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# Poisson geometry and first integrals of geostrophic equations

Boris Khesin, Paul Lee*

*Department of Mathematics, University of Toronto, ON M5S 2E4, Canada*

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**Abstract**

We describe first integrals of geostrophic equations, which are similar to the enstrophy invariants of the Euler equation for an ideal incompressible fluid. We explain the geometry behind this similarity, give several equivalent definitions of the Poisson structure on the space of smooth densities on a symplectic manifold, and show how it can be obtained via the Hamiltonian reduction from a symplectic structure on the diffeomorphism group.


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## 1. Introduction

The Euler equation of an ideal incompressible $m$-dimensional fluid has a peculiar set of invariants: in addition to the energy conservation, the Euler equation has the helicity-type invariant in any odd $m$, and an infinite number of enstrophy-type invariants in any even $m$. Furthermore, these invariants are Casimir functions, which implies that they are invariants of the Euler equation for any choice of a Riemannian metric on the manifold filled by the fluid.

In this paper we show how the same enstrophy-type invariants appear in semi-geostrophic equations. These invariants are related to the Poisson geometry of the corresponding space of densities. Namely, for any Poisson manifold $M$ the space of densities on $M$ is also Poisson. The reason is that the space of functions on $M$ forms a Lie algebra with respect to the Poisson bracket, while densities are objects dual to functions, so their space forms a dual Lie algebra. Thus this dual space gets equipped with the linear Kirillov-Kostant (or Lie-Poisson) structure, see [1,7,11]. We show that this Poisson structure has several equivalent descriptions and relate it to the symplectic geometry of the diffeomorphism group of the manifold. In this paper we explore the role of Casimirs and the corresponding group actions in the Poisson geometry of these infinite-dimensional spaces.

Recall that Poisson manifolds are foliated by symplectic leaves, and Casimir functions are functions constant on symplectic leaves. Equivalently, Casimir functions are those Hamiltonians which correspond to everywhere vanishing Hamiltonian vector fields on a Poisson manifold. They are constants of motion for any Hamiltonian flow on the manifold. For instance, the hydrodynamic Euler equation on an odd-dimensional manifold has a helicity-type Casimir, which generalizes the 3-dimensional helicity integral

$$I(v) = \int_M (v, \operatorname{curl} v) \mathrm{d}\mu.$$

For an even-dimensional manifold $M$ ($m = 2n$) the Euler equation has an infinite number of enstrophy-type invariants:

$$I_h(v) = \int_M h\left(\frac{(\operatorname{curl} v)^n}{\mathrm{d}\mu}\right) \mathrm{d}\mu,$$

where $\operatorname{curl} v$ is the vorticity 2-form for the velocity field $v$ on $2n$-dimensional manifold $M$ and $h$ is any function $\mathbb{R} \to \mathbb{R}$, see [9,10]. The latter integral turns out to be similar to Casimir functions found on the space of densities on both even- and odd-dimensional manifold, as we discuss below.

We should mention that the information on Casimir functions is useful for the study of the stability of Hamiltonian

* Corresponding author. Tel.: +1 4163355047.
*E-mail addresses:* khesin@math.toronto.edu (B. Khesin), plee@math.toronto.edu (P. Lee).

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flows, e.g. via the energy-Casimir method. Note that the existence of an infinite number of Casimirs for a given Hamiltonian system does not mean its complete integrability. These invariants merely single out the symplectic leaf where the dynamics takes place, but do not specify the dynamics along this leaf, cf. [2].

Below we also describe explicitly two natural Hamiltonian reductions leading to the Poisson structure on densities. A Hamiltonian reduction is a two-step procedure for reducing the dimension of a Hamiltonian system with symmetry: restriction to a given level set of first integrals, and taking the quotient along the symmetry group action. We prove that the Poisson structure on densities can be obtained by the reduction from the symplectic structure considered in [4]. More precisely, let $\mathcal{M}ap$ be the space of all maps from a manifold $M$ equipped with a volume form $\mu$ to a symplectic manifold $N$. The symplectic structure on $\mathcal{M}ap$ is given by averaging the pullbacks of the one on $N$ against the volume form $\mu$, see [4]. In Section 3 we describe what this general construction gives for diffeomorphism groups of symplectic manifolds and densities on them.

