



Coherent sets for nonautonomous dynamical systems

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ABSTRACT

We describe a mathematical formalism and numerical algorithms for identifying and tracking slowly mixing objects in nonautonomous dynamical systems. In the autonomous setting, such objects are variously known as almost-invariant sets, metastable sets, persistent patterns, or strange eigenmodes, and have proved to be important in a variety of applications. In this current work, we explain how to extend existing autonomous approaches to the nonautonomous setting. We call the new time-dependent slowly mixing objects *coherent sets* as they represent regions of phase space that disperse very slowly and remain coherent. The new methods are illustrated via detailed examples in both discrete and continuous time.

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1. Introduction

The study of transport and mixing in dynamical systems has received considerable attention in the last two decades; see e.g. [1–4] for discussions of transport phenomena. In particular, the detection of very slowly mixing objects, known variously as almost-invariant sets, metastable sets, persistent patterns, or strange eigenmodes, has found wide application in fields such as fluid dynamics [5–7], ocean dynamics [8,9], astrodynamics [10], and molecular dynamics [11,12]. A shortcoming of this prior work, based around eigenfunctions of Perron–Frobenius operators (or transfer operators, or evolution operators) is the restriction to autonomous systems or periodically forced systems. In this work, we extend the notions of almost-invariant sets, metastable sets, persistent patterns, and strange eigenmodes to time-dependent *Lagrangian coherent sets*. These coherent sets form a time parameterised family of sets that approximately follow the flow and disperse very slowly; in other words *they stay coherent*. Coherent sets are the natural nonautonomous analogue to almost-invariant sets.

The standard dynamical system model of transport assumes that the motion of passive particles is completely determined

by either an autonomous or a time-dependent vector field. Traditional approaches to understanding transport are based upon the determination of the location of geometric objects such as invariant manifolds. In the autonomous setting, an invariant manifold of one dimension less than the ambient space will form an impenetrable transport barrier that locally partitions the ambient space. In the periodically forced setting, primarily in two-dimensional flows, it has been shown that slow mixing in the neighbourhood of invariant manifolds is sometimes controlled by “lobe dynamics” [13,14,2]. In the truly nonautonomous, or aperiodically forced setting, finite time hyperbolic material lines [15] and surfaces [16] have been proposed as generalisations of invariant manifolds that form barriers to mixing. These material lines and surfaces are known as *Lagrangian coherent structures*; see also [17] for an alternative definition. The geometric approach can often be used to find co-dimension 1 sets (coherent structures) that form boundaries of coherent sets.

An alternative to the geometric approach is the ergodic-theoretic approach, which attempts to locate almost-invariant sets (or metastable sets) directly, rather than inferring their location indirectly from their boundaries. The basic tool is the Perron–Frobenius operator (or transfer operator). Real eigenvalues of this operator close to 1 correspond to eigenmodes that decay at slow (exponential) rates. Almost-invariant sets are heuristically determined from the corresponding eigenfunctions f as sets of the form $\{f > c\}$ or $\{f < c\}$ for thresholds $c \in \mathbb{R}$. Such an approach arose in the context of smooth autonomous maps and flows on

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subsets of \mathbb{R}^d [18,19] about a decade ago. Further theoretical and computational extensions have since been constructed [20–22]. A parallel series of work specific to time-symmetric Markov processes and applied to identifying molecular conformations was developed in [23,24,11] and surveyed in [12].

There have been some recent studies of the connections between slow mixing in *periodically* driven fluid flow and eigenfunctions of Perron–Frobenius operators. Liu and Haller [6] observe via simulation a transient “strange eigenmode” as predicted by classical Floquet theory. Pikovsky and Popovych [5,7] numerically integrated an advection–diffusion equation to simulate the evolution of a passive scalar, observing that it is the sub-dominant eigenfunction of the Perron–Frobenius operator that describes the most persistent deviation from the unique steady state.

The Perron–Frobenius operator based approach has been successful in a variety of application areas, however, as the key mathematical object is an *eigenfunction*, there is no simple extension of the method to systems that have *nonperiodic* time-dependence.¹

Indeed, Liu and Haller [6] state that:

“...strange eigenmodes may also be viewed as eigenfunctions of an appropriate Frobenius–Perron operator.... This fresh approach offers an alternative view on scalar mixing, but leaves the questions of completeness and general time-dependence open.”

It is this question of general time-dependence that we address in the current work. We extend a standard formalism for random dynamical systems to the level of Perron–Frobenius operators to create a Perron–Frobenius operator framework for general time-dependence. We also state an accompanying numerical algorithm, and demonstrate its effectiveness in identifying strange eigenmodes and coherent sets.

An outline of the paper is as follows. In Section 2 we formalise the notions of nonautonomous systems in both discrete and continuous time. In Section 3 we describe a Galerkin projection method that we will use to produce finite matrix representations of Perron–Frobenius operators. In Section 4 we define the critical constructions for the nonautonomous setting. We show that the nonautonomous analogues of strange eigenmodes are described by the “Oseledets subspaces” or “Lyapunov vectors” corresponding to compositions of the projected Perron–Frobenius operators. In Section 5 we describe in detail a numerical algorithm to practically compute these slowly decaying modes, and demonstrate that in the continuous time setting, these modes vary continuously in time. Our numerical approach is illustrated firstly in the discrete time setting with an aperiodic composition of interval maps, and secondly in the continuous time setting with an aperiodically forced flow on a cylinder. Section 6 provides some further background on almost-invariant sets and coherent sets and Section 7 describes a new heuristic to extract coherent sets from slowly decaying modes in the nonautonomous setting. This heuristic is then illustrated using the examples from Section 5.

2. Nonautonomous dynamical systems

We will treat time-dependent dynamical systems on a smooth compact d -dimensional manifold $M \subset \mathbb{R}^D$, $D \geq d$ in both discrete and continuous time. In order to keep track of “time” we use a probability space $(\Omega, \mathcal{H}, \mathbb{P})$, with the passing of time controlled by an ergodic automorphism $\theta : \Omega \rightarrow \Omega$ preserving \mathbb{P} (i.e., $\mathbb{P} = \mathbb{P} \circ \theta^{-t}$ for all $t \geq 0$). We require this somewhat more complicated description of time for technical reasons: to run the ergodic-theoretic arguments in Theorem 1 that guarantee the existence of the nonautonomous analogues of strange eigenmodes. The requirement that \mathbb{P} be an ergodic probability measure rules out obvious choices for Ω and θ : (i) in discrete time, $\Omega = \mathbb{Z}$ and $\theta^s(t) = t + s$, and (ii) in continuous time, $\Omega = \mathbb{R}$ and $\theta^s(t) = t + s$. In both (i) and (ii), there is no ergodic probability measure on Ω preserved by θ . In the next two sections, we will introduce suitable examples of Ω and θ and describe the nonautonomous systems they generate.

2.1. Discrete time-maps

In the discrete time setting, we will think of $\Omega \subset (\mathbb{Z})^{\mathbb{Z}}$, and θ as a left shift σ on Ω defined by $(\sigma\omega)_i = \omega_{i+1}$, where $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) \in \Omega$. We assume that σ is ergodic with respect to \mathbb{P} . Let $\mathcal{T} = \{T_{\omega_0}\}_{\omega_0 \in \mathbb{Z}}$ be a collection of (possibly non-invertible) piecewise differentiable maps on a compact manifold M . For brevity, we will sometimes write T_ω in place of T_{ω_0} . We will define a nonautonomous dynamical system by map compositions of the form $T_{\sigma^{k-1}\omega} \circ \dots \circ T_{\sigma\omega} \circ T_\omega$. Define

$$\Phi(k, \omega, x) := \begin{cases} T_{\sigma^{k-1}\omega} \circ \dots \circ T_{\sigma\omega} \circ T_\omega(x), & k > 0; \\ \text{Id}, & k = 0; \\ T_{\sigma^{-k}\omega}^{-1} \circ \dots \circ T_{\sigma^{-2}\omega}^{-1} \circ T_{\sigma^{-1}\omega}^{-1}(x), & k < 0. \end{cases}$$

For $k \geq 0$ (resp. $k < 0$), $\Phi(k, \omega, x)$ represents the forward time (resp. backward time) k -fold application of the nonautonomous dynamics to the point x initialised at “time” ω . Whenever T_ω is non-invertible, $T_\omega^{-1}(x)$ will represent the finite set of all preimages of x . We call Φ a *map cocycle*.

