Physica D 411 (2020) 132546

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Generalized symmetries, conservation laws and Hamiltonian structures of an isothermal no-slip drift flux model

Stanislav Opanasenko [a,b,*], Alexander Bihlo [a], Roman O. Popovych [b,c,d], Artur Sergyeyev [d]

[a] Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (NL) A1C 5S7, Canada
[b] Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01024 Kyiv, Ukraine
[c] Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
[d] Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic

ARTICLE INFO

Article history:
Received 28 January 2020
Accepted 13 May 2020
Available online 28 May 2020
Communicated by D. Pelinovsky

Keywords:
Generalized symmetry
Local conservation law
Recursion operator
Hamiltonian structure
Hydrodynamic-type system
Isothermal no-slip drift flux

ABSTRACT

We study the hydrodynamic-type system of differential equations modeling isothermal no-slip drift flux. Using the facts that the system is partially coupled and its subsystem reduces to the (1+1)-dimensional Klein–Gordon equation, we exhaustively describe generalized symmetries, cosymmetries and local conservation laws of this system. A generating set of local conservation laws under the action of generalized symmetries is proved to consist of two zeroth-order conservation laws. The subspace of translation-invariant conservation laws is singled out from the entire space of local conservation laws. We also find broad families of local recursion operators and a nonlocal recursion operator, and construct an infinite family of Hamiltonian structures involving an arbitrary function of a single argument. For each of the constructed Hamiltonian operators, we obtain the associated algebra of Hamiltonian symmetries.

1. Introduction

The drift flux model introduced in [1] is a simplified model of a well-known two-phase flow phenomenon [2,3]. The former was thoroughly studied in [4–7], where several submodels easier to tackle but still real-world applicable were suggested. Amongst them is the isothermal no-slip drift flux model given by the system

$$\rho^1_t + u\rho^1_x + u_x\rho^1 = 0,$$
$$\rho^2_t + u\rho^2_x + u_x\rho^2 = 0,$$
$$(\rho^1+\rho^2)(u_t + uu_x) + a^2(\rho^1_x+\rho^2_x) = 0,$$

which we denote by $\mathcal{S}$. This model describes the mixing motion of liquids (or gases) rather than their individual phases. Here $u = u(t,x)$ is the common velocity, $\rho^1 = \rho^1(t,x)$ and $\rho^2 = \rho^2(t,x)$ are the densities of the liquids, and the constant parameter a can be set to 1 by scaling (x,u) with a. Any constraint meaning that ρ^1 and ρ^2 are proportional, e.g., $\rho^2 = \rho^1$ or $\rho^2 = 0$, reduces $\mathcal{S}$ to the system $\tilde{\mathcal{S}}_0$ describing one-dimensional isentropic gas flows with constant sound speed, cf. the system (3)–(4) with $\nu = 0$ in [8, Section 2.2.7]. The system $\mathcal{S}$ is a diagonalizable hydrodynamic-type system since it can be equivalently rewritten as

$$\mathfrak{r}^1_t + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathfrak{r}^1_x = 0, \tag{1a}$$
$$\mathfrak{r}^2_t + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathfrak{r}^2_x = 0, \tag{1b}$$
$$\mathfrak{r}^3_t + (\mathfrak{r}^1+\mathfrak{r}^2)\mathfrak{r}^3_x = 0 \tag{1c}$$

by changing the dependent variables (u,ρ^1,ρ^2) to the Riemann invariants $(\mathfrak{r}^1,\mathfrak{r}^2,\mathfrak{r}^3)$ via

$$\mathfrak{r}^1 = \frac{u+\ln(\rho^1+\rho^2)}{2},\quad \mathfrak{r}^2 = \frac{u-\ln(\rho^1+\rho^2)}{2},\quad \mathfrak{r}^3 = \frac{\rho^2}{\rho^1}.$$

The corresponding characteristic velocities

$$V^1 = \mathfrak{r}^1+\mathfrak{r}^2+1,\quad V^2 = \mathfrak{r}^1+\mathfrak{r}^2-1,\quad V^3 = \mathfrak{r}^1+\mathfrak{r}^2 \tag{2}$$

are distinct, meaning that the system $\mathcal{S}$ is strictly hyperbolic. Besides, the characteristic velocities satisfy the system

$$\partial_{\mathfrak{r}^i}\frac{V^k_{\mathfrak{r}^j}}{V^j-V^k} = \partial_{\mathfrak{r}^j}\frac{V^k_{\mathfrak{r}^i}}{V^i-V^k}\quad \text{for all}\ \ i,j,k\in\{1,2,3\}\ \ \text{with}\ \ i,j\neq k.$$

Thus, the system $\mathcal{S}$ is semi-Hamiltonian and, since $V^3_{\mathfrak{r}^3} = 0$, it is not genuinely nonlinear with respect to $\mathfrak{r}^3$; see [9] for related definitions. The system $\mathcal{S}$ is also partially coupled. The essential subsystem $\mathcal{S}_0$ consisting of Eqs. (1a)–(1b) coincides with the diagonalized form of the system $\tilde{\mathcal{S}}_0$ [8, Section 2.2.7, Eq. (16)].

* Corresponding author at: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's (NL) A1C 5S7, Canada.
E-mail addresses: sopanasenko@mun.ca (S. Opanasenko), abihlo@mun.ca (A. Bihlo), rop@imath.kiev.ua (R.O. Popovych), artur.sergyeyev@math.slu.cz (A. Sergyeyev).

https://doi.org/10.1016/j.physd.2020.132546

Hydrodynamic-type systems are extensively studied in the literature in view of their various physical applications in fluid mechanics, acoustics and gas and shock dynamics [8,10] and rich differential geometry [9,11–13]. See [14–22] and references therein for an assortment of examples.

In view of the above properties, the system $\mathcal{S}$ can be integrated in an implicit form. In [23] for this system we expressed the general solution in terms of the general solution of the $(1+1)$-dimensional Klein–Gordon equation using the generalized hodograph transformation [13] and described the entire set of local solutions via the linearization of the subsystem $\mathcal{S}_0$ to the same equation. Since the practical use of the derived representations for solutions of $\mathcal{S}$ is limited because of their implicit form and complicated structure, in [23] we also began the extended classical symmetry analysis of the system $\mathcal{S}$. In particular, for this system we constructed the maximal Lie invariance algebra $\mathfrak{g}$, the algebra of generalized symmetries of order not greater than one, the complete point symmetry group and group-invariant solutions. Thus, the algebra $\mathfrak{g}$ is spanned by the vector fields

$$\hat{\mathcal{D}} = t\partial_t + x\partial_x, \quad \hat{\mathcal{G}}_1 = t\partial_x + \partial_{\mathfrak{r}^1}, \quad \hat{\mathcal{G}}_2 = \partial_{\mathfrak{r}^1} - \partial_{\mathfrak{r}^2},$$
$$\hat{\mathcal{P}}^t = \partial_t, \quad \hat{\mathcal{P}}^x = \partial_x, \quad \hat{\mathcal{W}}(\Omega) = \Omega(\mathfrak{r}^3)\partial_{\mathfrak{r}^3},$$

where Ω runs through the set of smooth functions of $\mathfrak{r}^3$. The maximal Lie invariance algebra $\mathfrak{g}_0$ of the essential subsystem $\mathcal{S}_0$ is wider than the projection of the algebra $\mathfrak{g}$ to the space with the coordinates $(t, x, \mathfrak{r}^1, \mathfrak{r}^2)$ and is spanned by the vector fields

$$\check{\mathcal{D}} = t\partial_t + x\partial_x, \quad \check{\mathcal{G}}_1 = t\partial_x + \partial_{\mathfrak{r}^1}, \quad \check{\mathcal{G}}_2 = \partial_{\mathfrak{r}^1} - \partial_{\mathfrak{r}^2},$$
$$\check{\mathcal{P}}(\tau^0, \xi^0) = \tau(\mathfrak{r}^1, \mathfrak{r}^2)\partial_t + \xi(\mathfrak{r}^1, \mathfrak{r}^2)\partial_x,$$
$$\check{\mathcal{J}} = \left(\frac{1}{2}x - t(\mathfrak{r}^1 + \mathfrak{r}^2)\right)\partial_t + t\left(\mathfrak{r}^1 - \mathfrak{r}^2 - \frac{1}{2}(\mathfrak{r}^1 + \mathfrak{r}^2)^2 + \frac{1}{2}\right)\partial_x + \mathfrak{r}^1\partial_{\mathfrak{r}^1} - \mathfrak{r}^2\partial_{\mathfrak{r}^2},$$

where (τ, ξ) is a tuple of smooth functions of $(\mathfrak{r}^1, \mathfrak{r}^2)$, running through the solution set of the system $\xi_{\mathfrak{r}^1} = V^2\tau_{\mathfrak{r}^1}$, $\xi_{\mathfrak{r}^2} = V^1\tau_{\mathfrak{r}^2}$. In [23], for the system $\mathcal{S}$ we also found the zeroth-order local conservation laws using the direct method and, following [24], constructed the entire space of first-order conservation laws with (t, x)-translation-invariant densities and a subspace of (t, x)-translation-invariant conservation laws of arbitrarily high order. Building on the description of a subalgebra of generalized symmetries of order not greater than one, we obtained an infinite-dimensional subspace of generalized symmetries of arbitrarily high order for $\mathcal{S}$. (In the present paper we show that this subspace is an ideal in the entire algebra of generalized symmetries of the system $\mathcal{S}$.)

At the same time, the system $\mathcal{S}$ possesses two properties that allow us to exhaustively describe the entire spaces of generalized symmetries, cosymmetries and local conservation laws. Firstly, the system is partially coupled with the essential subsystem $\mathcal{S}_0$ being linearizable through the rank-two hodograph transformation to the $(1+1)$-dimensional Klein–Gordon equation, which was thoroughly studied in [25] from the point of view of generalized and variational symmetries and local conservation laws. Secondly, in addition to being not genuinely nonlinear with respect to $\mathfrak{r}^3$, the system $\mathcal{S}$ is decoupled with respect to $\mathfrak{r}^3$, and the third equation of $\mathcal{S}$ is linear in $\mathfrak{r}^3$. Thus, speaking of the degeneracy of the system $\mathcal{S}$, we mean both its linear degeneracy and decoupling with respect to $\mathfrak{r}^3$. Due to the dual nature of this degeneracy, the system $\mathcal{S}$ admits not only an infinite number of linearly independent conservation laws of arbitrarily high order, that are related to the degeneracy, cf. [24,26], but also similar generalized symmetries.

Substantially generalizing results of [23], in the present paper we comprehensively study generalized symmetries, cosymmetries and local conservation laws (see [27] for definitions) of the system $\mathcal{S}$. This includes both a description of the corresponding spaces and their interrelations, which are described in terms of recursion operators and Noether and Hamiltonian operators. Our *modus operandi* to study the system $\mathcal{S}$ is to select appropriate symmetry-like objects of the Klein–Gordon equation (generalized symmetries, cosymmetries and conservation laws), to find their counterparts for the system $\mathcal{S}$ and to complement these counterparts with the objects of the same kind that are related to the degeneracy of the system. Then we prove that the constructed objects span the entire spaces of objects of the corresponding kinds for the system $\mathcal{S}$. As a result, we obtain one more example, in addition to a few ones existing in the literature, cf. [25], where generalized symmetries and local conservation laws are exhaustively described for a model arising in real-world applications and possessing symmetry-like objects of arbitrarily high order.

The structure of this paper is as follows. In Section 2 we reduce the system $\mathcal{S}$ to the $(1+1)$-dimensional Klein–Gordon equation and show that any regular solution of the former is expressed in terms of solutions of the latter. In Section 3 we lay out notations and auxiliary results to be used throughout the remainder of the paper. It is proved in Section 4 that the algebra of reduced generalized symmetries of the system $\mathcal{S}$ is a (non-direct) sum of an ideal related to the degeneracy of $\mathcal{S}$ and consisting of generalized vector fields with zero $\mathfrak{r}^1$- and $\mathfrak{r}^2$-components and of a subalgebra stemming from generalized symmetries of the Klein–Gordon equation. At the same time, not all generalized symmetries of the Klein–Gordon equation have counterparts among those of the system $\mathcal{S}$, and we solve the problem on selecting appropriate elements of the algebra of generalized symmetries of the Klein–Gordon equation. This differs from cosymmetries and conservation laws of $\mathcal{S}$, for which there are injections from the corresponding spaces for the Klein–Gordon equation to those for the system $\mathcal{S}$, see Sections 5 and 6, respectively. The space of conservation laws of $\mathcal{S}$ is proved to be generated, under the action of generalized symmetries of $\mathcal{S}$, by two zeroth-order conservation laws. We also find the space of conservation-law characteristics of $\mathcal{S}$. The knowledge of them helps us to single out the conservation laws of orders zero and one as well as the (t, x)-translation-invariant ones. In Section 7 we construct a family of compatible hydrodynamic-type Hamiltonian operators for the system $\mathcal{S}$, parameterized by an arbitrary function of the degenerate Riemann invariant $\mathfrak{r}^3$. For each of these operators, we find the space of its distinguished (Casimir) functionals and the associated algebra of Hamiltonian symmetries. The partition of symmetry-like objects in accordance with the above two properties of the system $\mathcal{S}$ also manifests itself in recursion operators studied in Section 8. The independent local recursion operators constructed turn out either to nontrivially act in the subspace of generalized symmetries arising due to the degeneracy of $\mathcal{S}$ and annihilate the counterparts of generalized symmetries of the Klein–Gordon equation or vice versa. A nonlocal recursion operator is also found. Section 9 is left for the conclusions, where we underline the nontrivial features encountered in the course of the study of the system $\mathcal{S}$ in the present paper and discuss further problems to be considered for this system within the framework of symmetry analysis of differential equations.

2. Solution through linearization of the essential subsystem

Using the facts that the system $\mathcal{S}$ is partially coupled and the subsystem $\mathcal{S}_0$ can be linearized, we construct an implicit representation of the general solution for the diagonalized form (1) of the system $\mathcal{S}$ in terms of the general solution of the $(1+1)$-dimensional Klein–Gordon equation; cf. [23, Section 8].

At first, we reduce the system (1) by a point transformation to a system containing the $(1+1)$-dimensional Klein–Gordon equation. It is convenient to derive this transformation as a chain of simpler point transformations. We begin with the rank-two hodograph transformation, where

$y=\mathfrak{r}^1/2,\quad z=-\mathfrak{r}^2/2$ are the new independent variables and

$p=t,\quad \hat q=x,\quad s=\mathfrak{r}^3$ are the new dependent variables.

(For convenience of the presentation, we compose the hodograph transformation with scaling of $\mathfrak{r}^1$ and $\mathfrak{r}^2$.) This transformation maps the system (1) to the system

$$\hat q_z-(2y-2z+1)p_z=0, \tag{3a}$$

$$\hat q_y-(2y-2z-1)p_y=0, \tag{3b}$$

$$s_yp_z+s_zp_y=0. \tag{3c}$$

After representing Eq. (3a) in the form

$$\big(\hat q-(2y-2z+1)p\big)_z-2p=0,$$

it becomes natural to make the change $\check q=\hat q-(2y-2z+1)p$ of $\hat q$. Then Eqs. (3a) and (3b) take the form $p=\check q_z/2$ and $\check q_y+2p_y+2p=0$, respectively. Excluding p from the second equation in view of the first one, we obtain the second-order linear partial differential equation $\check q_{yz}+\check q_y+\check q_z=0$ in $\check q$, which reduces by the change $q=e^{y+z}\check q$ of $\check q$ to the $(1+1)$-dimensional Klein–Gordon equation for q in light-cone variables, $q_{yz}=q$. Carrying out this chain of two transformations in the whole system (3), we obtain the system $\mathcal{K}$, which reads

$$q_{yz}=q, \tag{4a}$$

$$K^1s_y=K^2s_z, \tag{4b}$$

where

$$K^1:=q_{zz}-2q_z+q,\quad K^2:=q_y+q_z-2q.$$

We have $K^1=(\mathrm{D}_z-1)^2q$ and, on solutions of (4a),

$$K^2=-(\mathrm{D}_y-1)(\mathrm{D}_z-1)q,\quad \mathrm{D}_yK^1=K^2,\quad \mathrm{D}_zK^2=K^1.$$

Here D_y and D_z are the total derivative operators with respect to y and z, respectively. We exclude p from the system (4) in view of the equation

$$p=\frac{1}{2}e^{-y-z}(q_z-q) \tag{5}$$

as well as we neglect this equation itself. The composition of the above three transformations is the transformation

$$\begin{aligned}\mathcal{T}\colon\ & y=\frac{\mathfrak{r}^1}{2},\quad z=-\frac{\mathfrak{r}^2}{2},\quad p=t,\\ & q=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big),\quad s=\mathfrak{r}^3.\end{aligned} \tag{6}$$

Therefore, to make the inverse transition from the system (4) to the system (1), we should attach Eq. (5) to the system (4), thus extending the tuple of dependent variables (q,s) by p, and carry out the inverse to the transformation (6),

$$\begin{aligned}\hat{\mathcal{T}}\colon\ & t=p,\quad x=e^{-y-z}q+(2y-2z+1)p,\\ & \mathfrak{r}^1=2y,\quad \mathfrak{r}^2=-2z,\quad \mathfrak{r}^3=s.\end{aligned} \tag{7}$$

It is convenient to collect the expressions for low-order derivatives of p and q and for their combinations in terms of the old variables in view of the system (1), which will be needed below:

$$p_y=-\frac{1}{\mathfrak{r}^1_x},\quad p_z=-\frac{1}{\mathfrak{r}^2_x},$$

$$K^1=-\frac{2}{\mathfrak{r}^2_x}e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2},\quad K^2=\frac{2}{\mathfrak{r}^1_x}e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2},\quad \frac{s_y}{K^2}=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\frac{\mathfrak{r}^3_x}{2},$$

$$q_y=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left(\frac{2}{\mathfrak{r}^1_x}+x-V^1t-2t\right),\quad q_z=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(x-V^2t),$$

$$q_{zz}=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\left(-\frac{2}{\mathfrak{r}^2_x}+x-V^2t+2t\right).$$

Following the procedure analogous to that in [23], we find the complete set of local solutions of the system (1) via the linearization of the subsystem (1a)–(1b).

We are allowed to make the point transformation (6) if and only if the nondegeneracy condition $\mathfrak{r}^1_t\mathfrak{r}^2_x-\mathfrak{r}^1_x\mathfrak{r}^2_t\neq0$ holds, which is equivalent, on solutions of (1), to the inequality $\mathfrak{r}^1_x\mathfrak{r}^2_x\neq0$. Therefore, $\mathfrak{r}^1_t\mathfrak{r}^2_t\neq0$ as well, and thus both Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are not constants. In this case, we introduce the "pseudopotential" Ψ defined by the potential system

$$\Psi_y=q-\Psi,\quad \Psi_z=q_z-\Psi$$

for Eq. (4a). In fact, this "pseudopotential" is a modification, $\Psi=e^{-y-z}\tilde\Psi$, of the standard potential $\tilde\Psi$ for Eq. (4a) associated with the conserved current $(e^{y+z}q_z,-e^{y+z}q)$ of this equation via the potential system

$$\tilde\Psi_y=e^{y+z}q,\quad \tilde\Psi_z=e^{y+z}q_z.$$

It is easily seen that the function Ψ satisfies the Klein–Gordon equation $\Psi_{yz}=\Psi$. Moreover, solutions of Eqs. (4a), (4b) and (5) are locally expressed in terms of Ψ,

$$q=\Psi_y+\Psi,\quad p=\frac{1}{2}e^{-y-z}(\Psi_z-\Psi_y),$$

$$s=W\left(e^{y+z}(\Psi_y+\Psi_z-2\Psi)\right).$$

Here and in what follows W is an arbitrary smooth function of its argument. Returning to the old coordinates, we obtain the regular family of solutions of the system (1), which is expressed in terms of the general solution of the Klein–Gordon equation. Note that the nondegeneracy condition for this inverse transformation is $K^1K^2\neq0$, where, in terms of Ψ,

$$K^1=\Psi_{zz}-\Psi_z+\Psi_y-\Psi,\quad K^2=\Psi_{yy}-\Psi_y+\Psi_z-\Psi.$$

In view of the Klein–Gordon equation $\Psi_{yz}=\Psi$, the inequalities $K^1\neq0$ and $K^2\neq0$ are equivalent to each other as well as to the condition

$$\Psi\notin\langle e^{-y-z},e^{y+z},(y-z)e^{y+z}\rangle.$$

If the nondegeneracy condition $\mathfrak{r}^1_t\mathfrak{r}^2_x-\mathfrak{r}^1_x\mathfrak{r}^2_t\neq0$ does not hold, then at least one of the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ is a constant. If only one Riemann invariant is a constant, we derive the singular family of solutions of (1). Let $\mathfrak{r}^1$ be a constant, $\mathfrak{r}^1=c$. Then Eq. (1a) is trivially satisfied, and we make the rank-one hodograph transformation

$$\bar t=t,\quad \bar z=\mathfrak{r}^2,\quad \bar q=x,\quad \bar s=\mathfrak{r}^3$$

in the two remaining Eqs. (1b) and (1c), exchanging the roles of x and $\mathfrak{r}^2$, that is, $\bar t$ and $\bar z$ are the new independent variables, $\bar q$ and $\bar s$ are the new dependent variables. This yields the system

$$\bar q_{\bar t}=\bar z+c-1,\quad \bar s_{\bar z}+\bar q_{\bar z}\bar s_{\bar t}=0.$$

Integrating the first equation to $\bar q=(\bar z+c-1)\bar t+e^{\bar z}\Theta^2_{\bar z}$, where Θ^2 is an arbitrary function of $\bar z$. It is chosen with a help of a hindsight to represent the general solution of the second equation in the form

$$\bar s=W(e^{-\bar z}\bar t-\Theta^2_{\bar z}-\Theta^2).$$

The consideration when $\mathfrak{r}^2$ being a constant is similar.

When the both Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are constants, we obtain an ultra-singular family of solutions of (1).

Theorem 1. *Any solution of the system (1) (locally) belongs to one of the following families; below W is an arbitrary function of its argument.*

1. The regular family, where both the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are not constants (the general solution):

$$t=-e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}(\Psi_{\mathfrak{r}^1}+\Psi_{\mathfrak{r}^2}),$$
$$x=e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\big((2\Psi_{\mathfrak{r}^1}+\Psi)-(\mathfrak{r}^1+\mathfrak{r}^2+1)(\Psi_{\mathfrak{r}^1}+\Psi_{\mathfrak{r}^2})\big),$$
$$\mathfrak{r}^3=W\big(e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}(\Psi_{\mathfrak{r}^1}-\Psi_{\mathfrak{r}^2}-\Psi)\big).$$

Here the function $\Psi=\Psi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the set of solutions of the Klein–Gordon equation $\Psi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Psi/4$ with

$$\Psi\notin\langle\, e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2},\ e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2},\ (\mathfrak{r}^1+\mathfrak{r}^2)e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\,\rangle.$$

2. The two singular families, where exactly one of the Riemann invariants $\mathfrak{r}^1$ and $\mathfrak{r}^2$ is a constant:

$$\mathfrak{r}^1=c,\quad x=(\mathfrak{r}^2+c-1)t+e^{\mathfrak{r}^2}\Theta^2_{\mathfrak{r}^2},\quad \mathfrak{r}^3=W(e^{-\mathfrak{r}^2}t-\Theta^2_{\mathfrak{r}^2}-\Theta^2);$$
$$\mathfrak{r}^2=c,\quad x=(\mathfrak{r}^1+c+1)t+e^{-\mathfrak{r}^1}\Theta^1_{\mathfrak{r}^1},\quad \mathfrak{r}^3=W(e^{\mathfrak{r}^1}t+\Theta^1_{\mathfrak{r}^1}-\Theta^1).$$

Here c is an arbitrary constant and $\Theta^1=\Theta^1(\mathfrak{r}^1)$ and $\Theta^2=\Theta^2(\mathfrak{r}^2)$ are arbitrary functions of their arguments.

3. The ultra-singular family, where $\mathfrak{r}^1$ and $\mathfrak{r}^2$ are arbitrary constants and $\mathfrak{r}^3=W(x-(\mathfrak{r}^1+\mathfrak{r}^2)t)$.

The regular, singular and ultra-singular families of solutions of the system $\mathcal{S}$ are associated with solutions of the subsystem $\mathcal{S}_0$ of rank 2, 1 and 0, respectively; cf. [28].

Alternatively, to get the subfamily of regular solutions with nonconstant parameter function W, one can employ the generalized hodograph transformation [13], see details in [23, Section 9].

3. Preliminaries

Given a system $\mathcal{L}$ of differential equations, we denote by $\mathcal{L}^{(\infty)}$ the manifold defined by the system $\mathcal{L}$ and its differential consequences in the associated jet space. A local object associated with $\mathcal{L}$ within the framework of symmetry analysis of differential equations, like a generalized symmetry, a conserved current of a local conservation law, a conservation-law characteristic or a cosymmetry, is called trivial if it vanishes on solutions of $\mathcal{L}$ or, equivalently, on $\mathcal{L}^{(\infty)}$. Two such local objects of the same kind are naturally assumed equivalent if their difference is trivial, and thus such local objects of the same kind in total are considered up to this equivalence relation.

