};\mathbf{s}). \tag{7.27}$$

Note that $Z^\kappa(\mathbf{t};\mathbf{s}) = 1 + \cdots$ and that $Z^\kappa(\mathbf{t};\mathbf{0}) = Z(\mathbf{t})$. Expanding in $s_1, s_2, \dots$ we have

$$A(z;\mathbf{s}) = \sum_{k=0}^\infty \sum_{\lambda\in\mathbb{Y}_k} A^\lambda(z)\, s_\lambda, \qquad A^{(0)}(z) = A^{WK}(z), \tag{7.28}$$

$$B(z;\mathbf{s}) = \sum_{k=0}^\infty \sum_{\lambda\in\mathbb{Y}_k} B^\lambda(z)\, s_\lambda, \qquad B^{(0)}(z) = B^{WK}(z) \tag{7.29}$$

for some functions $A^\lambda(z)$ and $B^\lambda(z)$. The formulæ (1.48), (1.49) are then obtained from Theorems 1.2, 1.7 and 7.1, and this proves Part II. The proof of Theorem 1.14 is complete. □

One can use the formulæ in Lemma 4.1 and the formula (4.20) recursively to obtain

$$\nabla(z_1)\dots\nabla(z_n)\Psi. \tag{7.30}$$

At present we do not have a closed form for these generating functions of "multi-point wave functions".

7.2. Examples of computations of higher Weil–Petersson volumes

In this section we give some explicit examples of calculations of higher Weil–Petersson volumes by using the main theorem and Theorem 1.14. The following corollary of Lemma 6.3 will be useful for these computations.

Corollary 7.6. *For the Witten–Kontsevich solution of the KdV hierarchy we have*

$$\mathcal{R}(z;\mathbf{0})=\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}z^{-6g},\qquad \mathcal{R}_x(z;\mathbf{0})=\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+4},\tag{7.31}$$

$$\mathcal{R}_{xx}(z;\mathbf{0})=\sum_{g=1}^{\infty}\frac{(6g-3)!!}{24^{g-1}\cdot(g-1)!}z^{-6g+2}.\tag{7.32}$$

Example 7.7 (*Weil–Petersson Volumes*)**.** Consider the special case $\mathbf{s}=(s,0,0,\dots)$. Then

$$\log Z^{\kappa}(x,\mathbf{0};s,\mathbf{0})=\sum_{g=0}^{\infty}\sum_{\substack{n=0\\3g-3+n\geq 0}}^{\infty}\frac{s^{3g-3+n}}{(3g-3+n)!\,n!}\langle\kappa_1^{3g-3+n}\tau_0^n\rangle_{g,n}\,x^n,\tag{7.33}$$

and by taking twice the x-derivatives we find

$$u_0(x;s):=\partial_x^2\log Z^{\kappa}(x,\mathbf{0};s,\mathbf{0})=\sum_{n\geq 0}\frac{x^n}{n!}\sum_{\substack{g=0\\3g-1+n\geq 0}}^{\infty}\frac{s^{3g-1+n}}{(3g-1+n)!}\langle\kappa_1^{3g-1+n}\tau_0^{n+2}\rangle_{g,n+2}.\tag{7.34}$$

The corresponding n-point functions take the following form

$$F_n^{WP}(z_1,\dots,z_n;s):=\nabla(z_1)\cdots\nabla(z_n)\log Z^{\kappa}(\mathbf{t};s,\mathbf{0})\,|_{\mathbf{t}=0}=\sum_{g=0}^{\infty}W_{g,n}^{WP}(z_1,\dots,z_n;s)\tag{7.35}$$

where $W_{g,n}^{WP}(z_1,\dots,z_n;s)$ are rational functions of the "Eynard–Orantin type":

$$W_{g,n}^{WP}(z_1,\dots,z_n;s)=\sum_{d=0}^{\infty}\sum_{d+k_1+\dots+k_n=3g-3+n}\langle\kappa_1^d\tau_{k_1}\dots\tau_{k_n}\rangle_g\frac{s^d}{d!}\prod_{i=1}^{n}\frac{(2k_i+1)!!}{z^{2k_i+2}}.\tag{7.36}$$

Define

$$b_{n,g}=\begin{cases}\dfrac{s^{3g-1+n}}{(3g-1+n)!}\langle\kappa_1^{3g-1+n}\tau_0^{n+2}\rangle_{g,n+2}, & \text{if } 3g-1+n\geq 0,\\ 0, & \text{otherwise.}\end{cases}\tag{7.37}$$

An algorithm for computing $u_0(x;s)$ has been given in [48], from a table of which we get

$$b_{0,0}=0,\qquad b_{0,1}=\frac{s^2}{8\cdot 2!},\qquad b_{0,2}=\frac{787s^5}{128\cdot 5!},\dots,\tag{7.38}$$

$$b_{1,0}=1,\qquad b_{1,1}=\frac{7s^3}{6\cdot 3!},\qquad b_{1,2}=\frac{1498069s^6}{5760\cdot 6!},\dots,\tag{7.39}$$

$$b_{2,0}=s,\qquad b_{2,1}=\frac{529s^4}{24\cdot 4!},\qquad b_{2,2}=\frac{10098059s^7}{640\cdot 7!},\dots,\tag{7.40}$$

$$b_{3,0}=5\frac{s^2}{2!},\qquad b_{3,1}=\frac{16751s^5}{24\cdot 5!},\qquad b_{3,2}=\frac{7473953867s^8}{5760\cdot 8!},\dots.\tag{7.41}$$

Substituting the above initial data into (2.10), (1.28) we obtain

$$W_{0,2}^{WP}=0,\qquad W_{1,2}^{WP}=\frac{s^2}{16z_1^2z_2^2}+s\left(\frac{1}{4z_1^4z_2^2}+\frac{1}{4z_2^4z_1^2}\right)+\frac{3}{8z_1^4z_2^4+\frac{5}{8z_1^6z_2^2}+\frac{5}{8z_1^2z_2^6}},\tag{7.42}$$

$$\begin{aligned}W_{2,2}^{WP}&=\frac{787s^5}{15000z_1^2z_2^2}+s^4\left(\frac{1085}{4608z_1^4z_2^2}+\frac{1085}{4608z_1^2z_2^4}\right)+s^3\left(\frac{551}{576z_1^6z_2^2}+\frac{7}{8z_1^4z_2^4}+\frac{551}{576z_1^2z_2^6}\right)\\&\quad+s^2\left(\frac{399}{128z_1^8z_2^2}+\frac{181}{64z_1^6z_2^4}+\frac{181}{64z_1^4z_2^6}+\frac{399}{128z_1^2z_2^8}\right)\\&\quad+s\left(\frac{231}{32z_1^{10}z_2^2}+\frac{203}{32z_1^8z_2^4}+\frac{105}{16z_1^6z_2^6}+\frac{203}{32z_1^4z_2^8}+\frac{231}{32z_1^2z_2^{10}}\right)+\frac{1155}{128z_1^{12}z_2^2}+\frac{945}{128z_1^{10}z_2^4}\\&\quad+\frac{1015}{128z_1^8z_2^6}+\frac{1015}{128z_1^6z_2^8}+\frac{945}{128z_1^4z_2^{10}}+\frac{1155}{128z_1^2z_2^{12}},\end{aligned}\tag{7.43}$$

