Accepted Manuscript

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PII: S0167-2789(17)30344-5
DOI: https://doi.org/10.1016/j.physd.2018.01.013
Reference: PHYSD 32007

To appear in: *Physica D*

Please cite this article as: Q. Jiu, M.C. Lopes Filho, D. Niu, H.J. Nussenzveig Lopes, The inviscid limit of the incompressible 3D Navier–Stokes equations with helical symmetry, *Physica D* (2018), https://doi.org/10.1016/j.physd.2018.01.013

THE INVISCID LIMIT OF THE INCOMPRESSIBLE 3D NAVIER-STOKES EQUATIONS WITH HELICAL SYMMETRY

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ABSTRACT. In this paper, we are concerned with the vanishing viscosity problem for the three-dimensional Navier-Stokes equations with helical symmetry, in the whole space. We choose viscosity-dependent initial $\mathbf{u}_0^\nu$ with helical swirl, an analogue of the swirl component of axisymmetric flow, of magnitude $\mathcal{O}(\nu)$ in the L^2 norm; we assume $\mathbf{u}_0^\nu \to \mathbf{u}_0$ in H^1. The new ingredient in our analysis is a decomposition of helical vector fields, through which we obtain the required estimates.

KEY WORDS: Navier-Stokes equations; Euler equations; Helical symmetry; Vanishing viscosity limit.

2000 MATHEMATICS SUBJECT CLASSIFICATION. 76B47; 35Q30.

Dedicated to Edriss S. Titi, on the occasion of his 60^{th} birthday.

1. INTRODUCTION

The initial-value problem for the three-dimensional incompressible Navier-Stokes equations with viscosity $\nu > 0$ is given by

$$\begin{cases} \partial_t \mathbf{u}^\nu + \mathbf{u}^\nu \cdot \nabla \mathbf{u}^\nu + \nabla p^\nu = \nu \Delta \mathbf{u}^\nu & (\mathbf{x}, t) \in \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} \mathbf{u}^\nu = 0 & (\mathbf{x}, t) \in \mathbb{R}^3 \times (0,\infty), \\ \mathbf{u}^\nu(t=0,\mathbf{x}) = \mathbf{u_0}^\nu & \mathbf{x} \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{u}^\nu = (u_1^\nu, u_2^\nu, u_3^\nu)$ is the velocity and p^ν is the pressure.

Formally, when $\nu = 0$, (1.1) becomes the classical incompressible Euler equations

$$\begin{cases} \partial_t \mathbf{u}^0 + \mathbf{u}^0 \cdot \nabla \mathbf{u}^0 + \nabla p^0 = 0 & (\mathbf{x}, t) \in \mathbb{R}^3 \times (0,\infty), \\ \operatorname{div} \mathbf{u}^0 = 0 & (\mathbf{x}, t) \in \mathbb{R}^3 \times (0,\infty), \\ \mathbf{u}^0(t=0,\mathbf{x}) = \mathbf{u}_0 & \mathbf{x} \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Global existence of weak solutions and local in time well-posedness of strong solutions for problem (1.1) is due to J. Leray, see [12]. There is a vast literature on existence, uniqueness and regularity of solutions of (1.1), see [17, 23] and references therein. Global existence of strong solutions and uniqueness of weak solutions remain open.

One direction of investigation has been to study the special case of axisymmetric flows, i.e. viscous flows which are invariant under rotation around a fixed symmetry axis. In particular, among axisymmetric flows, one distinguishes the no-swirl case. The axisymmetric velocity has three components, a component in the direction of the axis of symmetry, a radial component, which is orthogonal to the axis of symmetry, in any plane that contains it, and the azimuthal component, which points in the direction of the rotation around the axis. No-swirl means that the azimuthal component of velocity vanishes. Global well-posedness of strong, axisymmetric, solutions of the Navier-Stokes equations (1.1) in the no-swirl case, and in the swirl case when the domain avoids the symmetry axis, is due to Ladyzhenskaya, see [11]. If the domain contains the symmetry axis, global well-posedness is open, and singularities may occur, but only on the symmetry axis [3]. For blow-up criteria in this case, see [4].

Helical flows are another class of three-dimensional flows with an axis of symmetry. Flows with helical symmetry are an idealized model for the flow induced by rotating blades, such as propellers, helicopter rotors and wind turbines, see [20] and references therein.

Helical flows are invariant under a simultaneous rotation around a symmetry axis and translation along the same axis. The displacement along the axis after one full turn around the axis is an important parameter of helical symmetry, which, in this article, is assumed to be of unit length. This class of flows is preserved under both Navier-Stokes and Euler evolution. The mathematical literature on helical flows is much less extensive than that of axisymmetric flows, but there is growing recent interest. Well-posedness of strong solutions to three-dimensional Navier-Stokes with helical symmetry in bounded domain, was proved by Mahalov, Titi and Leibovich [16] with the initial helical velocity $\mathbf{u}_0^\nu \in H^1$. The key observation in [16] is that the helical flows inherit properties of the two-dimensional flow in the plane, to a greater extent than axisymmetric flows. Specifically, it is proved in [16] that, for a helical vector field $\mathbf{v}$, the following inequality holds true:

$$\|\mathbf{v}\|_{L^4(\Omega)} \leq C\|\mathbf{v}\|^{\frac{1}{2}}_{L^2(\Omega)}\|\mathbf{v}\|^{\frac{1}{2}}_{H^1(\Omega)}, \tag{1.3}$$

where $C > 0$ is a constant and $\Omega = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 < 1, 0 < z < 2\pi\}$ is a cylindrical domain. Recently, the results in [16] were generalized to the helical three-dimensional Navier-Stokes equations with partial viscosity, see [14]. Furthermore, it has been shown that Leray-Hopf weak solutions of viscous, helically symmetric flows, in a bounded domain, are stable in L^2, see Theorem 3.2 in [1].

In analogy with the notion of swirl in axisymmetric flows, we define the *helical swirl* of a helical vector field $\mathbf{v}$ as

$$\eta := \mathbf{v} \cdot \boldsymbol{\xi} \tag{1.4}$$

with $\boldsymbol{\xi} \equiv (y, -x, 1)^T$. Helical swirl plays an important role in global well-posedness of three-dimensional Euler equations with helical symmetry. In particular, the helical swirl component satisfies a transport equation and it is conserved along particle trajectories for Euler flow with helical symmetry. Assuming that the initial velocity field has vanishing helical swirl, Dutrifoy [6] proved the global existence and uniqueness of classical solutions of three-dimensional Euler equations with helical symmetry. Ettinger and Titi [7] obtained the global well-posedness of weak solutions with the initial vorticity belonging to L^∞, which is similar to Yudovich's well-known result for two-dimensional Euler. Recently, Bronzi, Lopes Filho and Nussenzveig Lopes [2] verified global existence of weak solutions when the initial vorticity belongs to $L^p, p > \frac{4}{3}$ with compact support. Subsequently, Jiu, Li and Niu [8] generalized this result to include initial vorticities in $L^1 \cap L^p, p > 1$. All of the aforementioned results assume the initial data has vanishing helical swirl; the problem of global existence for helical Euler with initial nonzero helical swirl remains open.

In this paper, we intend to focus on the inviscid limit of three-dimensional Navier-Stokes equations with helical symmetry in the whole space, and we allow initial data for the Navier-Stokes equations with nonzero helical swirl, but with controlled magnitude $\mathcal{O}(\nu)$, measured in L^2.

The vanishing viscosity problem has a long history. Among the early results, for flows in the full space, are the work of Swann [22] and of Kato [9, 10], where the inviscid limit was established under assumptions of high regularity of initial data (at least $H^3(\mathbb{R}^3)$) and for short time (see also Constantin [5] for a discussion of time-of-existence and vanishing viscosity). Masmoudi [19, 18] studied the inviscid limit in H^s, $s > 1 + N/2$, where N is the dimension of physical space; in our work one would need $s > 5/2$.

In this work we study the inviscid limit for flows with helical symmetry, with H^1 regularity, and our results are global in time, because we will assume that the Euler initial data has zero helical swirl. We will see however, that, for viscous flows, the helical swirl is not conserved along particle trajectories, and the vanishing of the helical swirl is not preserved under Navier-Stokes evolution. It is hence important to understand the behavior of the helical swirl with respect to both time and viscosity. In particular, controlling the magnitude of the swirl component of velocity is the key aspect of obtaining the vorticity estimates needed to carry out our analysis.

More precisely, for helical velocity fields $\mathbf{u}^\nu$ the vorticity has the form of

$$\boldsymbol{\omega}^\nu = \operatorname{curl} \mathbf{u}^\nu = \omega_3^\nu \boldsymbol{\xi} + \left(\frac{\partial \eta^\nu}{\partial y}, -\frac{\partial \eta^\nu}{\partial x}, 0 \right), \tag{1.5}$$

where $\omega_3^\nu = \partial_x u_2^\nu - \partial_y u_1^\nu$ is the third component of the vorticity and $\eta^\nu = \mathbf{u}^\nu \cdot \boldsymbol{\xi}$ is the helical swirl. The equation for ω_3^ν can be written as

$$\partial_t \omega_3^\nu + (\mathbf{u}^\nu \cdot \nabla)\omega_3^\nu + \partial_x \eta^\nu \partial_y u_3^\nu - \partial_y \eta^\nu \partial_x u_3^\nu = \nu \Delta \omega_3^\nu. \tag{1.6}$$

Clearly, vortex stretching terms appear in the above equations (see the third and forth terms on the left hand side) and we cannot control them uniformly with respect to the viscosity ν. To overcome this difficulty, we introduce a decomposition of helical vector fields to obtain the desired *a priori* estimates (see (2.14), Lemma 2.5, and Section 4 for more details). Before we investigate the convergence of the Navier-Stokes equations to the Euler equations, we prove global existence of weak, and of strong, helical solutions to the Navier-Stokes equations (1.1) provided that the initial velocity is helical and belongs to L^2 and H^1, respectively. This result is not included in the existence result of [16] because our fluid domain is the whole space.