The system $\mathcal{S}$ given by (1) is of the evolution type. The jet variables t, x and $\mathfrak{r}^i_\kappa=\partial^\kappa\mathfrak{r}^i/\partial x^\kappa$, $i=1,2,3$, $\kappa\in\mathbb{N}_0$, constitute the standard coordinates on the manifold $\mathcal{S}^{(\infty)}$. Therefore, up to the above equivalence relation on solutions of $\mathcal{S}$, for the coset of each of local symmetry-like objects associated with $\mathcal{S}$ we can consider a representative whose components do not depend on the derivatives of $\mathfrak{r}$ involving differentiation with respect to t.[1] A symbol with $[\mathfrak{r}]$, like $f[\mathfrak{r}]$, denotes a differential function of $\mathfrak{r}$ that depends at most on t, x and a finite number of derivatives of $\mathfrak{r}$ with respect to x, $f=f(t,x,\mathfrak{r}_0,\dots,\mathfrak{r}_\kappa)$, $\kappa\in\mathbb{N}_0$. Below we consider only such differential functions and assume that the components of any local symmetry-like objects associated with $\mathcal{S}$ are such differential functions. For $i\in\{1,2,3\}$, the order $\operatorname{ord}_{\mathfrak{r}^i}f[\mathfrak{r}]$ of a differential function $f[\mathfrak{r}]$ with respect to $\mathfrak{r}^i$ is defined to be equal $\max\{\kappa\in\mathbb{N}_0\mid f_{\mathfrak{r}^i_\kappa}\ne0\}$ unless this set is empty and $-\infty$ otherwise.

We restrict the total derivative operators D_x and D_t with respect to x and t to the set of above differential functions of $\mathfrak{r}$, and additionally exclude the derivatives of $\mathfrak{r}$ that involve differentiation with respect to t from D_t in view of the system $\mathcal{S}$, respectively obtaining the (commuting) operators

$$\mathscr{D}_x:=\partial_x+\sum_{\kappa=0}^{\infty}\sum_{i=1}^{3}\mathfrak{r}^i_{\kappa+1}\partial_{\mathfrak{r}^i_\kappa},\quad \mathscr{D}_t:=\partial_t-\sum_{\kappa=0}^{\infty}\sum_{i=1}^{3}\mathscr{D}_x^\kappa(V^i\mathfrak{r}^i_1)\partial_{\mathfrak{r}^i_\kappa}.$$

We also define the commuting operators

$$\mathscr{A}:=e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathscr{D}_x\quad\text{and}\quad \mathscr{B}:=\mathscr{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2)\mathscr{D}_x,\quad \mathscr{A}\mathscr{B}=\mathscr{B}\mathscr{A}.$$

It is convenient to introduce the modified coordinates t, x, $r^j_\kappa=\mathfrak{r}^j_\kappa$ and $\omega^\kappa:=\mathscr{A}^\kappa\mathfrak{r}^3$ for $\kappa\in\mathbb{N}_0$ and $j=1,2$ on the manifold $\mathcal{S}^{(\infty)}$ instead of the standard ones.[2] In this notation, we have

$$\mathscr{A}\omega^\kappa=\omega^{\kappa+1},\quad \mathscr{B}\omega^\kappa=0,\quad \kappa\in\mathbb{N}_0,\quad \mathscr{B}r^1=-r^1_1,\quad \mathscr{B}r^2=r^2_1,$$
$$\mathscr{D}_x=\partial_x+\sum_{\kappa=0}^{\infty}\Big(r^1_{\kappa+1}\partial_{r^1_\kappa}+r^2_{\kappa+1}\partial_{r^2_\kappa}+e^{r^1-r^2}\omega^{\kappa+1}\partial_{\omega^\kappa}\Big),$$
$$\mathscr{D}_t=\partial_t-\sum_{\kappa=0}^{\infty}\Big(\,\mathscr{D}_x^\kappa(V^1r^1_1)\partial_{r^1_\kappa}+\mathscr{D}_x^\kappa(V^2r^2_1)\partial_{r^2_\kappa}$$
$$+(r^1+r^2)e^{r^1-r^2}\omega^{\kappa+1}\partial_{\omega^\kappa}\,\Big)\,.$$

We define the orders $\operatorname{ord}_{r^j}f$, $j=1,2$, and $\operatorname{ord}_\omega f$ of a differential function $f=f[\mathfrak{r}]$ with respect to r^j and "ω" to be equal

$$\max\big\{\kappa\mid f_{r^j_\kappa}\ne0\big\}\quad\text{and}\quad \max\big\{\kappa\mid f_{\omega^\kappa}\ne0\big\},$$

respectively, unless the corresponding set is empty and $-\infty$ otherwise. Note that $\operatorname{ord}_\omega f=\operatorname{ord}_{\mathfrak{r}^3}f$. The notation like $f[r^1,r^2]$, or equivalently $f[\mathfrak{r}^1,\mathfrak{r}^2]$, denotes a differential function f of $(r^1,r^2)=(\mathfrak{r}^1,\mathfrak{r}^2)$.

Lemma 2. *A differential function $f=f[\mathfrak{r}]$ satisfies the equation $\mathscr{B}f=0$ if and only if it is a smooth function of a finite number of ω's, $f=f(\omega^0,\dots,\omega^\kappa)$ with $\kappa\in\mathbb{N}_0$.*

Proof. Provided f being a smooth function of a finite number of ω's, it satisfies the equation $\mathscr{B}f=0$ because of $\mathscr{B}\omega^\kappa=0$ for all $\kappa\in\mathbb{N}_0$.

Conversely, using the modified coordinates on $\mathcal{S}^{(\infty)}$ we denote $\kappa_j=\operatorname{ord}_{r^j}f$, $j=1,2$. Suppose that $\kappa_j\geqslant0$ for some j. Then collecting coefficients of $r^j_{\kappa_j+1}$ in the equation $\mathscr{B}f=0$ yields $\partial f/\partial r^j_{\kappa_j}=0$, which gives a contradiction. Hence the function f does not depend on r^j_κ, $\kappa\in\mathbb{N}_0$. The equation $\mathscr{B}f=0$ takes the form $f_t+(r^1+r^2)f_x=0$, splitting with respect to (r^1,r^2) to $f_t=f_x=0$. $\square$

As the standard coordinates on the manifold $\mathcal{K}^{(\infty)}$ associated with the system (4), we can take the jet variables y, z, $q_\iota=\partial^\iota q/\partial y^\iota$ if $\iota\geqslant0$ and $q_\iota=\partial^{-\iota}q/\partial z^{-\iota}$ if $\iota<0$, $\iota\in\mathbb{Z}$, $s_\kappa=\partial^\kappa s/\partial y^\kappa$, $\kappa\in\mathbb{N}_0$. In these coordinates, the restrictions of the total derivative operators with respect to y and z respectively take the form

$$\mathscr{D}_y=\partial_y+\sum_{\iota=-\infty}^{+\infty}q_{\iota+1}\partial_{q_\iota}+\sum_{\kappa=0}^{+\infty}s_{\kappa+1}\partial_{s_\kappa},$$
$$\mathscr{D}_z=\partial_z+\sum_{\iota=-\infty}^{+\infty}q_{\iota-1}\partial_{q_\iota}+\sum_{\kappa=0}^{+\infty}\mathscr{D}_y^\kappa\left(\frac{K^1}{K^2}s_1\right)\partial_{s_\kappa},$$

[1] Here, for conservation-law characteristics we need to use Lemma 3 in [29], see also [30, Lemma 4.28].

[2] The operator $\mathscr{A}$ and the modified coordinates are related to the degeneration of V^3 meaning, that $V^3_{\mathfrak{r}^3}=0$; cf. [24, Theorem 5.2].

where $K^1 := q_{-2} - 2q_{-1} + q_0$, $K^2 := q_1 + q_{-1} - 2q_0$. The infinite prolongation of the transformation (6) induces pushing forward of the operators $\mathcal{D}_t$, $\mathcal{D}_x$, $\mathcal{A}$ and $\mathcal{B}$ to the operators

$$\hat{\mathcal{D}}_t = -\frac{e^{y+z}}{K^2}(2y-2z+1)\mathcal{D}_y - \frac{e^{y+z}}{K^1}(2y-2z-1)\mathcal{D}_z,$$
$$\hat{\mathcal{D}}_x = \frac{e^{y+z}}{K^2}\mathcal{D}_y + \frac{e^{y+z}}{K^1}\mathcal{D}_z,$$
$$\hat{\mathcal{A}} = \frac{e^{-y-z}}{K^2}\mathcal{D}_y + \frac{e^{-y-z}}{K^1}\mathcal{D}_z, \quad \hat{\mathcal{B}} = -\frac{e^{y+z}}{K^2}\mathcal{D}_y + \frac{e^{y+z}}{K^1}\mathcal{D}_z,$$

and thus $\hat{\mathcal{A}}\hat{\mathcal{B}} = \hat{\mathcal{B}}\hat{\mathcal{A}}$.

A symbol with $[q,s]$, like $f[q,s]$, denotes a differential function of (q,s) that depends at most on y, z and a finite, but unspecified number of q_ι, $\iota \in \mathbb{Z}$, and s_κ, $\kappa \in \mathbb{N}_0$. The order $\operatorname{ord}_s f$ of a differential function $f = f[q,s]$ with respect to s is defined to be equal $\max\{\kappa \in \mathbb{N}_0 \mid f_{s_\kappa} \neq 0\}$ unless this set is empty and $-\infty$ otherwise. Analogously, a symbol with $[q]$, like $f[q]$, denotes a differential function of q that depends at most on y, z and a finite, but unspecified number of q_ι, $\iota \in \mathbb{Z}$.

We also use the modified coordinates y, z, $\hat{q}_\iota = q_\iota$, $\iota \in \mathbb{Z}$ and $\hat{\omega}^\kappa = \hat{\mathcal{A}}^\kappa s$, $\kappa \in \mathbb{N}_0$, on the manifold $\mathcal{K}^{(\infty)}$.

Lemma 2 implies the following assertion.

Corollary 3. *A differential function $f = f[q,s]$ satisfies the equation $\hat{\mathcal{B}}f = 0$, i.e.,*

$$K^1\mathcal{D}_y f = K^2\mathcal{D}_z f,$$

if and only if it is a smooth function of a finite number of $\hat{\omega}$'s, $f = f(\hat{\omega}^0, \dots, \hat{\omega}^\kappa)$ with $\kappa \in \mathbb{N}_0$.

The infinite prolongation of the transformation (7) induces pushing forward of the operators $\mathcal{D}_y$ and $\mathcal{D}_z$ to the (commuting) operators

$$\tilde{\mathcal{D}}_y := -\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big), \quad \tilde{\mathcal{D}}_z := -\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big).$$

4. Generalized symmetries

The following two facts allow us to exhaustively describe generalized symmetries of the system (1). Firstly, Eq. (1c) is partially coupled with Eqs. (1a) and (1b). Secondly, the subsystem (1a)–(1b) is linearized by the hodograph transformation, and the associated linear system reduces to the $(1+1)$-dimensional Klein–Gordon equation.

We denote by Σ the algebra of generalized symmetries of the system (1), and by $\Sigma^{\rm triv}$ the algebra of its trivial generalized symmetries, whose characteristics vanish on solutions of (1). The quotient algebra $\Sigma^{\rm q} = \Sigma/\Sigma^{\rm triv}$ can be identified, e.g., with the subalgebra of canonical representatives in the reduced evolutionary form,

$$\hat{\Sigma}^{\rm q} = \left\{\sum_{i=1}^{3}\eta^i[\mathfrak{r}]\partial_{\mathfrak{r}^i} \in \Sigma\right\}.$$

The criterion of invariance of the system (1) with respect to the generalized vector field $\sum_{i=1}^{3}\eta^i[\mathfrak{r}]\partial_{\mathfrak{r}^i}$ results in the system of three determining equations for the components η^i,

$$\mathcal{D}_t\eta^1 + (\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\eta^1 + \mathfrak{r}^1_x(\eta^1+\eta^2) = 0, \tag{8a}$$
$$\mathcal{D}_t\eta^2 + (\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\eta^2 + \mathfrak{r}^2_x(\eta^1+\eta^2) = 0, \tag{8b}$$
$$\mathcal{D}_t\eta^3 + (\mathfrak{r}^1+\mathfrak{r}^2)\mathcal{D}_x\eta^3 + \mathfrak{r}^3_x(\eta^1+\eta^2) = 0. \tag{8c}$$

Lemma 4. *For any generalized vector field $\sum_{i=1}^{3}\eta^i[\mathfrak{r}]\partial_{\mathfrak{r}^i}$ from $\hat{\Sigma}^{\rm q}$, its components η^1 and η^2 do not depend on derivatives of $\mathfrak{r}^3$, i.e., $\eta^1 = \eta^1[\mathfrak{r}^1,\mathfrak{r}^2]$ and $\eta^2 = \eta^2[\mathfrak{r}^1,\mathfrak{r}^2]$.*

Proof. Suppose that $\kappa_j := \operatorname{ord}_{\mathfrak{r}^3}\eta^j \geqslant 0$ for some $j \in \{1,2\}$. Collecting the coefficients of the jet variable $\mathfrak{r}^3_{\kappa_j+1}$ in the jth equation of (8) yields the equation $\partial\eta^j/\partial\mathfrak{r}^3_{\kappa_j} = 0$, which contradicts the assumption. Hence $\kappa_j = -\infty$ for any $j = 1, 2$. □

Lemma 4 is the manifestation of partial coupling of the system (1). In view of this lemma, the subalgebra $\hat{\Sigma}^{\rm q}_3$ of $\hat{\Sigma}^{\rm q}$ constituted by elements with vanishing η^1 and η^2 is an ideal of $\hat{\Sigma}^{\rm q}$, and the quotient algebra $\Sigma^{\rm q}_{12} := \hat{\Sigma}^{\rm q}/\hat{\Sigma}^{\rm q}_3$ is isomorphic to the subalgebra of reduced generalized symmetries of the subsystem (1a)–(1b) that admit local prolongations to $\mathfrak{r}^3$.

The ideal $\hat{\Sigma}^{\rm q}_3$ is described by the following corollary of Lemma 2.

Corollary 5. *A generalized vector field $\eta^3\partial_{\mathfrak{r}^3}$ belongs to $\hat{\Sigma}^{\rm q}$ if and only if the coefficient η^3 is a smooth function of a finite number of ω's.*

Proof. The invariance of the system (1) with respect to the generalized vector field $\eta^3\partial_{\mathfrak{r}^3}$ leads to the single determining equation $\mathcal{B}\eta^3 = 0$. Further we use Lemma 2. □

Therefore, the infinite prolongation of an element $f\partial_{\mathfrak{r}^3}$ of $\hat{\Sigma}^{\rm q}$ is equal to $\sum_{\iota=0}^{\infty}(\hat{\mathcal{A}}^\iota f)\partial_{\omega^\iota}$, and thus the commutator of elements $f^1\partial_{\mathfrak{r}^3}$ and $f^2\partial_{\mathfrak{r}^3}$ of $\hat{\Sigma}^{\rm q}$ is

$$\sum_{\iota=0}^{\infty}\big((\hat{\mathcal{A}}^\iota f^1)f^2_{\omega^\iota} - (\hat{\mathcal{A}}^\iota f^2)f^1_{\omega^\iota}\big)\partial_{\mathfrak{r}^3}, \quad \text{where} \quad \hat{\mathcal{A}} = \sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}.$$

We specify the form of canonical representatives of cosets of $\hat{\Sigma}^{\rm q}_3$.

Lemma 6. *Each coset of $\hat{\Sigma}^{\rm q}_3$ contains a generalized vector field of the form*

$$\eta^1[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^1} + \eta^2[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^2} + e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat{\eta}^3[\mathfrak{r}^1,\mathfrak{r}^2]\partial_{\mathfrak{r}^3}, \tag{9}$$

where the coefficients η^1, η^2 and $\hat{\eta}^3$ satisfy the system of determining Eqs. (8a), (8b) and

$$\mathcal{D}_t\hat{\eta}^3 + (\mathfrak{r}^1+\mathfrak{r}^2)\mathcal{D}_x\hat{\eta}^3 + e^{\mathfrak{r}^1-\mathfrak{r}^2}(\eta^1+\eta^2) = 0. \tag{10}$$

Proof. In view of Lemma 4 and Corollary 5, it suffices to show that the third components of canonical representatives for elements from the quotient algebra $\Sigma^{\rm q}_{12}$ can be chosen to be of the form $\eta^3 = e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat{\eta}^3[\mathfrak{r}^1,\mathfrak{r}^2]$. After substituting the representation $\eta^3 = e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\hat{\eta}^3[\mathfrak{r}]$ into Eq. (8c), we derive Eq. (10). We use the modified coordinates on the manifold $\mathcal{S}^{(\infty)}$. If the coefficient $\hat{\eta}^3$ depends on ω^κ for some $\kappa \in \mathbb{N}_0$, then a differential function of $(\mathfrak{r}^1,\mathfrak{r}^2)$ obtained from $\hat{\eta}^3$ by fixing values of all involved ω^κ's in the domain of $\hat{\eta}^3$ is also a solution of (10) for the same value of (η^1,η^2). □

The elements of the form (9) from the algebra $\hat{\Sigma}^{\rm q}$ constitute a subalgebra of this algebra, which we denote by $\bar{\Sigma}^{\rm q}_{12}$. Unfortunately, the algebras $\Sigma^{\rm q}_{12}$ and $\bar{\Sigma}^{\rm q}_{12}$ are not isomorphic. Although $\hat{\Sigma}^{\rm q} = \bar{\Sigma}^{\rm q}_{12} + \hat{\Sigma}^{\rm q}_3$, this sum is not direct since

$$\bar{\Sigma}^{\rm q}_{12} \cap \hat{\Sigma}^{\rm q}_3 = \langle e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\rangle.$$

The algebra $\Sigma^{\rm q}_{12}$ is naturally isomorphic to the quotient algebra $\bar{\Sigma}^{\rm q}_{12}/\langle e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\rangle$.

Deriving the exhaustive description of the algebra $\Sigma^{\rm q}_{12}$ is quite complicated. For this purpose, we reduce the system (1) to a system (4) containing the $(1+1)$-dimensional Klein–Gordon equation. Similarly to the system (1), we denote by $\mathfrak{S}$ the algebra of generalized symmetries of the system (4), and by $\mathfrak{S}^{\rm triv}$ the algebra of its trivial generalized symmetries, whose characteristics vanish on solutions of (4). The quotient algebra $\mathfrak{S}^{\rm q}=\mathfrak{S}/\mathfrak{S}^{\rm triv}$ can be identified, e.g., with the subalgebra of canonical representatives in the evolutionary form,

$$\hat{\mathfrak{S}}^{\rm q}=\{\chi[q,s]\partial_q+\theta[q,s]\partial_s\in\mathfrak{S}\}.$$

The Lie bracket on $\hat{\mathfrak{S}}^{\rm q}$ is defined as the modified Lie bracket of generalized vector fields in the jet space with the independent variables (y,z) and the dependent variables (q,s), where all arising mixed derivatives of q and all arising derivatives of s that involve differentiation with respect to y are substituted in view of the system (4) and its differential consequences. The system of determining equations for components of elements of $\hat{\mathfrak{S}}^{\rm q}$ is

$$\mathscr{D}_y\mathscr{D}_z\chi=\chi, \tag{11a}$$

$$s_1(\mathscr{D}_z-1)^2\chi+K_1\mathscr{D}_y\theta=\frac{K^1}{K^2}s_1(\mathscr{D}_y+\mathscr{D}_z-2)\chi+K^2\mathscr{D}_z\theta. \tag{11b}$$

The algebra $\hat{\mathfrak{S}}^{\rm q}$ is isomorphic to the algebra $\hat{\Sigma}^{\rm q}$. This isomorphism is induced by the pushforward of Σ onto $\mathfrak{S}$ that is generated by the point transformation (6), excluding the derivatives of p (including p itself) in view of Eq. (5) and its differential consequences and the successive projection of the obtained generalized vector fields to the jet space with the independent variables (y,z) and the dependent variables (q,s). To map $\mathfrak{S}$ into Σ, we need to prolong the elements of $\mathfrak{S}$ to p according Eq. (5) and make the pushforward by the point transformation (7).

Lemma 7. *The q-component of every element of $\hat{\mathfrak{S}}^{\rm q}$ does not depend on s and its derivatives.*

Proof. Suppose that $Q=\chi\partial_q+\theta\partial_s\in\hat{\mathfrak{S}}^{\rm q}$, and $\kappa:=\operatorname{ord}_s\chi\geqslant0$. Then invariance criterion for the equation $q_{yz}=q$ and the generalized vector field Q implies, after collecting coefficients of $s_{\kappa+2}$, the equation $\chi_{s_\kappa}=0$, which contradicts the assumption. This is why $\operatorname{ord}_s\chi=-\infty$. □

Remark 8. Only simultaneously nonvanishing of K^1 and K^2 is essential for Lemma 7, but not the specific form of these coefficients.

Lemma 7 is the counterpart of Lemma 4 for the system (4) and is the manifestation of partial coupling of this system. In view of Lemma 7, the subalgebra $\hat{\mathfrak{S}}^{\rm q}_s$ of $\hat{\mathfrak{S}}^{\rm q}$ constituted by elements with vanishing q-components is an ideal of $\hat{\mathfrak{S}}^{\rm q}$. In view of Corollary 3 (or Corollary 5), this ideal consists of generalized vector fields of the form $\theta\partial_s$, where θ is a smooth function of a finite, but unspecified number of $\hat{\omega}$'s. Since the ideal $\hat{\mathfrak{S}}^{\rm q}_s$ of $\hat{\mathfrak{S}}^{\rm q}$ corresponds to and is isomorphic to the ideal $\hat{\Sigma}^{\rm q}_3$ of $\hat{\Sigma}^{\rm q}$, for our purpose it suffices to describe the quotient algebra $\mathfrak{S}^{\rm q}_q:=\hat{\mathfrak{S}}^{\rm q}/\hat{\mathfrak{S}}^{\rm q}_s$.

Denote by $\hat{\mathfrak{K}}^{\rm q}$ the algebra of reduced generalized symmetries of the $(1+1)$-dimensional Klein–Gordon equation (4a),

$$\hat{\mathfrak{K}}^{\rm q}=\{\chi[q]\partial_q\mid\mathscr{D}_y\mathscr{D}_z\chi=\chi\}.$$

The quotient algebra $\mathfrak{S}^{\rm q}_q$ is naturally isomorphic to the subalgebra $\mathfrak{A}$ of $\hat{\mathfrak{K}}^{\rm q}$ that consists of elements of $\hat{\mathfrak{K}}^{\rm q}$ admitting local prolongations to s. It was proved in [25] that the algebra $\hat{\mathfrak{K}}^{\rm q}$ is the semi-direct sum of its subalgebra $\hat{\Lambda}^{\rm q}$ and its ideal $\hat{\mathfrak{K}}^{-\infty}$, $\hat{\mathfrak{K}}^{\rm q}=\hat{\Lambda}^{\rm q}\ltimes\hat{\mathfrak{K}}^{-\infty}$, where

$$\hat{\Lambda}^{\rm q}:=\langle(\mathscr{J}^\kappa q)\partial_q,\ (\mathscr{D}_y^\iota\mathscr{J}^\kappa q)\partial_q,\ (\mathscr{D}_z^\iota\mathscr{J}^\kappa q)\partial_q,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\rangle,$$

$$\hat{\mathfrak{K}}^{-\infty}:=\{f(y,z)\partial_q\mid f\in{\rm KG}\},$$

$\mathscr{J}:=y\mathscr{D}_y-z\mathscr{D}_z$, and KG denotes the solution set of the $(1+1)$-dimensional Klein–Gordon equation (4a), i.e., $f\in{\rm KG}$ means that $f_{yz}=f$.

Lemma 9. $\mathfrak{A}=\big\{Q^{\zeta,c}:=\big((\mathscr{D}_y+1)\zeta+cq\big)\partial_q\mid\zeta=\zeta[q]\colon\mathscr{D}_y\mathscr{D}_z\zeta=\zeta,\ c\in\mathbb{R}\big\}$, *and an appropriate prolongation of the generalized vector field $Q^{\zeta,c}$ to s is given by*

$$\theta=\frac{s_1}{K^2}(\mathscr{D}_y+\mathscr{D}_z-2)\zeta. \tag{12}$$

Proof. Denote

$$\tilde{\mathfrak{A}}=\big\{Q^{\zeta,c}:=\big((\mathscr{D}_y+1)\zeta+cq\big)\partial_q\mid\zeta=\zeta[q]\colon\mathscr{D}_y\mathscr{D}_z\zeta=\zeta,\ c\in\mathbb{R}\big\}.$$

Note that here the form of ζ is defined up to summands proportional to e^{-y-z}.

For any solution ζ of the equation $\mathscr{D}_y\mathscr{D}_z\zeta=\zeta$, the differential functions $\chi=(\mathscr{D}_y+1)\zeta$ and θ defined by (12) satisfy the system (11). The tuple $(\chi,\theta)=(q,0)$ is a solution of (11) as well. Hence $\mathfrak{A}\supseteq\tilde{\mathfrak{A}}$.