Definition 1. Endow M with the Borel σ -algebra and let μ be a probability measure on M . We call μ an *invariant measure* if $\mu \circ \Phi(-1, \omega, \cdot) = \mu$ for all $\omega \in \Omega$.

This definition of an invariant measure is stricter than is usual for random or nonautonomous dynamical systems (e.g. [25, Definition 1.4.1]). More generally, one may allow sample measures $\mu = \mu_\omega$ and insist that $\mu_{\sigma^{-1}\omega} \circ \Phi(-1, \omega, \cdot) = \mu_\omega$ for all $\omega \in \Omega$.

Example 1 (Aperiodic Map Cocycle). We construct a map cocycle Φ by the composition of maps T_i from a collection \mathcal{T} according to sequences of indices $\omega \in \Omega$. The collection $\mathcal{T} := \{T_1, T_2, T_3, T_4\}$ consists of expanding maps of the circle S^1 , which we think of as $[0, 1]$ with endpoints identified. The sequence space $\Omega \subset \{1, 2, 3, 4\}^{\mathbb{Z}}$ is given by

$$\Omega = \{\omega \in \{1, 2, 3, 4\}^{\mathbb{Z}} : \forall i \in \mathbb{Z}, M_{\omega_i \omega_{i+1}} = 1\},$$

with adjacency matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Elements of Ω correspond to bi-infinite paths in the graph Fig. 1. The shift $\sigma : \Omega \rightarrow \Omega$ is a subshift of finite type. A Borel σ -algebra \mathcal{H} is generated by the length-one cylinder sets $C_i = \{\omega : \omega_0 = i\}$,

¹ A relevant analogy to see this is the following. Consider repeated application of a single matrix A . The eigenvectors of A provide information on directions of exponential growth/decay specified by the corresponding eigenvalues. Similarly, the eigenvectors of a product of matrices $A_k \dots A_2 A_1$ describe directions of exponential growth/decay, specified by the eigenvalues of the product, under repeated application of this matrix product. However, the directions of exponential growth/decay under a *non-repeating* product $\dots A_k \dots A_2 A_1$ cannot in general be found as eigenvectors of some matrix.

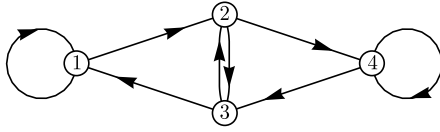


Fig. 1. Graph of the sequence space Ω .

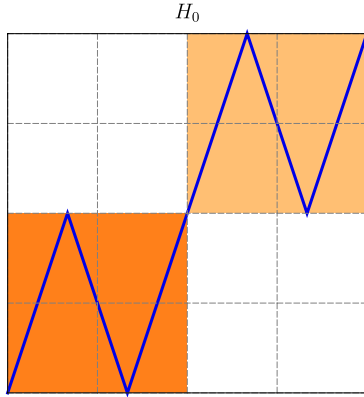


Fig. 2. The map H_0 has invariant sets $[0, 0.5]$ and $[0.5, 1]$; that is, $H_0^{-1}([0, 0.5]) = [0, 0.5]$ and $H_0^{-1}([0.5, 1]) = [0.5, 1]$.

$i = 1, \dots, 4$, and by giving equal measure to these four cylinder sets, we generate a shift-invariant probability measure \mathbb{P} .

The maps of \mathcal{T} are defined in terms of a continuous piecewise linear map $H_a : S^1 \rightarrow S^1$, which has almost-invariant sets (see Definition 5, Section 6) $[0, 0.5]$ and $[0.5, 1]$ for a close to zero. Define

$$H_a(x) = \begin{cases} +3x - a & 0 \leq x < \frac{1}{6} + \frac{2}{3}a, \\ -3x + 3a + 1 & \frac{1}{6} + \frac{2}{3}a \leq x < \frac{1}{3} + \frac{1}{2}a, \\ +3x - 1 & \frac{1}{3} + \frac{1}{2}a \leq x < \frac{2}{3} + \frac{1}{2}a, \\ -3x + 3a + 3 & \frac{2}{3} + \frac{1}{2}a \leq x < \frac{5}{6} + \frac{2}{3}a, \\ +3x - 2 - a & \frac{5}{6} + \frac{2}{3}a \leq x \leq 1, \end{cases}$$

where values are taken modulo 1. Fig. 2 shows a graph of H_0 . Let $a_i \in \mathbb{R}$, $i = 1, \dots, 4$, be close to zero, for example $(a_1, a_2, a_3, a_4) = (\pi, 2\sqrt{2}, \sqrt{3}, e)/40$. We now construct the map T_i from H_{a_i} , for $i = 1, \dots, 4$ as follows:

$$T_1 = H_{a_1}(x)$$

$$T_2 = R \circ H_{a_2}(x)$$

$$T_3 = H_{a_3} \circ R^{-1}$$

$$T_4 = R \circ H_{a_4} \circ R^{-1},$$

where $R : S^1 \rightarrow S^1$ is the rotation $R(x) = x + 1/4 \pmod{1}$; see Fig. 3.

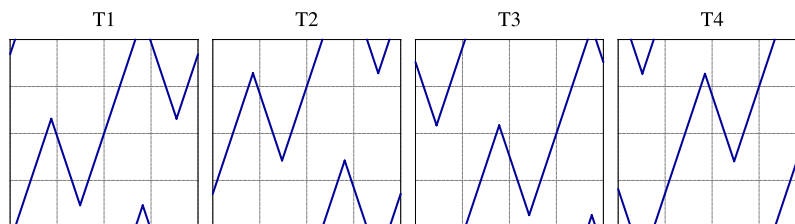


Fig. 3. Graphs of T_i for $i = 1, \dots, 4$.

Let m denote normalised Lebesgue measure on M . To each map T_ω we associate a Perron–Frobenius operator $\mathcal{P}_\omega : L^1(M, m) \rightarrow L^1(M, m)$ defined by $\mathcal{P}_\omega f = \sum_{y \in T_\omega^{-1}x} f(y) / |\det DT_\omega(y)|$. The operator \mathcal{P}_ω is a linear operator that acts on integrable functions in analogy to the action of T_ω on points. If $f \in L^1(M, m)$ represents a density function for an ensemble of initial conditions, then $\mathcal{P}_\omega f$ represents the density function of the ensemble after the action of T_ω has been applied to the ensemble. The map cocycle Φ naturally generates a Perron–Frobenius cocycle $\mathcal{P}_\omega^{(k)} = \mathcal{P}_{\sigma^{k-1}\omega} \circ \dots \circ \mathcal{P}_{\sigma\omega} \circ \mathcal{P}_\omega$. This composition of k Perron–Frobenius operators capture the action on a function f after k iterations of the nonautonomous system.

2.2. Continuous time-flows

Let $F : \Omega \times M \rightarrow \mathbb{R}^d$ be a sufficiently regular vector field. More precisely, we suppose that F satisfies the conditions of [25, Theorem 2.2.2], which will guarantee the existence of a classical solution of the nonautonomous ODE $\dot{x}(t) = F(\theta^t \omega, x(t))$, $t \in \mathbb{R}$.

To be concrete about the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in the continuous time setting, we may set $\Omega = \mathcal{E} \subset \mathbb{R}^{d_1}$, $d_1 \geq 3$, and consider an autonomous ODE $\dot{z} = g(z)$ on \mathcal{E} . Denote the flow for this ODE by $\xi : \mathbb{R} \times \mathcal{E} \rightarrow \mathcal{E}$ and suppose that ξ preserves the probability measure \mathbb{P} ; that is, $\mathbb{P} \circ \xi(-t, \cdot) = \mathbb{P}$ for all $t \in \mathbb{R}$. Thus, the autonomous, aperiodic flow ξ drives the nonautonomous ODE

$$\dot{x}(t) = F(\theta^t \omega, x(t)) = F(\xi(t, z), x(t)). \quad (1)$$

We think of points $z \in \mathcal{E}$ as representing generalized time. We assume that $(\mathcal{E}, \xi, \mathbb{P})$ is ergodic in the sense that if $\xi(-t, \tilde{\mathcal{E}}) = \tilde{\mathcal{E}}$ for some $\tilde{\mathcal{E}} \subset \mathcal{E}$ and for all $t \geq 0$ then $\mathbb{P}(\tilde{\mathcal{E}}) = 0$ or 1.