A quantity of keen interest in turbulence theory, which is conserved for incompressible inviscid flows, is *helicity*, $\mathcal{H} \equiv \int \mathbf{u} \cdot \boldsymbol{\omega} dx$. In view of (1.5) it is clear that, for helical flows with vanishing helical swirl, the helicity vanishes.

This paper is organized as follows. In Section 2 we recall some useful facts about helical flows and state our main result. In Section 3 we present global existence of weak, and strong, solutions to the three-dimensional helical Navier-Stokes equations in full space, with L^2, and H^1 initial velocity, respectively. The key *a priori* estimates and the proof of our main result will be given in Section 4.

2. Preliminaries and main result

We begin this section by recalling basic definitions, taken from [7], regarding helical symmetry. Denote by R_θ the rotation by an angle θ around the z-axis:

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

The helical symmetry group G^κ is a one-parameter group of isometries of $\mathbb{R}^3$ given by

$$G^\kappa = \{ S_\theta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \mid \theta \in \mathbb{R} \}, \tag{2.2}$$

where

$$S_\theta(\mathbf{x}) = R_\theta(\mathbf{x}) + \begin{pmatrix} 0 \\ 0 \\ \kappa\theta \end{pmatrix} \equiv \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z + \kappa\theta \end{pmatrix}, \tag{2.3}$$

for $\mathbf{x} = (x, y, z)$. Above, κ is a fixed nonzero constant length scale. The transformation S_θ corresponds to the superposition of a simultaneous rotation around the z-axis and a translation along the same z-axis. A *scalar function* $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is said to be helical if

$$f(S_\theta(\mathbf{x})) = f(\mathbf{x}),\ \forall\theta \in \mathbb{R}. \tag{2.4}$$

A *vector field* $\mathbf{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is said to be helical, if

$$\mathbf{v}(S_\theta(\mathbf{x})) = R_\theta \mathbf{v}(\mathbf{x}),\ \forall\theta \in \mathbb{R}. \tag{2.5}$$

Clearly, helical functions and helical vector fields are periodic in the z direction, with period $2\pi\kappa$.

For simplicity, we will henceforth assume that $\kappa = 1$. By virtue of the periodicity of helical functions with respect to the third variable z, it is enough to work in the fundamental domain $\mathcal{D} := \mathbb{R}^2 \times [-\pi, \pi]$. Let $L^2(\mathcal{D})$ denote the square-integrable functions on $\mathcal{D}$ and let $H^1_{per}(\mathcal{D})$ be the usual L^2-based Sobolev space H^1, periodic with respect to z, with period 2π; we use the notation $H^2_{per}(\mathcal{D})$ in an analogous manner. We also use the subscript *loc* to denote Sobolev spaces which are local with respect to the horizontal variables x and y.

Hereafter we use the notation c and C for generic constants which are independent of ν.

Below, we state equivalent definitions of helical functions and helical vector fields; we refer the reader to Claim 2.3 and Claim 2.5 of [7] for the corresponding proofs.

Set

$$\boldsymbol{\xi} \equiv (y, -x, 1)^T. \tag{2.6}$$

Lemma 2.1. *A C^1 scalar function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is helical if and only if*

$$y\partial_x f - x\partial_y f + \partial_z f \equiv \boldsymbol{\xi} \cdot \nabla f = \partial_{\boldsymbol{\xi}} f = 0. \tag{2.7}$$

Lemma 2.2. *A C^1 vector field $\mathbf{v} = (v_1, v_2, v_3)^T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is helical if and only if it the following relations hold true:*

$$\partial_{\boldsymbol{\xi}} v_1 = v_2, \tag{2.8}$$

$$\partial_{\boldsymbol{\xi}} v_2 = -v_1, \tag{2.9}$$

$$\partial_{\boldsymbol{\xi}} v_3 = 0. \tag{2.10}$$

Next we recall the relation between three-dimensional helical vector fields and their two-dimensional traces on "slices" z $=$constant, as discussed in [15]. Recall that we are assuming $\kappa = 1$ so, in the notation of [15], $\sigma = 2\pi$.

Lemma 2.3. *Set* $\mathbf{x} = (x_1, x_2, x_3)$. *Let* $\mathbf{v} = \mathbf{v}(\mathbf{x})$*, be a smooth helical vector field and let* $p = p(\mathbf{x})$ *be a smooth helical function. Then there exist* unique $\mathbf{w} = (w^1, w^2, w^3) = (w^1, w^2, w^3)(y_1, y_2)$ *and* $q = q(y_1, y_2)$ *such that*

$$\mathbf{v}(\mathbf{x}) = R_{x_3}\mathbf{w}(\mathbf{y}(\mathbf{x})), \quad p = p(\mathbf{x}) = q(\mathbf{y}(\mathbf{x})), \tag{2.11}$$

with R_θ *given in* (2.1)*, and*

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 & -\sin x_3 \\ \\ \sin x_3 & \cos x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix}. \tag{2.12}$$

Conversely, if $\mathbf{v}$ *and* p *are defined through* (2.11) *for some* $\mathbf{w} = \mathbf{w}(y_1, y_2)$*,* $q = q(y_1, y_2)$*, then* $\mathbf{v}$ *is a helical vector field and* p *is a helical scalar function.*

This is precisely Proposition 2.1 in [15], in the case $\sigma = 2\pi$, to which we refer the reader for the proof.

Next we will formally introduce the helical swirl, a quantity which plays an important role in helical flows.

Definition 2.1. Let $\mathbf{v}$ be a helical vector field. The helical swirl is defined to be

$$\eta \equiv \mathbf{v} \cdot \boldsymbol{\xi}.$$

Vorticity, the curl of the velocity field, is a key object in the study of incompressible fluid flow. For helical vector fields, vorticity has a special form.

Lemma 2.4. *Let* $\mathbf{v}$ *be a helical vector field. Then its curl,* $\boldsymbol{\omega} =$ *curl* $\mathbf{v} = (\omega_1, \omega_2, \omega_3)$*, is given by*

$$\boldsymbol{\omega} = \omega_3 \boldsymbol{\xi} + (\partial_y \eta, -\partial_x \eta, 0). \tag{2.13}$$

Proof. The result follows by a straightforward calculation. □

Remark 2.1. We note that, in view of Lemma 2.4, if $\mathbf{v}$ is a helical vector field for which the helical swirl vanishes then

$$\text{curl } \mathbf{v} = (\text{ curl } \mathbf{v})_3 \boldsymbol{\xi} \equiv (\partial_x v_2 - \partial_y v_1)\boldsymbol{\xi}.$$

Let $\mathbf{v}$ be a helical vector field. We introduce a decomposition of $\mathbf{v}$ into two other helical vector fields, one of which is orthogonal to the symmetry lines of the helical symmetry group G^1. Let $\mathbf{V}$ be defined through the equation

$$\mathbf{v} \equiv \mathbf{V} + \eta \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}, \tag{2.14}$$

where η is the helical swirl introduced in Definition 2.1.

Lemma 2.5. *Let* $\mathbf{v}$ *be a helical vector field and consider the decomposition* (2.14)*. Then* $\mathbf{V}$ *is also a helical vector field. In addition,* $\mathbf{V}$ *satisfies*

$$\mathbf{V} \cdot \boldsymbol{\xi} = 0.$$

Moreover, if $\mathbf{v}$ *is divergence free,* $\mathbf{V}$ *is also divergence free.*

Proof. As $\mathbf{v}$ is helical, we have, thanks to Lemma 2.2, $\partial_{\boldsymbol{\xi}} \mathbf{v} = (v_2, -v_1, 0)^T$. Now, a direct calculation using Lemma 2.2, together with the expression for $\boldsymbol{\xi}$, yields

$$\partial_{\boldsymbol{\xi}} \eta = \partial_{\boldsymbol{\xi}} (\mathbf{v} \cdot \boldsymbol{\xi}) = (\partial_{\boldsymbol{\xi}} \mathbf{v}) \cdot \boldsymbol{\xi} + \mathbf{v} \cdot \partial_{\boldsymbol{\xi}} \boldsymbol{\xi} = 0.$$

Hence, by Lemma 2.1, η is also helical.

Furthermore, we have

$$\begin{aligned} &\partial_{\boldsymbol{\xi}} \left(\eta \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right) \\ &= \frac{\eta}{|\boldsymbol{\xi}|^2} \partial_{\boldsymbol{\xi}} \boldsymbol{\xi} + \boldsymbol{\xi} \partial_{\boldsymbol{\xi}} \left(\frac{\eta}{|\boldsymbol{\xi}|^2} \right) \\ &= \frac{\eta}{|\boldsymbol{\xi}|^2} \partial_{\boldsymbol{\xi}} \boldsymbol{\xi} = \left(\frac{(\eta \boldsymbol{\xi})_2}{|\boldsymbol{\xi}|^2}, -\frac{(\eta \boldsymbol{\xi})_1}{|\boldsymbol{\xi}|^2}, 0 \right). \end{aligned}$$

Therefore, by Lemma 2.2, it follows that $\eta \boldsymbol{\xi} / |\boldsymbol{\xi}|^2$ is a helical vector field. Consequently, $\mathbf{V}$ is a helical vector field.