Suppose that a generalized vector field $\chi[q]\partial_q$ belongs to $\mathfrak{A}$. This means that there exists $\theta=\theta[q,s]$ such that $\chi\partial_q+\theta\partial_s\in\hat{\mathfrak{S}}^{\rm q}$. Then the tuple (χ,θ) satisfies the system (11). By the substitution $\theta=s_1(K^2)^{-1}\tilde{\theta}$, Eq. (11b) is reduced to

$$K^1(\mathscr{D}_y+1)\tilde{\theta}-K^2(\mathscr{D}_z+1)\tilde{\theta}=K^1(\mathscr{D}_y+\mathscr{D}_z-2)\chi-K^2(\mathscr{D}_z-1)^2\chi. \tag{13}$$

We use the modified coordinates on the manifold $\mathcal{K}^{(\infty)}$. If the function $\tilde{\theta}$ depends on ω^κ for some $\kappa\in\mathbb{N}_0$, then a differential function of q obtained from $\tilde{\theta}$ by fixing values of all involved $\hat{\omega}^\kappa$'s in the domain of $\tilde{\theta}$ is also a solution of (13) for the same value of χ. Therefore, without loss of generality we can assume that $\tilde{\theta}=\tilde{\theta}[q]$. Then Eq. (13) rewritten in the form

$$K^1\big((\mathscr{D}_y+1)\tilde{\theta}-(\mathscr{D}_y+\mathscr{D}_z-2)\chi\big)=K^2\big((\mathscr{D}_z+1)\tilde{\theta}-(\mathscr{D}_z-1)^2\chi\big)$$

implies that there exists a differential function $\mu=\mu[q]$ such that

$$\begin{aligned}&(\mathscr{D}_y+1)\tilde{\theta}-(\mathscr{D}_y+\mathscr{D}_z-2)\chi=\mu K^2,\\&(\mathscr{D}_z+1)\tilde{\theta}-(\mathscr{D}_z-1)^2\chi=\mu K^1.\end{aligned} \tag{14}$$

We exclude $\tilde{\theta}$ from these equations by acting the operators $\mathscr{D}_z+1$ and $\mathscr{D}_y+1$ on the first and the second equations, respectively, and subtracting the first obtained equation from the second one, which gives the equation on μ alone, $K^1\mathscr{D}_y\mu=K^2\mathscr{D}_z\mu$. In view of Corollary 3, μ is a constant, and hence Eqs. (14) can be rewritten as

$$\begin{aligned}&(\mathscr{D}_y+1)\tilde{\theta}=(\mathscr{D}_y+\mathscr{D}_z-2)(\chi+\mu q),\\&(\mathscr{D}_z+1)\tilde{\theta}=(\mathscr{D}_z-1)^2(\chi+\mu q).\end{aligned} \tag{15}$$

We subtract the second equation from the result of acting the operator $\mathscr{D}_z$ on the first equation and thus derive the equation $(\mathscr{D}_y\mathscr{D}_z-1)\tilde{\theta}=0$. Then the differential function $\zeta=\zeta[q]$ that is defined by $\zeta:=-\frac14\big(\tilde{\theta}-(\mathscr{D}_z+1)(\chi+\mu q)\big)$ satisfies the same equation, $(\mathscr{D}_y\mathscr{D}_z-1)\zeta=0$. We express $\tilde{\theta}$ from the equality defining ζ,

$$\tilde{\theta}=-4\zeta+(\mathscr{D}_z+1)(\chi+\mu q),$$

and substitute the obtained expression into (15), deriving the equations

$$-4(\mathscr{D}_y+1)\zeta=-4(\chi+\mu q),\quad -4(\mathscr{D}_z+1)\zeta=-4\mathscr{D}_z(\chi+\mu q).$$

The first of these equations gives the required representation for χ, $\chi=(\mathscr{D}_y+1)\zeta-\mu q$. The second equation is identically satisfied in view of the above representation for χ and the equation $\mathscr{D}_y\mathscr{D}_z\zeta=\zeta$. We also get

$$\tilde{\theta}=-4\zeta+(\mathscr{D}_z+1)(\mathscr{D}_y+1)\zeta=(\mathscr{D}_y+\mathscr{D}_z-2)\zeta.$$

Therefore, $\mathfrak{A}\subseteq\tilde{\mathfrak{A}}$, i.e., $\mathfrak{A}=\tilde{\mathfrak{A}}$, and the equality (12) defines an appropriate prolongation of $Q^{\zeta,c}\in\mathfrak{A}$ to s. □

In other words, Lemma 9 implies that an element of $\hat\Lambda^{\mathfrak q}$ can be mapped to a generalized symmetry of the system (1) if and only if the associated operator belongs to the subspace

$$\langle\, 1,\ (\mathcal D_y+1)\mathcal D_y^\iota \mathcal J^\kappa,\ (\mathcal D_z+1)\mathcal D_z^\iota \mathcal J^\kappa,\ \kappa,\iota\in\mathbb N_0\,\rangle.$$

In particular, this subspace contains all polynomials of $\mathcal D_y$ and all polynomials of $\mathcal D_z$. A complement subspace to it in the entire space of operators associated with elements of $\hat\Lambda^{\mathfrak q}$ is $\langle\,\mathcal J^\kappa,\ \kappa\in\mathbb N\,\rangle$. Elements of $\hat\Lambda^{\mathfrak q}$ associated with operators from the complement subspace are mapped to *nonlocal* symmetries of the system (1). Such nonlocal symmetries are generalized symmetries of certain potential systems for the system (1) that are related to potential systems for the $(1+1)$-dimensional Klein–Gordon equation (4a). We plan to study generalized potential symmetries of (1) in the sequel of the present paper.

Completing the above consideration, we prove the following theorem.

Theorem 10. *The quotient algebra $\Sigma^{\mathfrak q}$ of generalized symmetries of the system (1) is naturally isomorphic to the algebra $\hat\Sigma^{\mathfrak q}$ spanned by the generalized vector fields*

$$\check{\mathcal W}(\Omega)=\Omega\partial_{\mathfrak r^3},$$

$$\check{\mathcal P}(\Phi)=e^{(\mathfrak r^2-\mathfrak r^1)/2}\left((\Phi+2\Phi_{\mathfrak r^1})\mathfrak r^1_x\partial_{\mathfrak r^1}+(\Phi-2\Phi_{\mathfrak r^2})\mathfrak r^2_x\partial_{\mathfrak r^2}+2\Phi\mathfrak r^3_x\partial_{\mathfrak r^3}\right),$$

$$\check{\mathcal D}=\big(x-(\mathfrak r^1+\mathfrak r^2+1)t\big)\mathfrak r^1_x\partial_{\mathfrak r^1}+\big(x-(\mathfrak r^1+\mathfrak r^2-1)t\big)\mathfrak r^2_x\partial_{\mathfrak r^2}+\big(x-(\mathfrak r^1+\mathfrak r^2)t\big)\mathfrak r^3_x\partial_{\mathfrak r^3},$$

$$\check{\mathcal R}(\Gamma)=e^{(\mathfrak r^2-\mathfrak r^1)/2}\left((\tilde{\mathcal D}_y\Gamma+\Gamma)\mathfrak r^1_x\partial_{\mathfrak r^1}+(\tilde{\mathcal D}_z\Gamma+\Gamma)\mathfrak r^2_x\partial_{\mathfrak r^2}+2\Gamma\mathfrak r^3_x\partial_{\mathfrak r^3}\right),$$

where Γ runs through the set $\{\tilde{\mathcal J}^\kappa\tilde q,\ \tilde{\mathcal D}_y^\iota\tilde{\mathcal J}^\kappa\tilde q,\ \tilde{\mathcal D}_z^\iota\tilde{\mathcal J}^\kappa\tilde q,\ \kappa\in\mathbb N_0,\ \iota\in\mathbb N\}$ with

$$\tilde{\mathcal D}_y:=-\frac{1}{\mathfrak r^1_x}\big(\mathcal D_t+(\mathfrak r^1+\mathfrak r^2-1)\mathcal D_x\big),$$

$$\tilde{\mathcal D}_z:=-\frac{1}{\mathfrak r^2_x}\big(\mathcal D_t+(\mathfrak r^1+\mathfrak r^2+1)\mathcal D_x\big),$$

$$\tilde{\mathcal J}:=\frac{\mathfrak r^1}{2}\tilde{\mathcal D}_y+\frac{\mathfrak r^2}{2}\tilde{\mathcal D}_z,\quad \tilde q:=e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(x-(\mathfrak r^1+\mathfrak r^2+1)t\big),$$

the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$, and the parameter function Ω runs through the set of smooth functions of a finite, but unspecified number of $\omega^\kappa:=(e^{\mathfrak r^2-\mathfrak r^1}\mathcal D_x)^\kappa\mathfrak r^3$, $\kappa\in\mathbb N_0$.

Proof. For computing the counterpart of an element $Q=\chi\partial_q+\theta\partial_s\in\hat{\mathfrak G}^{\mathfrak q}$ in $\hat\Sigma^{\mathfrak q}$, one should make the following steps:

- prolong the generalized vector field Q to p in view of (5),
- push forward the prolonged vector field by an appropriate prolongation of the transformation (7),
- convert the obtained image to the evolutionary form and
- substitute for all derivatives of $\mathfrak r$ with differentiation with respect to t in view of the system (1) and its differential consequences.

This procedure gives the generalized vector field

$$\tilde Q=-e^{(\mathfrak r^2-\mathfrak r^1)/2}\tilde\chi\mathfrak r^1_x\partial_{\mathfrak r^1}-e^{(\mathfrak r^2-\mathfrak r^1)/2}(\tilde{\mathcal D}_z\tilde\chi)\mathfrak r^2_x\partial_{\mathfrak r^2}+\big(\theta-\tfrac12 e^{(\mathfrak r^2-\mathfrak r^1)/2}(\tilde{\mathcal D}_z\tilde\chi+\tilde\chi)\mathfrak r^3_x\big)\partial_{\mathfrak r^3}.$$

Here and in what follows tildes mark the counterparts of involved operators and differential functions that are computed according to the procedure.

The ideal $\hat{\mathfrak G}^{\mathfrak q}_s$ of $\hat{\mathfrak G}^{\mathfrak q}$ corresponds to and is isomorphic to the ideal $\hat\Sigma^{\mathfrak q}_3$ of $\hat\Sigma^{\mathfrak q}$, and the form of elements of $\hat\Sigma^{\mathfrak q}_3$, $\check{\mathcal W}(\Omega)$, is already known. The generalized vector field $q\partial_q$ is mapped to $-\check{\mathcal D}$. We also prolong each generalized vector field of the form $Q^{\zeta,0}:=(\mathcal D_y+1)\zeta\partial_q$ from $\mathfrak A$ to s according to (12) and then employ the above procedure, getting the generalized vector field

$$\tilde Q^{\zeta,0}=-e^{(\mathfrak r^2-\mathfrak r^1)/2}\left(\mathfrak r^1_x(\tilde{\mathcal D}_y+1)\tilde\zeta\partial_{\mathfrak r^1}+\mathfrak r^2_x(\tilde{\mathcal D}_z+1)\tilde\zeta\partial_{\mathfrak r^2}+2\mathfrak r^3_x\tilde\zeta\partial_{\mathfrak r^3}\right),$$

where $\zeta=\zeta[q]$ runs through the characteristics of elements of $\hat{\mathfrak K}^{\mathfrak q}$ and is defined up to summands proportional to e^{-y-z}, and $\tilde\zeta$ denotes the pullback of ζ by the infinite prolongation of the transformation (6). According to the splitting $\hat{\mathfrak K}^{\mathfrak q}=\hat\Lambda^{\mathfrak q}\Subset\hat{\mathfrak K}^{-\infty}$, for $\zeta\partial_q\in\hat\Lambda^{\mathfrak q}$ and $\zeta\partial_q\in\hat{\mathfrak K}^{-\infty}$ we obtain generalized vector fields of the forms $-\check{\mathcal R}(\Gamma)$ and $-\check{\mathcal P}(\Phi)$, respectively, where $\Gamma\partial_q$ can be assumed to run through the chosen basis of $\hat\Lambda^{\mathfrak q}$, and the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$ and is defined up to summands proportional to $e^{(\mathfrak r^2-\mathfrak r^1)/2}$. □

Remark 11. The subspaces $\mathcal I^1$ and $\mathcal I^2$ that consist of all generalized vector fields of the forms $\check{\mathcal P}(\Phi)$ and $\check{\mathcal W}(\Omega)$ from the algebra $\hat\Sigma^{\mathfrak q}$, respectively, are (infinite-dimensional) ideals of $\hat\Sigma^{\mathfrak q}$. Moreover, the ideal $\mathcal I^1$ is commutative. Since $\check{\mathcal P}(e^{\mathfrak r^2-\mathfrak r^1})=\check{\mathcal W}(\omega^1)=e^{\mathfrak r^2-\mathfrak r^1}\mathfrak r^3_x\partial_{\mathfrak r^3}$, these ideals are not disjoint, $\mathcal I^1\cap\mathcal I^2=\langle e^{\mathfrak r^2-\mathfrak r^1}\mathfrak r^3_x\partial_{\mathfrak r^3}\rangle$, which displays the above indeterminacy of Φ.

Remark 12. The algebra of first-order reduced generalized symmetries of the system (1) can be identified with the subspace of $\hat\Sigma^{\mathfrak q}$ spanned by $\check{\mathcal D}$, $\check{\mathcal R}(\tilde q)$, $\check{\mathcal R}(\tilde{\mathcal D}_z\tilde q)$, $\check{\mathcal P}(\Phi)$, $\check{\mathcal W}(\Omega)$, where the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ runs through the solution set of the Klein–Gordon equation $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$, and the parameter function Ω runs through the set of smooth functions of $\omega^0=\mathfrak r^3$ and $\omega^1=e^{\mathfrak r^2-\mathfrak r^1}\mathfrak r^3_x$. As was noted in [23, Remark 19], this subspace is closed with respect to the Lie bracket of generalized vector fields, and thus we can call it an algebra. The indicated property is shared by all strictly hyperbolic diagonalizable hydrodynamic-type systems. In the notation of [23, Theorem 18],

$$\check{\mathcal R}(\tilde q)=2(\check{\mathcal D}-\check{\mathcal G}_1),\quad \check{\mathcal R}(\tilde{\mathcal D}_z\tilde q)=2(\check{\mathcal D}+\check{\mathcal G}_1+\check{\mathcal G}_2),$$

where $\check{\mathcal G}_1=(t\mathfrak r^1_x-1)\partial_{\mathfrak r^1}+t\mathfrak r^2_x\partial_{\mathfrak r^2}+t\mathfrak r^3_x\partial_{\mathfrak r^3}$ and $\check{\mathcal G}_2=\partial_{\mathfrak r^1}-\partial_{\mathfrak r^2}$. Moreover, the generalized vector fields

$$\check{\mathcal D},\quad \check{\mathcal G}_1,\quad \check{\mathcal G}_2,\quad \check{\mathcal P}\big((\mathfrak r^1+\mathfrak r^2)e^{(\mathfrak r^1-\mathfrak r^2)/2}\big),\quad \check{\mathcal P}\big(e^{(\mathfrak r^1-\mathfrak r^2)/2}\big),\quad \check{\mathcal W}(\Omega)\qquad(16)$$

with an arbitrary Ω depending on $\mathfrak r^3$ only are the evolutionary forms of Lie-symmetry vector fields $-\hat{\mathcal D}$, $-\hat{\mathcal G}_1$, $\hat{\mathcal G}_2$, $2\hat{\mathcal P}^t$, $-2\hat{\mathcal P}^x$ and $\hat{\mathcal W}(\Omega)$ of the system (1), respectively, which span the entire Lie invariance algebra of this system. Therefore, any element of $\hat\Sigma^{\mathfrak q}$ that does not belong to the span of (16) is a genuinely generalized symmetry of the system (1).

5. Cosymmetries

The space Υ of cosymmetries of the system (1) can be computed in a way that is similar to the computation of generalized symmetries and involves the partial coupling of this system and the linearizability of the subsystem (1a)–(1b) by the hodograph transformation. Let $\Upsilon^{\rm triv}\subset\Upsilon$ denote the space of trivial cosymmetries of the system (1), which vanish on solutions thereof. The quotient space $\Upsilon^{\mathfrak q}=\Upsilon/\Upsilon^{\rm triv}$ can be identified, e.g., with the subspace that consists of canonical representatives of cosymmetries, $\hat\Upsilon^{\mathfrak q}=\big\{(\lambda^i[\mathfrak r],\,i=1,2,3)\in\Upsilon\big\}$. This subspace coincides with the solution space of the system

$$\mathcal D_t\lambda^1+(\mathfrak r^1+\mathfrak r^2+1)\mathcal D_x\lambda^1=\mathfrak r^2_x(\lambda^2-\lambda^1)+\mathfrak r^3_x\lambda^3,\qquad(17a)$$

$$\mathcal D_t\lambda^2+(\mathfrak r^1+\mathfrak r^2-1)\mathcal D_x\lambda^2=\mathfrak r^1_x(\lambda^1-\lambda^2)+\mathfrak r^3_x\lambda^3,\qquad(17b)$$

$$\mathcal D_t\lambda^3+(\mathfrak r^1+\mathfrak r^2)\mathcal D_x\lambda^3+(\mathfrak r^1_x+\mathfrak r^2_x)\lambda^3=0,\qquad(17c)$$

which is formally adjoint to the system (8) for generalized symmetries of (1). The substitution $(\lambda^1,\lambda^2,\lambda^3)=e^{\mathfrak r^1-\mathfrak r^2}(\tilde\lambda^1,\tilde\lambda^2,\tilde\lambda^3)$ reduces the system (17) to

$$\mathrm D_t\tilde\lambda^1+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\tilde\lambda^1=\mathfrak r^2_x(\tilde\lambda^1+\tilde\lambda^2)+\mathfrak r^3_x\tilde\lambda^3, \tag{18a}$$
$$\mathrm D_t\tilde\lambda^2+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\tilde\lambda^2=\mathfrak r^1_x(\tilde\lambda^1+\tilde\lambda^2)+\mathfrak r^3_x\tilde\lambda^3, \tag{18b}$$
$$\mathrm D_t\tilde\lambda^3+(\mathfrak r^1+\mathfrak r^2)\mathrm D_x\tilde\lambda^3=0. \tag{18c}$$

We again use the modified coordinates on $\mathcal S^{(\infty)}$. In view of Lemma 2, any solution of the last equation, which can be shortly rewritten as $\mathcal B\tilde\lambda^3=0$, is a smooth function of a finite number of ω's, $\tilde\lambda^3=\tilde\lambda^3(\omega^0,\dots,\omega^{\kappa_0})$ with $\kappa_0\in\mathbb N_0$. The subsystem (18a)–(18b) implies the representation

$$\tilde\lambda^3=\frac{\hat{\mathcal A}\Omega}{\omega^1}\quad\text{with}\quad \hat{\mathcal A}=\sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}$$

and a smooth function $\Omega=\Omega(\omega^0,\dots,\omega^{\kappa_0})$. Thus, $(\Omega,-\Omega)$ is a particular solution of this subsystem as a linear inhomogeneous system with respect to $(\tilde\lambda^1,\tilde\lambda^2)$, and its general solution can be represented in the form

$$\tilde\lambda^1=\tilde\lambda^{1\mathrm h}+\Omega,\quad \tilde\lambda^2=\tilde\lambda^{2\mathrm h}-\Omega.$$

The tuple $(\tilde\lambda^{1\mathrm h},\tilde\lambda^{2\mathrm h})$ is the general solution of the corresponding homogeneous system,

$$\mathrm D_t\tilde\lambda^{1\mathrm h}+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\tilde\lambda^{1\mathrm h}=\mathfrak r^2_x(\tilde\lambda^{1\mathrm h}+\tilde\lambda^{2\mathrm h}),$$
$$\mathrm D_t\tilde\lambda^{2\mathrm h}+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\tilde\lambda^{2\mathrm h}=\mathfrak r^1_x(\tilde\lambda^{1\mathrm h}+\tilde\lambda^{2\mathrm h}).$$

Following the proof of Lemma 4, we can show that any solution of this system does not depend on ω's, i.e., $\tilde\lambda^{j\mathrm h}=\tilde\lambda^{j\mathrm h}[\mathfrak r^1,\mathfrak r^2]$, $j=1,2$. The counterpart $(\lambda^{1\mathrm h},\lambda^{2\mathrm h})=e^{\mathfrak r^1-\mathfrak r^2}(\tilde\lambda^{1\mathrm h},\tilde\lambda^{2\mathrm h})$ of $(\tilde\lambda^{1\mathrm h},\tilde\lambda^{2\mathrm h})$ satisfies the system (17a)–(17b) with $\lambda^3=0$,

$$\mathrm D_t\lambda^{1\mathrm h}+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\lambda^{1\mathrm h}=\mathfrak r^2_x(\lambda^{2\mathrm h}-\lambda^{1\mathrm h}), \tag{19a}$$
$$\mathrm D_t\lambda^{2\mathrm h}+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\lambda^{2\mathrm h}=\mathfrak r^1_x(\lambda^{1\mathrm h}-\lambda^{2\mathrm h}). \tag{19b}$$

Therefore, the triple $\lambda=(\lambda^1,\lambda^2,\lambda^3)$ belongs to $\hat\Upsilon^{\mathrm q}$ if and only if it can be represented, in the above notation, in the form

$$\lambda=e^{\mathfrak r^1-\mathfrak r^2}\big(\Omega,-\Omega,(\hat{\mathcal A}\Omega)/\omega^1\big)+(\lambda^{1\mathrm h},\lambda^{2\mathrm h},0).$$

The substitution $(\lambda^{1\mathrm h},\lambda^{2\mathrm h})=e^{(\mathfrak r^1-\mathfrak r^2)/2}(\hat\lambda^1,\hat\lambda^2)$ reduces the system (19) to the system

$$\mathrm D_t\hat\lambda^1+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\hat\lambda^1=\mathfrak r^2_x\hat\lambda^2,$$
$$\mathrm D_t\hat\lambda^2+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\hat\lambda^2=\mathfrak r^1_x\hat\lambda^1,$$

which can be rewritten in terms of the operators $\tilde{\mathrm D}_y$ and $\tilde{\mathrm D}_z$,

$$\tilde{\mathrm D}_z\hat\lambda^1=-\hat\lambda^2,\quad \tilde{\mathrm D}_y\hat\lambda^2=-\hat\lambda^1.$$

Therefore, both the components $\hat\lambda^1$ and $\hat\lambda^2$ satisfy the image of Eq. (11a) under the transformation (7) and thus are the reduced forms of the pullbacks of the characteristics of elements of $\hat{\mathfrak K}^{\mathrm q}$ by this transformation.

As a result, we have proved the following assertion.

Theorem 13. *The space $\hat\Upsilon^{\mathrm q}$ of canonical representatives of cosymmetries is spanned by cosymmetries from three families,*

1. $e^{\mathfrak r^1-\mathfrak r^2}\big(\Omega,-\Omega,(\hat{\mathcal A}\Omega)/\omega^1\big)$ with the operator $\hat{\mathcal A}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}$ and with Ω running through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(e^{\mathfrak r^2-\mathfrak r^1}\mathrm D_x)^\kappa\mathfrak r^3$, $\kappa\in\mathbb N_0$.

2. $e^{(\mathfrak r^1-\mathfrak r^2)/2}(-2\Phi_{\mathfrak r^1},\Phi,0)$, with the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ running through the solution space of the Klein–Gordon equation $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$.

3. $e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(-\tilde{\mathrm D}_y\tilde{\mathfrak Q}\tilde q,\ \tilde{\mathfrak Q}\tilde q,\ 0\big)$, where the operator $\tilde{\mathfrak Q}$ runs through the set $\big\{\tilde{\mathfrak J}^\kappa,\ \tilde{\mathfrak J}^\kappa\tilde{\mathrm D}_y^\iota,\ \tilde{\mathfrak J}^\kappa\tilde{\mathrm D}_z^\iota,\ \kappa\in\mathbb N_0,\ \iota\in\mathbb N\big\}$, and

$$\tilde{\mathrm D}_y:=-\frac{1}{\mathfrak r^1_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\big),$$
$$\tilde{\mathrm D}_z:=-\frac{1}{\mathfrak r^2_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\big),$$
$$\tilde{\mathfrak J}:=\frac{\mathfrak r^1}{2}\tilde{\mathrm D}_y+\frac{\mathfrak r^2}{2}\tilde{\mathrm D}_z,\quad \tilde q:=e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(x-(\mathfrak r^1+\mathfrak r^2+1)t\big).$$

Remark 14. The first and the second families from Theorem 13, which are linear spaces, are not disjoint in the sense of linear spaces. Their intersection is one-dimensional and is spanned by the cosymmetry $e^{\mathfrak r^1-\mathfrak r^2}(1,-1,0)$ corresponding to $\Omega=1$ and $\Phi=-e^{(\mathfrak r^1-\mathfrak r^2)/2}$. The span of these two families has the zero intersection with the span of the third family.