$$W_{0,3}^{WP} = \frac{1}{z_1^2 z_2^2 z_3^2}, \tag{7.44}$$

$$\begin{aligned} W_{1,3}^{WP} &= \frac{7s^3}{36z_1^2z_2^2z_3^2} + s^2\left(\frac{13}{16z_1^4z_2^2z_3^2} + \frac{13}{16z_1^2z_2^4z_3^2} + \frac{13}{16z_1^2z_2^2z_3^4}\right) \\ &\quad + s\left(\frac{5}{2z_1^6z_2^2z_3^2} + \frac{9}{4z_1^4z_2^4z_3^2} + \frac{9}{4z_1^4z_2^2z_3^4} + \frac{5}{2z_1^2z_2^6z_3^2} + \frac{9}{4z_1^2z_2^4z_3^4} + \frac{5}{2z_1^2z_2^2z_3^6}\right) \\ &\quad + \frac{35}{8z_1^8z_2^2z_3^2} + \frac{15}{4z_1^6z_2^4z_3^2} + \frac{15}{4z_1^6z_2^2z_3^4} + \frac{15}{4z_1^4z_2^6z_3^2} + \frac{9}{4z_1^4z_2^4z_3^4} + \frac{15}{4z_1^4z_2^2z_3^6} \\ &\quad + \frac{35}{8z_1^2z_2^8z_3^2} + \frac{15}{4z_1^2z_2^6z_3^4} + \frac{15}{4z_1^2z_2^4z_3^6} + \frac{35}{8z_1^2z_2^2z_3^8}. \end{aligned} \tag{7.45}$$

The expressions for $W_{1,2}^{WP}, W_{0,3}^{WP}$ coincide with those derived in [41]; the function $W_{0,2}^{WP}$ in [41] is not vanishing since some natural supplementary definitions for intersection numbers are used in [41]. Let $Vol_{g,n}(L_1, \ldots, L_n) := Vol(\mathcal{M}_{g,n}(L_1, \ldots, L_n))$ denote the Weil–Petersson volumes and let

$$v_{g,n}(L_1, \ldots, L_n) := \frac{Vol_{g,n}(L_1, \ldots, L_n)}{(2\pi)^{3g-3+n}}. \tag{7.46}$$

It was shown in [46,9] that

$$v_{g,n}(L_1, \ldots, L_n) = \sum_{d+k_1+\cdots k_n=3g-3+n} \frac{\langle \kappa_1^d \tau_{k_1} \ldots \tau_{k_n}\rangle_g}{d!\, k_1! \cdots k_n!} L_1^{2k_1} \cdots L_n^{2k_n}. \tag{7.47}$$

We remark that the relationship between $v_{g,n}$ and $W_{g,n}^{WP}$ is a Laplace transform [43,47,40]:

$$W_{g,n}^{WP}(z_1, \ldots, z_n; s = 1) = 2^n \int_0^\infty \cdots \int_0^\infty L_1 \cdots L_n \cdot v_{g,n}(L_1, \ldots, L_n) \cdot e^{-\sum_{i=1}^n \sqrt{2}z_i L_i} dL_1 \ldots dL_n. \tag{7.48}$$

Note that in the above example we have not used Theorem 1.14. More generally, given any initial value problem of the KdV hierarchy with an initial data $u(x, \mathbf{0}) = u_0(x) \in \mathbb{C}[[x]]$ one can compute the resolvent function $\mathcal{R}(z; x)$ of the Lax operator by solving the ODE (2.11). Then, with the help of the Main Theorem, the generating functions of multipoint correlators readily follow.

Let us now give examples of application of Theorem 1.14.

Example 7.8 (*Linear Insertion of κ-Classes*)**.** For $n = 0$, $\lambda = (j)$ we have

$$\langle \kappa_j \rangle = \operatorname*{res}_{w=\infty} \frac{-w^{2j+3}}{(2j+3)!!} F_1(w)\, dw = \operatorname*{res}_{w=\infty} \frac{-w^{2j+3}}{(2j+3)!!} \sum_{g=1}^\infty \frac{(6g-3)!!}{24^g g!} w^{-(6g-2)} dw. \tag{7.49}$$

So we have

$$\langle \kappa_{3g-3}\rangle_{g,0} = \frac{1}{24^g \cdot g!}. \tag{7.50}$$

For $n = 1$, $\lambda = (j)$ we have

$$\begin{aligned} \sum_{k_1\ge 0} \langle \kappa_j \tau_{k_1}\rangle \frac{(2k_1+1)!!}{z^{2k_1+2}} &= \operatorname*{res}_{w=\infty} \frac{-w^{2j+3}}{(2j+3)!!} F_2(w, z)\, dw & (7.51) \\ &= \operatorname*{res}_{w=\infty} \frac{-w^{2j+3}}{(2j+3)!!} \left(\frac{\operatorname{Tr}(M(w)M(z))}{(w^2-z^2)^2} - \frac{w^2+z^2}{(w^2-z^2)^2}\right) dw & (7.52) \\ &= \frac{1}{(2j+3)!!} \operatorname{Tr}\left(\frac{1}{2z}\left[\partial_z\left(z^{2j+2}M(z)\right)\right]_+ \cdot M(z)\right) - \frac{1}{(2j+1)!!} z^{2j+2}. & (7.53) \end{aligned}$$

Here "+" means taking the polynomial part in the z expansion at infinity. Particularly, if $n = 1$, $j = 1$ then we have

$$\frac{1}{2z}\left[\partial_z\left(z^4 M(z)\right)\right]_+ = \begin{pmatrix} -\frac{1}{2} & -2z^2 \\ -3z^4 & \frac{1}{2} \end{pmatrix}; \tag{7.54}$$

hence

$$\begin{aligned}\sum_{k\geq 0}\langle\kappa_1\tau_k\rangle\frac{(2k+1)!!}{z^{2k+2}} &= \frac{1}{5!!}\mathrm{Tr}\left(\frac{1}{2z}\left[\partial_z\left(z^4M(z)\right)\right]_+\cdot M(z)\right)-\frac{1}{3!!}z^4\\ &= \frac{1}{5!!}\sum_{g=1}^{\infty}\left(\frac{1}{2}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}-2\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g\cdot g!}+3\frac{(6g-1)!!}{24^g\cdot g!}\right)z^{-6g+4}\\ &= 3\sum_{g=1}^{\infty}\frac{(12g^2-12g+5)(6g-5)!!}{5!!\cdot 24^g\cdot g!}z^{-6g+4}. \end{aligned}\tag{7.55}$$

This means that

$$\langle\kappa_1\tau_{3g-3}\rangle_{g,1}=3\frac{12g^2-12g+5}{5!!\cdot 24^g\cdot g!},\quad g\geq 1. \tag{7.56}$$

If $n=1,\ j=2$ then we have

$$\frac{1}{2z}\left[\partial_z\left(z^6M(z)\right)\right]_+=\begin{pmatrix}-z^2 & -3z^4\\ \frac{7}{8}-4z^6 & z^2\end{pmatrix}; \tag{7.57}$$

whence a straightforward manipulation of series shows

$$\begin{aligned}\sum_{k\geq 0}\langle\kappa_2\tau_k\rangle\frac{(2k+1)!!}{z^{2k+2}} &= \frac{1}{7!!}\mathrm{Tr}\left(\frac{1}{2z}\left[\partial_z\left(z^6M(z)\right)\right]_+\cdot M(z)\right)-\frac{1}{5!!}z^6\\ &= 3\sum_{g=1}^{\infty}\frac{(72g^3-132g^2+95g-35)(6g-7)!!}{7!!\cdot 24^g\cdot g!}z^{-6g+6}. \end{aligned}\tag{7.58}$$