In addition, a simple calculation yields

$$\mathbf{V} \cdot \boldsymbol{\xi} = (\mathbf{v} - \eta \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}) \cdot \boldsymbol{\xi} = 0.$$

Finally, suppose that $\mathbf{v}$ is divergence free. We have that

$$\operatorname{div} \left(\eta \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \right) = \partial_{\boldsymbol{\xi}} \left(\frac{\eta}{|\boldsymbol{\xi}|^2} \right) + \frac{\eta}{|\boldsymbol{\xi}|^2} \operatorname{div} \boldsymbol{\xi} = 0.$$

Thus we obtain that $\mathbf{V}$ is divergence free as well.

□

Remark 2.2. Suppose that $\mathbf{v}$ is a helical vector field and let $\mathbf{V}$ be as in (2.14). Then, since $\mathbf{V}$ is helical and has vanishing helical swirl, it follows that its vorticity, $\operatorname{curl} \mathbf{V} = \boldsymbol{\Omega}$ is given by

$$\boldsymbol{\Omega} = \operatorname{curl} \mathbf{V} = (\partial_x V_2 - \partial_y V_1) \boldsymbol{\xi}.$$

See Remark 2.1 for details. Therefore it follows from (2.14) that the third component of the vorticity $\operatorname{curl} \mathbf{v} = \boldsymbol{\omega}$ is given by

$$\omega_3 = \Omega_3 + \left(\operatorname{curl} \left(\frac{\eta}{|\boldsymbol{\xi}|^2} \boldsymbol{\xi} \right) \right)_3$$

$$= \Omega_3 + \partial_x \left(\frac{-\eta x}{|\boldsymbol{\xi}|^2} \right) - \partial_y \left(\frac{\eta y}{|\boldsymbol{\xi}|^2} \right),$$

i.e.

$$\omega_3 = (\partial_x V_2 - \partial_y V_1) - \partial_x \left(\frac{\eta x}{|\boldsymbol{\xi}|^2} \right) - \partial_y \left(\frac{\eta y}{|\boldsymbol{\xi}|^2} \right). \tag{2.15}$$

We will make use of the following Ladyzhenskaya inequality, valid for helical vector fields, see also [14], [11] and [16]. We give a sketch of the proof for the sake of completeness.

Lemma 2.6. *There exists a constant $C > 0$ such that, for every helical vector field $\mathbf{v} \in H^1_{per}(\mathcal{D})$, it holds that*

$$\|\mathbf{v}\|_{L^4(\mathcal{D})} \leq C \|\mathbf{v}\|^{\frac{1}{2}}_{L^2(\mathcal{D})} \|\nabla \mathbf{v}\|^{\frac{1}{2}}_{L^2(\mathcal{D})}. \tag{2.16}$$

Proof of Lemma 2.6. Let $\mathbf{v} \in H^1_{per}(\mathcal{D})$ and consider the vector field $\mathbf{w}$ given in Lemma 2.3, satisfying (2.11). Since R_{x_3} is an orthogonal matrix, we find

$$|\mathbf{v}(x_1, x_2, x_3)|^2 = |\mathbf{w}(y_1, y_2)|^2,$$

and, hence,

$$\|\mathbf{w}\|_{L^p(\mathbb{R}^2)} = \frac{1}{\sqrt[p]{2\pi}} \|\mathbf{v}\|_{L^p(\mathbb{R}^2 \times [-\pi,\pi])}. \tag{2.17}$$

Therefore, using the two-dimensional Ladyzhenskaya inequality (see [11], [21]), we obtain

$$\|\mathbf{v}\|_{L^4(\mathbb{R}^2 \times [-\pi,\pi])} = \sqrt[4]{2\pi} \|\mathbf{w}\|_{L^4(\mathbb{R}^2)} \leq c [\|\mathbf{w}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)} \|\nabla_y \mathbf{w}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}]. \tag{2.18}$$

Thus, to prove (2.16), it suffices to note that, for each $x_3 \in (-\pi, \pi)$, relation (2.11) and (2.12) can be inverted, so that

$$\mathbf{w}(\mathbf{y}) = R^{-1}_{x_3} \mathbf{v}(\mathbf{x}(\mathbf{y})), \tag{2.19}$$

with

$$R^{-1}_{x_3} = \begin{bmatrix} \cos x_3 & -\sin x_3 & 0 \\ \sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.20}$$

and

$$\mathbf{x}(\mathbf{y}) = \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 & \sin x_3 \\ \\ -\sin x_3 & \cos x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix}. \tag{2.21}$$

Hence, in view of (2.19)-(2.21), it follows that, for some $C > 0$,

$$\|\nabla_y \mathbf{w}\|_{L^2(\mathbb{R}^2)} \leq C \|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2 \times [-\pi,\pi])}. \tag{2.22}$$

We conclude by substituting (2.17) and (2.22) into (2.18).

$\square$

Throughout this paper we will make use of the following estimate.

Lemma 2.7. *Let* $\mathbf{v} \in H^1_{per}(\mathcal{D})$ *be a helical vector field. Then*

$$\|\nabla \mathbf{v}\|_{L^2(\mathcal{D})} \leq \|\operatorname{div} \mathbf{v}\|_{L^2(\mathcal{D})} + \|\operatorname{curl} \mathbf{v}\|_{L^2(\mathcal{D})}. \tag{2.23}$$

Proof. Without loss of generality we may assume that $\mathbf{v}$ is a smooth vector field, compactly supported with respect to x and y , periodic with respect to z. The following is a well-known calculus identity:

$$\Delta \mathbf{v} = \nabla(\operatorname{div} \mathbf{v}) - \nabla \times (\operatorname{curl} \mathbf{v}). \tag{2.24}$$

Take the inner product of (2.24) with $-\mathbf{v}$ and integrate in $\mathcal{D}$ to obtain

$$\int_{\mathcal{D}} |\nabla \mathbf{v}|^2 \, dx = \int_{\mathcal{D}} (\operatorname{div} \mathbf{v})^2 \, dx + \int_{\mathcal{D}} |\operatorname{curl} \mathbf{v}|^2 \, dx.$$

This clearly yields the desired estimate.

□

Our objective, in this work, is to show that, under certain assumptions, the vanishing viscosity limit of viscous, helical flows is a helical weak solution of the Euler equations (1.2); below we give a precise definition of such a weak solution.

Definition 2.2. Fix $T > 0$. Let $\mathbf{u}_0 \in H^1_{per,loc}(\mathcal{D})$. We say $\mathbf{u} \in C^0(0,T; L^2(\mathcal{D})) \cap L^\infty(0,T; H^1_{per,loc}(\mathcal{D}))$ is a *helical weak solution* of the incompressible Euler equations (1.2) with initial velocity $\mathbf{u}_0$ if the following hold true:

(1) At each time $0 \leq t < T$, $\mathbf{u}(\cdot, t)$ is a helical vector field;
(2) For every test vector field $\Phi \in C^\infty_c([0,T) \times \overline{\mathcal{D}})$, periodic in z with period 2π, divergence free, the following identity is valid:

$$\int_0^T \int_{\mathcal{D}} \partial_t \Phi \cdot \mathbf{u} + [(\mathbf{u} \cdot \nabla)\Phi] \cdot \mathbf{u} \, d\mathbf{x} dt + \int_{\mathcal{D}} \Phi_0 \cdot \mathbf{u}_0 \, d\mathbf{x} = 0;$$

(3) At each time $0 \leq t < T$, $\operatorname{div} \mathbf{u}(\cdot, t) = 0$ in the sense of distributions.

As is usual, it is possible to recover the scalar pressure by means of the Hodge decomposition.

Remark 2.3. The requirement in Definition 2.2 that $\mathbf{u} \in L^\infty(0,T; H^1_{per,loc}(\mathcal{D}))$ is not needed to make sense of the terms in the weak formulation. We note, however, that a weak solution as in Definition 2.2 satisfies, additionally, a weak form of the inviscid vorticity equation. Definition 2.2 excludes, hence, all known examples of *wild solutions*.

We will conclude this section with the statement of our main result.

Theorem 2.8. *Let $\{\mathbf{u}_0^\nu\}_{\nu>0} \subset H^1_{per}(\mathcal{D})$ be divergence free, helical vector fields and let $\eta_0^\nu = \mathbf{u}_0^\nu \cdot \boldsymbol{\xi}$ denote their respective helical swirls.*

Let $\mathbf{u}_0 \in H^1_{per}(\mathcal{D})$ be a divergence free, helical vector field, such that $\mathbf{u}_0$ has vanishing helical swirl, i.e., $\mathbf{u}_0 \cdot \boldsymbol{\xi} = 0$.

Assume that:

(1)
$$\|\mathbf{u}_0^\nu - \mathbf{u}_0\|_{H^1(\mathcal{D})} \to 0 \text{ as } \nu \to 0;$$

(2) *there exists a constant $C > 0$ such that*
$$\|\eta_0^\nu\|_{L^2(\mathcal{D})} \leq C\nu.$$

Fix $T > 0$. Let $\mathbf{u}^\nu \in L^\infty(0,T;H^1_{per}(\mathcal{D}))$ denote the strong solution of the incompressible Navier-Stokes equations (1.1) *with initial velocity $\mathbf{u}_0^\nu$. Then, there exists $\mathbf{u}^0 \in C^0(0,T;L^2(\mathcal{D})) \cap L^\infty(0,T;H^1_{per,loc}(\mathcal{D}))$ such that, passing to subsequences as needed, we have*

$$\mathbf{u}^\nu \to \mathbf{u}^0 \text{ strongly in } L^2(0,T;L^2_{loc}(\mathcal{D})), \tag{2.25}$$

and $\mathbf{u}^0$ is a helical weak solution of the incompressible Euler equations, with initial velocity $\mathbf{u}_0$, and with vanishing helical swirl at any time $0 \leq t < T$.