Denote by $\phi : \mathbb{R} \times \mathcal{E} \times M \rightarrow M$ the flow for (1). The flow ϕ satisfies $\frac{d}{dt} \phi(t, z, x) = F(\xi(t, z), \phi(t, z, x))$.

Definition 2. Endow M with the Borel σ -algebra and let μ be a probability measure on M . We call μ an *invariant measure* if $\mu \circ \phi(-t, z, \cdot) = \mu$ for all $z \in \mathcal{E}$ and $t \in \mathbb{R}$.

Remark 1. In Definition 2 we are insisting that μ is preserved at all “time instants”. As in the discrete time setting, more generally one may allow $\mu = \mu_z$ and insist that $\mu_{\xi(-t, z)} \circ \phi(-t, z, \cdot) = \mu_z$. However, as we will soon begin to focus on coherent sets rather than invariant measures, we will restrict the invariant measure to a “time independent” measure for clarity of presentation. This is perfectly reasonable for one of the main applications we have in mind, namely, aperiodically driven fluid flow where $\mu \equiv$ Lebesgue, and volume is preserved by the flow at all times.

Example 2. Consider the following nonautonomous system on a cylinder $M = S^1 \times [0, \pi]$. Let $\xi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ denote the flow for the driving system generated by the Lorenz system of ODEs (2)–(4) with standard parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$.

$$\dot{z}_1 = \sigma(z_2 - z_1)/\tau \quad (2)$$

$$\dot{z}_2 = (\rho z_1 - z_2 - z_1 z_3)/\tau \quad (3)$$

$$\dot{z}_3 = (-\beta z_3 + z_1 z_2)/\tau. \quad (4)$$

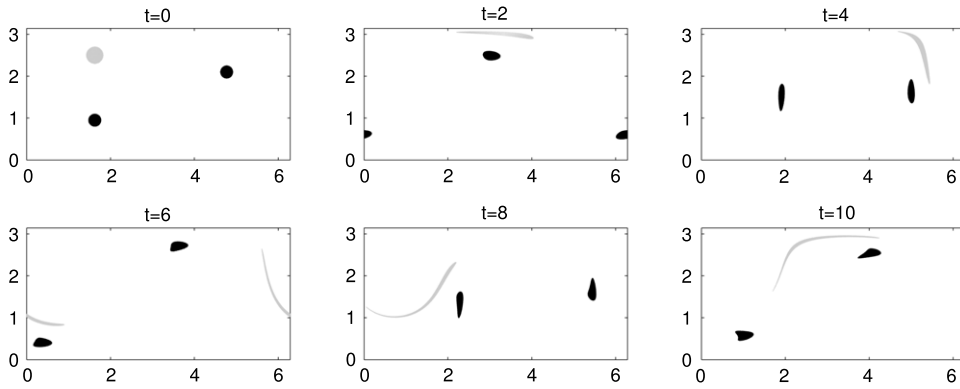


Fig. 4. Trajectory of the time-dependent systems (5)–(6) driven by the Lorenz system at generalized times $\xi(t, z)$.

It is well known that this Lorenz flow possesses an SBR measure \mathbb{P} [26]. Let the time-dependent vector field $F : \mathbb{R} \times S^1 \times [0, \pi] \rightarrow S^1 \times [0, \pi]$, where we now think of S^1 as the interval $[0, 2\pi]$ with the endpoints identified, generate our nonautonomous ODE $(\dot{x}(t), \dot{y}(t)) = F(\xi(t, z), x(t), y(t))$. Explicitly,

$$\dot{x} = c - A \sin(x - \nu z_1(t)) \cos(y) \pmod{2\pi} \quad (5)$$

$$\dot{y} = A \cos(x - \nu z_1(t)) \sin(y), \quad (6)$$

with $c = 0.5$, $A = 1$, $\nu = 0.25$. We set initial condition $z(0) = (0, 1, 1.5)$ and take the z_1 -coordinate of the Lorenz driving system to represent the generalized time for the vector field $F(\xi(t, z), x(t), y(t))$. We use a scaling factor of $\tau = 6.6685$ so that the temporal and spatial variation of $z_1(t)$ is similar to that of the “actual” time t . Since $F(\xi(t, z), x, y)$ is differentiable and bounded on M for all t , classical solutions of the nonautonomous ODE (5)–(6) exist. The system (2)–(6) uniquely generates an RDS, see [25, Theorem 2.2.2]. In Fig. 4 we demonstrate a trajectory of three different initial points.

We may define a family of Perron–Frobenius operators as $\mathcal{P}_z^{(t)} f(x) = f(\phi(-t, \xi(t, z), x)) \cdot |\det D\phi(-t, \xi(t, z), x)|$ for $t \geq 0$. This family is a semigroup in t as $\mathcal{P}_z^{(t_1+t_2)} f = \mathcal{P}_{\xi(t_1, z)}^{(t_2)} \mathcal{P}_z^{(t_1)} f$.

3. Galerkin projection and matrix cocycles

Let $\mathcal{B}_n = \text{sp}\{\chi_{B_i} : B_i \in \mathfrak{B}\}$ where $\mathfrak{B} = \{B_1, \dots, B_n\}$ is a partition of M into connected sets of positive Lebesgue measure. Define a projection $\pi_n : L^1(M, m) \rightarrow \mathcal{B}_n$ as

$$\pi_n f = \sum_{i=1}^n \frac{\int_{B_i} f \, dm}{m(B_i)} \chi_{B_i}. \quad (7)$$

Following Ulam [27], hereafter we will consider the finite rank operators $\pi_n \mathcal{P}_\omega^{(1)} : L^1(M, m) \rightarrow \mathcal{B}_n$ and $\pi_n \mathcal{P}_z^{(1)} : L^1(M, m) \rightarrow \mathcal{B}_n$, and the matrix representations of the restrictions of $\pi_n \mathcal{P}_\omega^{(1)}$ and $\pi_n \mathcal{P}_z^{(1)}$ to \mathcal{B}_n . We denote these matrix representations (under multiplication on the right) by $P(\omega)$ and $P(z)$. Extending Lemma 2.3 [28] in a straightforward way to the nonautonomous setting, one has

$$P(\omega)_{ij} = \frac{m(B_j \cap \Phi(-1, \sigma\omega, B_i))}{m(B_i)} \quad (8)$$

and

$$P(z)_{ij} = \frac{m(B_j \cap \phi(-1, \xi(1, z), B_i))}{m(B_i)}. \quad (9)$$

In particular, these matrices are numerically accessible.

Remark 2. Note we do not concern ourselves at all with the relationship between $\mathcal{P}_\omega^{(1)}$ and $\pi_n \mathcal{P}_\omega^{(1)}$; this is a subtle relationship and beyond the scope of this work. See [28–34] for work in this direction.

The matrices $P(\omega)$ and $P(z)$ generate matrix cocycles

$$P^{(k)}(\omega) := P(\sigma^{k-1}\omega) \cdots P(\sigma\omega) \cdot P(\omega) \quad (10)$$

and

$$P^{(k)}(z) := P(\xi(k-1, z)) \cdots P(\xi(1, z)) \cdot P(z). \quad (11)$$

4. Discretised Oseledets functions and the Multiplicative Ergodic Theorem

In periodically driven flows, Liu and Haller [6] and Pikovsky and Popovych [5], observed that certain tracer patterns persisted for long times before eventually relaxing to the equilibrium tracer distribution. Pikovsky and Popovych [5] recognised these patterns as graphs of eigenfunctions of a Perron–Frobenius operator corresponding to an eigenvalue $L < 1$. These eigenfunctions decay over time and the closer L is to 1, the slower the decay and the more slowly an initial tracer distribution will relax to equilibrium. We now develop a framework for the considerably more difficult aperiodic setting.