## 2. The Poisson structures on the density spaces and their Casimirs

Let $M$ be a compact Poisson manifold with a Poisson bracket $\{,\}$, and let Dens be the set of smooth volume forms on $M$ with total integral 1. (The set Dens can be given a smooth topology and regarded as an infinite-dimensional smooth manifold, see [6]. It is also a dense subset in the $L^2$-Wasserstein space of Borel probability measures on $M$.) Any smooth function $f$ on $M$ defines a linear functional on Dens whose value at a point $\mathrm{d}\nu \in$ Dens is given by the formula

$$F_f(\mathrm{d}\nu) := \int_M f\,\mathrm{d}\nu.$$

(For a noncompact $M$, e.g. for $M = \mathbb{R}^n$, we can consider functions $f$ with compact support.)

**Definition 2.1.** Let $\mathcal{P} = \{F_f : \mathrm{Dens} \to \mathbb{R} \mid f \in C^\infty(M)\}$ be the set of linear functionals $F_f$. Define the bracket on $\mathcal{P}$ by

$$\{F_f, F_g\}_{\mathrm{Dens}}(\mathrm{d}\nu) := F_{\{f,g\}}(\mathrm{d}\nu) = \int_M \{f, g\}\,\mathrm{d}\nu.$$

**Proposition 2.2** (*See e.g. [7,11]*).

1. *The bracket $\{,\}_{Dens}$ defines a Poisson structure on the density space Dens.*
2. *The symplectic leaves on Dens are orbits of the natural action of the group of Hamiltonian diffeomorphisms on densities on $M$.*

As we discussed in Introduction, the Poisson structure on the density space comes from the Poisson structure on the underlying manifold, as the Poisson–Lie structure on the dual of the Lie algebra of Hamiltonian functions on $M$. The statement (2) is proved in [7] for a symplectic $M$, but the proof extends verbatim to a general Poisson manifold $M$, provided that the group of Hamiltonian diffeomorphisms is understood as that generated by flows of Hamiltonian fields on $M$.

### 2.1. The symplectic case

First consider in more detail the case of a *symplectic* manifold $M$ of dimension $2n$. Let $\omega$ be a symplectic structure on $M$, which generates the Poisson bracket $\{\ ,\ \}$. Note that in this case the Liouville form $\omega^n$ can be regarded as a natural choice for a reference density $\mathrm{d}\mu = \omega^n$.

**Proposition 2.3.** *The Poisson bracket $\{,\}_{Dens}$ admits infinitely many functionally independent Casimirs. Namely, for any function $h : \mathbb{R} \to \mathbb{R}$, the functional on Dens defined by*

$$C_h(\mathrm{d}\nu) := \int_M h\left(\frac{\mathrm{d}\nu}{\omega^n}\right)\omega^n$$

*is a Casimir, i.e. it is constant on symplectic leaves of this bracket in the density space Dens.*

**Proof.** Since the symplectic leaves of the Poisson structure $\{,\}_{\mathrm{Dens}}$ are orbits of the action of Hamiltonian flows on the smooth Wasserstein space Dens, it suffices to check that the functions $C_h$ are invariant under this action. Now we have:

$$\begin{aligned} C_h(\phi^*\mathrm{d}\nu) &= \int_M h\left(\frac{\phi^*\mathrm{d}\nu}{\omega^n}\right)\omega^n \\ &= \int_M h\left(\frac{\phi^*\mathrm{d}\nu}{\phi^*\omega^n}\right)\phi^*\omega^n \\ &= C_h(\mathrm{d}\nu), \end{aligned}$$

where the last identity follows from the change of variable formula, and the second one follows from conservation of $\omega$ under the Hamiltonian action: $\phi^*\omega = \omega$. □

Geometrically, these Casimirs capture all moments of the relative density $\mathrm{d}\nu$ with respect to the reference density $\mathrm{d}\mu$. The ratio function $\theta = \mathrm{d}\nu/\mathrm{d}\mu$ is preserved by any Hamiltonian flow, and hence so are all its moments over the manifold $M$.