6. Conservation laws

Theorem 15. *The space of conservation laws of the system (1) is naturally isomorphic to the space spanned by the following conserved currents of this system:*

1. $\big(e^{\mathfrak r^1-\mathfrak r^2}\Omega,\ (\mathfrak r^1+\mathfrak r^2)e^{\mathfrak r^1-\mathfrak r^2}\Omega\big)$, where the parameter function Ω runs through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(e^{\mathfrak r^2-\mathfrak r^1}\mathrm D_x)^\kappa\mathfrak r^3$, $\kappa\in\mathbb N_0$, and such two functions should be assumed equivalent if their difference belongs to the image of the operator $\hat{\mathcal A}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}$.

2. $\big(e^{(\mathfrak r^1-\mathfrak r^2)/2}(2\Phi_{\mathfrak r^1}+\Phi),\ e^{(\mathfrak r^1-\mathfrak r^2)/2}(2(\mathfrak r^1+\mathfrak r^2+1)\Phi_{\mathfrak r^1}+(\mathfrak r^1+\mathfrak r^2-1)\Phi)\big)$, where the parameter function $\Phi=\Phi(\mathfrak r^1,\mathfrak r^2)$ runs through the solution space of the Klein–Gordon equation $\Phi_{\mathfrak r^1\mathfrak r^2}=-\Phi/4$.

3. $\big(\mathfrak r^2_x\tilde\rho+\mathfrak r^1_x\tilde\sigma,\ (\mathfrak r^1+\mathfrak r^2-1)\mathfrak r^2_x\tilde\rho+(\mathfrak r^1+\mathfrak r^2+1)\mathfrak r^1_x\tilde\sigma\big)$ with $\tilde\rho=-\tilde q\tilde{\mathrm D}_z\tilde{\mathfrak Q}\tilde q$, $\tilde\sigma=(\tilde{\mathrm D}_y\tilde q)\tilde{\mathfrak Q}\tilde q$, where the operator $\tilde{\mathfrak Q}$ runs through the set

$$\Big\{\tilde{\mathfrak J}^{\kappa'},\ \kappa'\in2\mathbb N_0+1,\ (\tilde{\mathfrak J}+\iota/2)^\kappa\tilde{\mathrm D}_y^\iota,\ (\tilde{\mathfrak J}-\iota/2)^\kappa\tilde{\mathrm D}_z^\iota,\ \kappa\in\mathbb N_0,\ \iota\in\mathbb N,\ \kappa+\iota\in2\mathbb N_0+1\Big\},$$

and

$$\tilde{\mathrm D}_y:=-\frac{1}{\mathfrak r^1_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2-1)\mathrm D_x\big),$$
$$\tilde{\mathrm D}_z:=-\frac{1}{\mathfrak r^2_x}\big(\mathrm D_t+(\mathfrak r^1+\mathfrak r^2+1)\mathrm D_x\big),$$
$$\tilde{\mathfrak J}:=\frac{\mathfrak r^1}{2}\tilde{\mathrm D}_y+\frac{\mathfrak r^2}{2}\tilde{\mathrm D}_z,\quad \tilde q:=e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(x-(\mathfrak r^1+\mathfrak r^2+1)t\big).$$

Proof. We compute the space of local conservation laws of the system (1) combining the direct method of finding conservation laws [31,32], which is based on the definition of conserved currents, with using the linearization of the essential subsystem (1a)–(1b) to the $(1+1)$-dimensional Klein–Gordon equation. Up to the equivalence of conserved currents, meaning that they coincide on the solution set of the corresponding system of differential equations, it suffices to consider only reduced conserved currents of the system (1), which are of the form (ρ,σ), where $\rho=\rho[\mathfrak r]$ and $\sigma=\sigma[\mathfrak r]$. A tuple $(\rho[\mathfrak r],\sigma[\mathfrak r])$ is a conserved current of the system (1) if and only if $\mathrm D_t\rho+\mathrm D_x\sigma=0$. We should also take into account the equivalence of conserved currents up to adding null divergences, which means that conserved currents $(\rho[\mathfrak r],\sigma[\mathfrak r])$ and $(\rho'[\mathfrak r],\sigma'[\mathfrak r])$ belong to the same conservation law if and only if there exists a differential function $f=f[\mathfrak r]$ such that $\rho'=\rho+\mathrm D_xf$ and $\sigma'=\sigma-\mathrm D_tf$.

We associate each reduced conserved current $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ of the system (1) with the modified density $\breve\rho := e^{\mathfrak{r}^2-\mathfrak{r}^1}\rho$ and the modified flux $\breve\sigma = \sigma-(\mathfrak{r}^1+\mathfrak{r}^2)\rho$, i.e.,

$$(\rho,\sigma)=\left(e^{\mathfrak{r}^1-\mathfrak{r}^2}\breve\rho,\,(\mathfrak{r}^1+\mathfrak{r}^2)e^{\mathfrak{r}^1-\mathfrak{r}^2}\breve\rho+\breve\sigma\right),$$

and $\mathcal{D}_t\rho+\mathcal{D}_x\sigma = e^{\mathfrak{r}^1-\mathfrak{r}^2}(\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma)$. Therefore, the equality $\mathcal{D}_t\rho+\mathcal{D}_x\sigma=0$ for conserved currents is equivalent to the equality $\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma=0$ for modified conserved currents, and the equivalence of conserved currents up to adding a null divergence is modified to $\breve\rho'=\breve\rho+\mathcal{A}f$ and $\breve\sigma'=\breve\sigma-\mathcal{B}f$.

Fixing a reduced conserved current $(\rho[\mathfrak{r}],\sigma[\mathfrak{r}])$ and using the modified coordinates on $\mathcal{S}^{(\infty)}$, we define

$$\kappa := \max(\operatorname{ord}_\omega\breve\rho,\operatorname{ord}_\omega\breve\sigma)$$

and prove by mathematical induction with respect to $\kappa\in\{-\infty\}\cup\mathbb{N}_0$ that up to adding a modified null divergence we have the representation $\breve\rho=\breve\rho^1[r^1,r^2]+\breve\rho^0(\omega^0,\dots,\omega^\kappa)$ for some differential functions $\breve\rho^0=\breve\rho^0(\omega^0,\dots,\omega^\kappa)$ and $\breve\rho^1=\breve\rho^1[r^1,r^2]$, and $\breve\sigma=\breve\sigma[r^1,r^2]$.

The base case $\kappa=-\infty$ is obvious.

For the inductive step, we fix $\kappa\in\mathbb{N}_0$, suppose that the above claim is true for all $\kappa'<\kappa$ and prove it for κ. Collecting coefficients of $\omega^{\kappa+1}$ in the equality $\mathcal{B}\breve\rho+\mathcal{A}\breve\sigma=0$, we derive $\breve\sigma_{\omega^\kappa}=0$, i.e., in fact $\operatorname{ord}_\omega\breve\sigma<\kappa$. Then we differentiate the same equality twice with respect to ω^κ, which leads to $\mathcal{B}\breve\rho_{\omega^\kappa\omega^\kappa}=0$. In view of Lemma 2, this means that the $\breve\rho_{\omega^\kappa\omega^\kappa}$ can depend at most on $(\omega^0,\dots,\omega^\kappa)$. Therefore, there exist differential functions $\breve\rho^{10}=\breve\rho^{10}(\omega^0,\dots,\omega^\kappa)$, $\breve\rho^{11}=\breve\rho^{11}[\mathfrak{r}]$ and $\breve\rho^{12}=\breve\rho^{12}[\mathfrak{r}]$ such that $\operatorname{ord}_\omega\breve\rho^{11}<\kappa$, $\operatorname{ord}_\omega\breve\rho^{12}<\kappa$ and

$$\breve\rho=\breve\rho^{12}[\mathfrak{r}]\omega^\kappa+\breve\rho^{11}[\mathfrak{r}]+\breve\rho^{10}(\omega^0,\dots,\omega^\kappa).$$

Since $\mathcal{B}\breve\rho^{10}=0$, the tuple $(\breve\rho^{10},0)$ is a modified conserved current of the system (1). Hence the tuple $(\breve\rho^{12}\omega^\kappa+\breve\rho^{11},\breve\sigma)$ is a modified conserved current of this system as well. Adding the modified null divergence $(-\mathcal{A}\int\breve\rho^{12}\,\mathrm{d}\omega^{\kappa-1},\mathcal{B}\int\breve\rho^{12}\,\mathrm{d}\omega^{\kappa-1})$ to the latter modified conserved current, we obtain an equivalent modified conserved current $(\breve\rho',\breve\sigma')$ with $\max(\operatorname{ord}_\omega\breve\rho',\operatorname{ord}_\omega\breve\sigma')<\kappa$. The induction hypothesis implies that up to adding a modified null divergence, the component $\breve\rho'$ admits the representation

$$\breve\rho'=\breve\rho^{21}[r^1,r^2]+\breve\rho^{20}(\omega^0,\dots,\omega^\kappa)$$

for some differential functions $\breve\rho^{20}=\breve\rho^{20}(\omega^0,\dots,\omega^\kappa)$ and $\breve\rho^{21}=\breve\rho^{21}[r^1,r^2]$, and $\breve\sigma'=\breve\sigma'[r^1,r^2]$. Setting $\breve\rho^0=\breve\rho^{10}+\breve\rho^{20}$, $\breve\rho^1=\breve\rho^{21}$ and $\breve\sigma=\breve\sigma'$, we complete the inductive step.

In other words, we have proved that up to adding a null divergence, any conserved current of the system (1) can be represented as the sum of a conserved current from the first theorem's family and of a conserved current of the form $(\rho[r^1,r^2],\sigma[r^1,r^2])$. The subspace of conserved currents of the latter forms is the pullback of the space of reduced conserved currents of the essential subsystem (1a)–(1b) by the projection

$$(t,x,\mathfrak{r})\to(t,x,\mathfrak{r}^1,\mathfrak{r}^2),$$

cf. [33, Proposition 3]. The latter space is naturally isomorphic to the space of conservation laws of the essential subsystem (1a)–(1b), which is the pullback of the space of conservation laws of the Klein–Gordon equation (4a) with respect to the composition of the restriction of the transformation (6) to the space with coordinates $(t,x,\mathfrak{r}^1,\mathfrak{r}^2)$ (i.e., the s-component of this transformation should be neglected) with the projection

$$(y,z,q,p)\to(y,z,q).$$

We take the space of conservation laws of the (1+1)-dimensional Klein–Gordon equation, which was constructed in [25], and perform the above pullbacks,

$$\rho=-\frac12(\mathfrak{r}^2_x\tilde\rho_{\rm KG}+\mathfrak{r}^1_x\tilde\sigma_{\rm KG}),\quad \sigma=-\frac12(V^2\mathfrak{r}^2_x\tilde\rho_{\rm KG}+V^1\mathfrak{r}^1_x\tilde\sigma_{\rm KG}),$$

where $\tilde\rho_{\rm KG}$ and $\tilde\sigma_{\rm KG}$ are, as differential functions, the pullbacks of the density $\rho_{\rm KG}$ and the flux $\sigma_{\rm KG}$ of a conserved current of (4a), respectively; see [31, Section III] or [34, Proposition 1]. As a result, we obtain, up to the equivalence on solutions of the system (1) and up to rescaling of conserved currents, the other families of the conserved currents of this system that are presented in the theorem.

More specifically, Eq. (4a) is the Euler–Lagrange equation for the Lagrangian

$$\mathrm{L}=-\frac12(q_yq_z+q^2).$$

Hence characteristics of generalized symmetries of this equation are also its cosymmetries, and vice versa. The quotient algebra $\mathfrak{K}^{\rm q}=\mathfrak{K}/\mathfrak{K}^{\rm triv}$ of generalized symmetries of (4a), where $\mathfrak{K}$ and $\mathfrak{K}^{\rm triv}$ are the algebra (of evolutionary representatives) of generalized symmetries of the Lagrangian (4a) and its ideal of trivial generalized symmetries, is naturally isomorphic to the algebra $\tilde{\mathfrak{K}}^{\rm q}=\tilde\Lambda^{\rm q}\Subset\tilde{\mathfrak{K}}^{-\infty}$, which is the semidirect sum of the subalgebra

$$\tilde\Lambda^{\rm q}:=\langle(\mathrm{J}^\kappa q)\partial_q,\ (\mathrm{J}^\kappa\mathrm{D}_y^\iota q)\partial_q,\ (\mathrm{J}^\kappa\mathrm{D}_z^\iota q)\partial_q,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\rangle$$

with the abelian ideal

$$\tilde{\mathfrak{K}}^{-\infty}:=\{f(y,z)\partial_q\mid f\in\mathrm{KG}\}$$

[25, Theorem 4]. Here D_y and D_z are the operators of total derivatives in y and z, respectively, and $\mathrm{J}:=y\mathrm{D}_y-z\mathrm{D}_z$. Denote by $\varUpsilon$, $\varUpsilon^{\rm triv}$ and $\varUpsilon^{\rm q}$ the algebra (of evolutionary representatives) of variational symmetries of the Lagrangian L, its ideal of trivial variational symmetries and the quotient algebra of variational symmetries of this Lagrangian, i.e., $\varUpsilon\subset\mathfrak{K}$, $\varUpsilon^{\rm triv}:=\varUpsilon\cap\mathfrak{K}^{\rm triv}$ and $\varUpsilon^{\rm q}:=\varUpsilon/\varUpsilon^{\rm triv}$. The quotient algebra $\varUpsilon^{\rm q}$ is naturally isomorphic to the algebra $\tilde\varUpsilon^{\rm q}=\tilde\Lambda^{\rm q}_-\Subset\tilde\Sigma^{-\infty}$, where

$$\begin{aligned}\tilde\Lambda^{\rm q}_-:=\big\langle&(\mathfrak{Q}_{\kappa'0}q)\partial_q,\ \kappa'\in2\mathbb{N}_0+1,\\&(\mathfrak{Q}_{\kappa\iota}q)\partial_q,\ (\bar{\mathfrak{Q}}_{\kappa\iota}q)\partial_q,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N},\ \kappa+\iota\in2\mathbb{N}_0+1\big\rangle\end{aligned}$$

with

$$\mathfrak{Q}_{\kappa\iota}=\left(\mathrm{J}+\frac{\iota}{2}\right)^\kappa\mathrm{D}_y^\iota,\quad \kappa,\iota\in\mathbb{N}_0,$$
$$\bar{\mathfrak{Q}}_{\kappa\iota}=\left(\mathrm{J}-\frac{\iota}{2}\right)^\kappa\mathrm{D}_z^\iota,\quad \kappa\in\mathbb{N}_0,\quad \iota\in\mathbb{N},$$

is the subspace of $\tilde\Lambda^{\rm q}$ that is associated with the space of formally skew-adjoint differential operators generated by D_y, D_z and J. Note that in the context of Noether's theorem, we need to consider the algebra $\tilde{\mathfrak{K}}^{\rm q}$ instead of the algebra $\hat{\mathfrak{K}}^{\rm q}$ of reduced generalized symmetries of (4a), which is mentioned in Section 4, since cosets of $\varUpsilon^{\rm triv}$ in $\varUpsilon$ do not necessarily intersect the algebra $\hat{\mathfrak{K}}^{\rm q}$, see Remark 9 in [25]. The space of conservation laws of (4a) is naturally isomorphic to the space spanned by the conserved currents

$$\bar{\mathrm{C}}^0_f=(-f_zq,fq_y),\quad \mathrm{C}_{\mathfrak{Q}}=(-q\mathrm{D}_z\mathfrak{Q}q,\ q_y\mathfrak{Q}q),$$

where the parameter function $f=f(y,z)$ runs through the solution set of (4a), and the operator $\mathfrak{Q}$ runs through the basis of $\tilde\Lambda^{\rm q}_-$ [25, Proposition 10]. The conserved current $\bar{\mathrm{C}}^0_f$ is equivalent to the conserved current

$$\mathrm{C}^0_f=(fq_z,-f_yq).$$

We map conserved currents of the form $\mathrm{C}_{\mathfrak{Q}}$, where $\mathfrak{Q}q\partial_q$ runs through the basis of $\tilde\Lambda^{\rm q}_-$, to conserved currents of the system (1), which leads to the third family of the theorem. Possible modifications of the form of these conserved currents up to recombining them and adding null divergences are discussed in Remark 23.

At the same time, it is convenient to modify conserved currents of the form $\bar{\mathrm C}^0_f$ before their mapping in order to directly obtain hydrodynamic conservation laws.[3] We reparameterize these conserved currents, representing the parameter function f in the form $f=\bar f_y+\bar f_z+2\bar f$, where the function $\bar f=\bar f(y,z)$ also runs through the solution set of the $(1+1)$-dimensional Klein–Gordon equation (4a). Then $f_z=\bar f_{zz}+2\bar f_z+\bar f$. Adding the null divergence $(\mathrm D_zR,-\mathrm D_yR)$ with $R:=\bar fq_z-\bar f_zq-2\bar fq$ to $-\bar{\mathrm C}^0_f$, we obtain the equivalent conserved current $(\bar fK_1,-\bar f_yK_2)$, which is mapped to the conserved current from the second family with

$$\Phi=\bar f(\mathfrak r^1/2,-\mathfrak r^2/2).$$

Note that the first and second theorem's families are in fact subspaces in the space of conserved currents of the system (1). Analyzing the equivalence of modified conserved currents, we see that conserved currents from the first theorem's family are equivalent if and only if the difference of corresponding Ω's belongs to the image of the operator $\hat{\mathcal A}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}$. The intersection of the first and the second families is one-dimensional and spanned by the conserved current $\big(e^{\mathfrak r^1-\mathfrak r^2},\,(\mathfrak r^1+\mathfrak r^2)e^{\mathfrak r^1-\mathfrak r^2}\big)$. The sum of these two families does not intersect the span of the third family. The equivalence of conserved currents within the span of all the three families is generated by the equivalence of conserved currents within the first family. □

Remark 16. The kernel $\ker\mathsf E$ of the operator

$$\mathsf E=\sum_{\kappa=1}^\infty\sum_{\kappa'=0}^{\kappa-1}\omega^{\kappa-\kappa'}(-\hat{\mathcal A})^{\kappa'}\partial_{\omega^\kappa}-1$$

is contained in the kernel $\ker\mathsf E'$ of the operator

$$\mathsf E'=\sum_{\kappa=0}^\infty(-\hat{\mathcal A})^\kappa\partial_{\omega^\kappa},$$

$\ker\mathsf E\subset\ker\mathsf E'$, since the operator identity $\hat{\mathcal A}\mathsf E=-\omega^1\mathsf E'$ holds. In view of [30, Theorem 4.26], Theorem 18 implies that (locally) the image of the operator $\hat{\mathcal A}$ coincides with $\ker\mathsf E\cap\ker\mathsf E'=\ker\mathsf E$. The kernel $\ker\mathsf E'$ of $\mathsf E'$ is spanned by the constant function 1 and the image of $\hat{\mathcal A}$. Hence $\operatorname{im}\hat{\mathcal A}=\ker\mathsf E\subsetneq\ker\mathsf E'$.

Remark 17. The conserved currents from Theorem 15 that are associated with

$$\Omega=\frac{\mathfrak r^3}{\mathfrak r^3+1},\quad \Omega=\frac{1}{\mathfrak r^3+1},\quad \Omega=1,$$
$$\Phi=e^{(\mathfrak r^1-\mathfrak r^2)/2}(\mathfrak r^1+\mathfrak r^2-1),\quad \Phi=\frac18e^{(\mathfrak r^1-\mathfrak r^2)/2}\big((\mathfrak r^1+\mathfrak r^2)^2-4\mathfrak r^2\big)$$

correspond to the conservation of masses of the both individual phases and of mixture mass as well as the conservation of mixture momentum and of energy in the drift flux model, respectively, cf. [2, Chapter 13]. The related equations in conserved form are

$$\rho^1_t+(\rho^1u)_x=0,\quad \rho^2_t+(\rho^2u)_x=0,$$
$$(\rho^1+\rho^2)_t+\big((\rho^1+\rho^2)u\big)_x=0,$$
$$\big((\rho^1+\rho^2)u\big)_t+\big((\rho^1+\rho^2)(u^2+1)\big)_x=0,$$
$$\big((\rho^1+\rho^2)\left(u^2/2+\ln(\rho^1+\rho^2)\right)\big)_t+\big((\rho^1+\rho^2)\left(u^2/2+\ln(\rho^1+\rho^2)+1\right)u\big)_x=0.$$

In particular, the magnitude $\ln(\rho^1+\rho^2)$ can be interpreted as (proportional to) the internal mixture energy. The first, second and fourth equations constitute the conserved form of the system $\mathcal S$ in the original variables (ρ^1,ρ^2,u).

[3] Recall that a conservation law is called *hydrodynamic* if its density ρ is a function of dependent variables only.

Theorem 18. *In the notation of Theorem 15, the associated reduced conservation-law characteristics of the system (1) are respectively*

1. $$e^{\mathfrak r^1-\mathfrak r^2}\Bigg(\Omega-\sum_{\kappa=1}^\infty\sum_{\kappa'=0}^{\kappa-1}\omega^{\kappa-\kappa'}(-\hat{\mathcal A})^{\kappa'}\Omega_{\omega^\kappa},\ -\Omega+\sum_{\kappa=1}^\infty\sum_{\kappa'=0}^{\kappa-1}\omega^{\kappa-\kappa'}(-\hat{\mathcal A})^{\kappa'}\Omega_{\omega^\kappa},\ \sum_{\kappa=0}^\infty(-\hat{\mathcal A})^\kappa\Omega_{\omega^\kappa}\Bigg).$$
2. $e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(2\Phi_{\mathfrak r^1\mathfrak r^1}+2\Phi_{\mathfrak r^1}+\tfrac12\Phi,\ \Phi_{\mathfrak r^2}-\Phi_{\mathfrak r^1}-\Phi,\ 0\big)$.
3. $e^{(\mathfrak r^1-\mathfrak r^2)/2}\big(-\tilde{\mathfrak D}_y\tilde{\mathfrak Q}\tilde q,\ \tilde{\mathfrak Q}\tilde q,\ 0\big)$.[4]

The space spanned by these characteristics is naturally isomorphic to the quotient space of conservation-law characteristics of the system (1).

Proof. Since the system (1) is a system of evolution equations, its conservation-law characteristics can be found from reduced densities of the associated conservation laws by acting the Euler operator,

$$\mathsf E=\left(\sum_{\kappa=0}^\infty(-\mathrm D_x)^\kappa\partial_{\mathfrak r^i_\kappa},\ i=1,2,3\right),$$

see e.g. [35, Proposition 7.41]. This perfectly works for characteristics related to the second family of conserved currents presented in Theorem 15 but does not give reasonable representations for characteristics related to the first and third families, for which we use different methods.

Characteristics related to the third family can be obtained from conservation-law characteristics of the $(1+1)$-dimensional Klein–Gordon equation (4a). A characteristic of the conservation law of (4a) containing the conserved current $\mathrm C_{\mathfrak Q}$ is

$$\lambda=(\mathfrak Q-\mathfrak Q^\dagger)q=2\mathfrak Qq$$

for $(\mathfrak Qq)\partial_q\in\tilde\Lambda^{\mathrm q}_-$. It is trivially prolonged to the conservation-law characteristic $(\lambda,0)$ of the system (4a), (5). Denote by R^1, R^2, L^1 and L^2 the differential functions associated with Eqs. (1a), (1b), (4a) and (5), respectively,

$$R^1:=\mathfrak r^1_t+(\mathfrak r^1+\mathfrak r^2+1)\mathfrak r^1_x,\quad R^2:=\mathfrak r^2_t+(\mathfrak r^1+\mathfrak r^2-1)\mathfrak r^2_x,$$
$$L^1:=q_{yz}-q,\quad L^2:=p-\frac12e^{-y-z}(q_z-q).$$

These differential functions are related via the transformation $\mathcal T$, $\hat{\mathcal T}^*(R^1,R^2)^{\mathsf T}=\mathfrak M(L^1,L^2)^{\mathsf T}$ with

$$\mathfrak M=\begin{pmatrix}0 & -\dfrac4\Delta\\ \dfrac2\Delta e^{-y-z} & \dfrac4\Delta(\mathrm D_y+1)\end{pmatrix},$$

and

$$\mathfrak M^\dagger=\begin{pmatrix}0 & \dfrac2\Delta e^{-y-z}\\ -\dfrac4\Delta & -(\mathrm D_y-1)\circ\dfrac4\Delta\end{pmatrix},$$

where

$$\Delta=(\mathrm D_y\hat{\mathcal T}^t)(\mathrm D_z\hat{\mathcal T}^x)-(\mathrm D_z\hat{\mathcal T}^t)(\mathrm D_y\hat{\mathcal T}^x),\quad \mathcal T^*\Delta=-4(\mathfrak r^1_t\mathfrak r^2_x-\mathfrak r^1_x\mathfrak r^2_t).$$

The conservation-law characteristic (λ^1,λ^2) of the system (1a), (1b) that is associated with the conservation-law characteristic $(\lambda,0)$ of the system (4a), (5) is defined by

$$\mathfrak M^\dagger(\Delta\hat{\mathcal T}^*\lambda^1,\Delta\hat{\mathcal T}^*\lambda^2)^{\mathsf T}=(\lambda,0)^{\mathsf T}.$$

[4] Here we omitted the multiplier -2, which is needed for the direct correspondence between these conservation-law characteristics and conserved currents from the third family of Theorem 15.