Consider some suitable Banach space $(\mathcal{F}, \|\cdot\|)$ of real valued functions; \mathcal{F} is the function class in which we search for slowly decaying functions. Suppose that the norm is chosen so that for each $\omega \in \Omega$ and $k \geq 0$, the operator $\mathcal{P}_\omega^{(k)}$ is Markov; that is, $\|\mathcal{P}_\omega^{(k)}\| = 1$ for all ω and $k \geq 0$. For $f \in \mathcal{F}$, we calculate the following limit:

$$\lambda(\omega, f) = \limsup_{k \rightarrow \infty} \frac{1}{k} \log \|\mathcal{P}_\omega^{(k)} f\|. \quad (12)$$

We refer to $\lambda(\omega, f) \leq 0$ as the *Lyapunov exponent* of f . If f decays under the action of the Perron–Frobenius operators at a geometric rate of r^k , $0 < r < 1$, then $\lambda(\omega, f) = \log r$. The closer r is to 1, the slower the decay. The extreme case of $r = 1$ (no decay) is exhibited when f is the density of the invariant measure μ that is common to all maps in our nonautonomous dynamical system. We define the *Lyapunov spectrum* $\Lambda(\mathcal{P}, \omega) := \{\lambda(\omega, f) : f \in \mathcal{F}\}$. In the aperiodic setting the new mathematical objects that are analogous to strange eigenmodes and persistent patterns will be called *Oseledets functions*.

Definition 3. *Oseledets functions* correspond to f for which (i) $\lambda(\omega, f)$ is near zero and (ii) the value $\lambda(\omega, f)$ is an isolated point in the *Lyapunov spectrum*.

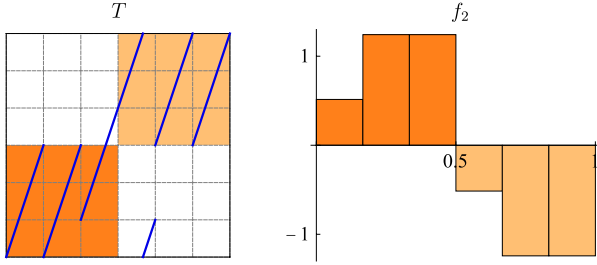


Fig. 5. Graph of T and Oseledets function f_2 .

By considering $(\mathcal{F}, \|\cdot\|) = (\mathcal{B}_n, \|\cdot\|_1)$, the actions of $\mathcal{P}_\omega^{(k)}$ and $\mathcal{P}_z^{(k)}$ are described by $P^{(k)}(\omega)$ and $P^{(k)}(z)$, respectively. We may replace $\mathcal{P}_\omega^{(k)}$ and $\mathcal{P}_z^{(k)}$ in (12) by $P^{(k)}(\omega)$ and $P^{(k)}(z)$, respectively, to obtain a standard setting where the possible values of $\lambda(\omega, f)$ are the Lyapunov exponents of cocycles of $n \times n$ matrices, and

$$\Lambda(P, \omega) := \left\{ \lim_{k \rightarrow \infty} \frac{1}{k} \log \|P^{(k)}(\omega)f\|_1 : f \in \mathcal{B}_n \right\} \quad (13)$$

and

$$\Lambda(P, z) := \left\{ \lim_{k \rightarrow \infty} \frac{1}{k} \log \|P^{(k)}(z)f\|_1 : f \in \mathcal{B}_n \right\}, \quad (14)$$

exist for \mathbb{P} almost-all $\omega \in \Omega$, and consist of at most n isolated points, $\lambda_n < \dots < \lambda_1 = 0$. Of particular interest to us is the function $f_2(\omega)$ (or $f_2(z)$) in \mathcal{B}_n , which represents the function that decays at the slowest possible geometric rate λ_2 .

Remark 3. In certain settings, this matrix cocycle *exactly* captures all large isolated Lyapunov exponents of the operator cocycle $\mathcal{P} : (\text{BV}, \|\cdot\|_{\text{BV}}) \rightarrow (\text{BV}, \|\cdot\|_{\text{BV}})$. One such setting is a map cocycle formed by composition of piecewise linear expanding maps with a common Markov partition $\mathfrak{B} = \{B_1, \dots, B_n\}$; see [35].

The following example illustrates the concept of Lyapunov spectrum and Oseledets functions in the familiar autonomous setting. For the remainder of this section, we adopt the discrete time notation of σ and ω .

Example 3 (“Autonomous” Single Map). In [36] individual maps are constructed for which the Perron–Frobenius operator has at least one non-unit isolated eigenvalue when acting on the Banach space $(\text{BV}, \|\cdot\|_{\text{BV}})$. A single autonomous map may be regarded as a cycle over a one-point space $\Omega = \{\omega\}$, and so we may drop the dependence on ω in notation. Keller [37] shows that for a piecewise expanding map T of the interval I , the spectrum of the associated Perron–Frobenius operator \mathcal{P} has an essential spectral radius $\rho_{\text{ess}}(\mathcal{P})$ equal to the asymptotic local expansion rate $\sup_{x \in I} \lim_{k \rightarrow \infty} |1/DT^k(x)|^{1/k}$, and that there are at most countably many spectral points, each isolated, of modulus greater than $\rho_{\text{ess}}(\mathcal{P})$. In order to have an isolated spectral point, we construct a map of S^1 which has an almost-invariant set (see Definition 5). The relation between almost-invariant sets and isolated eigenvalues was noted in [18]. Consider the partition $\mathfrak{B} = \{B_i : i = 1, \dots, 6\}$, where $B_i = ((i-1)/6, i/6)$. Given $a \in \mathbb{Z}^6$, any map $T : S^1 \rightarrow S^1$ defined by

$$T(x) = 3x - (i-1)/2 + a_i/6 \pmod{1}, \quad x \in B_i \quad (15)$$

is Markov with respect to \mathfrak{B} . Here we take $a = (0, 0, 1, 4, 3, 3)$; see Fig. 5. Notice that there is a low transfer of mass between the two intervals $[0, 1/2]$ and $[1/2, 1]$. Since \mathfrak{B} is a Markov partition for T , the space of characteristic functions $\mathcal{B}_6 = \{\chi_{B_i} : i = 1, \dots, 6\}$ is an invariant subspace of BV for the Perron–Frobenius operator \mathcal{P}

of T . Thus the action of $\mathcal{P}_\omega = \mathcal{P}$ on $\mathcal{F} = \mathcal{B}_6$ is represented by the matrix

$$P = P(\omega) = \frac{1}{3} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad (16)$$

which has non-zero eigenvalues $L_1 = 1$, $L_2 = (1 + \sqrt{2})/3$, $L_3 = (1 - \sqrt{2})/3$. The map T is piecewise affine with constant slope 3 and so the logarithm of the local expansion rate is $\log(1/3)$.

The eigenvalue $L_2 \approx 0.805$ of P thus gives rise to an isolated point $\lambda_2 \approx \log 0.805$ in the Lyapunov spectrum $\Lambda(\mathcal{P})$. The corresponding Oseledets function f_2 is given by $f_2(x) = \sum_{i=1}^6 w_{2,i} \chi_{B_i}(x)$, where w_2 is the eigenvector of P corresponding to the eigenvalue L_2 , see Fig. 5. Since $|L_3| \approx 0.138 < 1/3$, this means that $\log L_2$ is the unique isolated Lyapunov exponent in $\Lambda(\mathcal{P})$. Note that the set $\{f_2 > 0\}$ corresponds to the set $[0, 1/2]$. We will discuss this property further in Section 7.