**Remark 2.4.** Similar Casimirs arise in the case of the Euler equation

$$\partial_t v + v \cdot \nabla v = -\nabla p$$

for a divergence-free vector field $v$ on any even-dimensional Riemannian manifold $M$ with volume form $\mathrm{d}\mu$. Namely, one considers the vorticity 2-form $\mathrm{d}u$ for the 1-form $u$ which is related to the vector field $v$ by means of the metric on $M$. Then for any function $h : \mathbb{R} \to \mathbb{R}$, the functional on vorticities defined by

$$I_h(\mathrm{d}u) = \int_M h\left(\frac{(\mathrm{d}u)^n}{\mathrm{d}\mu}\right)\mathrm{d}\mu$$

is a Casimir for the action of diffeomorphisms preserving the "reference density" $\mathrm{d}\mu$, see [9,10] and introduction. These Casimirs also measure relative density of the generalized vorticity $(\mathrm{d}u)^n$, which is frozen into the ideal fluid, with respect to the volume form $\mathrm{d}\mu$.

Conjecturally, a complete set of Casimirs is encoded in the (Morse) graph with measure, associated to the function $\theta$ on $M$. Its vertices correspond to critical points of $\theta$ on $M$, and the edges correspond to pairs of critical points which can be connected via nonsingular levels, while $\mathrm{d}\theta$ defines the measure on the graph. This construction has been used for regular vorticity function in the 2D Euler equation (cf. [2]), and is applicable to symplectic leaves in the density space for any dimension.

**Example 2.5.** Consider the following semi-geostrophic equation in (a domain of) $\mathbb{R}^2$:

$$\partial_t v_g + v \cdot \nabla v_g + Jv + \nabla f = 0,$$

where $J$ is the 90°-rotation operator on $\mathbb{R}^2$, $v$ is a divergence-free velocity field, $v_g$ is the geostrophic velocity field "defined by" the relation $\nabla f = J v_g$ for a potential $f$ in the domain, see [3]. (This system is obtained from the two-dimensional Euler equation in the rotating frame, where we assume the Coriolis force to be constant in the domain, and make the semi-geostrophic approximation, see e.g. [8].)

Introduce the new potential $\tilde{f}(t,x) := |x|^2/2 + f(t,x)$. Consider the map $\phi_t(x) = \nabla\tilde{f}(t, \varphi_t(x))$, where $\varphi_t$ is the flow of the divergence-free vector field $v(t,\cdot)$ solving the above semi-geostrophic equation, and assume that $\phi_t$ is a diffeomorphism for $t$ in some interval. Then the family $\phi_t$ descends to the following Hamiltonian system on the density space by tracing how it pushes the reference density $\mathrm{d}\mu$. Namely, the form $\mathrm{d}\nu_t := (\phi_t)_*\mathrm{d}\mu$ satisfies the Hamiltonian system on the space Dens with respect to the Poisson structure $\{,\}_{\mathrm{Dens}}$ and the Hamiltonian $H^{\mathrm{Dens}}$ given by

$$H^{\mathrm{Dens}}(\mathrm{d}\nu) = -\mathrm{Wass}^2(\mathrm{d}\mu, \mathrm{d}\nu)/2,$$

where *Wass* is the Wasserstein $L^2$-distance on Dens.

The relative density $\mathrm{d}\nu/\mathrm{d}\mu$ discussed in Proposition 2.3 becomes

$$\theta := \frac{\phi_*\mathrm{d}\mu}{\mathrm{d}\mu} = \frac{(\nabla\tilde{f})_*\mathrm{d}\mu}{\mathrm{d}\mu} = \det(\mathrm{Hess}\ \tilde{f}) = \det(I + \mathrm{Hess}\ f),$$

where Hess $f$ is the Hessian matrix of the function $f$. The latter expression for $\theta$ is known as the potential vorticity in the semi-geostrophic equation, and is known to be frozen into semi-geostrophic flow, similar to the standard vorticity of an ideal two-dimensional fluid, see [8].

Thus Proposition 2.3 is a generalization of the Casimir property of the potential vorticity to higher dimensions and to other Riemannian metrics. Its frozenness property is shown to be related to the geometry of the underlying Poisson structure $\{,\}_{\mathrm{Dens}}$ on the density space, rather than to the the specific Hamiltonian equation.

### 2.2. *The Poisson case*

Assume now that $M$ is a *Poisson* manifold whose symplectic leaves are of codimension $\geq 1$, and $\lambda : M \to \mathbb{R}$ is a smooth nonconstant Casimir function on $M$. It turns out that in this case symplectic leaves of the Poisson bracket $\{,\}_{\mathrm{Dens}}$ still have infinite codimension in Dens, similar to the case of a symplectic $M$.