3. Global existence of Navier-Stokes equation with helical symmetry

In this section we discuss well-posedness results for (1.1). In particular, we prove the global existence of weak helical solutions provided the initial velocity belongs to $L^2(\mathcal{D})$ and is helically symmetric, and we prove global existence and uniqueness of strong solutions when the initial data, additionally, belongs to H^1_{per}. These results are not included in [16] because our fluid domain is unbounded.

First we introduce a basic mollifier, adapted to the helical symmetry. Let $\rho_1 = \rho_1(|\mathbf{x}'|) \in C_c^\infty(\mathbb{R}^2)$ be a radially symmetric function satisfying that $\rho_1 \geq 0$ and $\int_{\mathbb{R}^2} \rho_1(\mathbf{x}')d\mathbf{x}' = 1$, where $\mathbf{x}' = (x,y)$; let also $\rho_2 = \rho_2(z)$ in $[-\pi,\pi]$ be a nonnegative, periodic, smooth function with $\int_{-\pi}^{\pi} \rho_2(z)dz = 1$. Set $J_\epsilon \mathbf{u}$ to be the mollification of a helical vector field $\mathbf{u} \in L^p(\mathcal{D}), 1 \leq p \leq \infty$, given by

$$J_\epsilon \mathbf{u} = J_\epsilon \mathbf{u}(\mathbf{x}) \equiv \int_{\mathcal{D}} \rho^\epsilon(\mathbf{x}-\mathbf{y})\mathbf{u}(\mathbf{y})d\mathbf{y},\ \epsilon > 0, \tag{3.1}$$

where $\rho^\epsilon(\mathbf{x}) = \epsilon^{-3}\rho(\frac{\mathbf{x}}{\epsilon})$ and $\rho = \rho(\mathbf{x}) = \rho_1(\mathbf{x}')\rho_2(z)$.

The following lemma provides some basic properties of these mollifiers.

Lemma 3.1. *Let J_ϵ be the mollifier defined in* (3.1)*. Then, for each $\mathbf{u} \in L^p(\mathcal{D})$, $1 \le p \le \infty$, $J_\epsilon \mathbf{u}$ is a C^∞ function and*

(1) $$J_\epsilon[\mathbf{u}(\mathbf{y}+\mathbf{h})](\mathbf{x}) = J_\epsilon[\mathbf{u}(\mathbf{y})](\mathbf{x}+\mathbf{h}), \quad \forall \mathbf{h} \in \mathcal{D}, \tag{3.2}$$

(2) $$J_\epsilon[\mathbf{u}(R_\theta \mathbf{y})](\mathbf{x}) = J_\epsilon[\mathbf{u}(\mathbf{y})](R_\theta \mathbf{x}), \quad \text{where } R_\theta \text{ is defined in (2.1)}, \tag{3.3}$$

(3) $$J_\epsilon[\mathbf{u}(S_\theta \mathbf{y})](\mathbf{x}) = J_\epsilon[\mathbf{u}(\mathbf{y})](S_\theta \mathbf{x}), \quad \text{where } S_\theta \text{ is defined in (2.3)}, \tag{3.4}$$

(4) $$D_x^\alpha(J_\epsilon[\mathbf{u}(\mathbf{y})])(\mathbf{x}) = J_\epsilon[D_y^\alpha \mathbf{u}(\mathbf{y})](\mathbf{x}), \quad |\alpha| \le m, \ \mathbf{u} \in H^m. \tag{3.5}$$

Proof of Lemma 3.1. The proof of (3.2) is easily obtained from the definition of J_ϵ, (3.1). Item (3.4) follows directly from (3.2) and (3.3), while (3.5) can be found in [17]. Item (3.3) follows by a straightforward calculation. □

Let us briefly recall that a weak solution of the Navier-Stokes equations has the regularity $L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1_{per}(\mathcal{D}))$, whereas a strong solution belongs to $L^\infty(0,T;H^1_{per}(\mathcal{D})) \cap L^2(0,T;H^2_{per}(\mathcal{D}))$.

We can now state and prove a basic result on existence of weak and strong helical solutions to the Navier-Stokes equations (1.1).

Theorem 3.2. *Fix $\nu > 0$. Let $\mathbf{u}_0^\nu \in L^2(\mathcal{D})$ be a divergence free and helical vector field. Fix, also, $T > 0$.*

(1) *There exists $\mathbf{u}^\nu \in L^\infty(0,T;L^2(\mathcal{D})) \cap L^2(0,T;H^1_{per}(\mathcal{D}))$ which is a helical weak solution to the three-dimensional Navier-Stokes equations* (1.1)*. In addition, $\mathbf{u}^\nu$ satisfies the following inequality*

$$\|\mathbf{u}^\nu(t)\|^2_{L^\infty(0,T;L^2(\mathcal{D}))} + \nu\|\nabla \mathbf{u}^\nu\|^2_{L^2(0,T;L^2(\mathcal{D}))} \le \|\mathbf{u}_0^\nu\|^2_{L^2(\mathcal{D})}. \tag{3.6}$$

(2) *If, in addition, $\mathbf{u}_0^\nu \in H^1_{per}(\mathcal{D})$, then the three-dimensional Navier-Stokes equations* (1.1) *has a unique and global strong solution $\mathbf{u}^\nu \in L^\infty(0,T;H^1_{per}(\mathcal{D})) \cap L^2(0,T;H^2_{per}(\mathcal{D}))$ which is helically symmetric.*

Proof of Theorem 3.2. We will begin by establishing (1); the proof will be divided into four steps. As much of this proof is standard, we will be brief.

Step I As in [17], we construct approximate solutions $\mathbf{u}^{\nu,\epsilon}$ to the Navier-Stokes equations by solving

$$\begin{cases} \mathbf{u}_t^{\nu,\epsilon} + J_\epsilon[(J_\epsilon \mathbf{u}^{\nu,\epsilon} \cdot \nabla)(J_\epsilon \mathbf{u}^{\nu,\epsilon})] + \nabla p^{\nu,\epsilon} = \nu J_\epsilon(J_\epsilon \Delta \mathbf{u}^{\nu,\epsilon}), \\ \operatorname{div} \mathbf{u}^{\nu,\epsilon} = 0, \\ \mathbf{u}^{\nu,\epsilon}(t=0,\mathbf{x}) = \mathbf{u}_0^{\nu,\epsilon}, \end{cases} \tag{3.7}$$

where $\mathbf{u}_0^{\nu,\epsilon}(\mathbf{x}) := J_\epsilon \mathbf{u}_0^\nu(\mathbf{x})$, with J_ϵ defined in (3.1). By the Picard theorem (see e.g. [17]), there exists a unique, global, smooth solution $\mathbf{u}^{\nu,\epsilon}$ for the regularized Navier-Stokes equations (3.7).

Step II Next, we show that the approximate solutions $\mathbf{u}^{\nu,\epsilon}$ preserve helical symmetry.

First we note that $\mathbf{u}_0^{\nu,\epsilon}$ is helical. More precisely, using (3.4) in Lemma 3.1 together with the fact that $\mathbf{u}_0^\nu$ is a helical vector field, we easily verify that

$$\begin{aligned} R_\theta^{-1}\mathbf{u}_0^{\nu,\epsilon}(S_\theta\mathbf{x}) &:= R_\theta^{-1}J_\epsilon\mathbf{u}_0^\nu(S_\theta\mathbf{x}) = R_\theta^{-1}J_\epsilon[\mathbf{u}_0^\nu(S_\theta\mathbf{y})](\mathbf{x}) \\ &:= R_\theta^{-1}J_\epsilon[R_\theta\mathbf{u}_0^\nu(\mathbf{y})](\mathbf{x}) = J_\epsilon\mathbf{u}_0^\nu(\mathbf{x}) := \mathbf{u}_0^{\nu,\epsilon}(\mathbf{x}). \end{aligned} \tag{3.8}$$

Consider $\bar{\mathbf{u}}(\mathbf{x},t) = R_\theta^{-1}\mathbf{u}^{\nu,\epsilon}(S_\theta\mathbf{x},t)$ and $\bar{p}^{\nu,\epsilon}(\mathbf{x},t) = p^{\nu,\epsilon}(S_\theta\mathbf{x},t)$. Direct calculations give that the pair $(\bar{\mathbf{u}}, \bar{p}^{\nu,\epsilon})$ is a solution of (3.7) with initial data $\bar{\mathbf{u}}(\mathbf{x},0) = R_\theta^{-1}\mathbf{u}_0^{\nu,\epsilon}(S_\theta\mathbf{x}) = \mathbf{u}_0^{\nu,\epsilon}(\mathbf{x})$. Hence, by uniqueness of smooth solutions $\mathbf{u}^{\nu,\epsilon}$ of (3.7), we obtain that

$$\mathbf{u}^{\nu,\epsilon}(\mathbf{x},t) \equiv R_\theta^{-1}\mathbf{u}^{\nu,\epsilon}(S_\theta\mathbf{x},t), \tag{3.9}$$

i.e., $\mathbf{u}^{\nu,\epsilon}$ is a helical vector field.