This means that

$$\langle\kappa_2\tau_{3g-4}\rangle_{g,1}=3\frac{72g^3-132g^2+95g-35}{7!!\cdot 24^g\cdot g!},\quad g\geq 2. \tag{7.59}$$

Similar computations lead to

$$\langle\kappa_3\tau_{3g-5}\rangle_{g,1}=\frac{1296g^4-3888g^3+4482g^2-2835g+945}{9!!\cdot 24^g\cdot g!},\quad g\geq 2. \tag{7.60}$$

In general, we have

$$\langle\kappa_j\tau_{3g-j-2}\rangle_{g,1}\sim\frac{6^{j+1}g^{j+1}}{(2j+3)!!\cdot 24^g\cdot g!},\quad g\to\infty. \tag{7.61}$$

For $n\geq 2,\ \lambda=(j)$ we have

$$\sum_{k_1,\dots,k_n\geq 0}\langle\kappa_j\tau_{k_1}\dots\tau_{k_n}\rangle\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n+1)!!}{z_n^{2k_n+2}}=\operatorname*{res}_{w=\infty}\frac{-w^{2j+3}}{(2j+3)!!}F_{n+1}(w,z_1,\dots,z_n)\,dw \tag{7.62}$$

$$=\operatorname*{res}_{w=\infty}\frac{w^{2j+3}}{(2j+3)!!}\sum_{r\in S_n}\frac{\mathrm{Tr}\left(M(w)M(z_{r_1})\cdots M(z_{r_n})\right)}{(w^2-z_{r_1}^2)(z_{r_n}^2-w^2)\prod_{j=1}^{n-1}(z_{r_j}^2-z_{r_{j+1}}^2)} \tag{7.63}$$

$$=\frac{1}{(2j+3)!!}\operatorname*{res}_{w=\infty}\sum_{r\in S_n}\frac{\mathrm{Tr}\left(M(w)M(z_{r_1})\cdots M(z_{r_n})\right)}{\prod_{j=1}^{n}(z_{r_j}^2-z_{r_{j+1}}^2)}\left(\frac{w^{2j+3}}{w^2-z_{r_1}^2}-\frac{w^{2j+3}}{w^2-z_{r_n}^2}\right)dw \tag{7.64}$$

$$=-\frac{1}{(2j+3)!!}\sum_{r\in S_n}\frac{\mathrm{Tr}\left(\left((z_{r_1}^{2j+2}M(z_{r_1}))_+-(z_n^{2j+2}M(z_{r_n}))_+\right)M(z_{r_1})\cdots M(z_{r_n})\right)}{\prod_{j=1}^{n}(z_{r_j}^2-z_{r_{j+1}}^2)}. \tag{7.65}$$

So we can also collect the following generating function for linear insertion of κ-classes:

$$\sum_{j\geq 1}\sum_{k_1,\dots,k_n\geq 0}\langle\kappa_j\tau_{k_1}\dots\tau_{k_n}\rangle\frac{(2j+3)!!}{w^{2j}}\frac{(2k_1+1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n+1)!!}{z_n^{2k_n+2}}=\left[w^4F_{n+1}(w,z_1,\dots,z_n)\right]_- \tag{7.66}$$

where "−" means taking the negative part in the w expansion at ∞.

Example 7.9 (*Linear Deformation of the Wave Function*)**.** For $j \geq 1$, we have

$$\begin{aligned} A^{(j)}(z) &= -\frac{z^{2j+3}}{(2j+3)!!}A^{(0)}(z) - \sum_{|\mu|=j} L_{(j),\mu}\frac{(-1)^{\ell(\mu)}}{m(\mu)!}\partial_{t_{\mu_1+1}}\ldots\partial_{t_{\mu_{\ell(\mu)}+1}}\psi^{WK}(z;\mathbf{t})|_{\mathbf{t=0}} \\ &= -\frac{z^{2j+3}}{(2j+3)!!}c(z) + \psi^{WK}_{t_{j+1}}(z;\mathbf{0}). \end{aligned} \tag{7.67}$$

By using Lemma 4.1 and Corollary 7.6 we obtain

$$\begin{aligned} \nabla(w)\psi^{WK}(z;\mathbf{t})|_{\mathbf{t=0}} &= \frac{2S(w;\mathbf{0})B^{WK}(z) - \mathcal{R}_x(w;\mathbf{0})A^{WK}(z)}{2\,(w^2-z^2)} \\ &= \frac{2\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}w^{-6g}z\,q(z) - \sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}w^{-6g+4}c(z)}{2\,(w^2-z^2)} \\ &= \frac{z\,q}{w^2} + \frac{2z^3q-c}{2\,w^4} + \frac{4z^5q-2z^2c}{4\,w^6} + \frac{(8z^7+5z)\,q-4z^4c}{8\,w^8} + \frac{(16z^9+10z^3)q-(8z^6+35)c}{16\,w^{10}} + \mathcal{O}(w^{-12}). \end{aligned} \tag{7.68}$$

So we have

$$A^{(1)}(z) = -\frac{z^5}{5!!}c + \frac{z^5}{5!!}q - \frac{z^2}{2\cdot 5!!}c = -\frac{1}{24}z^{-1} + \frac{77}{576}z^{-4} + \mathcal{O}(z^{-7}), \tag{7.69}$$

$$A^{(2)}(z) = -\frac{z^7}{7!!}c + \frac{8z^7+5z}{8\cdot 7!!}q - \frac{z^4}{2\cdot 7!!}c = \frac{1}{48}z^{-2} - \frac{13}{144}z^{-5} + \mathcal{O}(z^{-8}), \tag{7.70}$$

$$A^{(3)}(z) = -\frac{z^9}{9!!}c + \frac{16z^9+10z^3}{16\cdot 9!!}q - \frac{8z^6+35}{16\cdot 9!!}c = -\frac{11}{1152}z^{-3} + \frac{1639}{27648}z^{-6} + \mathcal{O}(z^{-9}), \tag{7.71}$$

etc. In the above formulæ it is understood that $q = q(z)$, $c = c(z)$ are Faber–Zagier series. As it was expected, indeed, all deformation terms $A^{\lambda}(z)$, $\lambda \neq (0)$ contain only negative powers in their expansions at $z \to \infty$. Similarly, we have

$$B^{(j)}(z) = -\frac{z^{2j+4}}{(2j+3)!!}q(z) + \psi^{WK}_{t_0t_{j+1}}(z;\mathbf{0}), \tag{7.72}$$

$$\begin{aligned} \nabla(w)\psi^{WK}_x(z;\mathbf{t})|_{\mathbf{t=0}} &= \frac{\mathcal{R}_x(w;\mathbf{0})B^{WK}(z) - \left[\mathcal{R}_{xx}(w;\mathbf{0}) - 2S(w;\mathbf{0})z^2\right]A^{WK}(z)}{2\,(w^2-z^2)} \\ &= \frac{\sum_{g=1}^{\infty}\frac{(6g-5)!!}{24^{g-1}\cdot(g-1)!}w^{-6g+4}z\,q(z) - \left[\sum_{g=1}^{\infty}\frac{(6g-3)!!}{24^{g-1}\cdot(g-1)!}w^{-6g+2} - 2\sum_{g=0}^{\infty}\frac{(6g-1)!!}{24^g\cdot g!}w^{-6g}z^2\right]c(z)}{2\,(w^2-z^2)} \\ &= \frac{z^2c}{w^2} + \frac{2z^4c+zq}{2w^4} + \frac{(4z^6-6)c+2z^3q}{4w^6} + \frac{(8z^8-7z^2)c+4z^5q}{8w^8} + \mathcal{O}(w^{-10}). \end{aligned} \tag{7.73}$$