**Proposition 2.6.** *The Poisson bracket* $\{,\}_{Dens}$ *admits infinitely many functionally independent Casimirs. Namely, for any function* $h : \mathbb{R} \to \mathbb{R}$*, the functional*

$$C_{h,\lambda}(\mathrm{d}\nu) := \int_M (h \circ \lambda)\mathrm{d}\nu$$

*is a Casimir on the density space Dens.*

**Proof.** We check that the functionals $C_{h,\lambda}$ are invariant under the Hamiltonian action:

$$\begin{aligned} C_{h,\lambda}(\phi^*\mathrm{d}\nu) &= \int_M (h\circ\lambda)(x)\phi^*\mathrm{d}\nu(x) \\ &= \int_M (h\circ\lambda)(\phi(x))\phi^*\mathrm{d}\nu(x) \\ &= C_{h,\lambda}(\mathrm{d}\nu), \end{aligned}$$

where we used the Casimir property of $\lambda$ on $M$: $\lambda(\phi(x)) = \lambda(x)$ for a Hamiltonian diffeomorphism $\phi$. □

Note that for symplectic leaves of codimension 1 on $M$, one can think of invariants $C_{h,\lambda}$ as measuring the relative volume for the volume form $d\lambda \wedge \omega^n$ with respect to the reference density $\mathrm{d}\mu$, where $\omega$ stands for symplectic structure on the leaves in $M$. This, in turn, is similar to the helicity-type invariants for the Euler equation on odd-dimensional manifolds, with the important distinction, though, that for the density space Dens one has not only one, but an infinite number of Casimirs regardless of the dimension of the manifold $M$.

## 3. Symplectic structure on the diffeomorphism group of a symplectic manifold

Let $(M,\omega)$ be a $2n$-dimensional symplectic manifold and let $\mathcal{D}$ be the space of all orientation preserving diffeomorphisms of $M$. This is an infinite-dimensional Lie group with the Lie algebra $\mathfrak{X}$ of all smooth vector fields on the manifold $M$. The tangent space to the group $\mathcal{D}$ at a point $\phi$ consists of right translations of vector fields to $\phi$: $T_\phi\mathcal{D} = \{X \circ \phi \mid X \in \mathfrak{X}\}$. Fix the reference volume form $\mathrm{d}\mu = \omega^n$ on $M$.

**Definition 3.1.** The diffeomorphism group $\mathcal{D}$ can be equipped with the following natural symplectic form $W^{\mathcal{D}}$: given two tangent vectors $X\circ\phi$ and $Y\circ\phi$ at $\phi \in \mathcal{D}$ we set

$$W^{\mathcal{D}}(X\circ\phi, Y\circ\phi) := \int_M \omega(X\circ\phi(x), Y\circ\phi(x))\mathrm{d}\mu(x)$$
$$= \int_M \omega(X,Y)(\phi^{-1})^*\mathrm{d}\mu = \int_M \omega(X,Y)\phi_*\mathrm{d}\mu.$$

As before, let Dens be the (smooth Wasserstein) space of all volume forms on the manifold $M$ with total integral 1. The tangent space to this infinite-dimensional manifold Dens at a point $\mathrm{d}\nu$ consists of smooth $2n$-forms on the manifold $M$ with zero integral. Denote the tangent bundle of the smooth Wasserstein space by $T$ Dens.

Consider the natural projection $\pi : \mathcal{D} \to$ Dens of diffeomorphisms into the volume forms on $M$, according to how the diffeomorphisms move the reference density $\mathrm{d}\mu$: $\pi(\phi) = \phi_*(\mathrm{d}\mu)$. This way the diffeomorphism group $\mathcal{D}$ can be regarded as the total space of the principal bundle over the base Dens with the structure group $\mathcal{D}_\mu$ of all diffeomorphisms preserving the volume form $\mathrm{d}\mu$.

**Theorem 3.2.** *The symplectic structure $W^{\mathcal{D}}$ on the diffeomorphism group $\mathcal{D}$ descends to the Poisson structure $\{\,,\}_{Dens}$ on the Wasserstein space Dens.*

**Proof.** The symplectic form $W^{\mathcal{D}}$ is invariant under the $\mathcal{D}_\mu$-action of volume-preserving diffeomorphisms and hence under the map $\pi$ it descends to a certain Poisson structure on the density space Dens. We would like to show that the corresponding quotient Poisson structure coincides with $\{\,,\}_{\mathrm{Dens}}$.