Step III In this step we discuss uniform, in ϵ, estimates. Take the L^2-inner product of the regularized momentum equations (3.7) with $\mathbf{u}^{\nu,\epsilon}$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^{\nu,\epsilon}\|^2_{L^2(\mathcal{D})} + \nu\|\nabla J_\epsilon\mathbf{u}^{\nu,\epsilon}\|^2_{L^2(0,T;L^2(\mathcal{D}))} \le 0. \tag{3.10}$$

Integrate (3.10) in time, from 0 to T, to find

$$\|\mathbf{u}^{\nu,\epsilon}\|^2_{L^\infty(0,T;L^2(\mathcal{D}))} + \nu\|\nabla J_\epsilon\mathbf{u}^{\nu,\epsilon}\|^2_{L^2(0,T;L^2(\mathcal{D}))} \le \|\mathbf{u}_0^{\nu,\epsilon}\|^2_{L^2(\mathcal{D})}. \tag{3.11}$$

In view of (3.11) it is standard that $\{J_\epsilon\mathbf{u}^{\nu,\epsilon}\}_{\epsilon>0}$ is a compact subset of $L^2(0,T;L^2(\mathcal{D}))$ and hence, passing to subsequences as needed and using properties of mollifiers, we find that $\mathbf{u}^{\nu,\epsilon}$ is a convergent sequence in $L^2(0,T;L^2(\mathcal{D}))$, as $\epsilon \to 0$. We easily obtain that the limit $\mathbf{u}^\nu$ satisfies (1.1) in the sense of distributions. From the uniform bound in $L^\infty(0,T;L^2(\mathcal{D}))$, (3.11), we obtain that $\mathbf{u}^\nu \in L^\infty(0,T;L^2(\mathcal{D}))$; similarly, we find that $\mathbf{u}^\nu \in L^2(0,T;H^1_{per}(\mathcal{D}))$. Since, from Step II, we deduced that $\mathbf{u}^{\nu,\epsilon}$ is a helical vector field, it follows easily that the limit $\mathbf{u}^\nu$ is also helically symmetric. Therefore, there exists a helical weak solution $\mathbf{u}^\nu \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ of (1.1). The energy inequality (3.6) follows by weak convergence in $L^2(0,T;H^1_{per}(\mathcal{D}))$.

Step IV Finally, we establish item (2), the existence and uniqueness of a strong solution if the initial data is smoother. From above, we have a weak helical solution in $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ to the system (1.1). We will show, by energy estimates, that the regularity of $\mathbf{u}^\nu$ can be improved to $L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. Although the estimates below are formal, they can be made rigorous using the regularized equation (3.7) in a similar way to what was done in Step III.

Taking the L^2-inner product of (1.1) with $\Delta\mathbf{u}^\nu$ we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\nabla\mathbf{u}^\nu\|^2_{L^2(\mathcal{D})} + \nu\|\Delta\mathbf{u}^\nu\|^2_{L^2(0,T;L^2(\mathcal{D}))}\\ &\le \left|\int_{\mathcal{D}}(\mathbf{u}^\nu\cdot\nabla)\mathbf{u}^\nu\cdot\Delta\mathbf{u}^\nu dx\right|\\ &\le \|\mathbf{u}^\nu\|_{L^4(\mathcal{D})}\|\nabla\mathbf{u}^\nu\|_{L^4(\mathcal{D})}\|\Delta\mathbf{u}^\nu\|_{L^2(\mathcal{D})}. \end{aligned}\tag{3.12}$$

Now, since $\mathbf{u}^\nu$ is a helical vector field, it follows from Lemma 2.6 that

$$\|\mathbf{u}^\nu\|_{L^4(\mathcal{D})} \le \|\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}\|\nabla\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}. \tag{3.13}$$

Let us examine $\nabla\mathbf{u}^\nu$. Recall that, from Lemma 2.6, there exists a unique vector field $\mathbf{w} = \mathbf{w}(y_1, y_2)$ such that the relation in (2.11) holds true, with $\mathbf{y} = \mathbf{y}(\mathbf{x})$ as in (2.12). We write $\nabla\mathbf{u}^\nu = (\nabla_H\mathbf{u}^\nu, \partial_{x_3}\mathbf{u}^\nu)$, where ∇_H refers to the *horizontal* derivatives, i.e. derivatives with respect to x_1, x_2. In view of (2.11), (2.12) an easy calculation yields, for each $1 \le p \le \infty$, $m \in \mathbb{N}$, the existence of constants $C_{p,m}$, $c_{p,m} > 0$ such that

$$c_{p,m}\|\nabla^m_y\mathbf{w}\|_{L^p(\mathbb{R}^2)} \le \|\nabla^m_H\mathbf{u}^\nu\|_{L^p(\mathcal{D})} \le C_{p,m}\|\nabla^m_y\mathbf{w}\|_{L^p(\mathbb{R}^2)}. \tag{3.14}$$

Since $\nabla\mathbf{w}$ is a function of two independent variables we may use the two dimensional Ladyzhenskaya inequality for $\nabla\mathbf{w}$ to find

$$\|\nabla_y\mathbf{w}\|_{L^4(\mathbb{R}^2)} \le C\|\nabla_y\mathbf{w}\|^{\frac12}_{L^2(\mathbb{R}^2)}\|\nabla^2_y\mathbf{w}\|^{\frac12}_{L^2(\mathbb{R}^2)},$$

from which, together with (3.14), it follows that

$$\|\nabla_H\mathbf{u}^\nu\|_{L^4(\mathcal{D})} \le C\|\nabla_H\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}\|\nabla^2_H\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}. \tag{3.15}$$

Next, we consider $\partial_{x_3}\mathbf{u}^\nu$. Recall the criteria in Lemma 2.2 for a vector field to be helical: $\partial_{\boldsymbol{\xi}}v_1 = v_2$, $\partial_{\boldsymbol{\xi}}v_2 = -v_1$, $\partial_{\boldsymbol{\xi}}v_3 = 0$. Note that $\partial_{x_3}\partial_{\boldsymbol{\xi}} = \partial_{\boldsymbol{\xi}}\partial_{x_3}$. Therefore, since $\mathbf{u}^\nu$ is a helical vector field, we deduce that

$$\partial_{\boldsymbol{\xi}}\partial_{x_3}u_1^\nu = \partial_{x_3}u_2^\nu; \quad \partial_{\boldsymbol{\xi}}\partial_{x_3}u_2^\nu = -\partial_{x_3}u_1^\nu; \quad \partial_{\boldsymbol{\xi}}\partial_{x_3}u_3^\nu = 0.$$

Hence, $\partial_{x_3}\mathbf{u}^\nu$ is a helical vector field and, therefore, in view of Lemma 2.6,

$$\|\partial_{x_3}\mathbf{u}^\nu\|_{L^4(\mathcal{D})} \le C\|\partial_{x_3}\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}\|\nabla\partial_{x_3}\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}. \tag{3.16}$$

Notice that both the right-hand-side of (3.15) and of (3.16) are bounded by $C\|\nabla\mathbf{u}^\nu\|^{1/2}_{L^2(\mathcal{D})}\|\Delta\mathbf{u}^\nu\|^{1/2}_{L^2(\mathcal{D})}$, since, from elliptic regularity theory, we know that all second derivatives are bounded, in L^2, by the Laplacian.

We obtain, from (3.15) and (3.16),

$$\|\nabla\mathbf{u}^\nu\|_{L^4(\mathcal{D})} \le C\|\nabla\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}\|\Delta\mathbf{u}^\nu\|^{\frac12}_{L^2(\mathcal{D})}. \tag{3.17}$$

Substituting (3.13) and (3.17) into (3.12) yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}^\nu\|^2_{L^2(\mathcal{D})} + \nu\|\Delta\mathbf{u}^\nu\|^2_{L^2(0,T;L^2(\mathcal{D}))}$$

$$\le \|\mathbf{u}^\nu\|_{L^2(\mathcal{D})}^{\frac{1}{2}} \|\nabla \mathbf{u}^\nu\|_{L^2(\mathcal{D})} \|\Delta \mathbf{u}^\nu\|_{L^2(\mathcal{D})}^{\frac{3}{2}}$$

$$\le \frac{\nu}{4} ||\Delta \mathbf{u}^\nu\|_{L^2(\mathcal{D})}^2 + C\nu^{-3} \|\mathbf{u}^\nu\|_{L^2(\mathcal{D})}^2 \|\nabla \mathbf{u}^\nu\|_{L^2(\mathcal{D})}^4, \tag{3.18}$$

where we used Young's inequality to obtain the last inequality. From (3.6) we have

$$\nu \int_0^T \|\nabla \mathbf{u}^\nu\|_{L^2(\mathcal{D})}^2 dt \le \|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}^2,$$

so that, by Gronwall's lemma, we obtain

$$\|\nabla \mathbf{u}^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))}^2 \le \|\nabla \mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}^2 \exp\left\{ \frac{C\|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}^4}{\nu^4} \right\}. \tag{3.19}$$

Thus $\mathbf{u}^\nu \in L^\infty(0,T;H^1_{per}(\mathcal{D}))$. That $\mathbf{u}^\nu \in L^2(0,T;H^2_{per}(\mathcal{D}))$ follows immediately upon revisiting (3.18) and integrating in time.