And we have

$$B^{(1)}(z) = -\frac{z^6}{5!!}q + \frac{z^3}{2\cdot 5!!}q + \frac{4z^6-6}{4\cdot 5!!}c = -\frac{1}{24} + \frac{79}{576}z^{-3} + \frac{18095}{27648}z^{-6} + \mathcal{O}(z^{-9}), \tag{7.74}$$

$$B^{(2)}(z) = -\frac{z^8}{7!!}q + \frac{z^5}{2\cdot 7!!}q + \frac{8z^8-7z^2}{8\cdot 7!!}c = -\frac{1}{48}z^{-1} - \frac{55}{576}z^{-4} - \frac{31603}{55296}z^{-7} + \mathcal{O}(z^{-10}). \tag{7.75}$$

These expressions agree with expression derived from a less straightforward method in Appendix B.

Example 7.10 (*Higher Deformation of the Wave Function*)**.** Consider deformation of the wave function associated to partitions of form (1^k), $k \geq 1$. For $k = 1$, it has been solved above. Let $k = 2$; we have

$$\begin{aligned} A^{(1^2)}(z) &= \left(\frac{z^7}{2\cdot 7!!} + \frac{z^{10}}{2\cdot 5!!^2}\right)c - \frac{z^5}{5!!}\psi^{WK}_{t_2}(z;\mathbf{0}) + \frac{1}{2!}\sum_{|\mu|=2} L_{(1^2),\mu}\frac{(-1)^{\ell(\mu)}}{m(\mu)!}\partial_{t_{\mu_1+1}}\ldots\partial_{t_{\mu_{\ell(\mu)}+1}}\psi^{WK}(z;\mathbf{t})|_{\mathbf{t=0}} \\ &= \left(\frac{z^7}{2\cdot 7!!} + \frac{z^{10}}{2\cdot 5!!^2}\right)c - \frac{z^5}{5!!}\frac{4z^5q-2z^2c}{4\cdot 5!!} + \frac{1}{2!}\left(-\frac{(8z^7+5z)\,q-4z^4c}{8\cdot 7!!} + \psi^{WK}_{t_2t_2}(z;\mathbf{0})\right) \end{aligned} \tag{7.76}$$

and

$$\begin{aligned} B^{(1^2)}(z) &= \left(\frac{z^8}{2\cdot 7!!} + \frac{z^{11}}{2\cdot 5!!^2}\right)q - \frac{z^5}{5!!}\psi^{WK}_{t_0t_2}(z;\mathbf{0}) + \frac{1}{2!}\sum_{|\mu|=2} L_{(1^2),\mu}\frac{(-1)^{\ell(\mu)}}{m(\mu)!}\partial_{t_{\mu_1+1}}\ldots\partial_{t_{\mu_{\ell(\mu)}+1}}\psi^{WK}_x(z;\mathbf{t})|_{\mathbf{t=0}} \\ &= \left(\frac{z^8}{2\cdot 7!!} + \frac{z^{11}}{2\cdot 5!!^2}\right)q - \frac{z^5}{5!!}\frac{(4z^6-6)c+2z^3q}{4\cdot 5!!} + \frac{1}{2!}\left(-\frac{(8z^8-7z^2)c+4z^5q}{8\cdot 7!!} + \psi^{WK}_{t_0t_2t_2}(z;\mathbf{0})\right). \end{aligned} \tag{7.77}$$

By using (4.1) one can derive that

$$\psi^{WK}_{t_2t_2}(z;\mathbf{0}) = \frac{4z^{10}-5z^4}{4\cdot 5!!^2}c + \frac{z}{4\cdot 5!!}q, \qquad \psi^{WK}_{t_0t_2t_2}(z;\mathbf{0}) = \frac{z^{11}q}{5!!^2} - \frac{z^5 q}{180} - \frac{z^2 c}{60}. \tag{7.78}$$

So we have

$$\begin{aligned} A^{(1^2)}(z) &= \left(\frac{z^{10}}{225} + \frac{11z^7}{1575} - \frac{z^4}{2520}\right)c + \left(-\frac{z^{10}}{225} - \frac{z^7}{210} + \frac{3z}{560}\right)q \\ &= \frac{37}{1152}z^{-2} - \frac{28249}{138240}z^{-5} + \mathcal{O}(z^{-8}), \end{aligned} \tag{7.79}$$

$$\begin{aligned} B^{(1^2)}(z) &= \left(-\frac{z^{11}}{225} - \frac{z^8}{210} + \frac{z^5}{150} - \frac{z^2}{240}\right)c + \left(\frac{z^{11}}{225} + \frac{4z^8}{1575} - \frac{13z^5}{2520}\right)q \\ &= -\frac{35}{1152}z^{-1} + \frac{29051}{138240}z^{-4} + \mathcal{O}(z^{-7}). \end{aligned} \tag{7.80}$$

Using Part I of Theorem 1.14 we find

$$\langle \kappa_1^2 \tau_2\rangle_{2,1} = \frac{139}{11520}, \qquad \langle \kappa_1^2 \tau_5\rangle_{3,1} = \frac{3781}{2903040}, \qquad \langle \kappa_1^2 \tau_8\rangle_{4,1} = \frac{48689}{928972800} \tag{7.81}$$

and

$$\begin{aligned} F_2^{\kappa}(z,w;s_1=s,\mathbf{0}) = s^2 &\left(\frac{300825}{1024w^8z^8} + \frac{399}{128w^8z^2} + \frac{181}{64w^6z^4} + \frac{181}{64w^4z^6} + \frac{399}{128w^2z^8} + \frac{1}{16w^2z^2} + \cdots\right) \\ &+ s\left(\frac{231}{32w^{10}z^2} + \frac{203}{32w^8z^4} + \frac{105}{16w^6z^6} + \frac{203}{32w^4z^8} + \frac{1}{4w^4z^2} + \frac{231}{32w^2z^{10}} + \frac{1}{4w^2z^4} + \cdots\right) \\ &+ \left(\frac{1015}{128w^8z^6} + \frac{1015}{128w^6z^8} + \frac{5}{8w^6z^2} + \frac{3}{8w^4z^4} + \frac{5}{8w^2z^6} + \cdots\right) + o(s^2). \end{aligned} \tag{7.82}$$

An alternative method for computing $A(z;\mathbf{s})$, $B(z;\mathbf{s})$ is presented in Appendix B.