Let $f : M \to \mathbb{R}$ be a function on the manifold $M$ and $F_f(\mathrm{d}\nu) = \int_M f\,\mathrm{d}\nu$ the corresponding linear functional on Dens. Consider the pullback $\bar{F}_f := \pi^* F_f$ of this functional $F_f$ to the diffeomorphism group $\mathcal{D}$ by the map $\pi$. Explicitly it is given by

$$\bar{F}_f(\phi) = \int_M f\,\mathrm{d}\nu = \int_M f\,\phi_*(\mathrm{d}\mu) = \int_M (f\circ\phi)(\mathrm{d}\mu).$$

Let $X_f$ be the Hamiltonian vector field for the Hamiltonian function $f$ on the symplectic manifold $(M,\omega)$ and let $X_f^{\mathcal{D}}$ be the Hamiltonian vector field of the pullback functional $\bar{F}_f$ on $(\mathcal{D}, W^{\mathcal{D}})$, the diffeomorphism group $\mathcal{D}$ equipped with the symplectic structure $W^{\mathcal{D}}$.

**Lemma 3.3.** *The Hamiltonian vector fields $X_f$ on $(M,\omega)$ and $X_f^{\mathcal{D}}$ on $(\mathcal{D}, W^{\mathcal{D}})$ are related in the following way:*

$$X_f^{\mathcal{D}}(\phi) = X_f\circ\phi.$$

**Proof.** By the definition of the Hamiltonian field $X_{\bar{F}_f}$ at a point $\phi\in\mathcal{D}$,

$$W^{\mathcal{D}}(X_f^{\mathcal{D}}(\phi), Y\circ\phi) = \langle d\bar{F}_f, Y\circ\phi\rangle$$

for any vector field $Y\in\mathfrak{X}$. On the other hand, by employing the definition of the pullback and changing the variable, we rewrite the latter expression as follows:

$$\int_M \langle \mathrm{d}f_{\phi(x)}, Y\circ\phi(x)\rangle\,\mathrm{d}\mu(x) = \int_M \langle \mathrm{d}f_x, Y(x)\rangle\,\phi_*(\mathrm{d}\mu)(x).$$

Now by the definition of the Hamiltonian field $X_f$ on $M$ this is equal to

$$\int_M \omega(X_f(x), Y(x))\phi_*(\mathrm{d}\mu)(x) = \int_M \omega(X_f\circ\phi, Y\circ\phi)(\mathrm{d}\mu),$$

which completes the proof of the lemma, due to arbitrariness of the field $Y$. □

Returning to the proof of the theorem, we are going to compute the Poisson bracket of the pullback functions $\bar{F}_f$ and $\bar{F}_g$. By the definition, the value of the Poisson bracket $\{,\}^{\mathcal{D}}$, which is dual to the symplectic structure $W^{\mathcal{D}}$ on the diffeomorphism group, for these two functions is

$$\{\bar{F}_f, \bar{F}_g\}^{\mathcal{D}}(\phi) = W^{\mathcal{D}}(X_{\bar{F}_f}(\phi), X_{\bar{F}_g}(\phi)).$$

By using the lemma above and the change of variable, the right-hand side above becomes

$$\begin{aligned}\int_M \omega(X_f\circ\phi, X_g\circ\phi)\mathrm{d}\mu &= \int_M \omega(X_f, X_g)\phi_*(\mathrm{d}\mu)\\ &= \int_M \{f,g\}\,\phi_*(\mathrm{d}\mu)\\ &= \int_M \{f,g\}\,\mathrm{d}\nu,\end{aligned}$$

as required. □

**Remark 3.4.** This symplectic structure $W^{\mathcal{D}}$ on the diffeomorphism group can be viewed as a particular case of that considered in [4]. More generally, let $S$ be a compact manifold with a fixed volume form $d\sigma$, while $(M,\omega)$ is a symplectic manifold. The space $\mathcal{M}ap$ of all maps $\rho : S\to M$ (of some fixed homotopy class) has a natural symplectic structure. Namely, the tangent space to $\mathcal{M}ap$ at a point $\rho\in\mathcal{M}ap$ is the space of sections of the bundle $\rho^*(TM)$ over $S$ and the symplectic structure is