Uniqueness is easily obtained under the regularity of $\mathbf{u}^\nu$. We omit the details. $\square$

4. Proof of main result

We will begin this section by obtaining an evolution equation for the helical swirl. Hereafter we assume that $\mathbf{u}_0^\nu \in H^1_{per}(\mathcal{D})$ is a divergence free, helical vector field and $\mathbf{u}^\nu \in L^\infty(0,T;H^1_{per}(\mathcal{D})) \cap L^2(0,T;H^2_{per}(\mathcal{D}))$ is the strong, helically symmetric, solution of (1.1) with initial velocity $\mathbf{u}_0^\nu$, given in Theorem 3.2. Let $\eta^\nu \equiv \mathbf{u}^\nu \cdot \boldsymbol{\xi}$. Multiply the momentum equation in (1.1) by $\boldsymbol{\xi}$ to obtain, after direct calculations,

$$\begin{cases} \partial_t \eta^\nu + (\mathbf{u}^\nu \cdot \nabla)\eta^\nu = \nu\Delta\eta^\nu + 2\nu(\partial_{x_1} u_2^\nu - \partial_{x_2} u_1^\nu), \\ \eta^\nu(t=0,\mathbf{x}) = \eta_0^\nu. \end{cases} \tag{4.1}$$

Clearly, in the case of the Euler equations ($\nu = 0$), the helical swirl $\eta^0 := \mathbf{u}^{\mathbf{0}} \cdot \boldsymbol{\xi}$ satisfies a transport equation and is conserved along particle paths. This is not the case if $\nu > 0$.

Nevertheless, we may still obtain a uniform bound, with respect to ν, for the helical swirl η^ν.

Lemma 4.1. *Fix $T > 0$. Let $\mathbf{u}_0^\nu \in H^1_{per}(\mathcal{D})$ and $\eta_0^\nu = \mathbf{u}_0^\nu \cdot \boldsymbol{\xi}$. Then there exists a constant $c = c(T) > 0$, independent of ν, such that*

$$\|\eta^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \sqrt{\nu}\|\nabla\eta^\nu\|_{L^2(0,T;L^2(\mathcal{D}))} \le c(\|\eta_0^\nu\|_{L^2(\mathcal{D})} + \sqrt{\nu}\|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}). \tag{4.2}$$

Proof of Lemma 4.1. Multiply both sides of (4.1) by η^ν, integrate the resulting equation in $\mathcal{D}$ and use that $\operatorname{div} \mathbf{u}^\nu = 0$ to obtain that

$$\frac{1}{2}\frac{d}{dt}\|\eta^\nu\|_{L^2(\mathcal{D})}^2 + \nu\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}^2 \le 2\nu\|\mathbf{u}^\nu\|_{L^2(\mathcal{D})}\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}$$

$$\leq \frac{\nu}{2}\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} + C\nu\|\mathbf{u}^\nu\|^2_{L^2(\mathcal{D})}. \tag{4.3}$$

It follows from integration over the time from 0 to T, together with inequality (3.6), that

$$\|\eta^\nu\|^2_{L^\infty(0,T;L^2(\mathcal{D}))} + \nu\|\nabla\eta^\nu\|^2_{L^2(0,T;L^2(\mathcal{D}))} \leq C(\|\eta^\nu_0\|^2_{L^2(\mathcal{D})} + T\nu\|\mathbf{u}^\nu_0\|^2_{L^2(\mathcal{D})}).$$

Clearly, this concludes the proof.

□

Using the decomposition (2.14), we introduce

$$\mathbf{U}^\nu \equiv \mathbf{u}^\nu - \eta^\nu\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}, \text{ and } \boldsymbol{\Omega}^\nu \equiv \operatorname{curl}\mathbf{U}^\nu. \tag{4.4}$$

Then $\mathbf{U}^\nu \cdot \boldsymbol{\xi} = 0$ and $\mathbf{U}^\nu$ is helical due to Lemma 2.5. As noted in Remark 2.2, we have

$$\partial_x u_2^\nu - \partial_y u_1^\nu = \Omega_3^\nu + \partial_x\left(\frac{-x\eta^\nu}{|\boldsymbol{\xi}|^2}\right) - \partial_y\left(\frac{y\eta^\nu}{|\boldsymbol{\xi}|^2}\right). \tag{4.5}$$

Moreover, direct calculations give

$$\begin{cases} \partial_t\mathbf{U}^\nu + \mathbf{U}^\nu\cdot\nabla\mathbf{U}^\nu + \nabla p^\nu - \nu\Delta\mathbf{U}^\nu \\ \quad = -\dfrac{\eta^\nu}{|\boldsymbol{\xi}|^2}\partial_{\boldsymbol{\xi}}\mathbf{U}^\nu - \mathbf{U}^\nu\cdot\nabla\left(\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\eta^\nu - \dfrac{(\eta^\nu)^2}{|\boldsymbol{\xi}|^2}\partial_{\boldsymbol{\xi}}\left(\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) \\ \qquad +2\nu\nabla\eta^\nu\cdot\nabla\left(\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) + \nu\eta^\nu\Delta\left(\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right) - 2\nu\Omega_3^\nu\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \\ \qquad\quad -2\nu\left[\operatorname{curl}\left(\dfrac{\eta^\nu\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right)\right]_3\dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}, \\ \operatorname{div}\mathbf{U}^\nu = 0. \end{cases} \tag{4.6}$$

By Lemma 2.4, $\boldsymbol{\Omega}^\nu \equiv \Omega_3^\nu \boldsymbol{\xi}$, where $\Omega_3^\nu = \partial_x U_2^\nu - \partial_y U_1^\nu$. Direct calculation leads to the following equation for Ω_3^ν:

$$\begin{cases} \partial_t \Omega_3^\nu + \mathbf{U}^\nu \cdot \nabla \Omega_3^\nu - \nu \Delta \Omega_3^\nu = \\ \quad -2\left[\partial_x\left(\dfrac{\eta^\nu(x^2 U_1^\nu + xyU_2^\nu)}{|\boldsymbol{\xi}|^4}\right) + \partial_y\left(\dfrac{\eta^\nu(xyU_1^\nu + y^2 U_2^\nu)}{|\boldsymbol{\xi}|^4}\right)\right] \\ \quad +2\left[\partial_x\left(\dfrac{\eta^\nu U_1^\nu}{|\boldsymbol{\xi}|^4}\right) + \partial_y\left(\dfrac{\eta^\nu U_2^\nu}{|\boldsymbol{\xi}|^4}\right)\right] - \partial_z\left(\dfrac{(\eta^\nu)^2}{|\boldsymbol{\xi}|^4}\right) \\ \quad -2\nu\left[\partial_x\left(\dfrac{\partial_x \eta^\nu}{|\boldsymbol{\xi}|^2}\right) + \partial_y\left(\dfrac{\partial_y \eta^\nu}{|\boldsymbol{\xi}|^2}\right)\right] \\ \quad +2\nu\left[\partial_x\left(\dfrac{x^2\partial_x\eta^\nu + xy\partial_y\eta^\nu}{|\boldsymbol{\xi}|^4}\right) + \partial_y\left(\dfrac{xy\partial_x\eta^\nu + y^2\partial_y\eta^\nu}{|\boldsymbol{\xi}|^4}\right)\right] \\ \quad +4\nu\left[\partial_x\left(\dfrac{x\eta^\nu}{|\boldsymbol{\xi}|^6}\right) + \partial_y\left(\dfrac{y\eta^\nu}{|\boldsymbol{\xi}|^6}\right)\right] + 2\nu\left[\partial_x\left(\Omega_3^\nu \dfrac{x}{|\boldsymbol{\xi}|^2}\right) + \partial_y\left(\Omega_3^\nu \dfrac{y}{|\boldsymbol{\xi}|^2}\right)\right], \\ \Omega_3^\nu(t=0,\mathbf{x}) = \Omega_{3,0}^\nu. \end{cases} \tag{4.7}$$

The following is a key estimate which will be used to obtain the compactness of the family of solutions to the Navier-Stokes equations (1.1), $\nu > 0$.

Lemma 4.2. *Fix $T > 0$. Let $\nu \le 1$. Assume that $\mathbf{u}_0^\nu \in H^1_{per}(\mathcal{D})$, and $\eta_0^\nu \in L^2(\mathcal{D})$ with $\|\eta_0^\nu\|_{L^2(\mathcal{D})} \le c\nu$. Then, there exists $c = c(T, \|\mathbf{u}_0^\nu\|_{H^1_{per}(\mathcal{D})}) > 0$ such that*

$$\|\Omega_3^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} \le c. \tag{4.8}$$

Furthermore,

$$\|\eta^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \sqrt{\nu}\|\nabla\eta^\nu\|_{L^2(0,T;L^2(\mathcal{D}))} \le C\nu, \tag{4.9}$$

for some constant $C = C(T, \|\mathbf{u}_0^\nu\|_{H^1_{per}(\mathcal{D})}) > 0$ which is independent of ν

Proof of Lemma 4.2. Let $t \in [0, T)$ and set

$$Y = Y(t) := \int_0^t \|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}. \tag{4.10}$$

Then

$$Y'(t) = \|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}, \quad Y''(t) = \frac{d}{dt}\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}.$$

We claim that

$$\|\eta^\nu\|_{L^\infty(0,t;L^2(\mathcal{D}))} + \sqrt{\nu}\|\nabla\eta^\nu\|_{L^2(0,t;L^2(\mathcal{D}))} \leq C\nu(1+Y(t))^{\frac{1}{2}}, \tag{4.11}$$

where C depends on T but is independent of ν.