8. Further remarks

It would be interesting to prove directly the equivalence between the formula (1.36), the (explicit) integral/recursive formulæ of "n-point functions" given by Okounkov [24], Liu–Xu [49,50], Brézin–Hikami [51,52], and Kontsevich's main identity [6]. We indicate the relation between these three as follows. Recall that, given a formal (divergent) series of the form

$$f(z) = \sum_{k=0}^{\infty} v_k \frac{(2k+1)!!}{z^{2k+2}}, \tag{8.1}$$

it can be re-summed by a suitable version of the Borel summation method. Noting that

$$(2k+1)!! = \frac{2^{k+1}}{\sqrt{\pi}}\int_0^\infty u^{\frac{2k+1}{2}} \mathrm{e}^{-u} du, \quad k \ge 0 \tag{8.2}$$

we have, integrating term-by-term

$$\sum_{k=0}^{\infty} v_k \frac{(2k+1)!!}{z^{2k+2}} = \frac{2z}{\sqrt{\pi}}\int_0^\infty s^{\frac{1}{2}} \hat f(s)\, \mathrm{e}^{-sz^2}\, ds \tag{8.3}$$

where $\hat f(s) := \sum_{k=0}^{\infty} v_k (2s)^k$ is the Borel re-summation of $f(z)$. In general we say that $f(z)$ is Borel summable if the function $\hat f(s)$ (the "Borel transform" of f) has a nonzero radius of convergence, and it can be extended analytically along a strip surrounding the real positive axis. It then follows that the relation between the generating functions of intersection numbers that we have constructed in this paper and Okounkov's generating functions is precisely that the latter are Borel transforms of the former, or, which is the same, that the former are asymptotic expansions of the Laplace transforms of the latter. The relation between our generating function and Kontsevich's is simpler. To be precise, let us introduce the following notations

$$F_n^{OK}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n=0}^{\infty} \langle \tau_{k_1}\dots\tau_{k_n}\rangle\, x_1^{k_1}\cdots x_n^{k_n}, \tag{8.4}$$

$$F_n^{K}(z_1,\dots,z_n) = \sum_{k_1,\dots,k_n=0}^{\infty} \frac{(2k_1-1)!!}{z_1^{2k_1+2}}\cdots\frac{(2k_n-1)!!}{z_n^{2k_n+2}}\langle \tau_{k_1}\dots\tau_{k_n}\rangle. \tag{8.5}$$

Then we have

$$F_n^{WK}(z_1,\ldots,z_n)=\frac{2^n z_1\cdots z_n}{\pi^{\frac{n}{2}}}\int_0^\infty\cdots\int_0^\infty (x_1\cdots x_n)^{\frac12}F_n^{OK}(2x_1,\ldots,2x_n)\,\mathrm{e}^{-x_1z_1^2-\cdots-x_nz_n^2}dx_1\cdots dx_n, \tag{8.6}$$

$$F_n^{WK}(z_1,\ldots,z_n)=(-1)^n\left(\prod_{j=1}^n\frac{\partial}{\partial z_j}z_j\right)\left(F_n^K(z_1,\ldots,z_n)\right). \tag{8.7}$$

For $n=1$, one can directly verify (8.6) (actually in the case $n=1$ the generating series of Okounkov and ours are both well-known). Indeed,

$$F_1^{OK}(2x_1)=\frac{1}{4x_1^2}\left(\mathrm{e}^{\frac{x_1^3}{3}}-1\right)=\sum_{g=1}^\infty\frac{1}{24^g g!}(2x_1)^{3g-2}. \tag{8.8}$$

Recall that

$$F_1^{WK}(z_1)=\sum_{g=1}^\infty\frac{(6g-3)!!}{24^g g!}z_1^{-(6g-2)}. \tag{8.9}$$

So it is straightforward to verify (8.6) for $n=1$. However, it appears that for $n\geq 2$ a direct verification of (8.6) is not trivial.

The final remark is that the formula (1.28) for multi-point correlation functions possesses certain universality in tau-symmetric integrable systems. Indeed, it can be generalized to the Gelfand–Dickey hierarchy and more generally to the Drinfeld–Sokolov hierarchy, as well as to the Jimbo–Miwa–Ueno isomonodromic problems [15,14], which will be presented in a separate publication [53]. It would be interesting to investigate whether the formula works for all integrable hierarchies of topological (or cohomological) type associated to semisimple Frobenius manifolds [10,12]: the first nontrivial examples in this investigation would be the intermediate long wave hierarchy [54], the discrete KdV hierarchy [12] and the extended Toda hierarchy [55].

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Appendix A. Some useful formulæ

The wave functions associated to the Witten–Kontsevich tau-function $Z(\mathbf{t})$ at $t_1=t_2=\cdots=0$ satisfy

$$\psi(z;t_0,0,0,\ldots)=\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}2^{\frac13}\,\mathrm{Ai}(\xi), \tag{A.1}$$

$$\psi^*(z;t_0,0,0,\ldots)=\mathrm{e}^{\frac{\pi i}{6}}\sqrt{2\pi z}\,\mathrm{e}^{-\frac{z^3}{3}}2^{\frac13}\,\mathrm{Ai}(\omega\xi), \tag{A.2}$$

where $\xi=2^{-\frac23}\left(z^2-2\,t_0\right),\ \omega=\mathrm{e}^{\frac{2\pi\sqrt{-1}}{3}}$. Noting that

$$\xi_x=-2^{\frac13},\qquad \xi_z=2^{\frac13}z \tag{A.3}$$

we have

$$\psi_x(z;t_0,0,0,\ldots)=-\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}2^{\frac23}\,\mathrm{Ai}'(\xi), \tag{A.4}$$

$$\psi_x^*(z;t_0,0,0,\ldots)=-\mathrm{e}^{\frac56\pi\sqrt{-1}}\sqrt{2\pi z}\,\mathrm{e}^{-\frac{z^3}{3}}2^{\frac23}\,\mathrm{Ai}'(\omega\xi), \tag{A.5}$$

$$\psi_z(z;t_0,0,0,\ldots)=\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}2^{\frac13}\left[\left(\frac{1}{2z}+z^2\right)\mathrm{Ai}(\xi)+2^{\frac13}z\,\mathrm{Ai}'(\xi)\right], \tag{A.6}$$

$$\psi_z^*(z;t_0,0,0,\ldots)=-\sqrt{2\pi z}\,\mathrm{e}^{\frac{-z^3}{3}}2^{\frac13}\left[\mathrm{e}^{\frac16\pi i}\left(\frac{1}{-2z}+z^2\right)\mathrm{Ai}(\omega\xi)-\mathrm{e}^{\frac56\pi i}2^{\frac13}z\,\mathrm{Ai}'(\omega\xi)\right], \tag{A.7}$$

$$\psi_{zx}(z;t_0,0,0,\ldots)=-\sqrt{2\pi z}\,\mathrm{e}^{\frac{z^3}{3}}2^{\frac23}\left[\left(\frac{1}{2z}+z^2\right)\mathrm{Ai}'(\xi)+2^{-\frac13}z\left(z^2-2t_0\right)\mathrm{Ai}(\xi)\right], \tag{A.8}$$

$$\psi_{zx}^*(z;t_0,0,0,\ldots)=\sqrt{2\pi z}\,\mathrm{e}^{\frac{-z^3}{3}}2^{\frac23}\left[\mathrm{e}^{\frac56\pi i}\left(\frac{1}{-2z}+z^2\right)\mathrm{Ai}'(\omega\xi)-\mathrm{e}^{\frac16\pi i}2^{-\frac13}z\left(z^2-2t_0\right)\mathrm{Ai}(\omega\xi)\right]. \tag{A.9}$$

Appendix B. Generalized Kac–Schwarz operator and higher Weil–Petersson volumes

Let us first compute $A(z;\mathbf{s}) = \psi^\kappa(z;\mathbf{0};\mathbf{s})$, $B(z;\mathbf{s}) = \psi^\kappa_x(z;\mathbf{0};\mathbf{s})$ associated with higher Weil–Petersson volumes by using the technique of generalized Kac–Schwarz operators explained above.