$$\Omega_f(v,w) := \int_M \rho^*\omega(v,w)d\sigma$$

for a pair of sections $v, w$ of $\rho^*(TM)$. The group of volume-preserving diffeomorphisms of $S$ defines a symplectic group action on $\mathcal{M}ap$. Donaldson considers in [4] the corresponding moment map and the Hamiltonian reduction of the space $\mathcal{M}ap$ under this group action. In our case, the two manifolds $S$ and $M$ coincide, while the volume form $d\sigma$ is the symplectic volume form $\mathrm{d}\mu = \omega^n$. Then the diffeomorphism group $\mathcal{D}$ is an open subset of $\mathcal{M}ap$ with the symplectic structure described above, and we consider the action of the subgroup $\mathcal{D}_\mu$ of volume-preserving diffeomorphisms on it.

**Remark 3.5.** The same Poisson structure on Dens was also defined in [1] in slightly different terms, cf. [7]. For a symplectic manifold $(M,\omega)$ we fix a Riemannian metric $\langle,\rangle$ and an almost complex structure $J$ compatible with the metric: $\omega(u,v) = \langle u, Jv\rangle$. Let $f$ be a function on the manifold $M$ and $\nabla f$ its gradient with respect to the metric $\langle,\rangle$. The Hamiltonian field on $M$ for the Hamiltonian $f$ is $X_f = J\nabla f$.

Consider the distribution $\tau$ on the smooth Wasserstein space defined at a point $\mathrm{d}\nu\in$ Dens by all possible infinitesimal shifts of $\mathrm{d}\nu$ by Hamiltonian fields: $\tau_\nu := \{L_{X_f}\mathrm{d}\nu \mid f\in C^\infty(M)\}$, where $L$ denotes the Lie derivative along a vector field on $M$. Define a 2-form on the distribution $\tau$ by

$$\omega^\tau(L_{X_f}\mathrm{d}\nu, L_{X_g}\mathrm{d}\nu) = \int_M \omega(\nabla f, \nabla g)\mathrm{d}\nu.$$

In [1] it is shown that the distribution $\tau_\nu$ is integrable on the smooth Wasserstein space, and this 2-form is a well-defined symplectic structure on the integral leaves of this distribution. One can see that these leaves are exactly the symplectic leaves of the Poisson structure $\{\,,\}_{\mathrm{Dens}}$ on the density space, while the symplectic structure $\omega^\tau$ is dual to the Poisson structure discussed above:

$$\omega^\tau(L_{X_f}\mathrm{d}\nu, L_{X_g}\mathrm{d}\nu) = \int_M \omega(X_f, X_g)\mathrm{d}\nu = \{F_f, F_g\}_{\mathrm{Dens}}(\mathrm{d}\nu).$$

## 4. The two-dimensional case and geostrophic equations

### 4.1. The Noether theorem for an extra symmetry on the plane

Return to the two-dimensional $M$ and consider the smooth density space Dens for $M = \mathbb{R}^2$ with the standard symplectic structure $\omega = dx_1 \wedge dx_2$. This induces the Poisson structure on Dens, as described above. There is the natural $SO(2)$-action by rotations on densities: $\mathrm{d}\nu \mapsto \varphi_*(\mathrm{d}\nu)$, where $\mathrm{d}\nu \in$ Dens is a measure and $\varphi \in SO(2)$.

Recall that for the standard measure $\omega$, the semi-geostrophic equation is the Hamiltonian equation on Dens with the Hamiltonian function $H(\mathrm{d}\nu) = -\mathrm{Wass}^2(\omega, \mathrm{d}\nu)/2$, where Wass is the Wasserstein distance on densities.

**Proposition 4.1.** *The functional $K(\mathrm{d}\nu) := \int_{\mathbb{R}^2} |x|^2\, \mathrm{d}\nu$ is a first integral of the semi-geostrophic equation.*

**Proof.** First we note that the $SO(2)$-action is Hamiltonian with the Hamiltonian function given by the functional $K(\mathrm{d}\nu)$ on Dens. Indeed, take the generator of the rotation group with the Hamiltonian $\kappa(x) = |x|^2$ on $\mathbb{R}^2$. Then the corresponding action on densities in Dens is generated by the field with Hamiltonian $K(\mathrm{d}\nu) := \int_{\mathbb{R}^2} \kappa\, \mathrm{d}\nu = \int_{\mathbb{R}^2} |x|^2\, \mathrm{d}\nu$, while the corresponding action on diffeomorphisms in $\mathcal{D}$ is generated by the Hamiltonian $\bar{K} = \pi^* K$, cf. Lemma 3.3.