Therefore, the conservation-law characteristic λ of (4a) is mapped to the conservation-law characteristic $\frac{1}{2}e^{y+z}(-\mathcal{D}_y\lambda,\lambda,0)$ of the system $\mathcal{S}$, where all values should be expressed in terms of the variables $(t,x,\mathfrak{r})$. This gives a conservation-law characteristic from the third family of the theorem.

Characteristics related to the first family are found following the procedure of defining them via the formal integration by parts, cf. [30, p. 266]. We denote by A and B the counterparts of the operators $\mathcal{A}$ and $\mathcal{B}$, respectively, in the complete total derivative operators with respect to t and x,

$$\mathrm{A}:=e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathrm{D}_x,\quad \mathrm{B}:=\mathrm{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2)\mathrm{D}_x.$$

Then

$$\begin{aligned}\mathrm{D}_t\big(e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\big)&+\mathrm{D}_x\big((\mathfrak{r}^1+\mathfrak{r}^2)e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\big)\\&=e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\mathcal{E}^1-e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\mathcal{E}^2+\sum_{\kappa=0}^{\infty}e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^\kappa.\end{aligned}\tag{20}$$

Here $\mathcal{E}^k$ denotes the left-hand side of the kth equation of the system (1), $\mathcal{E}^k=\mathfrak{r}^k_t+V^k\mathfrak{r}^k_x$, $k=1,2,3$. Note that $\mathcal{E}^3=\mathrm{B}\mathfrak{r}^3$. Since Ω depends on a finite number of ω's, there is no issue with convergence.

We derive using the mathematical induction with respect to ι that

$$\mathrm{B}\omega^\kappa=\mathrm{A}^\kappa\mathcal{E}^3+\sum_{\kappa'=0}^{\kappa-1}\mathrm{A}^{\kappa'}\big(\omega^{\kappa-\kappa'}(\mathcal{E}^2-\mathcal{E}^1)\big).\tag{21}$$

Indeed, for the base case $\kappa=0$, we have $\mathrm{B}\omega^0=\mathrm{B}\mathfrak{r}^3=\mathcal{E}^3$. The induction step follows from the equality

$$\mathrm{B}\omega^{\kappa+1}=\mathrm{BA}\omega^\kappa=\mathrm{AB}\omega^\kappa+\omega^{\kappa+1}(\mathcal{E}^2-\mathcal{E}^1).$$

Using again the mathematical induction with respect to κ, we prove the counterpart of the Lagrange identity in terms of the operator A,

$$e^{\mathfrak{r}^1-\mathfrak{r}^2}F\mathrm{A}^\kappa G=e^{\mathfrak{r}^1-\mathfrak{r}^2}\big((-\mathrm{A})^\kappa F\big)G+\mathrm{D}_x\sum_{\kappa'=0}^{\kappa-1}\big((-\mathrm{A})^{\kappa'}F\big)\mathrm{A}^{\kappa-\kappa'-1}G,$$

for any $\kappa\in\mathbb{N}_0$ and any differential functions F and G of $\mathfrak{r}$. We apply this identity to each summand of the expression $e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^\kappa$ expanded in view of (21), which gives

$$\begin{aligned}e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^\kappa&=e^{\mathfrak{r}^1-\mathfrak{r}^2}\big((-\mathrm{A})^\kappa\Omega_{\omega^\kappa}\big)\mathcal{E}^3\\&\quad+e^{\mathfrak{r}^1-\mathfrak{r}^2}\sum_{\kappa'=0}^{\kappa-1}\big((-\mathrm{A})^{\kappa'}\Omega_{\omega^\kappa}\big)\omega^{\kappa-\kappa'}(\mathcal{E}^2-\mathcal{E}^1)+\mathrm{D}_xH,\end{aligned}$$

where H is a differential function of $\mathfrak{r}$ that vanishes on the manifold $\mathcal{S}^{(\infty)}$ and whose precise form is not essential. When acting on functions of ω's, the operator A can be replaced by the operator $\hat{\mathcal{A}}=\sum_{\kappa=0}^{\infty}\omega^{\kappa+1}\partial_{\omega^\kappa}$. Substituting the derived expression for $e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega_{\omega^\kappa}\mathrm{B}\omega^\kappa$ into (20) and collecting coefficients of $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$, we obtain a characteristic from the first family of the theorem. □

Remark 19. Since the common element $e^{\mathfrak{r}^1-\mathfrak{r}^2}(1,-1,0)$ of cosymmetry families, which is mentioned in Remark 14, is a conservation-law characteristic of the system $\mathcal{S}$, it was expected that the families of conserved currents and of conservation-law characteristics from Theorems 15 and 18 have the same properties as the properties of cosymmetry families indicated in Remark 14. Thus, the above conservation-law characteristic, which spans the intersection of the first and the second families from Theorem 18, corresponds to the conserved current $e^{\mathfrak{r}^1-\mathfrak{r}^2}(1,\mathfrak{r}^1+\mathfrak{r}^2)$ spanning the intersection of the respective families from Theorem 15, cf. the end of the proof of this theorem.

Remark 20. The second family of cosymmetries presented in Theorem 13 coincides with the second family of conservation-law characteristics from Theorem 18 up to reparameterization. In other words, each cosymmetry in this family is a conservation-law characteristic. This is not the case for the first[5] and third families of cosymmetries from Theorem 13, which properly contain the first and third families of conservation-law characteristics from Theorem 18, respectively.

Theorem 21. *Under the action of generalized symmetries of the system (1) on its space of conservation laws, a generating set of conservation laws of this system is constituted by the two zeroth-order conservation laws respectively containing the conserved currents*

$$e^{\mathfrak{r}^1-\mathfrak{r}^2}\big(\mathfrak{r}^3,\ (\mathfrak{r}^1+\mathfrak{r}^2)\mathfrak{r}^3\big),\tag{22a}$$

$$e^{\mathfrak{r}^1-\mathfrak{r}^2}\big(x-V^3t,V^3(x-V^3t)-t\big)\quad\textit{with}\quad V^3:=\mathfrak{r}^1+\mathfrak{r}^2.\tag{22b}$$

Proof. The action of the generalized symmetry $\Omega\partial_{\mathfrak{r}^3}$ on the conserved current (22a) gives the conserved current

$$\big(e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega,\ (\mathfrak{r}^1+\mathfrak{r}^2)e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega\big).$$

Varying the parameter function Ω through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3$, $\kappa\in\mathbb{N}_0$, we obtain the first family of conserved currents from Theorem 15.

Conserved currents from the other two families are constructed by mapping conserved currents of the (1+1)-dimensional Klein–Gordon equation (4a) in the way described in the proof of Theorem 15. In view of [25, Corollary 11], a generating set of conservation laws of (4a) is constituted, under the action of generalized symmetries of (4a) on conservation laws thereof, by the single conservation law containing the conserved current $(q_z^2,-q^2)$. The counterpart of this conserved current for the system (1) is the conserved current

$$e^{\mathfrak{r}^1-\mathfrak{r}^2}\Big(\mathfrak{r}^2_x(x-V^2t)^2-\mathfrak{r}^1_x(x-V^1t)^2,\ V^2\mathfrak{r}^2_x(x-V^2t)^2-V^1\mathfrak{r}^1_x(x-V^1t)^2\Big),$$

which is equivalent to the conserved current (22b) multiplied by 2. It follows from Lemma 9 that not all generalized symmetries of (4a) can be naturally mapped to those of the system (1). This is why we need to carefully analyze the result on generating conservation laws of (4a) before adopting it for the system (1).

The conserved current $\mathrm{C}^0_f=(fq_z,-f_yq)$ of Eq. (4a) can be obtained by acting the generalized symmetry $\frac{1}{2}f_y\partial_q\in\hat{\mathfrak{K}}^{-\infty}$ of this equation on the chosen conserved current $(q_z^2,-q^2)$. Here the parameter function $f=f(y,z)$ runs through the solution set of (4a). Each conserved current from the second family of Theorem 15 is the image of a conserved current of the form C^0_f, and each Lie symmetry vector field $f\partial_q$ of (4a) is mapped to an element of the ideal $\mathcal{I}^1$ of the algebra $\hat{\Sigma}^{\mathrm{q}}$. Therefore, the second family of conserved currents from Theorem 15 is generated by acting the elements of $\mathcal{I}^1$ on the conserved current (22b).

The action of the generalized symmetry $\frac{1}{2}(\mathrm{D}_y\mathfrak{Q}q)\partial_q$, where $(\mathfrak{Q}q)\partial_q\in\tilde{\Lambda}^{\mathrm{q}}$, on the conserved current $(q_z^2,-q^2)$ gives the conserved current $(q_z\mathrm{D}_y\mathrm{D}_z\mathfrak{Q}q,-q\mathrm{D}_y\mathfrak{Q}q)$, which is equivalent to the conserved currents $(q_z\mathfrak{Q}q,-q\mathrm{D}_y\mathfrak{Q}q)$ and, therefore, to

$$\mathrm{C}_{\mathfrak{Q}}=(-q\mathrm{D}_z\mathfrak{Q}q,\ q_y\mathfrak{Q}q).$$

[5] In the notation of Remark 16, upon formally interpreting ω^0 as a single dependent variable of a single independent variable, say ς, and ω^1, $\omega^2,\dots$ as the successive derivatives of ω^0, the operators $\partial_\varsigma+\hat{\mathcal{A}}$ and E' become the total derivative operator with respect to ς and the Euler operator with respect to ω^0, respectively. Suppose that a smooth function Ω of a finite number of ω's belongs to $\operatorname{im}\mathsf{E}$. Then $(\hat{\mathcal{A}}\Omega)/\omega^1\in\operatorname{im}\mathsf{E}'$ and thus the Fréchet derivative of $(\hat{\mathcal{A}}\Omega)/\omega^1$ with respect to ω^0 is a formally self-adjoint operator. This is not the case for any Ω of even positive order. Therefore, any cosymmetry from the first family of Theorem 13 with Ω of even positive order is not a conservation-law characteristic of the system (1).

The conservation law containing the obtained conserved currents has the characteristic $(\mathfrak{Q}-\mathfrak{Q}^\dagger)q$.

We denote by $\mathfrak{V}$ the subalgebra of $\tilde{\Lambda}^{\rm q}$ constituted by the elements of $\tilde{\Lambda}^{\rm q}$ that have counterparts among generalized symmetries of the system (1), and

$$\mathfrak{J}:=\langle(\mathrm{J}^\kappa q)\partial_q,\ \kappa\in\mathbb{N}\rangle.$$

We also introduce the corresponding spaces $\mathfrak{V}_-$ and $\mathfrak{J}_-$ of linear generalized symmetries associated with formally skew-adjoint counterparts $\frac12(\mathfrak{Q}-\mathfrak{Q}^\dagger)$ of operators $\mathfrak{Q}$ from $\mathfrak{V}$ and $\mathfrak{J}$, respectively. Note that $\mathfrak{V}_-\supsetneq\mathfrak{V}\cap\tilde{\Lambda}^{\rm q}_-$ and $\mathfrak{J}_-=\mathfrak{J}\cap\tilde{\Lambda}^{\rm q}_-$. In view of Lemma 9, $(\mathfrak{Q}q)\partial_q\in\mathfrak{V}$ if and only if the operator $\mathfrak{Q}$ is represented in the form

$$\mathfrak{Q}=(\mathrm{D}_y+1)\mathfrak{Q}_1+(\mathrm{D}_z+1)\mathfrak{Q}_2+c$$

for some $\mathfrak{Q}_1\in\langle\mathrm{D}_y^\iota\mathrm{J}^\kappa,\ \kappa,\iota\in\mathbb{N}_0\rangle$, some $\mathfrak{Q}_2\in\langle\mathrm{D}_z^\iota\mathrm{J}^\kappa,\ \kappa,\iota\in\mathbb{N}_0\rangle$ and some $c\in\mathbb{R}$. Hence $\tilde{\Lambda}^{\rm q}$ is the direct sum of $\mathfrak{V}$ and $\mathfrak{J}$ as vector spaces, $\tilde{\Lambda}^{\rm q}=\mathfrak{V}+\mathfrak{J}$, and thus $\tilde{\Lambda}^{\rm q}_-=\mathfrak{V}_-+\mathfrak{J}_-$, where the sum is not direct by now. We are going to show that $\mathfrak{V}_-\supset\mathfrak{J}_-$, which implies that $\tilde{\Lambda}^{\rm q}_-=\mathfrak{V}_-$. Indeed, for any $\mathfrak{Q}:=(\mathrm{D}_y+1)(\mathrm{J}-1/2)^\kappa$ with $\kappa\in2\mathbb{N}_0+1$ we have

$$\begin{aligned}\mathfrak{Q}-\mathfrak{Q}^\dagger&=(\mathrm{D}_y+1)(\mathrm{J}-1/2)^\kappa-(\mathrm{J}+1/2)^\kappa(\mathrm{D}_y-1)\\&=(\mathrm{J}-1/2)^\kappa+(\mathrm{J}+1/2)^\kappa,\end{aligned}$$

i.e., $\big((\mathrm{J}-1/2)^\kappa q+(\mathrm{J}+1/2)^\kappa q\big)\partial_q\in\mathfrak{V}_-$ since $(\mathfrak{Q}q)\partial_q\in\mathfrak{V}$. Therefore,

$$\begin{aligned}\mathfrak{J}_-&=\big\langle(\mathrm{J}^\kappa q)\partial_q,\ \kappa\in2\mathbb{N}_0+1\big\rangle\\&=\big\langle\big((\mathrm{J}-1/2)^\kappa q+(\mathrm{J}+1/2)^\kappa q\big)\partial_q,\ \kappa\in2\mathbb{N}_0+1\big\rangle\subset\mathfrak{V}_-.\end{aligned}$$

As a result, for any $(\mathfrak{Q}q)\partial_q\in\tilde{\Lambda}^{\rm q}$ the conserved current $\mathrm{C}_\mathfrak{Q}$ is equivalent to a conserved current of (4a) that is obtained by the action of a generalized symmetry from $\mathfrak{V}$ on the chosen conserved current $(q_z^2,-q^2)$. For the system (1), this means that the third family of conserved currents from Theorem 15 is generated by acting the generalized symmetries of the form $\check{\mathcal{R}}(\Gamma)$ on the conserved current (22b). □

Remark 22. The conserved currents from the second family of Theorem 15 can be represented in a more symmetrical form. Reparameterizing them in terms of the potential $\bar{\Phi}$ defined via Φ by the system

$$\bar{\Phi}_{\mathfrak{r}^1}+\frac12\bar{\Phi}=2\Phi_{\mathfrak{r}^1},\quad -\bar{\Phi}_{\mathfrak{r}^2}+\frac12\bar{\Phi}=\Phi,$$

cf. Section 2, we obtain another representation for these conserved currents,

$$\begin{aligned}e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\,\big(&\bar{\Phi}_{\mathfrak{r}^1}-\bar{\Phi}_{\mathfrak{r}^2}+\bar{\Phi},\\&(\mathfrak{r}^1+\mathfrak{r}^2+1)\bar{\Phi}_{\mathfrak{r}^1}-(\mathfrak{r}^1+\mathfrak{r}^2-1)\bar{\Phi}_{\mathfrak{r}^2}+(\mathfrak{r}^1+\mathfrak{r}^2)\bar{\Phi}\,\big),\end{aligned}$$

where the parameter function $\bar{\Phi}=\bar{\Phi}(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution space of the Klein–Gordon equation $\bar{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2}=-\bar{\Phi}/4$ as well. The successive point transformation

$$\tilde{\Phi}=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\bar{\Phi}$$

reduces the above representation to

$$\big(\tilde{\Phi}_{\mathfrak{r}^1}-\tilde{\Phi}_{\mathfrak{r}^2},\ (\mathfrak{r}^1+\mathfrak{r}^2+1)\tilde{\Phi}_{\mathfrak{r}^1}-(\mathfrak{r}^1+\mathfrak{r}^2-1)\tilde{\Phi}_{\mathfrak{r}^2}\big),$$

where the parameter function $\tilde{\Phi}=\tilde{\Phi}(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution space of the equation $2\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2}=\tilde{\Phi}_{\mathfrak{r}^2}-\tilde{\Phi}_{\mathfrak{r}^1}$. It is the last representation that was employed in [23, Theorem 22]. In terms of $\tilde{\Phi}$, the associated characteristics take the form

$$(\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^1}-\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2},\tilde{\Phi}_{\mathfrak{r}^1\mathfrak{r}^2}-\tilde{\Phi}_{\mathfrak{r}^2\mathfrak{r}^2},0).$$

Remark 23. The advantage of using conserved currents of the form $\mathrm{C}_\mathfrak{Q}$ for mapping to conserved currents of the system $\mathcal{S}$ is that we obtain a uniform representation for elements of the third family of Theorem 15. At the same time, it is not obvious how to find equivalent conserved currents of minimal order for elements of this family or how to single out conserved currents in this family that are equivalent to ones not depending on (t,x) explicitly. The former problem can be solved by replacing conserved currents of the form $\mathrm{C}_\mathfrak{Q}$ in the mapping by equivalent conserved currents

$$\mathrm{C}^1_{\kappa\iota},\quad\kappa\in\mathbb{N}_0,\quad\iota\in\mathbb{N},\quad\bar{\mathrm{C}}^1_{\kappa\iota},\quad\mathrm{C}^2_{\kappa\iota},\quad\bar{\mathrm{C}}^2_{\kappa\iota},\quad\kappa,\iota\in\mathbb{N}_0,$$

presented in [25, Section 4] although an additional "integration by parts" may still be needed for lowest values of (κ,ι) after the mapping, cf. the proof of Theorem 21. For solving the latter problem, we use an analog of the trick used in the proof of Theorem 15 for deriving the second family of conserved currents, which leads to Theorem 26.

Corollary 24. *(i) The space of hydrodynamic conservation laws of the system (1) is infinite-dimensional and is naturally isomorphic to the space spanned by the conserved currents from the second family of Theorem 15 and from the first family with Ω running through the space of smooth functions of $\omega^0:=\mathfrak{r}^3$.*

(ii) The space of zeroth-order conservation laws of the system (1) is naturally isomorphic to the space spanned by its hydrodynamic conserved currents and the conserved current (22b).

Proof. This assertion was proved in [23, Theorem 22] by the direct computation. At the same time, it is a simple corollary of Theorems 15 and 18. Indeed, when linearly combining conserved currents from different families of Theorem 15, the maximum of their orders is preserved. The selection of zeroth-order conserved currents from the first and the second families is obvious. Theorem 18 implies that the space of zeroth-order characteristics related to the third family is one-dimensional and spanned by the characteristic

$$e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(\tilde{q},\ -\tilde{\mathfrak{D}}_z\tilde{q},\ 0\big)$$

of the conservation law with the conserved current (22b). □

Corollary 25. *The space of zeroth- and first-order conservation laws of the system (1) is naturally isomorphic to the space spanned by the conserved currents from the second family of Theorem 15 and from the first family, where the parameter function Ω runs through the space of smooth functions of $(\omega^0,\omega^1):=(\mathfrak{r}^3,e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathfrak{r}^3_x)$ and such two functions should be assumed equivalent if their difference is of the form $f(\omega^0)\omega^1$, as well as the conserved currents from the third family, where the operator $\tilde{\mathfrak{Q}}$ runs through the set*

$$\big\{\tilde{\mathfrak{D}}_z,\ \tilde{\mathfrak{D}}_y,\ \tilde{\mathfrak{J}},\ \tilde{\mathfrak{D}}_z^3,\ (\tilde{\mathfrak{J}}-1)\tilde{\mathfrak{D}}_z^2\big\}.$$

Proof. In the same spirit as in the proof of Corollary 24, we select the zeroth- and first-order conserved currents equivalent to those listed in Theorem 15 using Theorem 18 for estimating the orders of the associated conservation laws. Thus, the selection of the conserved currents from the second family is again obvious since all of then are of order zero. The order of a conservation law related to the first family coincides with the minimal order of the associated Ω's. In general, for zeroth- and first-order conservation laws of the system (1), the order of corresponding reduced characteristics is not greater than two. This is why a conservation law related to the span of the third family is of order not greater than one if and only if it contains a conserved current corresponding to $\tilde{\mathfrak{Q}}\in\big\langle\tilde{\mathfrak{D}}_z,\ \tilde{\mathfrak{D}}_y,\ \tilde{\mathfrak{J}},\ \tilde{\mathfrak{D}}_z^3,\ (\tilde{\mathfrak{J}}-1)\tilde{\mathfrak{D}}_z^2\big\rangle$. □

Theorem 26. *The space of (t,x)-translation-invariant conservation laws of the system (1) is naturally isomorphic to the space spanned by the conserved currents from the first and second families of Theorem 15 as well as the conserved currents from the span of the third family that have the form $\tilde{\mathrm C}_{\tilde{\mathfrak Q}}$ of elements of this family,*

$$\begin{aligned}&\big(\mathfrak r^2_x\tilde\rho+\mathfrak r^1_x\tilde\sigma,\ (\mathfrak r^1+\mathfrak r^2-1)\mathfrak r^2_x\tilde\rho+(\mathfrak r^1+\mathfrak r^2+1)\mathfrak r^1_x\tilde\sigma\big)\\&\text{with}\quad \tilde\rho=-\tilde q\tilde{\mathfrak D}_z\tilde{\mathfrak Q}\tilde q,\quad \tilde\sigma=(\tilde{\mathfrak D}_y\tilde q)\tilde{\mathfrak Q}\tilde q,\end{aligned}\tag{23}$$

where the operator $\tilde{\mathfrak Q}$ runs through the set $\mathfrak T$ constituted by the operators

$$\begin{aligned}&\tilde{\mathfrak Z}_{\kappa\iota}:=(\tilde{\mathfrak D}_z+1)^2(\tilde{\mathfrak J}-\iota/2)^\kappa\tilde{\mathfrak D}_z^\iota(\tilde{\mathfrak D}_z-1)^2,\\&\tilde{\mathfrak Y}_{\kappa,\iota+4}:=(\tilde{\mathfrak D}_y+1)^2(\tilde{\mathfrak J}+\iota/2)^\kappa\tilde{\mathfrak D}_y^\iota(\tilde{\mathfrak D}_y-1)^2,\\&\qquad\kappa,\iota\in\mathbb N_0\ \text{with}\ \kappa+\iota\in2\mathbb N_0+1;\\&\tilde{\mathfrak Y}_{\kappa1}:=(\tilde{\mathfrak J}+1/2)^\kappa(\tilde{\mathfrak D}_y+\tilde{\mathfrak D}_z-2)\\&\qquad+(\tilde{\mathfrak D}_z+2)(\tilde{\mathfrak J}-1/2)^\kappa(\tilde{\mathfrak D}_z-1)^2,\quad\kappa\in2\mathbb N_0;\\&\tilde{\mathfrak Y}_{\kappa2}:=2\tilde{\mathfrak J}^\kappa(\tilde{\mathfrak D}_y+\tilde{\mathfrak D}_z-2)+(\tilde{\mathfrak J}+1)^\kappa(\tilde{\mathfrak D}_y-1)^2\\&\qquad+(\tilde{\mathfrak J}-1)^\kappa(\tilde{\mathfrak D}_z-1)^2,\quad\kappa\in2\mathbb N_0+1;\\&\tilde{\mathfrak Y}_{\kappa3}:=(\tilde{\mathfrak J}-1/2)^\kappa(\tilde{\mathfrak D}_y+\tilde{\mathfrak D}_z-2)\\&\qquad+(\tilde{\mathfrak D}_y+2)(\tilde{\mathfrak J}+1/2)^\kappa(\tilde{\mathfrak D}_y-1)^2,\quad\kappa\in2\mathbb N_0.\end{aligned}$$

Proof. Denote by $\bar{\mathfrak T}$ a complementary subspace of the span of $\mathfrak T$ in the span of the set run by $\tilde{\mathfrak Q}$ in the third family of Theorem 15. Since conserved currents from the first and second families of Theorem 15 are (t,x)-translation-invariant, it suffices to prove that conserved currents of the form (23) with $\tilde{\mathfrak Q}\in\mathfrak T$ (resp. with nonzero $\tilde{\mathfrak Q}\in\bar{\mathfrak T}$) are equivalent (resp. not equivalent) to (t,x)-translation-invariant ones.