Indeed, as in the proof of Lemma 4.1, we multiply the both sides of (4.1) by η^ν, integrate the resulting equation in $\mathcal{D}$, and use the divergence free condition, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\eta^\nu\|^2_{L^2(\mathcal{D})} + \nu\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} \\
&\qquad \leq 2\nu\|\partial_x u_2^\nu - \partial_y u_1^\nu\|_{L^2(\mathcal{D})}\|\eta^\nu\|_{L^2(\mathcal{D})} \\
&\qquad \leq 2\nu(\|\Omega_3^\nu\|_{L^2(\mathcal{D})} + \|\nabla\eta^\nu\|_{L^2(\mathcal{D})} + \|\eta^\nu\|_{L^2(\mathcal{D})})\|\eta^\nu\|_{L^2(\mathcal{D})} \\
&\qquad \leq C\nu^2\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} + \frac{\nu}{2}\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} + C\|\eta^\nu\|^2_{L^2(\mathcal{D})}.
\end{aligned} \tag{4.12}$$

where we have used identity (4.5) and Young's inequality. This gives the estimate

$$\frac{1}{2}\frac{d}{dt}\|\eta^\nu\|^2_{L^2(\mathcal{D})} + \frac{\nu}{2}\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} \leq C\nu^2\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} + C\|\eta^\nu\|^2_{L^2(\mathcal{D})}. \tag{4.13}$$

It follows from Gronwall's lemma, upon performing parabolic regularity estimates, that

$$\|\eta^\nu\|^2_{L^\infty(0,t;L^2(\mathcal{D}))} + \nu\|\nabla\eta^\nu\|^2_{L^2(0,t;L^2(\mathcal{D}))} \leq C(\|\eta_0^\nu\|^2_{L^2(\mathcal{D})} + \nu^2 Y(t)),$$

for some constant $C = C(T) > 0$. Finally, condition (2) in Theorem 2.8 yields that

$$\|\eta^\nu\|^2_{L^\infty(0,t;L^2(\mathcal{D}))} + \nu\|\nabla\eta^\nu\|^2_{L^2(0,t;L^2(\mathcal{D}))} \leq C\nu^2(1+Y(t)) \tag{4.14}$$

i.e.,

$$\|\eta^\nu\|_{L^\infty(0,t;L^2(\mathcal{D}))} + \sqrt{\nu}\|\nabla\eta^\nu\|_{L^2(0,t;L^2(\mathcal{D}))} \leq C\nu(1+Y(t))^{\frac{1}{2}}, \tag{4.15}$$

where $C = C(T) > 0$ is a constant which is independent of ν. We have established (4.11).

Next, we use (4.11) to derive estimate (4.8) for Ω_3^ν. Multiplying both sides of (4.7) by Ω_3^ν and integrating in $\mathcal{D}$, gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} + \nu\|\nabla\Omega_3^\nu\|^2_{L^2(\mathcal{D})} \\
&\leq 4\int_{\mathcal{D}}\left|\nabla\Omega_3^\nu\frac{\eta^\nu\mathbf{U}^\nu}{|\boldsymbol{\xi}|^4}\right|d\mathbf{x} + 2\int_{\mathcal{D}}\left|\nabla\Omega_3^\nu\frac{\eta^\nu\mathbf{U}^\nu}{|\boldsymbol{\xi}|^2}\right|d\mathbf{x} + \int_{\mathcal{D}}\left|\nabla\Omega_3^\nu\frac{|\eta^\nu|^2}{|\boldsymbol{\xi}|^4}\right|d\mathbf{x} \\
&\quad + 6\nu\int_{\mathcal{D}}\left|\nabla\Omega_3^\nu\frac{\nabla\eta^\nu}{|\boldsymbol{\xi}|^2}\right|d\mathbf{x} + 8\nu\int_{\mathcal{D}}\left|\nabla\Omega_3^\nu\frac{\eta^\nu}{|\boldsymbol{\xi}|^4}\right|d\mathbf{x} + 4\nu\int_{\mathcal{D}}|\Omega_3^\nu|^2 d\mathbf{x}.
\end{aligned}$$

Then, using Cauchy's inequality together with Young's inequality leads to

$$\begin{aligned}
&\frac{d}{dt}\|\Omega_3^\nu\|_{L^2(\mathcal{D})}^2 + \frac{\nu}{2}\|\nabla\Omega_3^\nu\|_{L^2(\mathcal{D})}^2 \\
&\qquad \le \frac{c}{\nu}\left\|\frac{\eta^\nu \mathbf{U}^\nu}{|\boldsymbol{\xi}|^4}\right\|_{L^2(\mathcal{D})}^2 + \frac{c}{\nu}\left\|\frac{\eta^\nu \mathbf{U}^\nu}{|\boldsymbol{\xi}|^2}\right\|_{L^2(\mathcal{D})}^2 + \frac{c}{\nu}\left\|\frac{(\eta^\nu)^2}{|\boldsymbol{\xi}|^4}\right\|_{L^2(\mathcal{D})}^2 \\
&\qquad\quad + c\nu\left\|\frac{\nabla\eta^\nu}{|\boldsymbol{\xi}|^2}\right\|_{L^2(\mathcal{D})}^2 + c\nu\left\|\frac{\eta^\nu}{|\boldsymbol{\xi}|^4}\right\|_{L^2(\mathcal{D})}^2 + c\nu\|\Omega_3^\nu\|_{L^2(\mathcal{D})}^2. \qquad (4.16)
\end{aligned}$$

From Lemma 2.6, together with Hölder's inequality and (4.11), it follows that, for any $\alpha > 1$,

$$\begin{aligned}
\left\|\frac{\eta^\nu \mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right\|_{L^2(\mathcal{D})}^2 &\le \|\eta^\nu\|_{L^4}^2 \left\|\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right\|_{L^4(\mathcal{D})}^2 \\
&\le \|\eta^\nu\|_{L^2(\mathcal{D})}\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}\left\|\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right\|_{L^2(\mathcal{D})}\left\|\nabla(\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha})\right\|_{L^2(\mathcal{D})} \qquad (4.17) \\
&\le C\nu(1+Y(t))^{\frac{1}{2}}\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}\|\mathbf{U}^\nu\|_{L^2(\mathcal{D})}\left\|\nabla(\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha})\right\|_{L^2(\mathcal{D})}.
\end{aligned}$$

Using the result in Lemma 2.7, we find

$$\begin{aligned}
\left\|\nabla\left(\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right)\right\|_{L^2(\mathcal{D})} &\le \left(\left\|\mathrm{curl}\left(\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right)\right\|_{L^2(\mathcal{D})} + \left\|\mathrm{div}\left(\frac{\mathbf{U}^\nu}{|\boldsymbol{\xi}|^\alpha}\right)\right\|_{L^2(\mathcal{D})}\right) \\
&\le \left\|\frac{\Omega_3^\nu}{|\boldsymbol{\xi}|^{\alpha-1}}\right\|_{L^2(\mathcal{D})} + \alpha\left\|\frac{\boldsymbol{\xi}\times\mathbf{U}^\nu}{|\boldsymbol{\xi}|^{\alpha+2}}\right\|_{L^2(\mathcal{D})} + \alpha\left\|\frac{\mathbf{U}^\nu\cdot\boldsymbol{\xi}}{|\boldsymbol{\xi}|^{\alpha+2}}\right\|_{L^2(\mathcal{D})} \\
&\le \|\Omega_3^\nu\|_{L^2(\mathcal{D})} + (1+\alpha)\|\mathbf{U}^\nu\|_{L^2(\mathcal{D})}. \qquad (4.18)
\end{aligned}$$

Substituting (4.18) into (4.17) together with the fact that
$\|\boldsymbol{U}^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} \le \|\mathbf{u}^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} + \|\eta^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} \le \|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})} + c$
from (3.6), (4.4), and (4.2), we have

$$\left\|\frac{\eta^\nu \mathbf{U}^\nu}{\boldsymbol{\xi}^\alpha}\right\|_{L^2(\mathcal{D})}^2 \le c\nu\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}(1+Y(t))^{\frac{1}{2}}(1+\|\Omega_3^\nu\|_{L^2(\mathcal{D})}), \qquad (4.19)$$

where c depends on $\|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}$, $\|\eta_0^\nu\|_{L^2(\mathcal{D})}$ and T, independent of ν.

Moreover, noting that

$$\left\|\frac{(\eta^\nu)^2}{|\boldsymbol{\xi}|^4}\right\|_{L^2(\mathcal{D})} \le \|\eta^\nu\|_{L^4(\mathcal{D})}^2 \le c\|\eta^\nu\|_{L^2(\mathcal{D})}\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}, \qquad (4.20)$$

we find, by substituting (4.19) and (4.20) into (4.16) and using (4.2) and Young's inequality, that

$$\begin{aligned}
&\frac{d}{dt}\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} + \frac{\nu}{2}\|\nabla\Omega_3^\nu\|^2_{L^2(\mathcal{D})} \\
&\quad\le C\|\nabla\eta^\nu\|_{L^2(\mathcal{D})}(1+Y(t))^{\frac{1}{2}}(1+\|\Omega_3^\nu\|_{L^2(\mathcal{D})}) + C\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} \\
&\qquad\qquad + C\nu\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})} + C\nu^2 + C\nu\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} \\
&\quad\le C\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})}(1+Y(t)) + C(1+\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}) \\
&\qquad\quad + C\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})} + C\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})},
\end{aligned} \tag{4.21}$$

since $\nu \le 1$.