Next, we see that the Wasserstein distance $H(\mathrm{d}\nu)$ from any measure $\mathrm{d}\nu$ to the standard measure $\omega$ is $SO(2)$-invariant, since so is $\omega$. Thus the $SO(2)$-action is a symmetry of the function $H(\mathrm{d}\nu) = -\mathrm{Wass}^2(\omega, \mathrm{d}\nu)/2$, i.e. the Hamiltonians $H$ and $K$ are in involution on the density space Dens with respect to the Poisson structure $\{,\}_{\mathrm{Dens}}$. In particular, $K(\mathrm{d}\nu)$ is a conserved quantity for the semi-geostrophic equation. □

This proposition naturally generalizes to any dimension: If the Hamiltonian field with a Hamiltonian function $\kappa$ generates an isometry of $M^{2n}$, and the reference density $\mathrm{d}\mu = \omega^n$ is invariant with respect to this isometry, then the Hamiltonian field for $H(\mathrm{d}\nu) = -\mathrm{Wass}^2(\mathrm{d}\mu, \mathrm{d}\nu)/2$ on Dens has the first integral $K(\mathrm{d}\nu) := \int_M \kappa\, \mathrm{d}\nu$.

### 4.2. More Hamiltonian reductions to the density space

Consider the case of a two-dimensional manifold $M$ in more detail. In this section we would like to compare the symplectic geometry of the diffeomorphism group $\mathcal{D}(M)$ with that of the cotangent bundle $T^*\mathcal{D}_\mu(M)$ of the group of area-preserving diffeomorphisms of the surface $M$.

As we discussed above, the group $\mathcal{D}$ for an oriented surface (or, for any symplectic manifold) $M$ can be equipped with a symplectic structure, which descends to the Poisson structure on Dens under the projection $\pi : \mathcal{D} \to$ Dens, or, more precisely, under the Hamiltonian reduction with respect to the $\mathcal{D}_\mu$-action. Note that the space Dens is a convex subset in the space $\Omega^2(M)$ of 2-forms on $M$: Dens $= \{\mathrm{d}\nu \in \Omega^2(M) \mid \mathrm{d}\nu > 0, \int_M \mathrm{d}\nu = 1\}$.

Now consider the group $\mathcal{D}_\mu$ and its cotangent bundle $T^*\mathcal{D}_\mu$. Identify $T^*\mathcal{D}_\mu \simeq \mathcal{D}_\mu \times \mathfrak{X}_\mu^*$ by means of right translations on the group. Note that the Lie algebra $\mathfrak{X}_\mu$ consists of divergence-free vector fields on the surface $M$. Such fields are described locally by their Hamiltonian (or, stream) functions. First we assume that $M$ is a two-dimensional sphere $S^2$, so that the fields are globally Hamiltonian. Then the algebra $\mathfrak{X}_\mu$ can be viewed as the Poisson algebra $C_0^\infty(M)$ of functions with zero mean on $M$ (with respect to the reference density $\mathrm{d}\mu = \omega$). The corresponding (smooth) dual space $\mathfrak{X}_\mu^*(M) = \Omega_0^2(M)$ consists of smooth 2-forms on $M$ with zero total integral.

By shifting this dual space to the reference density $\mathrm{d}\mu$ we can regard the (smooth) density space Dens as a convex subset in $\Omega_0^2(M)$:

$$\mathrm{Dens} = \{\mathrm{d}\mu + d\bar{\nu} \mid \mathrm{d}\mu + d\bar{\nu} > 0, d\bar{\nu} \in \Omega_0^2(M)\} \subset \mathrm{d}\mu + \Omega_0^2(M).$$

After this shift to the reference density $\mathrm{d}\mu$ the diffeomorphism group $\mathcal{D} \simeq \mathcal{D}_\mu \times$ Dens becomes a subset in the cotangent bundle $T^*\mathcal{D}_\mu \simeq \mathcal{D}_\mu \times \Omega_0^2$.