For each $\tilde{\mathfrak Q}\in\mathfrak T$ we explicitly construct a related (t,x)-translation-invariant conserved current. To this end, we consider the associated operator $\mathfrak Q$ in $\tilde\Lambda^{\mathrm q}_-$, choose an appropriate conserved current of the Klein–Gordon equation (4a) among those equivalent to $\mathrm C_{\mathfrak Q}$ and map it to a conserved current of the system (1). Each operator $\mathfrak Q\in\tilde\Lambda^{\mathrm q}_-$ associated with some $\tilde{\mathfrak Q}\in\mathfrak T$ is equivalent, on solutions of (4a), to an operator of the form

$$(\mathrm D_z+1)^2\mathfrak P(\mathrm D_z-1)^2\quad\text{with}\quad(\mathfrak Pq)\partial_q\in\tilde\Lambda^{\mathrm q}_-,$$

where the operator $\mathfrak P$ coincides with $(\mathrm J-\iota/2)^\kappa\mathrm D_z^\iota$, $(\mathrm J+\iota/2+2)^\kappa\mathrm D_z^{\iota+4}$, $(\mathrm J+1/2)^\kappa\mathrm D_z$, $(\mathrm J+1)^\kappa\mathrm D_z^2$, $(\mathrm J+3/2)^\kappa\mathrm D_z^3$ for $\tilde{\mathfrak Z}_{\kappa\iota}$, $\tilde{\mathfrak Y}_{\kappa,\iota+4}$, $\tilde{\mathfrak Y}_{\kappa1}$, $\tilde{\mathfrak Y}_{\kappa2}$ and $\tilde{\mathfrak Y}_{\kappa3}$, respectively. For such $\mathfrak Q$ we obtain

$$\begin{aligned}&\mathrm C_{\mathfrak Q}\sim(-K^1\mathfrak D_z\mathfrak PK^1,K^2\mathfrak PK^1)\\&\quad\mapsto\ 2e^{(\mathfrak r^1-\mathfrak r^2)/2}\bigg((\tilde{\mathfrak D}_z+1)\tilde{\mathfrak P}\frac{e^{(\mathfrak r^1-\mathfrak r^2)/2}}{\mathfrak r^2_x},\ (V^2\tilde{\mathfrak D}_z+V^1)\tilde{\mathfrak P}\frac{e^{(\mathfrak r^1-\mathfrak r^2)/2}}{\mathfrak r^2_x}\bigg),\end{aligned}$$

which is obviously a (t,x)-translation-invariant conserved current of the system (1).

As a subspace complementary to the span of $\mathfrak T$, we can choose

$$\bar{\mathfrak T}=\big\langle\mathrm J^{2\kappa+1},\ (\mathrm J+1)^{2\kappa+1}\mathrm D_z^2,\ (\mathrm J+1/2)^{2\kappa}\mathrm D_z,\ (\mathrm J+3/2)^{2\kappa}\mathrm D_z^3,\ \kappa\in\mathbb N_0\big\rangle.$$

We prove by contradiction that for any nonzero $\tilde{\mathfrak Q}\in\bar{\mathfrak T}$, i.e.,

$$\tilde{\mathfrak Q}=\sum_{\kappa=0}^N\big(\,c_{0\kappa}\mathrm J^{2\kappa+1}+c_{2\kappa}(\mathrm J+1)^{2\kappa+1}\mathrm D_z^2+c_{3\kappa}(\mathrm J+3/2)^{2\kappa}\mathrm D_z^3\,\big)$$

for some $N\in\mathbb N_0$ and some constants c's with $(c_{0N},c_{1N},c_{2N},c_{3N})\neq(0,0,0,0)$, the corresponding conserved current of the form (23) is not equivalent to a (t,x)-translation-invariant one. Suppose that this is not the case. If a conservation law of the system (1) is (t,x)-translation-invariant, then its characteristic is also (t,x)-translation-invariant. The conservation-law characteristic associated with $\tilde{\mathfrak Q}$ (see Theorem 18) does not depend on the variables x and t if and only if $(\tilde{\mathfrak Q}\tilde q)_x=\tilde{\mathfrak Q}e^{(\mathfrak r^1-\mathfrak r^2)/2}=0$ and $(\tilde{\mathfrak Q}\tilde q)_t=-\tilde{\mathfrak Q}\big((\mathfrak r^1+\mathfrak r^2+1)e^{(\mathfrak r^1-\mathfrak r^2)/2}\big)=0$. In the coordinates (6), these conditions, after re-combining, take the form

$$\mathfrak Qe^{y+z}=0,\quad\mathfrak Q\big((y-z)e^{y+z}\big)=\mathfrak Q\mathrm Je^{y+z}=0,$$

or, equivalently,

$$\begin{aligned}R^1:=\sum_{\kappa=0}^N&\big(\,c_{0\kappa}\mathrm J^{2\kappa+1}+c_{2\kappa}(\mathrm J+1)^{2\kappa+1}+c_{1\kappa}(\mathrm J+1/2)^{2\kappa}\\&+c_{3\kappa}(\mathrm J+3/2)^{2\kappa}\,\big)\,e^{y+z}=0,\\R^2:=\sum_{\kappa=0}^N&\big(\,c_{0\kappa}\mathrm J^{2\kappa+2}+c_{2\kappa}(\mathrm J+1)^{2\kappa+1}(\mathrm J-2)\\&+c_{1\kappa}(\mathrm J+1/2)^{2\kappa}(\mathrm J-1)+c_{3\kappa}(\mathrm J+3/2)^{2\kappa}(\mathrm J-3)\,\big)\,e^{y+z}=0.\end{aligned}$$

The left-hand sides of these equations, R^1 and R^2, are polynomials of $y-z$ and $y+z$ multiplied by e^{y+z}, and the highest degrees of $y-z$ correspond to the highest degrees of J. Recombining these equations to

$$\begin{aligned}R^2-\mathrm JR^1=-\sum_{\kappa=0}^N&\big(\,2c_{2\kappa}(\mathrm J+1)^{2\kappa+1}+c_{1\kappa}(\mathrm J+1/2)^{2\kappa}\\&+3c_{3\kappa}(\mathrm J+3/2)^{2\kappa}\,\big)\,e^{y+z}=0,\\R^2-(\mathrm J-2)R^1=\sum_{\kappa=0}^N&\big(\,2c_{0\kappa}\mathrm J^{2\kappa+1}+c_{1\kappa}(\mathrm J+1/2)^{2\kappa}\\&-c_{3\kappa}(\mathrm J+3/2)^{2\kappa}\,\big)\,e^{y+z}=0,\end{aligned}$$

we easily see that $c_{0N}=c_{2N}=0$ and thus also $c_{1N}=c_{2N}=0$, which contradicts the supposition that tuple $(c_{0N},c_{1N},c_{2N},c_{3N})$ has nonzero components. □

In order to construct a lowest-order (t,x)-translation-invariant conserved current for conservation laws associated with operators from $\mathfrak T$, for the respective operator $\mathfrak P$ we should take the respective (up to a constant multiplier) conserved current among $\mathrm C^1_{\kappa\iota}$, $\kappa\in\mathbb N_0$, $\iota\in\mathbb N$, $\bar{\mathrm C}^1_{\kappa\iota}$, $\mathrm C^2_{\kappa\iota}$, $\bar{\mathrm C}^2_{\kappa\iota}$, $\kappa,\iota\in\mathbb N_0$, presented in [25, Section 4], formally replace (x,y,u) by (y,z,K^1) and map the obtained conserved current. In particular, linearly independent (t,x)-translation-invariant inequivalent conserved currents up to order two from the span of the third family of Theorem 15 are exhausted by the following:

$$\begin{aligned}&\tilde{\mathfrak Q}=\tilde{\mathfrak Y}_{01}=\tilde{\mathfrak D}_z^3-2\tilde{\mathfrak D}_z+\tilde{\mathfrak D}_y\colon\ \ \mathfrak P=\mathfrak D_y,\ \ \mathrm C_{\mathfrak Q}\sim\big(-(K^1)^2,\,(K^2)^2\big)\\&\qquad\mapsto\ 2e^{\mathfrak r^1-\mathfrak r^2}\left(\frac{1}{\mathfrak r^2_x}-\frac{1}{\mathfrak r^1_x},\ \frac{V^2}{\mathfrak r^2_x}-\frac{V^1}{\mathfrak r^1_x}\right),\\&\tilde{\mathfrak Q}=\tilde{\mathfrak Y}_{03}=\tilde{\mathfrak D}_y^3-2\tilde{\mathfrak D}_y+\tilde{\mathfrak D}_z\colon\ \ \mathfrak P=\mathfrak D_y^3,\ \ \mathrm C_{\mathfrak Q}\sim\big((K^2)^2,\,-(\mathfrak D_yK^2)^2\big)\\&\qquad\mapsto\ \frac{2}{(\mathfrak r^1_x)^5}e^{\mathfrak r^1-\mathfrak r^2}\ \big(\,(2\mathfrak r^1_{xx}+\mathfrak r^1_x\mathfrak r^2_x)^2-\mathfrak r^2_x(\mathfrak r^1_x)^3,\\&\qquad\qquad V^1(2\mathfrak r^1_{xx}+\mathfrak r^1_x\mathfrak r^2_x)^2-V^2\mathfrak r^2_x(\mathfrak r^1_x)^3\,\big),\\&\tilde{\mathfrak Q}=\tilde{\mathfrak Z}_{01}=\tilde{\mathfrak D}_z^5-2\tilde{\mathfrak D}_z^3+\tilde{\mathfrak D}_z\colon\ \ \mathfrak P=\mathfrak D_z,\ \ \mathrm C_{\mathfrak Q}\sim\big((\mathfrak D_zK^1)^2,\,-(K^1)^2\big)\\&\qquad\mapsto\ \frac{-2}{(\mathfrak r^2_x)^5}e^{\mathfrak r^1-\mathfrak r^2}\ \big(\,(2\mathfrak r^2_{xx}-\mathfrak r^1_x\mathfrak r^2_x)^2-\mathfrak r^1_x(\mathfrak r^2_x)^3,\\&\qquad\qquad V^2(2\mathfrak r^2_{xx}-\mathfrak r^1_x\mathfrak r^2_x)^2-V^1\mathfrak r^1_x(\mathfrak r^2_x)^3\,\big),\\&\tilde{\mathfrak Q}=\tilde{\mathfrak Z}_{10}:=(\tilde{\mathfrak D}_z+1)^2\tilde{\mathfrak J}(\tilde{\mathfrak D}_z-1)^2\colon\quad\mathfrak P=\mathfrak J,\\&\quad\mathrm C_{\mathfrak Q}\sim\big(-y(K^1)^2-z(\mathfrak D_zK^1)^2,\ y(K^2)^2+z(K^1)^2\big)\\&\quad\mapsto-e^{\mathfrak r^1-\mathfrak r^2}(\mathfrak z^1+\mathfrak z^2,\ V^1\mathfrak z^1+V^2\mathfrak z^2),\\&\quad\mathfrak z^1:=\frac{\mathfrak r^1}{\mathfrak r^1_x}-\frac{\mathfrak r^2\mathfrak r^1_x}{(\mathfrak r^2_x)^2},\quad\mathfrak z^2:=\frac{\mathfrak r^2}{(\mathfrak r^2_x)^5}(2\mathfrak r^2_{xx}-\mathfrak r^1_x\mathfrak r^2_x)^2-\frac{\mathfrak r^1}{\mathfrak r^2_x}.\end{aligned}$$

7. Hamiltonian structures of hydrodynamic type

A system $\mathcal{E}$ of evolution differential equations $\mathbf{u}_t - K[\mathbf{u}] = 0$, where K is a tuple of functions of independent variables $(t, \mathbf{x})$ and spatial derivatives (including ones of order zero) of the dependent variables $\mathbf{u} = (u^1, \dots, u^n)^\mathsf{T}$, is called Hamiltonian if it can be represented in the form $\mathbf{u}_t = \mathfrak{H}\,\delta\mathcal{H}$. Here $\mathfrak{H}$ is a Hamiltonian differential operator, i.e. a formally skew-adjoint matrix differential operator, whose associated bracket $\{\cdot,\cdot\}$ defined by $\{\mathcal{I},\mathcal{J}\} = \int \delta\mathcal{I}\cdot\mathfrak{H}\,\delta\mathcal{J}\,\mathrm{d}\mathbf{x}$ for appropriate functionals $\mathcal{I}$ and $\mathcal{J}$, satisfies the Jacobi identity and thus is a Poisson bracket, δ stands for the variational derivative, and the functional $\mathcal{H}$ is called a Hamiltonian of $\mathcal{E}$ with respect to $\mathfrak{H}$, see [12].

A procedure for finding a Hamiltonian structure for the system $\mathcal{E}$ is as follows:

- For the left hand side $F := \mathbf{u}_t - K[\mathbf{u}]$ of the system $\mathcal{E}$, one defines the universal linearization operator ℓ_F of F [36] (also known as the Fréchet derivative of F [30]) and its formally adjoint $\ell_F^\dagger$ to determine the linearization of the system $\mathcal{E}$ and the system adjoint to the linearization,

  $$\ell_F(\eta) = 0, \quad \ell_F^\dagger(\lambda) = 0.$$

  The differential vector functions η and λ of $\mathbf{u}$, that is, vector functions of t, $\mathbf{x}$, $\mathbf{u}$ and their spatial derivatives (time derivatives are excluded in view of the evolutionary form of the equations), satisfying the above systems in view of the system $\mathcal{E}$ are nothing else but symmetries (more precisely, characteristic-tuples of generalized symmetries) and cosymmetries for the system $\mathcal{E}$, respectively.
- By making an ansatz one finds Noether operators, which are by definition matrix differential operators mapping cosymmetries of the system to its symmetries.
- One selects a Hamiltonian operator $\mathfrak{H}$ amongst Noether ones, by requiring that it is skew-adjoint and the associated bracket satisfies the Jacobi identity.
- Choosing an ansatz for a Hamiltonian $\mathcal{H}$, one finds it from the condition $\mathfrak{H}\delta\mathcal{H} = K$.

Skew-adjoint Noether operators of systems of evolution equations are believed to satisfy the Jacobi identity automatically except for first-order scalar equations [37, Theorem 5]. This result was rigorously proved for systems of evolution equations of order greater than one in [38], while the same assertion for non-scalar systems of first-order evolution equations was conjectured in [39]. In spite of the fact that, in general, the verification of this conjecture for the system $\mathcal{S}$ can be done directly, we use a geometrical interpretation of hydrodynamic-type Hamiltonian differential operators for hydrodynamic-type systems. Hereafter we consider a $(1+1)$-dimensional (translation-invariant) hydrodynamic-type system $\mathcal{E}$, the indices i, j, k and l run from 1 to n, and the Einstein summation convention is assumed for the index l. A matrix differential operator $\mathfrak{D} = (\mathfrak{D}^{ij})$ and the associated bracket are said to be *of hydrodynamic type* or *of Dubrovin–Novikov type* if the entries of $\mathfrak{D}$ are of the form $\mathfrak{D}^{ij} = g^{ij}(u)\mathrm{D}_x + b^{ij}_l(u)u^l_x$.

The cornerstone of the geometric interpretation of hydrodynamic-type Hamiltonian operators, discovered in the seminal paper [11], is the fact that under a point transformation $\tilde{\mathbf{u}} = U(\mathbf{u})$ of dependent variables only, the coefficients g^{ij} of $\mathfrak{D}$ are transformed as components of a second-order contravariant tensor on the space with the coordinates $\mathbf{u}$ and, if the tensor (g^{ij}) is nondegenerate (which is a perpetual assumption below), the coefficients b^{ij}_k are transformed so that Γ^j_{lk} defined by $g^{il}\Gamma^j_{lk} = -b^{ij}_k$ are the Christoffel symbols of a connection ∇ on this space. The bracket associated with $\mathfrak{D}$ is skew-symmetric if and only if the tensor (g^{ij}) is symmetric, i.e., $g = (g_{i'j'}) = (g^{ij})^{-1}$ is a (pseudo-)Riemannian metric, and the connection ∇ agrees with g, $\nabla g = 0$. The bracket satisfies the Jacobi identity if and only if the metric g is flat and the connection ∇ is the Levi-Civita connection of g, i.e., the curvature tensor of g and the torsion tensor of ∇ vanish.

Recall that two Hamiltonian operators are called compatible if any their linear combination is a Hamiltonian operator as well. Two nondegenerate hydrodynamic-type Hamiltonian operators for a hydrodynamic-type system are compatible if the Nijenhuis tensor $\mathcal{N}$ of the tensor (s^i_j) defined by $s^i_j = \tilde{g}^{il}g_{lj}$ vanishes,

$$\mathcal{N}^i_{jk} := s^l_j\partial_{u^l}s^i_k - s^l_k\partial_{u^l}s^i_j - s^i_l(\partial_{u^j}s^l_k - \partial_{u^k}s^l_j) = 0,$$

see [40,41]. Here g and $\tilde{g}$ are the metrics corresponding to the Hamiltonian operators. In terms of g and $\tilde{g}$, the condition of vanishing the Nijenhuis tensor $\mathcal{N}$ takes the form

$$\nabla^i\nabla^j\tilde{g}^{kl} + \nabla^k\nabla^l\tilde{g}^{ij} - \nabla^i\nabla^k\tilde{g}^{jl} - \nabla^j\nabla^l\tilde{g}^{ik} = 0. \tag{24}$$

The covariant differentiation in (24) corresponds to the metric g. The conditions (24) are preserved by the permutation of g and $\tilde{g}$, so that they are indeed the compatibility conditions of the two metrics.

When the tensor g degenerates at some point, the associated hydrodynamic-type system loses its geometric charm and one needs to proceed otherwise. To show that the bracket of a skew-symmetric Noether operator $\mathfrak{N}$ for $\mathcal{E}$ satisfies the Jacobi identity, one may equivalently check that the variational Schouten bracket $[\![\mathfrak{N},\mathfrak{N}]\!]$ vanishes. To show the compatibility of two hydrodynamic-type Hamiltonian operators $\mathfrak{H}_1$ and $\mathfrak{H}_2$, $\mathfrak{H}^{ij}_k = g^{ij}_k\mathrm{D}_x + b^{ij}_{kl}\mathfrak{r}^l_x$, $k = 1, 2$, one may check that $[\![\mathfrak{H}_1,\mathfrak{H}_2]\!] = 0$, cf. [27, Section 10.1]. Since $\mathcal{E}$ is a system of evolution equations, one may consider the cotangent covering $T^*\mathcal{E}$ of $\mathcal{E}$ (i.e., the joint system $F = 0$, $\ell_F^\dagger(\lambda) = 0$) and substitute the latter condition by the equivalent one

$$\mathsf{E}\sum_{j=1}^n\Big((\mathsf{E}_{u^j}\mathrm{F}_{\mathfrak{H}_1})(\mathsf{E}_{\lambda^j}\mathrm{F}_{\mathfrak{H}_2}) + (\mathsf{E}_{\lambda^j}\mathrm{F}_{\mathfrak{H}_1})(\mathsf{E}_{u^j}\mathrm{F}_{\mathfrak{H}_2})\Big) = 0,$$

where $\mathsf{E} = (\mathsf{E}_{u^1},\dots,\mathsf{E}_{u^n},\mathsf{E}_{\lambda^1},\dots,\mathsf{E}_{\lambda^n})$ is the Euler operator on $T^*\mathcal{E}$, and $\mathrm{F}_{\mathfrak{H}_k} = \sum_{i,j}\big(g^{ij}_k(\mathrm{D}_x\lambda^i)\lambda^j + b^{ij}_{kl}\mathfrak{r}^l_x\lambda^i\lambda^j\big)$, $\quad k = 1, 2$.

Theorem 27. *The system (1) admits an infinite family of compatible Hamiltonian structures $\mathfrak{H}_\Theta$ parameterized by a smooth function Θ of $\mathfrak{r}^3$,*

$$\mathfrak{H}_\Theta = e^{\mathfrak{r}^2-\mathfrak{r}^1}\operatorname{diag}\big(-1, 1, \Theta(\mathfrak{r}^3)e^{\mathfrak{r}^2-\mathfrak{r}^1}\big)\mathrm{D}_x - \frac{e^{\mathfrak{r}^2-\mathfrak{r}^1}}{2}\begin{pmatrix}\mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x\\ \mathfrak{r}^2_x-\mathfrak{r}^1_x & \mathfrak{r}^1_x-\mathfrak{r}^2_x & -2\mathfrak{r}^3_x\\ 2\mathfrak{r}^3_x & 2\mathfrak{r}^3_x & -2f^{33}e^{\mathfrak{r}^1-\mathfrak{r}^2}\end{pmatrix} \tag{25}$$

with the corresponding family of Hamiltonians $\mathcal{H}_{c_0,\Xi} = \int H_{c_0,\Xi}\,\mathrm{d}x$ defined by densities

$$H_{c_0,\Xi} = (\mathfrak{r}^1+\mathfrak{r}^2)^2e^{\mathfrak{r}^1-\mathfrak{r}^2} + c_0(\mathfrak{r}^1+\mathfrak{r}^2) + 2\big(\mathfrak{r}^1-\mathfrak{r}^2+\Xi(\mathfrak{r}^3)\big)e^{2(\mathfrak{r}^1-\mathfrak{r}^2)}. \tag{26}$$

Here $f^{33} := e^{2\mathfrak{r}^2-2\mathfrak{r}^1}\big((\mathfrak{r}^2_x-\mathfrak{r}^1_x)\Theta + \mathfrak{r}^3_x\Theta_{\mathfrak{r}^3}\big)$, c_0 is an arbitrary constant and the function Ξ of $\mathfrak{r}^3$ satisfies the auxiliary condition

$$|\Theta|^{1/2}\,\Xi_{\mathfrak{r}^3\mathfrak{r}^3} + \frac{1}{2}\Theta_{\mathfrak{r}^3}\Xi_{\mathfrak{r}^3} = c_0.$$

Proof. For the first step of the algorithm expounded above we need to consider the joint system of equations

$$\mathfrak{r}^k_t + V^k\mathfrak{r}^k_x = 0, \tag{27a}$$

$$\mathrm{D}_t\eta^k + V^k\mathrm{D}_x\eta^k + (\eta^1+\eta^2)\mathfrak{r}^k_x = 0, \tag{27b}$$

$$\mathrm{D}_t\lambda^k + \mathrm{D}_x(V^k\lambda^k) - \mathfrak{r}^l_x\lambda^l(\delta^1_k+\delta^2_k) = 0, \tag{27c}$$

where the system (27a) is the system $\mathcal{S}$ itself, the system (27b) is the linearization of $\mathcal{S}$, and the system (27c) is adjoint to (27b). If η^i and λ^j are differential functions of $\mathfrak{r}$, then the tuples $\eta = (\eta^1, \eta^2, \eta^3)^\mathsf{T}$ and $\lambda = (\lambda^1, \lambda^2, \lambda^3)$ are a symmetry characteristic and a cosymmetry of the system $\mathcal{S}$, respectively. Here and in what follows a summation with respect to i, j and l, which run through the set $\{1, 2, 3\}$, is assumed, there is no summation with respect to k, which is a fixed number from the set $\{1, 2, 3\}$, and δ^i_j stands for the Kronecker delta. We are looking for a Noether operator $\mathfrak{N} = (\mathfrak{N}^{ij})$ with entries of the form

$$\mathfrak{N}^{ij} = h^{ij}(\mathfrak{r})\mathrm{D}_x + f^{ij}(\mathfrak{r}, \mathfrak{r}_x) \tag{28}$$

for some smooth functions h^{ij} and f^{ij} of their arguments. By definition of Noether operators, for any solution $(\mathfrak{r}, \lambda)$ of (27a), (27c) the expressions $\eta^i := \mathfrak{N}^{ij}\lambda^j = h^{ij}\mathrm{D}_x\lambda^j + f^{ij}\lambda^j$ give a solution of (27b). This implies the system

$$\begin{aligned}
&-h^{kj}_{\mathfrak{r}^l}V^l\mathfrak{r}^l_x\mathrm{D}_x\lambda^j + h^{kj}\mathrm{D}_x\left(-\mathrm{D}_x(V^j\lambda^j) + \mathfrak{r}^l_x\lambda^l\left(\delta^1_j + \delta^2_j\right)\right)\\
&\quad -f^{kj}_{\mathfrak{r}^l_x}\left(V^l\mathfrak{r}^l_{xx} + (\mathfrak{r}^1_x + \mathfrak{r}^2_x)\mathfrak{r}^l_x\right)\lambda^j\\
&\quad -f^{kj}_{\mathfrak{r}^l}V^l\mathfrak{r}^l_x\lambda^j + f^{kj}\left(-\mathrm{D}_x(V^j\lambda^j) + \mathfrak{r}^l_x\lambda^l\left(\delta^1_j + \delta^2_j\right)\right)\\
&\quad + V^k\left(h^{kj}_{\mathfrak{r}^l}\mathfrak{r}^l_x\mathrm{D}_x\lambda^j + h^{kj}\mathrm{D}^2_x\lambda^j + f^{kj}_{\mathfrak{r}^l}\mathfrak{r}^l_x\lambda^j + f^{kj}_{\mathfrak{r}^l_x}\mathfrak{r}^l_{xx}\lambda^j + f^{kj}\mathrm{D}_x\lambda^j\right)\\
&\quad + \mathfrak{r}^k_x\left(h^{1j}\mathrm{D}_x\lambda^j + f^{1j}\lambda^j + h^{2j}\mathrm{D}_x\lambda^j + f^{2j}\lambda^j\right) = 0.
\end{aligned}$$

Collecting coefficients of $\mathrm{D}^2_x\lambda^j$ immediately leads to $h^{kj} = 0$ for all $j \neq k$, and further collecting coefficients of $\mathrm{D}_x\lambda^j$ yields

$$\begin{aligned}
&f^{21} = -f^{12} = \frac{1}{2}\left(h^{11}\mathfrak{r}^2_x + h^{22}\mathfrak{r}^1_x\right),\\
&f^{31} = -f^{13} = h^{11}\mathfrak{r}^3_x, \quad f^{23} = -f^{32} = h^{22}\mathfrak{r}^3_x,\\
&h^{11}_{\mathfrak{r}^3} = 0, \quad h^{11}_{\mathfrak{r}^2} = h^{11}, \quad h^{22}_{\mathfrak{r}^3} = 0, \quad h^{22}_{\mathfrak{r}^1} = -h^{22},\\
&h^{33}_{\mathfrak{r}^1} = -2h^{33}, \quad h^{33}_{\mathfrak{r}^2} = 2h^{33}.
\end{aligned}$$

Finally, splitting with respect to λ^j and derivatives of $\mathfrak{r}^l$ allows us to deduce that the operator $\mathfrak{N}$ is of the form (25) with

$$f^{33} = \Theta(\mathfrak{r}^3)e^{2\mathfrak{r}^2-2\mathfrak{r}^1}(\mathfrak{r}^2_x - \mathfrak{r}^1_x) + \Psi(\mathfrak{r}^3, \mathfrak{r}^3_x e^{\mathfrak{r}^2-\mathfrak{r}^1})e^{\mathfrak{r}^2-\mathfrak{r}^1},$$

where Θ and Ψ are arbitrary smooth functions of their arguments. To be qualified as a Hamiltonian operator, the operator $\mathfrak{N}$ should be formally skew-adjoint, i.e., $\mathfrak{N}^\dagger = -\mathfrak{N}$, yielding the Noether operator $\mathfrak{H}_\Theta$ of the form (25) with f^{33} as in the statement of the theorem.