Theorem 7.1 together with Eq. (6.3) implies that $Z^\kappa(\mathbf{t};\mathbf{s})$ satisfies the string equation

$$\sum_{k\geq 0}\tilde{t}_{k+1}\frac{\partial Z^\kappa}{\partial t_k}+\frac{\tilde{t}_0^2}{2}Z^\kappa=0 \tag{B.1}$$

where $\tilde{t}_0=t_0,\ \tilde{t}_k=t_k-h_{k-1}(-\mathbf{s}),\ k\geq 1$, namely,

$$c_0=0,\qquad c_k=h_{k-1}(-\mathbf{s}),\quad k\geq 1. \tag{B.2}$$

The corresponding generalized Kac–Schwarz operator (see Definition 5.1) reads as follows

$$S_z=\frac{1}{z}\partial_z-\frac{1}{2z^2}-z-\sum_{k=1}^{\infty}\frac{h_k(-\mathbf{s})}{(2k+1)!!}z^{2k+1}=S_z^{WK}-\sum_{k=1}^{\infty}\frac{h_k(-\mathbf{s})}{(2k+1)!!}z^{2k+1}. \tag{B.3}$$

As before let $f(z;x;\mathbf{s})=\psi^\kappa(z;x,\mathbf{0};\mathbf{s}),\ u_0(x;\mathbf{s})=\partial_x^2\log Z^\kappa(x,\mathbf{0};\mathbf{s})$. Then according to Proposition 5.3 we have

$$S_zf(z;x;\mathbf{s})=-f_x(z;x;\mathbf{s})-\sum_{k\geq 1}h_k(-\mathbf{s})\psi^{\kappa t_k}(z;x,\mathbf{0};\mathbf{s}), \tag{B.4}$$

$$S_zf_x(z;x;\mathbf{s})=-(z^2-2\,u_0(x;\mathbf{s}))f(z;x;\mathbf{s})-\sum_{k\geq 1}h_k(-\mathbf{s})\psi^\kappa_{t_0t_k}(z;x,\mathbf{0};\mathbf{s}), \tag{B.5}$$

$$\psi^{\kappa t_k}(z;x,\mathbf{0};\mathbf{s})=\frac{1}{(2k+1)!!}\sum_{i=0}^{k}(2i-1)!!\,z^{2k-2i}\left(\Omega_{i-1;0}|_{u\to u_0}f_x(z;x;\mathbf{s})-\frac{1}{2}f(z;x;\mathbf{s})\partial_x\Omega_{i-1;0}|_{u\to u_0}\right), \tag{B.6}$$

$$\begin{aligned}\psi^\kappa_{t_0t_k}(z;x,\mathbf{0}) \;=\; &\frac{1}{(2k+1)!!}\sum_{i=0}^{k}(2i-1)!!\,z^{2k-2i}\Big(\partial_x\Omega_{i-1;0}|_{u\to u_0}f_x(z;x)+\Omega_{i-1;0}|_{u\to u_0}\left(z^2-2u_0(x)\right)f(z;x)\\ &-\frac{1}{2}f_x(z;x)\partial_x\Omega_{i-1;0}|_{u\to u_0}-\frac{1}{2}f(z;x)\partial_x^2\Omega_{i-1;0}|_{u\to u_0}\Big).\end{aligned} \tag{B.7}$$

Taking $x=0$ in the above Eqs. (B.4), (B.5) we arrive at

Lemma B.1. *The functions $A(z;\mathbf{s})$ and $B(z;\mathbf{s})$ satisfy*

$$S_zA(z;\mathbf{s})=-B(z;\mathbf{s})-\sum_{k\geq 1}h_k(-\mathbf{s})\,\psi^{\kappa t_k}(z;\mathbf{0};\mathbf{s}), \tag{B.8}$$

$$S_zB(z;\mathbf{s})=-(z^2-2\,u_0(0;\mathbf{s}))A(z;\mathbf{s})-\sum_{k\geq 1}h_k(-\mathbf{s})\,\psi^\kappa_{t_0t_k}(z;\mathbf{0};\mathbf{s}). \tag{B.9}$$

To solve the above Eqs. (B.8), (B.9) we can expand them as formal series in $\mathbf{s}$ and compare the coefficients. At the linear approximation in $\mathbf{s}$, employing Lemma 6.2 and comparing the coefficients of $s_j,\ j\geq 1$, we get

$$\begin{aligned}\frac{z^{2j+1}}{(2j+1)!!}A^{(0)}(z)+S_z^{WK}A^{(j)}(z) \;=\; &-B^{(j)}(z)+\frac{z^{2j}}{(2j+1)!!}B^{(0)}(z)\\ &+\sum_{i=1}^{j}\frac{(2i-1)!!}{(2j+1)!!}z^{2j-2i}\left(\langle\tau_{i-1}\tau_0\rangle\,B^{(0)}(z)-\frac{1}{2}\langle\tau_{i-1}\tau_0^2\rangle A^{(0)}(z)\right),\end{aligned} \tag{B.10}$$

$$\begin{aligned}\frac{z^{2j+1}}{(2j+1)!!}B^{(0)}(z)+S_z^{WK}B^{(j)}(z)=&-z^2A^{(j)}(z)+2\langle\tau_0^2\kappa_j\rangle A^{(j)}(z)+\frac{z^{2j+2}}{(2j+1)!!}A^{(0)}(z)\\ &+\sum_{i=1}^{j}\frac{(2i-1)!!}{(2j+1)!!}z^{2j-2i}\left(\frac{1}{2}\langle\tau_0^2\tau_{i-1}\rangle B^{(0)}(z)+\langle\tau_0\tau_{i-1}\rangle z^2A^{(0)}(z)-\frac{1}{2}A^{(0)}(z)\langle\tau_0^3\tau_{i-1}\rangle\right).\end{aligned} \tag{B.11}$$

Also, taking derivatives w.r.t. x in the string equation (B.1) and taking $\mathbf{t}=0$ yields

Lemma B.2. *For any $m\geq 0$, we have*

$$\sum_{k\geq 0}h_k(-\mathbf{s})\frac{\partial^{m+1}\log Z^\kappa}{\partial t_0^m\partial t_k}(\mathbf{0};\mathbf{s})=0. \tag{B.12}$$

Comparing the coefficients of s_j in (B.12) we find

$$-\langle\tau_0^m\tau_j\rangle+\langle\tau_0^{m+1}\kappa_j\rangle=0. \tag{B.13}$$

Substituting this expression into (B.11) and using (1.35), (6.3) we can solve out $A^{(j)}(z)$, $B^{(j)}(z)$, e.g. in the case $j=1$ we have