Recall that the cotangent bundle $T^*\mathcal{D}_\mu$ has a natural symplectic structure (denoted later by $W^{T^*}$), which descends to the Poisson–Lie structure on the dual Lie algebra $\mathfrak{X}_\mu^* \simeq \Omega_0^2$. The latter is exactly the Poisson structure $\{\ ,\ \}_{\mathrm{Dens}}$ upon restriction to the density space Dens $\subset \Omega_0^2$.

**Conjecture 4.2.** *The natural symplectic structure $W^{T^*}$ on the cotangent bundle $T^*\mathcal{D}_\mu$ coincides with the symplectic structure $W^{\mathcal{D}}$ on the diffeomorphism group $\mathcal{D}$, understood as a subset of $T^*\mathcal{D}_\mu$ via the identification described above.*

In other words, not only coincide the Poisson structures on the density space Dens understood by itself or as a part of the dual $\mathfrak{X}_\mu^*$, but presumably so do the corresponding symplectic structures before the Hamiltonian reduction. We note that the convex subset Dens of positive densities is preserved under the diffeomorphism action on $\Omega_0^2$, so the Poisson structure on $\Omega_0^2$ can indeed be restricted to this subset.

In the case of a general surface $M$, divergence-free fields on $M$ may have multivalued Hamiltonians: $\mathfrak{X}_\mu(M) = C_0^\infty(M) \oplus H_1(M)$. Respectively, the dual space $\mathfrak{X}_\mu^*(M)$ is a finite-dimensional extension of $\Omega_0^2(M)$, since $\mathcal{D}_\mu^*(M) \simeq \Omega^1(M)/d\Omega^0(M) \simeq \Omega_0^2(M) \oplus H^1(M)$. Now the density space Dens$(M)$ can be understood as a convex subset in a plane of finite codimension in the dual space $\mathfrak{X}_\mu^*(M)$.

Note also that for a higher-dimensional symplectic $M$, there is a natural map from the dual $\mathfrak{X}_\mu^* \simeq \Omega^1(M)/d\Omega^0(M) \simeq \Omega_0^2(M) \oplus H^1(M)$ to the space $\Omega^{2n}(M)$ of $2n$-forms: $\rho : [u] \mapsto (\mathrm{d}u)^n$, where $[u] \in \mathfrak{X}_\mu^*$ is a 1-form $u$ modulo addition of an exact 1-form on $M$. This map commutes with the natural $\mathcal{D}_\mu$-action of volume-preserving diffeomorphisms on forms, which explains the common origin of the Casimirs on the space Dens of volume forms and on the dual space $\mathfrak{X}_\mu^*$.

Finally, we note that the Euler equation of an ideal fluid on the two-dimensional $M$ is the Hamiltonian equation on $\Omega_0^2$ with respect to the Poisson–Lie structure $\{\ ,\ \}_{\text{Dens}}$ whose Hamiltonian function is the *energy* quadratic form on $\Omega_0^2$. It is interesting to compare this with (the projection of) the semi-geostrophic equation, where the Hamiltonian function is $H^{\text{Dens}}(\mathrm{d}\nu) = -\text{Wass}^2(\mathrm{d}\mu, \mathrm{d}\nu)/2$, the square of the Wasserstein distance on Dens $\subset \mathrm{d}\mu + \Omega_0^2$. This shift of the quadratic form to the reference density $\mathrm{d}\mu$ is similar to the shift observed in the infinite conductivity equation, and in the $f$-plane and $\beta$-plane geostrophic equations, see [2,5,12].

Before the Hamiltonian reduction, the semi-geostrophic equation is a Hamiltonian system on the diffeomorphism group with respect to the symplectic structure $W^{\mathcal{D}}$ and the Hamiltonian function $H^{\mathcal{D}}$ evaluating how far our diffeomorphism is from being area-preserving:

$$H^{\mathcal{D}}(\phi) = \frac{1}{2}\text{dist}^2(\phi, \mathcal{D}_\mu),$$

where *dist* is the distance in the "flat" $L^2$-type metric on the diffeomorphism group $\mathcal{D}$ from $\phi \in \mathcal{D}$ to the subgroup $\mathcal{D}_\mu$ of diffeomorphisms preserving the standard area form $\omega = \mathrm{d}\mu$ on $\mathbb{R}^2$, see [3]. This Hamiltonian is invariant under the action of the group $\mathcal{D}_\mu$ of area-preserving diffeomorphisms, and so it descends to the above semi-geostrophic Hamiltonian system on the density space Dens.

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