First consider the case when Θ is nonvanishing. The operator $\mathfrak{H}_\Theta$ is of hydrodynamic type with the pseudo-Riemannian metric

$$g = \mathrm{diag}\big(-e^{\mathfrak{r}^2-\mathfrak{r}^1},\ e^{\mathfrak{r}^2-\mathfrak{r}^1},\ \Theta(\mathfrak{r}^3)e^{2\mathfrak{r}^2-2\mathfrak{r}^1}\big). \tag{29}$$

It is easy to show that the coordinates $\mathfrak{r}$ are Liouville ones, cf. [12]. The connection ∇ associated with $\mathfrak{H}_\Theta$ in the sense discussed above is the Levi-Civita connection for g. Thus we should check that the corresponding Riemann curvature tensor vanishes. Due to its symmetries, we only need to verify that $R^i_{jij} = 0$ for $i \neq j$. This is easily computed to be true. Thus the Noether operator $\mathfrak{H}_\Theta$ is a Hamiltonian one.

Finally, we are looking for a Hamiltonian $\mathcal{H} = \int H(\mathfrak{r})\,\mathrm{d}x$ of $\mathcal{S}$ with respect to $\mathfrak{H}_\Theta$. It satisfies the condition

$$\mathfrak{H}_\Theta\frac{\delta\mathcal{H}}{\delta\mathfrak{r}} = -\begin{pmatrix} V^1\mathfrak{r}^1_x\\ V^2\mathfrak{r}^2_x\\ V^3\mathfrak{r}^3_x\end{pmatrix},$$

where $\delta\mathcal{H}/\delta\mathfrak{r}$ is the vector of variational derivatives of $\mathcal{H}$ with respect to the Riemann invariants $\mathfrak{r}^1$, $\mathfrak{r}^2$ and $\mathfrak{r}^3$, $\delta\mathcal{H}/\delta\mathfrak{r} = (H_{\mathfrak{r}^1}, H_{\mathfrak{r}^2}, H_{\mathfrak{r}^3})^\mathsf{T}$ due to the fact that H is a function of $\mathfrak{r}$ only. Expanding this condition we find the system of differential equations on H,

$$e^{\mathfrak{r}^2-\mathfrak{r}^1}\left(-\mathrm{D}_xH_{\mathfrak{r}^1} + \frac{\mathfrak{r}^2_x - \mathfrak{r}^1_x}{2}(H_{\mathfrak{r}^2} - H_{\mathfrak{r}^1}) + \mathfrak{r}^3_xH_{\mathfrak{r}^3}\right) = -2V^1\mathfrak{r}^1_x, \tag{30a}$$

$$e^{\mathfrak{r}^2-\mathfrak{r}^1}\left(\mathrm{D}_xH_{\mathfrak{r}^2} + \frac{\mathfrak{r}^2_x - \mathfrak{r}^1_x}{2}(H_{\mathfrak{r}^2} - H_{\mathfrak{r}^1}) + \mathfrak{r}^3_xH_{\mathfrak{r}^3}\right) = -2V^2\mathfrak{r}^2_x, \tag{30b}$$

$$\begin{aligned}
e^{2\mathfrak{r}^2-2\mathfrak{r}^1}\Big(&-\mathfrak{r}^3_x(H_{\mathfrak{r}^1} + H_{\mathfrak{r}^2})e^{\mathfrak{r}^1-\mathfrak{r}^2} + \mathrm{D}_x(H_{\mathfrak{r}^3})\Theta\\
&+ \left((\mathfrak{r}^2_x - \mathfrak{r}^1_x)\Theta + \mathfrak{r}^3_x\Theta_{\mathfrak{r}^3}\right)H_{\mathfrak{r}^3}\Big) = -2V^3\mathfrak{r}^3_x.
\end{aligned} \tag{30c}$$

Successively splitting with respect to $\mathfrak{r}^1_x$, $\mathfrak{r}^2_x$ and $\mathfrak{r}^3_x$ and solving the obtained overdetermined system of differential equations, we find the final form (26) for Hamiltonian densities and the auxiliary condition on Ξ.

For the system (1) the tensor (s^i_j) takes a particularly simple form, $(s^i_j) = \mathrm{diag}(1, 1, \tilde\Theta/\Theta)$, where Θ and $\tilde\Theta$ are functions of $\mathfrak{r}^3$ parameterizing the metrics g and $\tilde g$. It is trivial to verify that its Nijenhuis tensor vanishes. Since eigenvalues of (s^i_j) are not distinct, we need also to verify the conditions (24), and they also hold.

If Θ is a somewhere vanishing function, then the geometric reasoning for Hamiltonian operators is no longer available, and we should proceed by establishing that the corresponding variational Schouten brackets vanish, which is done symbolically. □

Remark 28. It is worth noting that provided $\Xi_{\mathfrak{r}^3} \neq 0$ the condition on Ξ can be equivalently represented as

$$\Theta = \frac{c_0\Xi + c_1}{\Xi_{\mathfrak{r}^3}^{\;2}},$$

where c_1 is an arbitrary constant.

For preliminary computations and testing the above results, we used the package Jets [42,43] for Maple.

Below we consider only canonical representatives of symmetry-type objects, where derivatives involving differentiations with respect to t are replaced by their expressions in view of the system $\mathcal{S}$, which is necessary for relating different kinds of such objects via Hamiltonian structures.

For any Hamiltonian operator $\mathfrak{H}_\Theta$ from Theorem 27, we can endow the space $\hat\Upsilon^\mathrm{q}$ of canonical representatives for cosymmetries of $\mathcal{S}$ with a Lie-algebra structure, cf. [44] and [45, Section 3.1], where the corresponding Lie bracket is defined by

$$[\gamma^1, \gamma^2]_{\mathfrak{H}_\Theta} = \ell_{\gamma^2}\mathfrak{H}_\Theta\gamma^1 + \ell^\dagger_{\mathfrak{H}_\Theta\gamma^1}\gamma^2 + (\ell_{\gamma^1} - \ell^\dagger_{\gamma^1})\mathfrak{H}_\Theta\gamma^2$$

for any $\gamma^1, \gamma^2 \in \hat\Upsilon^\mathrm{q}$. Here ℓ_γ and $\ell^\dagger_\gamma$ are the universal linearization operator of $\gamma \in \hat\Upsilon^\mathrm{q}$ and its formal adjoint, respectively. Denote the Lie algebra with the underlying space $\hat\Upsilon^\mathrm{q}$ and the Lie bracket $[\cdot,\cdot]_{\mathfrak{H}_\Theta}$ by $\hat\Upsilon^\mathrm{q}_\Theta$. The operator $\mathfrak{H}_\Theta$ establishes a homomorphism from the Lie algebra $\hat\Upsilon^\mathrm{q}_\Theta$ to the Lie algebra $\hat\Sigma^\mathrm{q}$. The image $\mathfrak{H}_\Theta\hat\Upsilon^\mathrm{q}_\Theta$ of this homomorphism is a proper subalgebra of $\hat\Sigma^\mathrm{q}$ of canonical representatives for generalized symmetries of the system $\mathcal{S}$. More specifically, the image $\mathfrak{H}_\Theta\hat\Upsilon^\mathrm{q}_\Theta$ is spanned by generalized symmetries from three families that are the images of the respective families from Theorem 13 and whose elements are, in the notation of Theorems 10 and 13, of the following form:

1. $\check{\mathcal{W}}(\bar\Omega^\Theta)$, where $\bar\Omega^\Theta = \hat{\mathcal{A}}\big((\hat{\mathcal{A}}\Omega)\Theta/\omega^1\big)$,

2. $\check{\mathcal{P}}(\bar\Phi)$, where $\bar\Phi = \Phi_{\mathfrak{r}^1} - \frac{1}{2}\Phi$, and thus the parameter function $\bar\Phi = \bar\Phi(\mathfrak{r}^1, \mathfrak{r}^2)$ runs through the solution space of the Klein–Gordon equation $\bar\Phi_{\mathfrak{r}^1\mathfrak{r}^2} = -\bar\Phi/4$ as well,

3. $\check{\mathcal{R}}(\bar\Gamma)$, where $\bar\Gamma = \frac{1}{2}(\tilde{\mathfrak{D}}_y - 1)\tilde{\mathfrak{Q}}\tilde q$.

For the nonvanishing function Θ, the kernel of the above homomorphism is two-dimensional and spanned by the cosymmetries $e^{\mathfrak{r}^1-\mathfrak{r}^2}(1,-1,0)$ and $e^{\mathfrak{r}^1-\mathfrak{r}^2}(\bar\Theta,-\bar\Theta,\bar\Theta_{\mathfrak{r}^3})$ with an antiderivative $\bar\Theta$ of $1/\Theta$, $\bar\Theta_{\mathfrak{r}^3}=1/\Theta$. The former cosymmetry is special due to being a single (up to linear independence) common element of the first and the second families from Theorem 13, see Remark 14. Both the cosymmetries are conservation-law characteristics and are associated with the conserved currents $e^{\mathfrak{r}^1-\mathfrak{r}^2}(1,\mathfrak{r}^1+\mathfrak{r}^2)$ and $e^{\mathfrak{r}^1-\mathfrak{r}^2}\big(\bar\Theta,(\mathfrak{r}^1+\mathfrak{r}^2)\bar\Theta\big)$, which belong to the first family of Theorem 15. As a result, the space of distinguished (Casimir) functionals of the Hamiltonian operator $\mathfrak{H}_\Theta$ is spanned by two functionals,

$$\mathcal{C}_1:=\int e^{\mathfrak{r}^1-\mathfrak{r}^2}\,\mathrm{d}x,\quad \mathcal{C}_2^\Theta:=\int e^{\mathfrak{r}^1-\mathfrak{r}^2}\bar\Theta(\mathfrak{r}^3)\,\mathrm{d}x.$$

In the degenerate case with $\Theta\equiv0$, the kernel of the above homomorphism is infinite-dimensional and coincides with the first family of Theorem 13. Elements of this family are conservation-law characteristics if and only if they belong to the first family of Theorem 18 and are thus associated with conserved currents from the first family of Theorem 15. This means that the space of distinguished (Casimir) functionals of the Hamiltonian operator $\mathfrak{H}_0$ consists of the functionals

$$\int e^{\mathfrak{r}^1-\mathfrak{r}^2}\Omega(\omega^0,\omega^1,\dots)\,\mathrm{d}x,$$

where the parameter function Ω runs through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3$, $\kappa\in\mathbb{N}_0$.

Consider the constraints that single out the space of canonical representatives conservation-law characteristics of $\mathcal{S}$, which is described in Theorem 18, from the space $\hat\Upsilon^{\rm q}$ of canonical representatives of cosymmetries of $\mathcal{S}$. Imposing these constraints on Ω and $\tilde{\mathfrak{Q}}$ that parameterize families spanning $\mathfrak{H}_\Theta\hat\Upsilon^{\rm q}$, we single out the algebra of Hamiltonian symmetries of $\mathcal{S}$ associated with the Hamiltonian operator $\mathfrak{H}_\Theta$.

Theorem 29. *Given a smooth function Θ of $\omega^0:=\mathfrak{r}^3$, the algebra of Hamiltonian symmetries of the system (1) for the Hamiltonian operator $\mathfrak{H}_\Theta$ is spanned by the generalized vector fields*

$$\check{\mathcal{W}}(\bar\Omega^\Theta)=\bar\Omega^\Theta\partial_{\mathfrak{r}^3},$$
$$\check{\mathcal{P}}(\Phi)=e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\Phi+2\Phi_{\mathfrak{r}^1})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+(\Phi-2\Phi_{\mathfrak{r}^2})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}+\Phi\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\right),$$
$$\check{\mathcal{R}}(\bar\Gamma)=e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\tilde{\mathfrak{D}}_y\bar\Gamma+\bar\Gamma)\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+(\tilde{\mathfrak{D}}_z\bar\Gamma+\bar\Gamma)\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}+\bar\Gamma\mathfrak{r}^3_x\partial_{\mathfrak{r}^3}\right),$$

where $\bar\Omega^\Theta=\hat{\mathcal{A}}\big(\Theta\sum_{\kappa=0}^\infty(-\hat{\mathcal{A}})^\kappa\Omega_{\omega^\kappa}\big)$ with the operator $\hat{\mathcal{A}}=\sum_{\kappa=0}^\infty\omega^{\kappa+1}\partial_{\omega^\kappa}$ and with Ω running through the space of smooth functions of a finite, but unspecified number of $\omega^\kappa=(e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x)^\kappa\mathfrak{r}^3$, $\kappa\in\mathbb{N}_0$, the parameter function $\Phi=\Phi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution space of the Klein–Gordon equation $\Phi_{\mathfrak{r}^1\mathfrak{r}^2}=-\Phi/4$, and $\bar\Gamma=\frac12(\tilde{\mathfrak{D}}_y-1)\tilde{\mathfrak{Q}}\tilde q$ with the operator $\tilde{\mathfrak{Q}}$ running through the set

$$\left\{\,\tilde{\mathfrak{J}}^{\kappa'},\ \kappa'\in2\mathbb{N}_0+1,\ (\tilde{\mathfrak{J}}+\iota/2)^\kappa\tilde{\mathfrak{D}}_y^\iota,\ (\tilde{\mathfrak{J}}-\iota/2)^\kappa\tilde{\mathfrak{D}}_z^\iota,\right.$$
$$\left.\kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N},\ \kappa+\iota\in2\mathbb{N}_0+1\,\right\}.$$

Here

$$\tilde{\mathfrak{D}}_y:=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big),$$
$$\tilde{\mathfrak{D}}_z:=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big),$$
$$\tilde{\mathfrak{J}}:=\frac{\mathfrak{r}^1}{2}\tilde{\mathfrak{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathfrak{D}}_z,\quad \tilde q:=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big).$$

8. Recursion operators

Some semi-Hamiltonian hydrodynamic-type systems admit Teshukov's recursion operators [46] which are specific first-order differential operators without pseudo-differential part. According to [47], such recursion operators exist if the Darboux rotation coefficients for an associated metric g, $\beta_{ik}:=\partial_{\mathfrak{r}^i}(\sqrt{|g_{kk}|})/\sqrt{|g_{ii}|}$ for $i\neq k$ and $\beta_{kk}:=0$, depend at most on pairwise differences of Riemann invariants. For the system $\mathcal{S}$, this condition is satisfied by the metric g of the form (29) with constant Θ. A canonical Teshukov's recursion operator for the system $\mathcal{S}$ and such a metric is easily computed, cf. [9, Eq. (8.1)],

$$\mathfrak{R}_{\rm T}=\mathrm{D}_x\circ\mathrm{diag}\left(\frac{1}{\mathfrak{r}^1_x},\frac{1}{\mathfrak{r}^2_x},\frac{1}{\mathfrak{r}^3_x}\right)+\begin{pmatrix}\dfrac{\mathfrak{r}^1_x-\mathfrak{r}^2_x}{2\mathfrak{r}^1_x} & \dfrac{\mathfrak{r}^2_x-\mathfrak{r}^1_x}{2\mathfrak{r}^2_x} & 0\\ \dfrac{\mathfrak{r}^2_x-\mathfrak{r}^1_x}{2\mathfrak{r}^1_x} & \dfrac{\mathfrak{r}^1_x-\mathfrak{r}^2_x}{2\mathfrak{r}^2_x} & 0\\ \dfrac{\mathfrak{r}^3_x-\mathfrak{r}^1_x}{\mathfrak{r}^1_x} & \dfrac{\mathfrak{r}^2_x-\mathfrak{r}^3_x}{\mathfrak{r}^2_x} & \dfrac{\mathfrak{r}^1_x-\mathfrak{r}^2_x}{\mathfrak{r}^3_x}\end{pmatrix}.$$

The operator $\mathfrak{R}_{\rm T}$ acts on the generalized vector fields spanning the algebra $\check\Sigma^{\rm q}$ as follows

$$\check{\mathcal{D}}\mapsto-2\check{\mathcal{G}}_1-\check{\mathcal{G}}_2+\check{\mathcal{W}}(1),\quad \check{\mathcal{R}}(\Gamma)\mapsto\frac12\check{\mathcal{R}}\big(\tilde{\mathfrak{D}}_y\Gamma-\tilde{\mathfrak{D}}_z\Gamma\big),$$
$$\check{\mathcal{P}}(\Phi)\mapsto\check{\mathcal{P}}\big(\Phi_{\mathfrak{r}^1}+\Phi_{\mathfrak{r}^2}\big),\quad \check{\mathcal{W}}(\Omega)\mapsto\check{\mathcal{W}}\left(\mathcal{A}(\Omega/\omega^1)\right).$$

At the same time, we can construct many more local recursion operators, including higher-order ones. For this purpose, we use the complete description of generalized symmetries of the system $\mathcal{S}$ that is presented in Theorem 10. Here the basic fact is again that the algebra $\hat\Sigma^{\rm q}$ is decomposed into a (non-direct) sum of its subalgebras $\bar\Sigma^{\rm q}_{12}$ and $\hat\Sigma^{\rm q}_3$. The subalgebra $\bar\Sigma^{\rm q}_{12}$ is a counterpart of the algebra of generalized symmetries of the $(1+1)$-dimensional Klein–Gordon equation (4a), and thus the recursion operators preserving $\bar\Sigma^{\rm q}_{12}$ are related to recursion operators of this equation. The ideal $\hat\Sigma^{\rm q}_3$ underlaid by the degeneracy of the system $\mathcal{S}$ is preserved by the operators of the form $\mathrm{diag}(0,0,\Omega\mathcal{A}^\kappa)$, where the coefficient Ω is a smooth function of a finite but unspecified number of $\omega^\iota=\mathcal{A}^\iota\mathfrak{r}^3$, $\iota\in\mathbb{N}_0$, and $\kappa\in\mathbb{N}_0$. The above gives a hint about the form of more local recursion operators for the system $\mathcal{S}$.

Theorem 30. *The system (1) admits recursion operators of the form*

$$\mathfrak{R}_{1,\mathfrak{Q}}=e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\begin{pmatrix}\mathfrak{r}^1_x(\tilde{\mathfrak{D}}_y+1) & 0 & 0\\ \mathfrak{r}^2_x(\tilde{\mathfrak{D}}_z+1) & 0 & 0\\ 2\mathfrak{r}^3_x & 0 & 0\end{pmatrix}\mathfrak{Q}\circ\frac{e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}}{\mathfrak{r}^1_x},$$

$$\mathfrak{R}_{2,\mathfrak{Q}}=e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\begin{pmatrix}0 & \mathfrak{r}^1_x(\tilde{\mathfrak{D}}_y+1) & 0\\ 0 & \mathfrak{r}^2_x(\tilde{\mathfrak{D}}_z+1) & 0\\ 0 & 2\mathfrak{r}^3_x & 0\end{pmatrix}\mathfrak{Q}\circ\frac{e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}}{\mathfrak{r}^2_x},$$

$$\mathfrak{R}_{3,\mathfrak{P}}=\mathfrak{P}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -1 & 1 & \mathcal{A}\circ(\omega^1)^{-1}\end{pmatrix},$$

where $\mathfrak{Q}\in\langle\tilde{\mathfrak{J}}^\kappa,\ \tilde{\mathfrak{D}}_y^\iota\tilde{\mathfrak{J}}^\kappa,\ \tilde{\mathfrak{D}}_z^\iota\tilde{\mathfrak{J}}^\kappa,\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\rangle$, $\mathfrak{P}=\sum_{\kappa=0}^N\Omega^\kappa\mathcal{A}^\kappa$ for some $N\in\mathbb{N}_0$, $\mathcal{A}:=e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x$, the coefficients Ω^κ are smooth functions of a finite but unspecified number of $\omega^\iota=\mathcal{A}^\iota\mathfrak{r}^3$, $\iota\in\mathbb{N}_0$, and

$$\tilde{\mathfrak{D}}_y:=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+V^2\mathcal{D}_x\big),\quad \tilde{\mathfrak{D}}_z:=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+V^1\mathcal{D}_x\big),$$
$$\tilde{\mathfrak{J}}:=\frac{\mathfrak{r}^1}{2}\tilde{\mathfrak{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathfrak{D}}_z.$$

Proof. We directly compute the action of the operators $\mathfrak{R}_{1,\mathfrak{Q}}$, $\mathfrak{R}_{2,\mathfrak{Q}}$ and $\mathfrak{R}_{3,\mathfrak{P}}$ on the generalized vector fields spanning the algebra $\hat{\Sigma}^{\rm q}$, obtaining

$$\begin{aligned}\mathfrak{R}_{1,\mathfrak{Q}}\colon\ &\check{\mathcal{D}}\mapsto\check{\mathcal{R}}(\mathfrak{Q}\tilde{q}),\quad \check{\mathcal{R}}(\Gamma)\mapsto\check{\mathcal{R}}\big(\mathfrak{Q}(\tilde{\mathfrak{D}}_y+1)\Gamma\big),\\ &\check{\mathcal{P}}(\Phi)\mapsto\check{\mathcal{P}}\big(\mathfrak{Q}(\Phi+2\Phi_{\mathfrak{r}^1})\big),\quad \check{\mathcal{W}}(\Omega)\mapsto 0.\end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{2,\mathfrak{Q}}\colon\ &\check{\mathcal{D}}\mapsto\check{\mathcal{R}}(\mathfrak{Q}\tilde{\mathfrak{D}}_z\tilde{q}),\quad \check{\mathcal{R}}(\Gamma)\mapsto\check{\mathcal{R}}\big(\mathfrak{Q}(\tilde{\mathfrak{D}}_z+1)\Gamma\big),\\ &\check{\mathcal{P}}(\Phi)\mapsto\check{\mathcal{P}}\big(\mathfrak{Q}(\Phi-2\Phi_{\mathfrak{r}^2})\big),\quad \check{\mathcal{W}}(\Omega)\mapsto 0.\end{aligned}$$

$$\begin{aligned}\mathfrak{R}_{3,\mathfrak{P}}\colon\ &\check{\mathcal{D}}\mapsto\check{\mathcal{W}}(\mathfrak{P}1)=\check{\mathcal{W}}(\Omega^0),\quad \check{\mathcal{R}}(\Gamma),\check{\mathcal{P}}(\Phi)\mapsto 0,\\ &\check{\mathcal{W}}(\Omega)\mapsto\check{\mathcal{W}}\big(\mathfrak{P}\mathcal{A}(\Omega/\omega^1)\big).\end{aligned}$$

This means that the above operators are recursion operators of the system $\mathcal{S}$. □

Remark 31. The action of the Teshukov's recursion operator $\mathfrak{R}_{\rm T}$ on symmetries of the system (1) coincides with that of the recursion operator $\frac{1}{2}\mathfrak{R}_{1,1}-\frac{1}{2}\mathfrak{R}_{2,1}+\mathfrak{R}_{3,1}$.