Lemma B.3. *$A^{(1)}(z)$ and $B^{(1)}(z)$ satisfy the following ODE system*

$$S_z^{WK}A^{(1)}(z)+B^{(1)}(z)=-\frac{1}{6}A^{(0)}(z)-\frac{z^3}{3}A^{(0)}(z)+\frac{z^2}{3}B^{(0)}(z), \tag{B.14}$$

$$S_z^{WK}B^{(1)}(z)+z^2A^{(1)}(z)=\frac{1}{6}B^{(0)}(z)-\frac{z^3}{3}B^{(0)}(z)+\frac{z^4}{3}A^{(0)}(z). \tag{B.15}$$

Moreover, the solution of this ODE system is unique under the boundary condition

$$A^{(1)}(z)=\mathcal{O}(z^{-1}),\quad z\to\infty. \tag{B.16}$$

Recall that $A^{(0)}(z)=c(z),\ B^{(0)}(z)=z\,q(z)$; we obtain the explicit solution of the ODE system (B.14), (B.15):

$$A^{(1)}(z)=\frac{1}{5}\sum_{g=1}^{\infty}g\,C_g\,z^{-3g+2}, \tag{B.17}$$

$$B^{(1)}(z)=\frac{1}{120}\sum_{g=0}^{\infty}(36g^2+48g-5)\,C_g\,z^{-3g}. \tag{B.18}$$

Similarly, we know from the system (B.8), (B.9) that for any partition λ,

$$S_z^{WK}A^{\lambda}(z)+B^{\lambda}(z)=\text{combinations of }A^{\mu}(z),B^{\mu}(z)\text{ with }|\mu|<|\lambda|, \tag{B.19}$$

$$S_z^{WK}B^{\lambda}(z)+z^2A^{\lambda}(z)=\text{combinations of }A^{\mu}(z),B^{\mu}(z)\text{ with }|\mu|<|\lambda|. \tag{B.20}$$

It is not difficult to see that the combination coefficients in the above formulæ can be obtained in a recursive procedure by applying Lemma B.2 and the *Main Theorem*. Details are omitted. So, in principle one can obtain all the initial data A^{λ}, B^{λ}, $|\lambda|>0$ from (B.8), (B.9), Lemma B.2 with the knowledge of $A^{(0)}$, $B^{(0)}$.

Now let $F_n^{\kappa}(z_1,\dots,z_n;\mathbf{s}):=\nabla(z_1)\cdots\nabla(z_n)\log Z^{\kappa}(\mathbf{t};\mathbf{s})\,|_{\mathbf{t}=0}$ be the n-point generating function corresponding to the partition function Z^{κ}. We have for $n=1$,

$$\begin{aligned}F_1^{\kappa}(z;\mathbf{s})&=\frac{1}{4z}\left(-A(z;\mathbf{s})\,B_z(-z;\mathbf{s})+B_z(z;\mathbf{s})\,A(-z;\mathbf{s})+B(z;\mathbf{s})\,A_z(-z;\mathbf{s})-A_z(z;\mathbf{s})\,B(-z;\mathbf{s})\right)\\&=F_1(z)+\frac{s_1}{4z}\Big(-A^{WK}(z)\,B_z^{(1)}(-z)-A^{(1)}(z)\,B_z^{WK}(-z)+B_z^{WK}(z)\,A^{(1)}(-z)+B_z^{(1)}(z)\,A^{WK}(-z)\\&\quad+B^{WK}(z)\,A_z^{(1)}(-z)+B^{(1)}(z)\,A_z^{WK}(-z)-A_z^{WK}(z)\,B^{(1)}(-z)-A_z^{(1)}(z)B^{WK}(-z)\Big)+\text{higher order terms}\\&=F_1(z)+s_1\sum_{g=1}^{\infty}\frac{(12g^2-12g+5)(6g-5)!!}{5\cdot 24^g\cdot g!}z^{-6g+4}+\text{higher order terms}.\end{aligned} \tag{B.21}$$

The last equality uses similar derivation as in Lemma 6.3. We read off from the above expression that

$$\langle\kappa_1\tau_{3g-3}\rangle_{g,1}=\frac{12g^2-12g+5}{5\cdot 24^g\cdot g!},\quad g\geq 1. \tag{B.22}$$

For $n\geq 2$, let $M^{\kappa}(z;\mathbf{s})=\Theta(z;\mathbf{0};\mathbf{s})$ then we have

$$F_n^{\kappa}(z_1,\dots,z_n;\mathbf{s})=-\frac{1}{n}\sum_{r\in S_n}\frac{\operatorname{Tr}\left(M^{\kappa}(z_{r_1};\mathbf{s})\cdots M^{\kappa}(z_{r_n};\mathbf{s})\right)}{\prod_{j=1}^{n}(z_{r_j}^2-z_{r_{j+1}}^2)}-\delta_{n,2}\frac{z_1^2+z_2^2}{(z_1^2-z_2^2)^2}. \tag{B.23}$$

Here the matrix-value function $M^{\kappa}(z;\mathbf{s})$ has the form

$$M^{\kappa}(z;\mathbf{s})=M(z)+s_1\begin{pmatrix}M_{11}^{(1)}(z) & M_{12}^{(1)}(z)\\ M_{21}^{(1)}(z) & M_{22}^{(1)}(z)\end{pmatrix}+\text{higher order terms} \tag{B.24}$$

with

$$M_{11}^{(1)}(z)=-\frac{1}{2}\left(B^{WK}(z)\,A^{(1)}(-z)+B^{(1)}(z)\,A^{WK}(-z)+A^{WK}(z)\,B^{(1)}(-z)+A^{(1)}(z)\,B^{WK}(-z)\right), \tag{B.25}$$