Recall that

$$Y(t) = \int_0^t \|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}, Y'(t) = \|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}, Y''(t) = \frac{d}{dt}\|\Omega_3^\nu\|^2_{L^2(\mathcal{D})}, \tag{4.22}$$

so that (4.21) implies that

$$Y''(t) \le C(1+\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})}Y(t)) + CY'(t) + C\|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})}. \tag{4.23}$$

Integrating (4.23) from 0 to t and using $Y(0) = 0$ and $Y'(0) = \|\Omega_{3,0}^\nu\|_{L^2(\mathcal{D})}$, we obtain

$$\begin{aligned}
&Y'(t) - Y'(0) \\
&\quad\le C\left(t + Y(t)\int_0^t \|\nabla\eta^\nu(s)\|^2_{L^2(\mathcal{D})}\,ds\right) + C\left(Y(t) + \int_0^t \|\nabla\eta^\nu\|^2_{L^2(\mathcal{D})}\,ds\right).
\end{aligned} \tag{4.24}$$

By virtue of (4.2), it follows that $\|\nabla\eta^\nu\|_{L^2(0,t;L^2(\mathcal{D}))} \le c$, for a constant $c = c(T, \|\eta_0^\nu\|_{L^2(\mathcal{D})}, \|\mathbf{u}_0^\nu\|_{L^2(\mathcal{D})}) > 0$. Then (4.24) becomes

$$Y'(t) - CY(t) \le Y'(0) + Ct + C. \tag{4.25}$$

Consequently,

$$Y(t) \le C(1+Y'(0))e^{-Ct},$$

i.e.,

$$\|\Omega_3^\nu\|_{L^2(0,T;L^2(\mathcal{D}))} \le C(1+\|\Omega_{3,0}^\nu\|_{L^2(\mathcal{D})}), \tag{4.26}$$

where $C > 0$ depends on T.

Combining (4.26) with (4.25), we get

$$\|\Omega_3^\nu\|_{L^\infty(0,T;L^2(\mathcal{D}))} \le c, \tag{4.27}$$

where the constant c only depends on T, $\|\Omega_{3,0}^\nu\|_{L^2(\mathcal{D})}$ and $\|\mathbf{u}_0^\nu\|_{H^1_{per}(\mathcal{D})}$, but not on ν. □

We are now ready to prove our main result.

Proof of Theorem 2.8. The proof will proceed in three broad steps. First we will show that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a compact subset of $L^2(0,T;L^2(\mathcal{D}))$. Then we will pass to subsequences as needed and show that there is a limit, $\mathbf{u}^0$, which is in $C(0,T;L^2)\cap L^2(0,T;H^1_{per,loc})$, which is helical, has vanishing helical swirl, and satisfies the weak formulation of the Euler equations. Finally, we will show that $\mathbf{u}^0\in L^\infty(0,T;H^1_{per,loc})$.

Recall (2.13) and (2.15). Then,

$$\operatorname{div}\mathbf{u}^\nu = 0 \tag{4.28}$$

$$\operatorname{curl}\mathbf{u}^\nu = \left[\Omega_3^\nu - \partial_x\left(\frac{\eta^\nu\, x}{|\boldsymbol{\xi}|^2}\right) - \partial_y\left(\frac{\eta^\nu\, y}{|\boldsymbol{\xi}|^2}\right)\right]\boldsymbol{\xi} + (\partial_y\eta^\nu, -\partial_x\eta^\nu, 0). \tag{4.29}$$

Condition (1) of Theorem 2.8 implies that $\|\mathbf{u}_0^\nu\|_{H^1_{per}(\mathcal{D})}\le C$ and, hence, from the energy inequality in Theorem 3.2, (3.6), it follows that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a bounded subset of $L^\infty(0,T;L^2(\mathcal{D}))$.

From condition (2) of Theorem 2.8 together with Lemma 4.2, (4.9), we find

$$\eta^\nu \to 0 \text{ strongly in } L^\infty(0,T;L^2(\mathcal{D})), \tag{4.30}$$

and

$$\nabla\eta^\nu \to 0 \text{ strongly in } L^2(0,T;L^2(\mathcal{D})). \tag{4.31}$$

In addition, from Lemma 4.2 we obtained a uniform estimate, with respect to ν, for Ω_3^ν in $L^\infty(0,T;L^2(\mathcal{D}))$. Putting these estimates together yields $\operatorname{curl}\mathbf{u}^\nu$ uniformly bounded in $L^2(0,T;L^2_{loc}(\mathcal{D}))$. (The subscript 'loc' is due to the growth of $\boldsymbol{\xi}$ at infinity.) Hence, from Lemma 2.7 it follows that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a bounded subset of $L^2(0,T;H^1_{per,loc}(\mathcal{D}))$.

Therefore, for any bounded sub-domain $\mathcal{U}\subset\mathcal{D}$, we have that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a bounded subset of $L^2(0,T;H^1(\mathcal{U}))$. In addition, we may use equation (1.1) to deduce that $\{\partial_t\mathbf{u}^\nu\}_{\nu>0}$ is a bounded subset of $L^2(0,T;H^{-1}(\mathcal{U}))$. It follows from the Aubin-Lions compactness theorem, see [13], that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a compact subset of $L^2(0,T;L^2(\mathcal{U}))$. We may now use a diagonal argument to pass to a subsequence, which we will not relabel, which converges strongly in $L^2([0,T];L^2_{loc}(\mathcal{D}))$. Passing to a further subsequence if needed, we may assume the convergence is also weak in $L^2(0,T;H^1_{per,loc}(\mathcal{D}))$.

It is standard that strong convergence in $L^2([0,T];L^2_{loc}(\mathcal{D}))$ is sufficient to show that the limit vector field, denoted $\mathbf{u}^0$, satisfies the weak formulation of the Euler equations in Definition 2.2.

The bounds in $L^2(0,T;H^1_{per,loc}(\mathcal{D}))$, for $\mathbf{u}^\nu$, and in $L^2(0,T;H^{-1}_{loc}(\mathcal{D})$, for $\partial_t\mathbf{u}^\nu$, imply that $\{\mathbf{u}^\nu\}_{\nu>0}$ is a bounded subset of $C^0(0,T;L^2(\mathcal{D}))$. (There is no need to localize this estimate, due to the previous uniform estimate in $L^\infty(0,T;L^2(\mathcal{D}))$.) It follows that $\mathbf{u}^0\in C^0(0,T;L^2(\mathcal{D}))$.

It is easy to see that $\mathbf{u}^0(\cdot,t)$ is a helical vector field, for each $0 \leq t < T$ and, also, that $\eta^0 \equiv \mathbf{u}^0 \cdot \boldsymbol{\xi} = 0$.

We have established all conditions of Definition 2.2 but one. It remains only to verify that $\mathbf{u}^0 \in L^\infty(0,T;H^1_{per,loc}(\mathcal{D}))$.

To see this we first note that, in view of (4.29),

$$\Omega_3^\nu = \operatorname{curl}\mathbf{u}^\nu \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} + \partial_x\left(\frac{\eta^\nu\, x}{|\boldsymbol{\xi}|^2}\right) + \partial_y\left(\frac{\eta^\nu\, y}{|\boldsymbol{\xi}|^2}\right) - (\partial_y\eta^\nu, -\partial_x\eta^\nu, 0)\cdot\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}.$$

Each of the terms on the right-hand-side converges, as $\nu \to 0$, weakly in $L^2(0,T;L^2(\mathcal{D}))$ and, in view of (4.30) and (4.31), the weak limit, in $L^2(0,T;L^2(\mathcal{D})$, of the right-hand-side is $\operatorname{curl}\mathbf{u}^0 \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}$.

In addition, since Ω_3^ν is bounded in $L^\infty(0,T;L^2(\mathcal{D}))$, we may assume, passing to further subsequences as needed, that the convergence of the right-hand-side is also weak-$*$ in $L^\infty(0,T;L^2(\mathcal{D}))$, so that

$$\operatorname{curl}\mathbf{u}^0 \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2} \in L^\infty(0,T;L^2(\mathcal{D})).$$

By virtue of $\eta^0 = 0$ and $\mathbf{u}^0$ being a helical vector field, we find $\operatorname{curl}\mathbf{u}^0 \equiv \boldsymbol{\omega}^0 = \omega_3^0\boldsymbol{\xi}$, see Lemma 2.4 and Remark 2.1. Therefore we deduce

$$\operatorname{curl}\mathbf{u}^0 \in L^\infty(0,T;L^2_{loc}(\mathcal{D})),$$

which, together with $\operatorname{div}\mathbf{u}^0 = 0$, imply

$$\mathbf{u}^0 \in L^\infty(0,T;H^1_{per,loc}(\mathcal{D})),$$

as desired.

This completes the proof.

□

In this article we have focused on the vanishing viscosity limit for helically symmetric flows. As we have discussed, helically symmetric solutions of the Navier-Stokes equations do not form singularities in finite time, whereas helical Euler is only known to have global solutions if the helical swirl vanishes. Furthermore, vanishing helical swirl is preserved by the Euler evolution, but not by Navier-Stokes. Given these distinctions, it seemed natural to explore the vanishing viscosity problem under helical symmetry. The key issue was to be able to control the helical swirl and to ensure that it vanishes as $\nu \to 0$.

The relevant problem which still remains open, in this direction, is global existence for helical Euler with nonzero helical swirl.

Acknowledgments: Quansen Jiu is partially supported by NSFC (No.11671273). The research of M. C. Lopes Filho was supported in part by CNPq Grant #306886/2014-6. The research of H. J. Nussenzveig Lopes was partially supported by CNPq Grant #307918/2014-9, and FAPERJ Grant #E-26/202.950/2015. The research of Dongjuan Niu was supported in part by FAPESP Grant #2008/09473-0; Dongjuan Niu is partially supported by NSFC (No.11471220) and by the Beijing Municipal Commission of Education grants (No. KM201610028001). This material is based upon work supported by the National Science Foundation under Grant No. DMS-1439786 while the second and fourth authors were in residence at the Institute for Computational and Experimental Research in Mathematics (ICERM) in Providence, RI, during the Spring 2017 semester. The authors thank Anna Mazzucato for useful discussions.

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Highlights

- Vanishing viscosity for flows with helical symmetry
- The effect of viscosity on the helical swirl
- Decomposition of helical vector fields parallel and orthogonal to the helix direction