Example 4 (Periodic Map Cocycle). We construct a periodic map cocycle from a collection of maps with a common Markov partition. The map cocycle is formed by cyclically composing three maps of S^1 . Consider the sequence space $\Omega = \{\omega \in \{1, 2, 3\}^{\mathbb{Z}} : \forall i \in \mathbb{Z}, M_{\omega_i, \omega_{i+1}} = 1\}$ where $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We consider $\mathcal{T} = \{T_j : j = 1, 2, 3\}$, where T_j is given by (15) with parameter $a^{(j)}$, where

$$\begin{aligned} a^{(1)} &= (3, 2, 2, 0, 5, 5), & a^{(2)} &= (2, 1, 4, 5, 4, 1), \\ a^{(3)} &= (1, 3, 3, 4, 0, 0), \end{aligned}$$

see Fig. 6. As in Example 3 we look for Lyapunov exponents that are strictly greater than the logarithm of the asymptotic local expansion rate

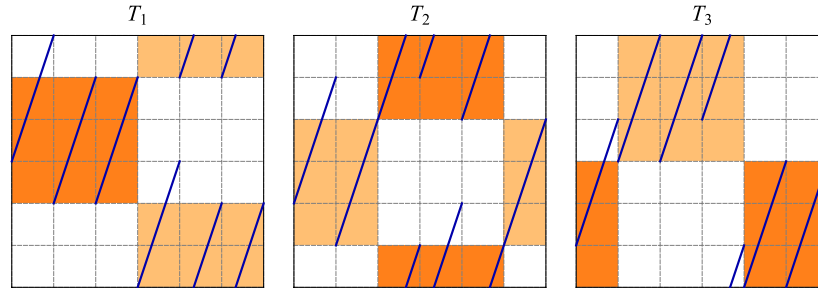
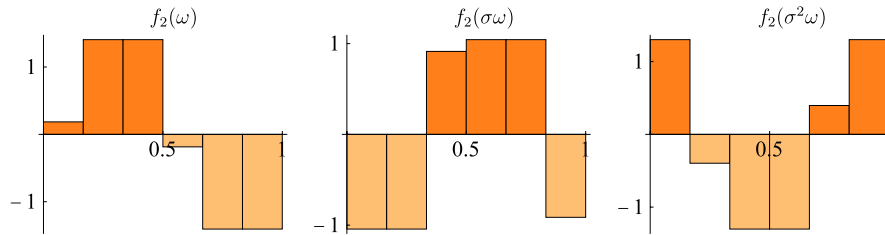
$$\sup_{x \in I} \lim_{k \rightarrow \infty} |1/D(T_3 \circ T_2 \circ T_1)^k|^{1/3k}. \quad (17)$$

As each map T_j is piecewise affine with constant slope 3, the logarithm of the local expansion rate is $\log(1/3)$. Note also that T_1 approximately maps $[0, 1/2]$ to $[1/3, 5/6]$, T_2 then maps $[1/3, 5/6]$ approximately to $[0, 1/6] \cup [2/3, 1]$, and finally T_3 maps $[2/3, 1/3]$ approximately back to $[0, 1/2]$. Each map T_j leaves the space \mathcal{B}_6 from Example 3 invariant, and thus the Perron–Frobenius operator \mathcal{P}_j of T_j restricted to \mathcal{B}_6 has matrix representation P_j , where $3P_j, j = 1, 2, 3$, are respectively

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}, \quad (18)$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The triple product $P^{(3)}(\omega) = P_3 P_2 P_1$ has non-zero eigenvalues $L_1 = 1$, $L_2 = (13 + \sqrt{233})/54$ and $L_3 = (13 - \sqrt{233})/54$. Since $L_2 \approx 0.523$, its associated eigenvector w_2 satisfies $\lambda(\omega, w_2) = \log \sqrt[3]{L_2} > \log(1/3)$. Since $P^{(3)}(\sigma^k \omega)$, $k = 1, 2$, are cyclic

Fig. 6. Graphs of T_1 , T_2 and T_3 .Fig. 7. Oseledets functions $f_2(\sigma^k \omega)$ for $k = 0, 1, 2$.

permutations of the factors of $P^{(3)}(\omega)$, they share the same eigenvalues, and in particular L_2 . Thus $(1/3) \log L_2$ is an isolated Lyapunov exponent of $\Lambda(\mathcal{P}, \omega)$ for each $\omega \in \Omega$. Associated to the eigenvalue L_2 , the matrices $P(\sigma^k \omega)$, $k = 0, 1, 2$, have corresponding eigenvectors $w_2(\sigma^k \omega)$. The three vectors $w_2(\sigma^k \omega)$, $k = 0, 1, 2$, generate the periodic Oseledets functions $f_2(\sigma^k \omega) = \sum_{i=1}^6 w_{2,i}(\sigma^{k \pmod{3}} \omega) \chi_{B_i}$, see Fig. 7. Note that the sets $\{f_2(\sigma^k \omega) > 0\}_{k=0,1,2}$ correspond to the sets $[0, 1/2]$, $[1/3, 5/6]$, and $[0, 1/6] \cup [2/3, 1]$, respectively. We will discuss this property further in Section 7. For another such example see [35]. See also [38] for a detailed example of similar calculations for a periodically driven flow.

In the nonautonomous setting, we can no longer easily construct Oseledets functions as eigenfunctions of a single operator, or eigenvectors of a single matrix. In fact, the Oseledets functions are themselves (aperiodically) time-dependent in the nonautonomous setting. Our model of Oseledets functions for nonautonomous systems is, as the name suggests, built around the Multiplicative Ergodic Theorem, see e.g. [25, Chapter 3, Section 4]. We now state a strengthened version [35] of the Multiplicative Ergodic Theorem that we require for our current purposes.

Theorem 1 ([35]). *Let σ be an invertible ergodic measure-preserving transformation of the space $(\Omega, \mathcal{H}, \mathbb{P})$. Let $P : \Omega \rightarrow M_n(\mathbb{R})$ be a measurable family of matrices satisfying*

$$\int \log^+ \|P(\omega)\| d\mathbb{P}(\omega) < \infty.$$

Then there exist $\lambda_1 > \lambda_2 > \dots > \lambda_\ell \geq -\infty$ and dimensions m_1, \dots, m_ℓ , with $m_1 + \dots + m_\ell = n$, and a measurable family of subspaces $W_j(\omega) \subseteq \mathbb{R}^n$ such that for almost every $\omega \in \Omega$ the following hold:

1. $\dim W_j(\omega) = m_j$;
2. $\mathbb{R}^n = \bigoplus_{j=1}^\ell W_j(\omega)$;
3. $P(\omega)W_j(\omega) \subseteq W_j(\sigma\omega)$ (with equality if $\lambda_j > -\infty$);
4. for all $v \in W_j(\omega) \setminus \{0\}$, one has

$$\lim_{k \rightarrow \infty} (1/k) \log \|P(\sigma^{k-1}\omega) \cdots P(\sigma\omega) \cdot P(\omega)v\| = \lambda_j.$$

The subspaces $W_j(\omega)$ are the general time-dependent analogues of the vectors w_2 and $w_2(\sigma^k \omega)$, $k = 0, 1, 2$ of Examples 3 and 4, respectively. We may explicitly construct a slowest decaying discrete Oseledets function as $f_2(\omega) := \sum_{i=1}^n w_{2,i}(\omega) \chi_{B_i}$, where $w_2(\omega) \in W_2(\omega)$. In what follows, for brevity we will often call $W_j(\omega)$ a subspace or a function, recognising its dual roles.

Remark 4. We remark that if $\ell \geq 2$, $m_2 = 1$, and $\lambda_2 > -\infty$, the family of vectors $\{f_2(\sigma^k \omega)\}_{k \geq 0}$ is the *unique*² (up to scalar multiples) family of vectors in \mathcal{B}_n with the properties that

1. $\lim_{k' \rightarrow \infty} (1/k') \log \|P^{(k')}(\omega) f_2(\sigma^k \omega)\|_1 = \lambda_2$, $k \geq 0$,
2. $\pi_n \mathcal{P}_\omega f_2(\sigma^k \omega) = \alpha_k f_2(\sigma^{k+1} \omega)$ for some $\alpha_k \neq 0$, $k \geq 0$.

Remark 5. Theorem 1 strengthens the standard version of the MET for one-sided time with non-invertible matrices (see e.g. [25, Theorem 3.4.1]) to obtain the conclusions of the two-sided time MET with invertible matrices (see e.g. [25, Theorem 3.4.11]). In [25, Theorem 3.4.1], the existence of only a *flag* $\mathbb{R}^n = V_1(\omega) \supset \dots \supset V_\ell(\omega)$ of Oseledets subspaces is guaranteed, while in [25, Theorem 3.4.11], the existence of a *splitting* $W_1(\omega) \oplus \dots \oplus W_\ell(\omega) = \mathbb{R}^n$ is guaranteed. Theorem 1 above demonstrates existence of an Oseledets *splitting* for two-sided time with *noninvertible* matrices. This is particularly important for our intended application as the projected Perron–Frobenius operator matrices are non-invertible. Recent further extensions [39] prove existence and uniqueness of Oseledets subspaces for cocycles of Lasota–Yorke maps.