$$M_{12}^{(1)}(z)=-A^{WK}(z)\,A^{(1)}(-z)-A^{(1)}(z)\,A^{WK}(-z), \tag{B.26}$$

$$M_{21}^{(1)}(z)=B^{WK}(z)\,B^{(1)}(-z)+B^{(1)}(z)\,B^{WK}(-z), \tag{B.27}$$

$$M_{22}^{(1)}(z)=-M_{12}^{(1)}(z). \tag{B.28}$$

References

[1] L.A. Dickey, Soliton Equations and Hamiltonian Systems (Vol. 26), second ed., World Scientific, 2003.
[2] M. Sato, Soliton equations as dynamical systems on a infinite dimensional grassmann manifolds (Random Systems and Dynamical Systems), RIMS Kokyuroku 439 (1981) 30–46.
[3] R. Hirota, Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27 (18) (1971) 1192–1194.
[4] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations, in: Proceedings of RIMS, 1981.
[5] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surv. Differ. Geom. 1 (1991) 243–310.
[6] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Comm. Math. Phys. 147 (1) (1992) 1–23.
[7] M. Kazarian, S. Lando, An algebro-geometric proof of Witten's conjecture, J. Amer. Math. Soc. 20 (4) (2007) 1079–1089.
[8] A. Okounkov, R. Pandharipande, Gromov–Witten theory, Hurwitz numbers, and matrix models. in: Proc. Symposia Pure Math., Vol. 80, Part 1, pp. 325–414, 2009.
[9] M. Mirzakhani, Weil–Petersson volumes and intersection theory on the moduli space of curves, J. Amer. Math. Soc. 20 (1) (2007) 1–23.
[10] B. Dubrovin, Y. Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants, ArXiv preprint, 2001. arXiv:math/0108160.
[11] B. Dubrovin, Gromov–Witten invariants and integrable hierarchies of topological type, in: Topology, Geometry, Integrable Systems, and Mathematical Physics: Novikov's Seminar 2012–2014 (Vol. 234), American Mathematical Soc, 2013, p. 141.
[12] B. Dubrovin, S.-Q. Liu, D. Yang, Y. Zhang, Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs, Adv. Math. 293 (2016) 382–435.
[13] G. Segal, G. Wilson, Loop groups and equations of KdV type, Publ. Math. Inst. Hautes Études Sci. 61 (1) (1985) 5–65.
[14] M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I. General theory and τ-function, Phys. D 2 (2) (1981) 306–352.
[15] M. Bertola, The dependence on the monodromy data of the isomonodromic tau-function, Comm. Math. Phys. 294 (2) (2010) 539–579.
[16] C. Itzykson, J.-B. Zuber, Combinatorics of the modular group. II. The Kontsevich integrals, Internat. J. Modern Phys. A 7 (23) (1992) 5661–5705.
[17] F. Balogh, D. Yang, Geometric interpretation of Zhou's explicit formula for the Witten–Kontsevich tau-function, ArXiv preprint, 2014. arXiv:1412.4419.
[18] J. Zhou, Explicit formula for Witten–Kontsevich tau-function. ArXiv preprint, 2013. arXiv:1306.5429.
[19] O. Babelon, D. Bernard, M. Talon, Introduction to Classical Integrable Systems, Cambridge University Press, 2003.
[20] B. Dubrovin, Periodic problems for the Korteweg–de Vries equation in the class of finite band potentials, Funct. Anal. Appl. 9 (3) (1975) 215–223.
[21] S.-Q. Liu, From Witten conjecture to DVV's loop equation. unpublished, 2015.
[22] J. Zhou, On absolute N-point function associated with Gelfand–Dickey polynomials, Preprint, 2015.
[23] C. Faber, A conjectural description of the tautological ring of the moduli space of curves, in: Moduli of Curves and Abelian Varieties, Vieweg + Teubner Verlag, 1999, pp. 109–129.
[24] A. Okounkov, Generating functions for intersection numbers on moduli spaces of curves, Int. Math. Res. Not. 18 (2002) 933–957.
[25] M. Bergère, B. Eynard, Determinantal formulae and loop equations, ArXiv preprint, 2009. arXiv:0901.3273.
[26] M. Bergère, G. Borot, B. Eynard, Rational differential systems, loop equations, and application to the q-th reductions of KP, in: Annales Henri Poincaré, Springer, Basel, 2013, pp. 1–70.
[27] E. Arbarello, M. Cornalba, Combinatorial and algebro-geometric cohomology classes on the moduli spaces of curves, J. Algebraic Geom. 5 (4) (1996) 705–750.
[28] D. Mumford, Towards an enumerative geometry of the moduli space of curves, in: Arithmetic and Geometry, Birkhäuser, Boston, 1983, pp. 271–328.
[29] A. Kabanov, T. Kimura, Intersection numbers and rank one cohomological field theories in genus one, Comm. Math. Phys. 194 (3) (1998) 651–674.
[30] R. Kaufmann, Y. Manin, D. Zagier, Higher Weil–Petersson volumes of moduli spaces of stable-pointed curves, Comm. Math. Phys. 181 (3) (1996) 763–787.
[31] I.G. Macdonald, Symmetric Functions and Hall Polynomials (Vol. 354), Oxford Clarendon press, 1995.
[32] B. Dubrovin, Geometry of 2D topological field theories, in: M. Francaviglia, S. Greco (Eds.), Integrable Systems and Quantum Groups, in: Lecture Notes in Math, vol. 1620, Springer, 1996, pp. 120–348.
[33] R. Dijkgraaf, H. Verlinde, E. Verlinde, Loop equations and Virasoro constraints in non-perturbative two-dimensional quantum gravity, Nuclear Phys. B 348 (3) (1991) 435–456.
[34] F. Balogh, M. Bertola, T. Bothner, D. Yang, (in preparation).
[35] V. Kac, A. Schwarz, Geometric interpretation of the partition function of 2D gravity, Phys. Lett. B 257 (3) (1991) 329–334.
[36] A. Buryak, Open intersection numbers and the wave function of the KdV hierarchy, ArXiv preprint, 2014. arXiv:1409.7957.
[37] M. Bertola, D. Yang, The partition function of the extended r-reduced Kadomtsev–Petviashvili hierarchy, J. Phys. A 48 (2015) (2015) 195205.
[38] J.-L. Lavoie, F. Grondin, A.K. Rathie, K. Arora, Generalizations of Dixon's theorem on the sum of a ${}_3F_2$, Math. Comp. 62 (205) (1994) 267–276.
[39] B. Eynard, N. Orantin, Invariants of algebraic curves and topological expansion, Commun. Number Theory Phys. 1 (2007) (2007) 347–452.
[40] J. Zhou, Intersection numbers on Deligne–Mumford moduli spaces and quantum Airy curve, ArXiv preprint, 2012. arXiv:1206.5896.
[41] J. Zhou, Topological recursions of Eynard–Orantin type for intersection numbers on moduli spaces of curves, Lett. Math. Phys. 103 (11) (2013) 1191–1206.
[42] Y.I. Manin, P. Zograf, Invertible cohomological field theories and Weil–Petersson volumes, ArXiv preprint, 1999.
[43] M. Mulase, B. Safnuk, Mirzakhani's recursion relations, Virasoro constraints and the KdV hierarchy, Indian J. Math. 50 (2008) (2006) 189–228.
[44] K. Liu, H. Xu, Recursion formulae of higher Weil–Petersson volumes, Int. Math. Res. Not (2008).
[45] M. Adler, M. Cafasso, P. van Moerbeke, Nonlinear PDEs for Fredholm determinants arising from string equations, in: Algebraic and Geometric Aspects of Integrable Systems and Random Matrices, Vol. 593, 2013, p. 1.
[46] M. Mirzakhani, Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces, Invent. Math. 167 (1) (2007) 179–222.
[47] B. Eynard, N. Orantin, Weil–Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models. ArXiv preprint, 2007. arXiv:0705.3600.
[48] P. Zograf, An algorithm for computing Weil–Petersson volumes of moduli spaces of curves, Institut Mittag-Leffler Preprints, 2006.
[49] K. Liu, H. Xu, A proof of the Faber intersection number conjecture, J. Differential Geom. 83 (2) (2009) 313–335.
[50] K. Liu, H. Xu, The n-point functions for intersection numbers on moduli spaces of curves, Adv. Theor. Math. Phys. 15 (5) (2011) 1201–1236.
[51] E. Brézin, S. Hikami, Vertices from replica in a random matrix theory, J. Phys. A 40 (45) (2007) 13545–13566.
[52] E. Brézin, S. Hikami, Random Matrix, Singularities and Open/Close Intersection Numbers. ArXiv preprint, 2015. arXiv:1502.01416.
[53] M. Bertola, B. Dubrovin, D. Yang, Simple Lie algebras and topological ODEs, ArXiv preprint, 2015. arXiv:1508.03750.
[54] A. Buryak, Dubrovin–Zhang hierarchy for the Hodge integrals, ArXiv preprint, 2013. arXiv:1308.5716.
[55] G. Carlet, B. Dubrovin, Y. Zhang, The extended Toda hierarchy, Mosc. Math. J. 4 (2) (2004) 313–332.