One can also find nonlocal recursion operators for the system (1). We construct an example of such an operator. Let $\eta=(\eta^1,\eta^2,\eta^3)^{\mathsf T}$ be an arbitrary solution of the system (27b). Consider a first-order pseudo-differential operator $\mathfrak{R}_4$ acting nonlocally on η as

$$\mathfrak{R}_4\eta=A(\mathfrak{r})\mathrm{D}_x\eta+B(\mathfrak{r},\mathfrak{r}_x)\eta+C(\mathfrak{r},\mathfrak{r}_x)Y,\tag{31}$$

where $A=(A^{ij})$ and $B=(B^{ij})$ are smooth 3×3 matrix functions of their arguments, C is a three-component column of smooth functions of $(\mathfrak{r},\mathfrak{r}_x)$ and Y is the potential associated with the conserved current $\big(\eta^1+\eta^2,V^1\eta^1+V^2\eta^2\big)$ of the system (27b), which is the linearized counterpart of the conserved current $\big(\mathfrak{r}^1+\mathfrak{r}^2,\frac{1}{2}(\mathfrak{r}^1+\mathfrak{r}^2)^2+\mathfrak{r}^1-\mathfrak{r}^2\big)$ of the system (1). In other words, the potential Y is defined by the system

$$\mathrm{D}_tY=-V^1\eta^1-V^2\eta^2,\quad \mathrm{D}_xY=\eta^1+\eta^2.\tag{32}$$

By definition, the operator $\mathfrak{R}_4$ is a recursion operator of the system (1) if for an arbitrary solution η of the system (27b), $\mathfrak{R}_4\eta$ is a solution of the same system. We successively substitute the ansatz (31) for $\mathfrak{R}_4\eta$, the expressions (32) for D_tY and D_xY and the expressions for $\mathrm{D}_t\eta$ in view of the system (27b) into the system (27b) for $\mathfrak{R}_4\eta$, which leads to the system

$$\begin{aligned}&(A^{kj}_{\mathfrak{r}^l}\mathrm{D}_x\eta^j+B^{kj}_{\mathfrak{r}^l}\eta^j+C^k_{\mathfrak{r}^l}Y)(V^k-V^l)\mathfrak{r}^l_x\\ &\quad+(B^{kj}_{\mathfrak{r}^l_x}\eta^j+C^k_{\mathfrak{r}^l_x}Y)\big((V^k-V^l)\mathfrak{r}^l_{xx}-(\mathfrak{r}^1_x+\mathfrak{r}^2_x)\mathfrak{r}^l_x\big)\\ &\quad-A^{kj}\mathrm{D}_x\big(V^j\mathrm{D}_x\eta^j+(\eta^1+\eta^2)\mathfrak{r}^j_x\big)-B^{kj}\big(V^j\mathrm{D}_x\eta^j+(\eta^1+\eta^2)\mathfrak{r}^j_x\big)\\ &\quad-C^k(V^1\eta^1+V^2\eta^2)+V^k\big(A^{kj}\mathrm{D}_x^2\eta^j+B^{kj}\mathrm{D}_x\eta^j+C^k(\eta^1+\eta^2)\big)\\ &\quad+\mathfrak{r}^k_x\big(A^{1j}\mathrm{D}_x\eta^j+B^{1j}\eta^j+C^1Y+A^{2j}\mathrm{D}_x\eta^j+B^{2j}\eta^j+C^2Y\big)=0.\end{aligned}$$

Here and in what follows the indices j, k, k' and l run from 1 to 3, we assume summation with respect to the repeated indices j and l, and there is no summation over k and k'. The splitting of the obtained system with respect to $\mathrm{D}_x^2\eta^{k'}$, $\mathrm{D}_x\eta^{k'}$, $\eta^{k'}$ and Y yields the system of determining equations for entries of A, B and C,

$$A^{kk'}(V^k-V^{k'})=0,$$

$$\begin{aligned}&A^{kk'}_{\mathfrak{r}^l}(V^k-V^l)\mathfrak{r}^l_x-A^{kk'}(\mathfrak{r}^1_x+\mathfrak{r}^2_x)-(\delta^1_{k'}+\delta^2_{k'})A^{kj}\mathfrak{r}^j_x+(V^k-V^{k'})B^{kk'}\\ &\quad+\mathfrak{r}^k_x(A^{1k'}+A^{2k'})=0,\end{aligned}$$

$$\begin{aligned}&B^{kk'}_{\mathfrak{r}^l}(V^k-V^l)\mathfrak{r}^l_x+B^{kk'}_{\mathfrak{r}^l_x}\big((V^k-V^l)\mathfrak{r}^l_{xx}-(\mathfrak{r}^1_x+\mathfrak{r}^2_x)\mathfrak{r}^l_x\big)+\mathfrak{r}^k_x(B^{1k'}+B^{2k'})\\ &\quad-(\delta^1_{k'}+\delta^2_{k'})\big(A^{kj}\mathfrak{r}^j_{xx}+B^{kj}\mathfrak{r}^j_x-(V^k-V^{k'})C^k\big)=0,\end{aligned}$$

$$C^k_{\mathfrak{r}^l}(V^k-V^l)\mathfrak{r}^l_x+C^k_{\mathfrak{r}^l_x}\big((V^k-V^l)\mathfrak{r}^l_{xx}-(\mathfrak{r}^1_x+\mathfrak{r}^2_x)\mathfrak{r}^l_x\big)+\mathfrak{r}^k_x(C^1+C^2)=0,$$

solving which, we prove the following proposition.

Proposition 32. *The system (1) admits the formally pseudo-differential recursion operator $\mathfrak{R}_4$ acting on a symmetry characteristic η as*

$$\mathfrak{R}_4\eta=B\eta+CY,\quad\text{where}\quad B=\begin{pmatrix}2&0&0\\0&-2&0\\0&0&0\end{pmatrix},\quad C=\begin{pmatrix}\mathfrak{r}^1_x\\ \mathfrak{r}^2_x\\ \mathfrak{r}^3_x\end{pmatrix},$$

and Y is the potential of the system (27b) that is defined by (32).

9. Conclusion

To study the diagonalized form (1) of the system $\mathcal{S}$, we heavily rely on its two primary features. The first feature is the degeneracy of $\mathcal{S}$ in the sense that this system is not genuinely nonlinear with respect to $\mathfrak{r}^3$ and, moreover, it is partially decoupled since the first two equations of $\mathcal{S}$ do not involve $\mathfrak{r}^3$. To take into account the degeneracy efficiently, we introduce the modified coordinates on $\mathcal{S}^{(\infty)}$, where derivatives of $\mathfrak{r}^3$ are replaced by ω's constituting a functional basis of the kernel of the operator $\mathcal{B}$. This operator is nothing else but the differential operator in the total derivatives that is associated with the equation on $\mathfrak{r}^3$. From another perspective, the infinite tuple of ω's, $\omega^0:=\mathfrak{r}^3$, $\omega^{\kappa+1}:=\mathcal{A}\omega^\kappa$, $\kappa\in\mathbb{N}_0$, can be seen to be generated by the differential operator $\mathcal{A}:=e^{\mathfrak{r}^2-\mathfrak{r}^1}\mathcal{D}_x$, commuting with $\mathcal{B}$, $[\mathcal{A},\mathcal{B}]=0$, cf. [24]. The introduction of the modified coordinates essentially simplifies computations of all kinds of symmetry-like objects for the system $\mathcal{S}$. Due to the partial decoupling of the system $\mathcal{S}$, we recognize its essential subsystem $\mathcal{S}_0$ constituted by Eqs. (1a), (1b). The second primary feature of $\mathcal{S}$ is the linearization of $\mathcal{S}_0$ to the $(1+1)$-dimensional Klein–Gordon equation, which was thoroughly studied from the point of view of generalized symmetries and conservation laws in [23].

In turn, these features allow us to describe symmetry-like objects for the system $\mathcal{S}$ by working within the following general approach. For a given kind of symmetry-like objects for $\mathcal{S}$, we show that the chosen space U of canonical representatives of equivalence classes of such objects is the sum of three subspaces,

$$U=U_1+U_2+U_3.$$

One of them, say, U_1, stems from the degeneracy of $\mathcal{S}$, and thus its elements are parameterized by an arbitrary function of a finite but unspecified number of ω's. The other two subspaces, U_2 and U_3, are related to the linearization of $\mathcal{S}_0$ to the $(1+1)$-dimensional Klein–Gordon equation (4a). Singling out these two subspaces is induced by decomposing the objects of the same kind for the Klein–Gordon equation as sums of those underlaid by linear superposition of solutions of (4a) and those associated with linear generalized symmetries of (4a). This is why the elements of the subspaces U_2 and U_3 are parameterized by an arbitrary solution of the $(1+1)$-dimensional Klein–Gordon equation and by characteristics of reduced linear generalized symmetries of this equation, respectively. Although $(U_1+U_2)\cap U_3=\{0\}$, the sum $U_1+U_2+U_3$ is not direct since the subspaces U_1 and U_2 are not disjoint, and their intersection is one-dimensional.

The first kind of objects we exhaustively describe for the system $\mathcal{S}$ is given by generalized symmetries. Not all generalized symmetries of the Klein–Gordon equation (4a) have counterparts among generalized symmetries of the system $\mathcal{S}$, which was also noted in [23] for first-order generalized symmetries. The most difficult problem here, which is solved in Lemma 9, is to single out the subalgebra $\mathfrak{A}$ of canonical representatives of generalized symmetries of the Klein–Gordon equation (4a) that have such counterparts. A complementary subalgebra to $\mathfrak{A}$ is

$$\bar{\mathfrak{A}}=\langle(\mathcal{J}^\kappa q)\partial_q,\ \kappa\in\mathbb{N}\rangle.$$

We conjecture that elements of $\bar{\mathfrak{A}}$ have counterparts among nonlocal, or specifically potential, symmetries of the system $\mathcal{S}$. To show this, we plan to study certain Abelian coverings and potential symmetries of the system $\mathcal{S}$ and of the Klein–Gordon equation (4a). We expect that the main role in this consideration will be played by the conservation laws of the Klein–Gordon equation (4a) with characteristics of the form $\mathcal{J}^\kappa e^{y+z}$, $\kappa\in\mathbb{N}_0$, and by their counterparts for the system $\mathcal{S}$.

Considering cosymmetries and local conservation laws, we do not need to make the selection among those for the Klein–Gordon equation (4a) since all of them have counterparts for the system $\mathcal{S}$. For conservation laws, this follows directly from the general assertion proved in [33, Theorem 1]. Amongst cosymmetries, local conservation laws and their characteristics, the complete description of the space of cosymmetries for the system $\mathcal{S}$ is the simplest, though it still requires utilizing a couple of nontrivial tricks within the framework of our general approach.

To construct the space of local conservation laws of $\mathcal{S}$, we have to make use of the direct method [31,32] whose essence is the direct construction of conserved currents canonically representing conservation laws using the definitions of conserved currents and of their equivalence. The standard approach [36] based on singling out conservation-law characteristics among cosymmetries is not effective for the system $\mathcal{S}$ since its application to $\mathcal{S}$ leads to too cumbersome computations. At the same time, we still need to know conservation-law characteristics for the system $\mathcal{S}$, in particular, to look for special-feature conservation laws, like low-order and translation-invariant ones. The known formula [35, Proposition 7.41] relating characteristics of conservation laws of systems in the extended Kovalevskaya form [48, Definition 4] to densities of these conservation laws gives suitable expressions only for characteristics of conservation laws from the second family of Theorem 15, which are of zeroth order. The other two families should be tackled differently. For the first family, we in fact derive an analogue of the above formula in terms of the operator $\mathcal{A}$ using the formal integration by parts. Characteristics of conservation laws from the third family are constructed from their counterparts being variational symmetries of the Klein–Gordon equation (4a). We also prove that under the action of generalized symmetries of the system $\mathcal{S}$ on its space of conservation laws, a generating set of conservation laws of this system is constituted by two zeroth-order conservation laws. One of them belongs to and generates the first subspace of conservation laws, which is related to the degeneracy of $\mathcal{S}$. The other is the counterpart of a single generating conservation law of the Klein–Gordon equation (4a). It belongs to the third subspace of conservation laws of $\mathcal{S}$ but generates the second subspace as well. The claim on generation of the entire third subspace is unexpected since only a proper part of linear generalized symmetries of the Klein–Gordon equation (4a) are naturally mapped to generalized symmetries of $\mathcal{S}$ but the amount of the images still suffices for generating all required conservation laws.

Interrelating generalized symmetries and cosymmetries, we construct a family of compatible Hamiltonian operators for the system $\mathcal{S}$ parameterized by an arbitrary function of $\mathfrak{r}^3$, and a Hamiltonian operator from this family is degenerate if the corresponding value of the parameter function vanishes at some point. This fundamentally differs from the case of genuinely nonlinear hydrodynamic-type systems, for which the number of local Hamiltonian operators of hydrodynamic type is known not to exceed $n+1$, where n is the number of dependent variables, see [49]. Note that the conjecture from [37] that skew-symmetric Noether operators for non-scalar systems of first-order evolution equations are Hamiltonian ones holds for Noether operators of $\mathcal{S}$ with entries of the form (28).

Finally, having the comprehensive description of the algebra of generalized symmetries of the system $\mathcal{S}$ at our disposal, we find *ad hoc* broad families of local recursion operators, which are presented in Theorem 30. The system $\mathcal{S}$ admits the canonical Teshukov's recursion operator $\mathfrak{R}_{\rm T}$ but this operator is equivalent to a linear combination of the three simplest local recursion operators from Theorem 30. We also construct a nonlocal recursion operator of $\mathcal{S}$. It is clear that one can construct many such operators, in particular, using the relation of the system $\mathcal{S}$ to the Klein–Gordon equation (4a), which will be a subject of our further studies.

We should like to emphasize that the local description of the solution set of the system $\mathcal{S}$ in Theorem 1 is implicit and involves the general solution of the $(1+1)$-dimensional Klein–Gordon equation. This is why it is difficult to further use this description, and thus it is still worthwhile to comprehensively study the system $\mathcal{S}$ within the framework of symmetry analysis of differential equations.

As the essential subsystem $\mathcal{S}_0$ coincides with the diagonalized form of the system describing one-dimensional isentropic gas flows with constant sound speed [8, Section 2.2.7, Eq. (16)], symmetry-like objects of $\mathcal{S}_0$ deserve a separate consideration but in fact they are implicitly described in the present paper. In contrast to the system $\mathcal{S}$, all the quotient spaces of symmetry-like objects of the subsystem $\mathcal{S}_0$ are isomorphic to their counterparts for the system (4a), (5) and thus to their counterparts for the Klein–Gordon equation (4a). Therefore, to construct an algebra of canonical representatives of generalized symmetries for the subsystem $\mathcal{S}_0$, we take the respective algebra for Eq. (4a) and follow the procedure given in the first paragraph of the proof of Theorem 10, just ignoring the $\mathfrak{r}^3$-components in the point transformation (7) and in the vector field $\tilde{Q}$. As a result, we obtain that the quotient algebra of generalized symmetries of the subsystem $\mathcal{S}_0$ is naturally isomorphic to the algebra spanned by the generalized vector fields

$$\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big)\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+\big(x-(\mathfrak{r}^1+\mathfrak{r}^2-1)t\big)\mathfrak{r}^2_x\partial_{\mathfrak{r}^2},$$
$$e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\big(\varGamma\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+\tilde{\mathcal{D}}_z\varGamma\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}\big),$$
$$e^{(\mathfrak{r}^2-\mathfrak{r}^1)/2}\left((\varPhi+2\varPhi_{\mathfrak{r}^1})\mathfrak{r}^1_x\partial_{\mathfrak{r}^1}+(\varPhi-2\varPhi_{\mathfrak{r}^2})\mathfrak{r}^2_x\partial_{\mathfrak{r}^2}\right),$$

where the parameter function $\varPhi=\varPhi(\mathfrak{r}^1,\mathfrak{r}^2)$ runs through the solution set of the Klein–Gordon equation $\varPhi_{\mathfrak{r}^1\mathfrak{r}^2}=-\varPhi/4$, $\varGamma$ runs through the set $\{\tilde{\mathcal{J}}^\kappa\tilde{q},\ \tilde{\mathcal{D}}_y^\iota\tilde{\mathcal{J}}^\kappa\tilde{q},\ \tilde{\mathcal{D}}_z^\iota\tilde{\mathcal{J}}^\kappa\tilde{q},\ \kappa\in\mathbb{N}_0,\ \iota\in\mathbb{N}\}$ with

$$\tilde{\mathcal{D}}_y:=-\frac{1}{\mathfrak{r}^1_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2-1)\mathcal{D}_x\big),$$
$$\tilde{\mathcal{D}}_z:=-\frac{1}{\mathfrak{r}^2_x}\big(\mathcal{D}_t+(\mathfrak{r}^1+\mathfrak{r}^2+1)\mathcal{D}_x\big),$$
$$\tilde{\mathcal{J}}:=\frac{\mathfrak{r}^1}{2}\tilde{\mathcal{D}}_y+\frac{\mathfrak{r}^2}{2}\tilde{\mathcal{D}}_z,\quad \tilde{q}:=e^{(\mathfrak{r}^1-\mathfrak{r}^2)/2}\big(x-(\mathfrak{r}^1+\mathfrak{r}^2+1)t\big),$$

and instead of the complete operators $\mathcal{D}_t$ and $\mathcal{D}_x$ defined in Section 3, one should use their restrictions to $(\mathfrak{r}^1,\mathfrak{r}^2)$,

$$\mathcal{D}_x:=\partial_x+\sum_{\kappa=0}^{\infty}\sum_{i=1}^{2}\mathfrak{r}^i_{\kappa+1}\partial_{\mathfrak{r}^i_\kappa},\quad \mathcal{D}_t:=\partial_t-\sum_{\kappa=0}^{\infty}\sum_{i=1}^{2}\mathcal{D}_x^\kappa(V^i\mathfrak{r}^i_1)\partial_{\mathfrak{r}^i_\kappa}.$$

The descriptions of cosymmetries and conservation laws of $\mathcal{S}_0$ are derived from those for the system $\mathcal{S}$ by excluding the first families of cosymmetries and conservation laws, which are related to the degeneracy of $\mathcal{S}$, in Theorems 13 and 15.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors thank Galyna Popovych for helpful discussions and interesting comments. The research of AB and SO was undertaken, in part, thanks to funding from the Canada Research Chairs program, the NSERC, Canada Discovery Grant program and the InnovateNL LeverageR&D program, Canada. The research of SO was supported by the NSERC's Alexander Graham Bell Canada Graduate Scholarship — Doctoral Fellowship. The research of ROP was supported by the Austrian Science Fund (FWF), projects P25064 and P29177, and by "Project for fostering collaboration in science, research and education" funded by the Moravian-Silesian Region, Czech Republic. The research of AS was supported in part by the Grant Agency of the Czech Republic (GA ČR) under grant P201/12/G028.

References

[1] N. Zuber, J.A. Findlay, Average volumetric concentration in two-phase flow systems, J. Heat Transf. 87 (1965) 357–372.
[2] M. Ishii, T. Hibiki, Thermo-Fluid Dynamics of Two-Phase Flow, Springer, New York, 2006.
[3] G. Yadigaroglu, Hewitt G.F. (Eds.), Introduction to multiphase flow. Basic concepts, applications and modelling, Zurich Lectures on Multiphase Flow, Springer, Cham, 2018.
[4] S. Evje, K. Fjelde, Relaxation schemes for the calculation of two-phase flow in pipes, Math. Comput. Modelling 36 (2002) 535–567.
[5] S. Evje, T. Flåtten, Weakly implicit numerical schemes for a two-fluid model, SIAM J. Appl. Math. 26 (2005) 1449–1484.
[6] S. Evje, T. Flåtten, On the wave structure of two phase flow models, SIAM J. Appl. Math. 67 (2007) 487–511.
[7] S. Evje, K.H. Karlsen, Global existence of weak solutions for a viscous two-phase model, J. Differential Equations 245 (2008) 2660–2703.
[8] B.L. Rozhdestvenskii, N.N. Janenko, Systems of Quasilinear Equations and their Applications to Gas Dynamics, American Mathematical Society, Providence, RI, 1983.
[9] S. Tsarev, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izv. 37 (1991) 397–419.
[10] G.B. Whitham, Linear and Nonlinear Waves, first ed., John Wiley–Interscience, New York, 1999.
[11] B. Dubrovin, S. Novikov, Hamiltonian formalism of one–dimensional systems of hydrodynamic type and Bogolyubov–Whitham averaging method, Sov. Math. Dokl. 270 (1983) 781–785 (in Russian).
[12] B. Dubrovin, S. Novikov, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, Russian Math. Surveys 44 (1989) 35–124.
[13] S. Tsarev, On Poisson brackets and one-dimensional systems of hydrodynamic type, Sov. Math. Dokl. 31 (1985) 488–491.
[14] M. Błaszak, A. Sergyeyev, A coordinate-free construction of conservation laws and reciprocal transformations for a class of integrable hydrodynamic-type systems, Rep. Math. Phys. 64 (2009) 341–354.
[15] G. Burde, A. Sergyeyev, Ordering of two small parameters in the shallow water wave problem, J. Phys. A 46 (2013) 075501.
[16] E. Ferapontov, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, Phys. Lett. A 158 (1991) 112–118.
[17] E. Ferapontov, Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, in: Topics in Topology and Mathematical Physics, Amer. Math. Soc. Transl. Ser. 2, vol. 170, Amer. Math. Soc, Providence, RI, 1995, pp. 33–58.
[18] A.M. Grundland, A.J. Hariton, Supersymmetric version of a hydrodynamic system in Riemann invariants and its solutions, J. Math. Phys. 49 (2008) 043502.
[19] A.M. Grundland, B. Huard, Conditional symmetries and Riemann invariants for hyperbolic systems of PDEs, J. Phys. A 40 (2007) 4093–4123.
[20] M. Pavlov, Integrable hydrodynamic chains, J. Math. Phys. 44 (2003) 4134–4156.
[21] A. Sergyeyev, A simple construction of recursion operators for multidimensional dispersionless integrable systems, J. Math. Anal. Appl. 454 (2017) 468–480.
[22] A. Sergyeyev, New integrable (3+1)-dimensional systems and contact geometry, Lett. Math. Phys. 108 (2018) 359–376.
[23] S. Opanasenko, A. Bihlo, R.O. Popovych, A. Sergyeyev, Extended symmetry analysis of isothermal no-slip drift model, Physica D 402 (2020) 132188.
[24] P.W. Doyle, Symmetry classes of quasilinear systems in one space variable, J. Nonlinear Math. Phys. 1 (1994) 225–266.
[25] S. Opanasenko, R.O. Popovych, Generalized Symmetries and Conservation Laws of (1 + 1)-Dimensional Klein–Gordon Equation, 2018, arXiv:1810.12434, 16 pp.
[26] M. Sheftel, Higher integrals and symmetries of semi-Hamiltonian systems, Differ. Equ. 29 (1994) 1548–1560.
[27] J. Krasil'shchik, A. Verbovetsky, R. Vitolo, The symbolic computation of integrability structures for partial differential equations, Texts & Monographs in Symbolic Computation, Springer, Cham, 2017.
[28] A.M. Grundland, B. Huard, Riemann invariants and rank-k solutions of hyperbolic systems, J. Nonlinear Math. Phys. 13 (2006) 393–419.
[29] L. Martínez Alonso, On the Noether map, Lett. Math. Phys. 3 (1979) 419–424.
[30] P.J. Olver, Applications of Lie groups to differential equations, second ed., Graduate Texts in Mathematics, vol. 107, Springer-Verlag, New York, 1993.
[31] R.O. Popovych, N.M. Ivanova, Hierarchy of conservation laws of diffusion–convection equations, J. Math. Phys. 46 (2005) 043502.
[32] T. Wolf, A comparison of four approaches to the calculation of conservation laws, European J. Appl. Math. 13 (2002) 129–152.
[33] M. Kunzinger, R.O. Popovych, Potential conservation laws, J. Math. Phys. 49 (2008) 103506.
[34] R.O. Popovych, M. Kunzinger, N.M. Ivanova, Conservation laws and potential symmetries of linear parabolic equations, Acta Appl. Math. 100 (2008) 113–185.
[35] T. Tsujishita, On variation bicomplexes associated to differential equations, Osaka J. Math. 19 (1982) 311–363.
[36] A.V. Bocharov, V.N. Chetverikov, S.V. Duzhin, N.G. Khor'kova, I.S. Krasil'shchik, A.V. Samokhin, Y.N. Torkhov, A.M. Verbovetsky, A.M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Providence, RI, 1999.
[37] P. Kersten, I. Krasil'shchik, A. Verbovetsky, A geometric study of the dispersionless Boussinesq type equation, Acta Appl. Math. 90 (2006) 143–178.
[38] D. Gessler, On the Vinogradov $\mathcal{C}$-spectral sequence for determined systems of differential equations, Differential Geom. Appl. 7 (1997) 303–324.
[39] P. Kersten, I. Krasil'shchik, A. Verbovetsky, On the integrability conditions for some structures related to evolution differential equations, Acta Appl. Math. 83 (2004) 167–173.
[40] E. Ferapontov, Compatible Poisson brackets of hydrodynamic type, J. Phys. A 34 (2001) 2377–2388.
[41] O. Mokhov, Compatible Poisson structures of hydrodynamic type and associativity equations, Proc. Steklov Inst. Math. 225 (1999) 269–284.
[42] H. Baran, M. Marvan, Jets. A software for differential calculus on jet spaces and diffieties. http://jets.math.slu.cz.
[43] M. Marvan, Sufficient set of integrability conditions of an orthonomic system, Found. Comput. Math. 9 (2009) 651–674.
[44] B. Fuchssteiner, The Lie algebra structure of degenerate Hamiltonian and bi-Hamiltonian systems, Progr. Theoret. Phys. 68 (1982) 1082–1104.
[45] M. Błaszak, Multi-Hamiltonian Theory of Dynamical Systems, Springer-Verlag, Berlin, 1998.
[46] V. Teshukov, Hyperbolic systems admitting nontrivial Lie–Bäcklund group, 1989, LIIAN preprint 106.
[47] M. Sheftel, Group analysis of defining equations – a method for finding recursion operators, Differ. Equ. 30 (1994) 412–423.
[48] R.O. Popovych, A. Bihlo, Inverse problem on conservation laws, Physica D 401 (2020) 132175.
[49] E.V. Ferapontov, M.V. Pavlov, Quasiclassical limit of coupled KdV equations. Riemann invariants and multi-Hamiltonian structure, Physica D 52 (1991) 211–219.