5. Numerical approximation of Oseledets functions

In the autonomous and periodic settings we have seen in Examples 3 and 4 that the subspaces $W_2(\omega) = \text{sp}\{w_2(\omega)\}$ were

² Assume there is another family $\{w'_2(\sigma^k \omega)\}_{k \geq 0} \neq \{w_2(\sigma^k \omega)\}_{k \geq 0}$ (up to scalar multiples) with these properties. Then $w'_2(\sigma^k \omega) = \sum_{j=2}^\ell \alpha_{k,j} w_j(\sigma^k \omega)$ for some $\alpha_{k,j}$, $j = 2, \dots, \ell$, with $\alpha_{k,2} \neq 0$. WLOG assume $\alpha_{k,2}, \alpha_{k,j'} \neq 0$ for some $2 < j' \leq \ell$ and all $k \geq 0$, but that $\alpha_{k,j} = 0$ for all $j \neq 2, j'$ and all $k \geq 0$. Then $m_2 \geq 2$ in Theorem 1, a contradiction.

one-dimensional, and that the vectors $w_2(\omega)$ could be simply determined as eigenvectors of matrices. For truly nonautonomous systems (those that are aperiodically driven), the Oseledets splittings are difficult to compute. In this section we outline a numerical algorithm to approximate the $W_j(\omega)$ subspaces from [Theorem 1](#). The algorithm is based on the push forward limit argument developed in the proof of [Theorem 1](#). To streamline notation, we describe the discrete time and continuous time setting separately.

5.1. Discrete time

We first describe a simple and efficient method to construct the matrix $P(\omega)$ defined in [\(8\)](#).

Algorithm 1 (Approximation of $P(\sigma^{-k}\omega)_{ij}$, $0 \leq k \leq N$).

1. Partition the state space M into a collection of connected sets $\{B_1, \dots, B_n\}$ of small diameter.
2. Fix i, j , and k and create a set of Q test points $x_{j,1}, \dots, x_{j,Q} \in B_j$ that are uniformly distributed over B_j .
3. For each $q = 1, \dots, Q$ calculate $y_{j,q} = T_{\sigma^{-k}\omega} x_{j,q}$.
4. Set

$$P(\sigma^{-k}\omega)_{ij} = \frac{\#\{q : y_{j,q} \in B_i\}}{Q}. \quad (19)$$

We now describe how to use the matrices $P(\omega)$ to approximate the subspaces $W_j(\omega)$. An intuitive description of the ideas behind [Algorithm 2](#) immediately follows the algorithm statement.

Algorithm 2 (Approximation of Oseledets Subspaces $W_j(\omega)$ at $\omega \in \Omega$).

1. Construct the Ulam matrices $P^{(M)}(\sigma^{-N}\omega)$ and $P^{(N)}(\sigma^{-N}\omega)$ from [\(19\)](#) and [\(10\)](#) for suitable M and N . The number M represents the number of iterates over which one measures the decay, while the number N represents how many iterates the resulting “initial vectors” are pushed forward to better approximate elements of the $W_j(\omega)$.
2. Form
$$\Psi^{(M)}(\sigma^{-N}\omega) := (P^{(M)}(\sigma^{-N}\omega)^\top P^{(M)}(\sigma^{-N}\omega))^{1/2M}$$
 as an approximation to the standard limiting matrix
$$B(\sigma^{-N}\omega) := \lim_{M \rightarrow \infty} (P^{(M)}(\sigma^{-N}\omega)^\top P^{(M)}(\sigma^{-N}\omega))^{1/2M}$$
 appearing in the Multiplicative Ergodic Theorem (see e.g. [\[25, Theorem 3.4.1\(i\)\]](#)).
3. Calculate the orthonormal eigenspace decomposition of $\Psi^{(M)}(\sigma^{-N}\omega)$, denoted by $U_j^{(M)}(\sigma^{-N}\omega)$, $j = 1, \dots, \ell$. We are particularly interested in low values of j , corresponding to large eigenvalues L_j .
4. Define $W_j^{(M,N)}(\omega) := P^{(N)}(\sigma^{-N}\omega)U_j^{(M)}(\sigma^{-N}\omega)$ via the push forward under the matrix cocycle.
5. $W_j^{(M,N)}(\omega)$ is our numerical approximation to $W_j(\omega)$.

Here is the idea behind the above algorithm. If we choose M large enough, the eigenspace $U_j^{(M)}(\sigma^{-N}\omega)$ should be close to the limiting ($M \rightarrow \infty$) eigenspace $U_j(\sigma^{-N}\omega)$. Vectors in the eigenspace $U_j(\sigma^{-N}\omega)$ experience stretching at a rate close to L_j . Note that the eigenspace $U_j^{(M)}(\sigma^{-N}\omega)$ is the j th singular vector of the matrix $P^{(M)}(\sigma^{-N}\omega)$, which experiences a “per unit time” average stretching from time $-N$ to $-N + M$ of L_j . Choose some arbitrary $v \in U_j(\sigma^{-N}\omega)$ and write $v = \sum_{j'=j}^\ell w_{j'}$ with $w_{j'} \in W_{j'}(\sigma^{-N}\omega)$. Pushing forward by $P^{(N)}(\sigma^{-N}\omega)$ for large enough N will result in $\|P^{(N)}(\sigma^{-N}\omega)w_j\|$ dominating $\|P^{(N)}(\sigma^{-N}\omega)w_{j'}\|$ for $j < j' \leq \ell$. Thus, for large M and N we expect $W_j^{(M,N)}(\omega)$ to be close to $W_j(\omega)$.

Remarks 1.

1. [Theorem 1](#) states that $W_j^{(\infty,N)}(\omega) \rightarrow W_j(\omega)$ as $N \rightarrow \infty$.
2. This method may also be used to calculate the Oseledets subspaces for two-sided linear cocycles, and may be more convenient, especially for large n , than the standard method of intersecting the relevant subspaces of flags of the forward and backward cocycles.

The numerical approximation of the Oseledets subspaces has been considered by a variety of authors in the context of (usually invertible) nonlinear differentiable dynamical systems, where the linear cocycle is generated by Jacobian matrices concatenated along trajectories of the nonlinear system. Froyland et al. [\[40\]](#) approximate the Oseledets subspaces in invertible two-dimensional systems by multiplying a randomly chosen vector by $P^{(N)}(\sigma^{-N}\omega)$ (pushing forward) or $P^{(-N)}(\sigma^N\omega)$ (pulling back, where $P^{(-N)}(\sigma^N\omega) = P^{-1}(\omega) \dots P^{-1}(\sigma^{N-1}\omega)$). Trevisan and Pancotti [\[41\]](#) calculate eigenvectors of $\Psi^{(M)}(\omega)$ for the three-dimensional Lorenz flow, increasing M until numerical convergence of the eigenvectors is observed. Ershov and Potapov [\[42\]](#) use an approach similar to ours, combining eigenvectors of a $\Psi^{(M)}$ with pushing forward under $P^{(N)}$. Ginelli et al. [\[43\]](#) embed the approach of [\[40\]](#) in a QR-decomposition methodology to estimate the Oseledets vectors in higher dimensions. In the numerical experiments that follow, we have found our approach to work very well, with fast convergence in terms of both M and N .

5.2. Continuous time

As our practical computations are necessarily over finite time intervals, from now on, when dealing with continuous time systems, we will compute $P^{(k)}(z)$ as $\pi_n \mathcal{P}_z^{(k)}$ rather than as $P(\xi(k-1, z)) \dots P(\xi(1, z)) \cdot P(z)$. If the computation of $\pi_n \mathcal{P}_z^{(k)}$ can be done accurately (this will be discussed further in [Section 5.5](#)), then this representation should be closer to $\mathcal{P}_z^{(k)}$ as there are fewer applications of π_n .

We first describe a simple and efficient method to construct the matrix $P(\omega)$ defined in [\(8\)](#).

Algorithm 3 (Approximation of $P^{(M)}(\xi(-N, z))$, $N \geq 0$).

1. Partition the state space M into a collection of connected sets $\{B_1, \dots, B_n\}$ of small diameter.
2. Fix i, j , and z and create a set of Q test points $x_{j,1}, \dots, x_{j,Q} \in B_j$ that are uniformly distributed over B_j .
3. For each $q = 1, \dots, Q$ calculate $y_{j,q} = \phi(M, \xi(-N, z), x_{j,q})$.
4. Set

$$P^{(M)}(\xi(-N, z))_{ij} = \frac{\#\{q : y_{j,q} \in B_i\}}{Q}. \quad (20)$$

The flow time M should be chosen long enough so that most test points leave their partition set of origin, otherwise at the resolution given by the partition $\{B_1, \dots, B_n\}$, the matrix $P^{(M)}(\xi(-N, z))$ matrix will be too close to the $n \times n$ identity matrix. If the action of ϕ separates nearby points, as is the case for chaotic systems, clearly the longer the flow duration M , the greater Q should be in order to maintain a good representation of the images $\phi(M, \xi(-N, z), B_i)$ by the test points.

Algorithm 4 (Approximation of Oseledets Subspaces $W_j(z)$ at $z \in \mathcal{E}$).

1. Construct the Ulam matrices $P^{(M)}(\xi(-N, z))$ and $P^{(N)}(\xi(-N, z))$ from [\(20\)](#) for suitable M and N . The number M represents the flow duration over which rate of decay is measured, while the number N represents the duration over which the resulting “initial vectors” are pushed forward to better approximate elements of the $W_j(z)$.

2. Form

$$\Psi^{(M)}(\xi(-N, z)) := (P^{(M)}(\xi(-N, z)))^\top P^{(M)}(\xi(-N, z))^{1/2M}$$

as an approximation to the standard limiting matrix

$$B(\xi(-N, z)) := \lim_{M \rightarrow \infty} (P^{(M)}(\xi(-N, z)))^\top P^{(M)}(\xi(-N, z))^{1/2M}$$

appearing in the Multiplicative Ergodic Theorem (see e.g. [25, Theorem 3.4.1(i)]).

3. Calculate the orthonormal eigenspace decomposition of $\Psi^{(M)}(\xi(-N, z))$, denoted by $U_j^{(M)}(\xi(-N, z))$, $j = 1, \dots, \ell$. We are particularly interested in low values of j , corresponding to large eigenvalues L_j .

4. Define $W_j^{(M,N)}(z) := P^{(N)}(\xi(-N, z))U_j^{(M)}(\xi(-N, z))$ via the push forward under the matrix cocycle.

5. $W_j^{(M,N)}(z)$ is our numerical approximation to $W_j(z)$.

5.3. Continuity of the Oseledets subspaces in continuous time

When treating continuous time systems, one may ask about the continuity properties of $W_j^{(M,N)}(z)$ in z . In the following we suppose that $W_2^{(M,N)}(z)$ is one-dimensional. For large M and N , $W_2^{(M,N)}(z)$ will approximate the most dominant Oseledets subspace at time z . Suppose that we are interested in how this subspace changes from time z to time $\xi(\delta, z)$ for small $\delta > 0$. There are two ways to obtain information at time $\xi(\delta, z)$. Firstly, we can simply push forward $W_2^{(M,N)}(z)$ slightly longer to produce $W_2^{(M,N+\delta)}(\xi(\delta, z))$. Secondly, we can compute $\Psi^{(M)}$ slightly later at time $\xi(\delta, z)$ to produce $W_2^{(M,N)}(\xi(\delta, z))$.

To compare the closeness of $W_2^{(M,N)}(z)$ to $W_2^{(M,N+\delta)}(\xi(\delta, z))$ and $W_2^{(M,N)}(\xi(\delta, z))$, we represent each as a function and make a comparison in the L^1 norm. We assume that $U_2^{(M)}(\xi(-N, z))$ is one-dimensional and define $f_{n,\xi(-N,z),M} = \sum_{i=1}^n (u_2^{(M)}(\xi(-N, z)))_i \chi_{B_i} \in L^1(M, m)$ where $u_2^{(M)}(\xi(-N, z)) \in U_2^{(M)}(\xi(-N, z))$ is scaled so that $\|f_{n,\xi(-N,z),M}\|_1 = 1$. Let $\hat{f}_{n,z,M,N} = \pi_n \mathcal{P}_{\xi(-N,z)}^{(N)} f_{n,\xi(-N,z),M}$. Note that $\hat{f}_{n,z,M,N} = \sum_{i=1}^n (w_2^{(M,N)}(z))_i \chi_{B_i}$ for some $w_2^{(M,N)}(z) \in W_2^{(M,N)}(z)$.

We firstly compare $\hat{f}_{n,\xi(\delta,z),M,N+\delta}$ and $\hat{f}_{n,z,M,N}$.

Proposition 1. $\|\hat{f}_{n,\xi(\delta,z),M,N+\delta} - \hat{f}_{n,z,M,N}\|_1 \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. Note that $\hat{f}_{n,\xi(\delta,z),M,N+\delta} = \pi_n \mathcal{P}_{\xi(-N,z)}^{(N+\delta)} f_{n,\xi(-N,z),M}$ while $\hat{f}_{n,z,M,N} = \pi_n \mathcal{P}_{\xi(-N,z)}^{(N)} f_{n,\xi(-N,z),M}$. The proof will follow from the result that $\mathcal{P}_t^{(\tau)}$ is a continuous semigroup; that is, $\lim_{\delta \rightarrow 0} \|\mathcal{P}_t^{(\delta)} f - f\|_1 = 0$ for all $t \in \mathbb{R}$, $f \in L^1(M, m)$.

Lemma 1. $\|\mathcal{P}_z^{(\delta)} f - f\|_1 \rightarrow 0$ as $\delta \rightarrow 0$ for all $z \in \mathcal{E}$ and $f \in L^1$.

Proof. The proof runs as a nonautonomous version of the discussion in Remark 7.6.2 [44]. Note that $\mathcal{P}_z^{(\delta)} f(x) = f(\phi(-\delta, \xi(\delta, z), x)) \cdot \det D\phi(-\delta, \xi(\delta, z), x)$, where $\phi(-\delta, \xi(\delta, z), \cdot)$ denotes the flow from $\xi(\delta, z)$ in reverse time for duration δ . For the moment consider continuous f . Since $x \mapsto \phi(s, z, x)$ is at least C^1 for each s, z (the derivative of ϕ wrt to x is continuous with respect to s and z for each fixed z) by [25, Theorem 2.2.2(iv)] and M is compact, $\mathcal{P}_z^{(\delta)} f(x) \rightarrow f(x)$ uniformly in x as $\delta \rightarrow 0$. Thus $\|\mathcal{P}_z^{(\delta)} f - f\|_1 \rightarrow 0$ as $\delta \rightarrow 0$. Since the continuous functions are dense in L^p , $1 \leq p < \infty$ as M is compact (see e.g. [45] Lemma IV.8.19), one can approximate any $f \in L^1$ by a continuous function and thus the result holds for all L^1 functions f . \square

Thus the result follows using Lemma 1 and the fact that $\|\pi_n\|_1 = 1$. \square

Now, let us compare $\hat{f}_{n,\xi(\delta,z),M,N}$ and $\hat{f}_{n,z,M,N}$.

Proposition 2. $\|\hat{f}_{n,\xi(\delta,z),M,N} - \hat{f}_{n,z,M,N}\|_1 \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. This result is more difficult to demonstrate as we need to firstly compare $\Psi^{(M)}(\xi(-N, z))$ with $\Psi^{(M)}(\xi(-N + \delta, z))$. To this end, consider

$$\begin{aligned} & \|\pi_n \mathcal{P}_{\xi(-N+\delta,z)}^{(M)} f - \pi_n \mathcal{P}_{\xi(-N,z)}^{(M)} f\|_1 \\ &= \|\pi_n \mathcal{P}_{\xi(\delta,z)-N}^{(M)} f - \pi_n \mathcal{P}_{\xi(-N,z)}^{(M+\delta)} f + \pi_n \mathcal{P}_{\xi(-N,z)}^{(M+\delta)} f - \pi_n \mathcal{P}_{\xi(-N,z)}^{(M)} f\|_1 \\ &\leq \|\pi_n\|_1 \left(\|\mathcal{P}_{\xi(-N+\delta,z)}^{(M)} f - \mathcal{P}_{\xi(-N,z)}^{(M+\delta)} f\|_1 \right. \\ &\quad \left. + \|\mathcal{P}_{\xi(-N,z)}^{(M+\delta)} f - \mathcal{P}_{\xi(-N,z)}^{(M)} f\|_1 \right) \\ &\leq \|\mathcal{P}_{\xi(-N+\delta,z)}^{(M)} (\text{Id} - \mathcal{P}_{\xi(-N,z)}^{(\delta)}) f\|_1 \\ &\quad + \|(\mathcal{P}_{\xi(-N,z)+M}^{(\delta)} - \text{Id}) \mathcal{P}_{\xi(-N,z)}^{(M)} f\|_1 \\ &\leq \|(\text{Id} - \mathcal{P}_{\xi(-N,z)}^{(\delta)}) f\|_1 + \|(\mathcal{P}_{\xi(-N,z)+M}^{(\delta)} - \text{Id}) \mathcal{P}_{\xi(-N,z)}^{(M)} f\|_1. \end{aligned}$$

The right hand side converges to zero as $\delta \rightarrow 0$ by Lemma 1. This result implies that $\|P^{(M)}(\xi(-N, z)) - P^{(M)}(\xi(-N + \delta, z))\| \rightarrow 0$ as $\delta \rightarrow 0$ in whatever matrix norm we choose. Thus $\|\Psi^{(M)}(\xi(-N, z))^{2M} - \Psi^{(M)}(\xi(-N + \delta, z))^{2M}\| = \|P^{(M)}(\xi(-N, z))^\top (P^{(M)}(\xi(-N, z)) - P^{(M)}(\xi(-N + \delta, z))) + (P^{(M)}(\xi(-N, z))^\top - P^{(M)}(\xi(-N + \delta, z))^\top) P^{(M)}(\xi(-N + \delta, z))\| \rightarrow 0$ as $\delta \rightarrow 0$. By standard perturbation results, see e.g. [46, Theorem II.5.1], this implies that eigenvectors $U_2^{(M)}(z)$ and $U_2^{(M)}(\xi(\delta, z))$ are close for sufficiently small δ . Thus $f_{n,\xi(-N,z),M}$ and $f_{n,\xi(-N+\delta,z),M}$ are close in L^1 norm. Now we need to push both of these forward by $\pi_n \mathcal{P}_{\xi(-N,z)}^{(N)}$. This will not increase the norm of the difference at all, so $\|\hat{f}_{n,\xi(\delta,z),M,N} - \hat{f}_{n,z,M,N}\|_1$ will also be small. \square

5.4. Oseledets functions for a 1D discrete time nonautonomous system

We now examine the Oseledets functions for the system defined in Example 1. We consider the approximation $\pi_{100} \mathcal{P}_\omega$ of rank 100, which we obtain by Galerkin projection. We denote by $P(\omega) \in \mathbb{R}^{100} \times \mathbb{R}^{100}$ the Ulam matrix representing the action of $\pi_{100} \mathcal{P}_\omega$ on functions $f \in \mathcal{B}_{100} := \text{sp}\{\chi_{[(i-1)/100, i/100]}, i = 1, \dots, 100\}$. The matrices $P(\sigma^{-k}\omega)$, $k = -10, \dots, 10$ are constructed by following Algorithm 1 using $Q = 100$.

We look for Oseledets functions for a particular aperiodic sequence ω . To generate an aperiodic sequence, let $\tau \in \{0, 1\}^{\mathbb{N}}$ be the binary expansion of $1/\sqrt{3}$. Extend τ to an element of $\{0, 1\}^{\mathbb{Z}}$ by setting $\tau_i = 0$ for all $i \leq 0$. Define $\omega_{i-25} = 1 + 2\tau_i + \tau_{i+1}$ for each i . Then $\omega \in \Omega$ and the central 21 terms of ω are

$$\omega = (\dots, 2, 3, 1, 2, 4, 4, 3, 2, 3, 1,$$

$$1, 2, 3, 2, 4, 3, 1, 2, 3, 1, 1, \dots), \quad (21)$$

where the dot denotes the zeroth term $\omega_0 = 1$.

We calculate the eigenvalues of $(P^{(20)}(\sigma^{-10}\omega))^\top \circ P^{(20)}(\sigma^{-10}\omega)^{1/40}$, where $P^{(20)}(\sigma^{-10}\omega)$ is defined as in (10), and find the top three to be

$$L_1 \approx 1.00, \quad L_2 \approx 0.84, \quad L_3 \approx 0.46.$$

As in Examples 3 and 4, the maps T_i are piecewise affine with constant slope three, and so $\rho(\omega) = 1/3$. Thus $\log L_2$ and $\log L_3$ may approximate isolated Lyapunov exponents in $\Lambda(\mathcal{P})$.

We follow Algorithm 2 to approximate the second Oseledets subspace $W_2^{(M,N)}(\sigma^k \omega)$ for $k = 0, \dots, 5$, using $(M, N) = (20, 10)$, see Fig. 8. In order to confirm the effectiveness of Algorithm 2 we calculate the L^1 distance $\Delta(N)$ between the normalisations of the

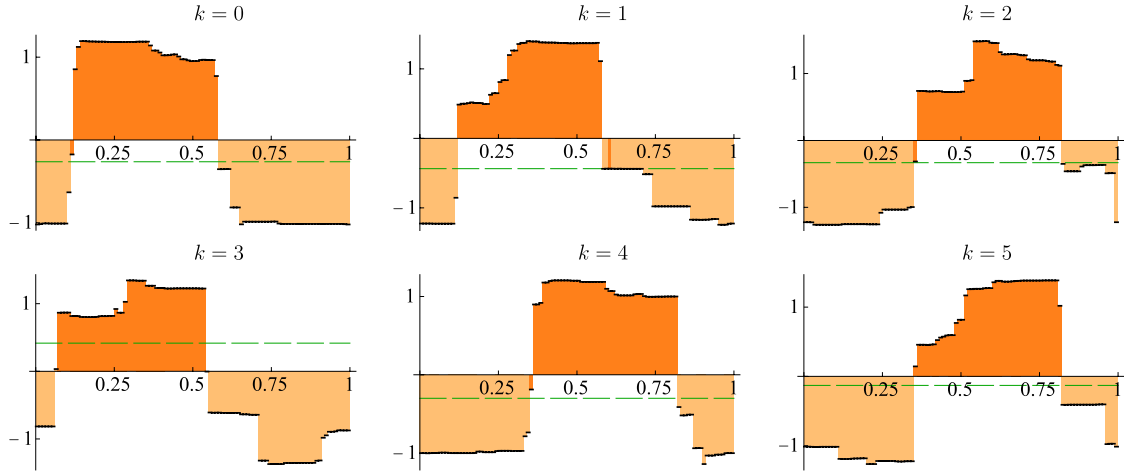


Fig. 8. The Oseledets function approximations $f_2^{(M,N)}(\sigma^k \omega)$ for $M = 20$, $N = 10$, and $k = 0, \dots, 5$, along with optimal thresholds (shown in dashed green), see Section 7.2.

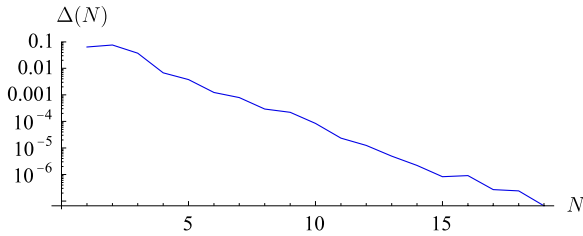


Fig. 9. A graph showing $\Delta(N)$ for $N = 1, \dots, 19$.

vectors $w_2^{(2N,N)}(\sigma \omega)$ and $P(\omega)w_2^{(2N,N)}(\omega)$, for $N = 2, \dots, 19$ with $M = 40$. By property 3 of [Theorem 1](#) this distance should be small if the family $W_2(\omega)$ is well approximated. A logarithmic plot of $\Delta(N)$ against N , see [Fig. 9](#), shows the fast convergence of $w_2^{(2N,N)}(\omega)$ to an Oseledets subspace. In Section 7.2 we will see how to extract coherent sets from these functions.

5.5. Oseledets functions in a 2D continuous time nonautonomous system

We consider the following nonautonomous system on $M = [0, 2\pi] \times [0, \pi]$, $t \in \mathbb{R}^+$:

$$\begin{aligned} \dot{x} &= c - A \sin(x - vt) \cos(y) \pmod{2\pi} \\ \dot{y} &= A \cos(x - vt) \sin(y). \end{aligned} \quad (22)$$

This equation describes a travelling wave in a stationary frame of reference with rigid boundaries at $y = 0$ and $y = \pi$, where the normal flow vanishes [[47,48](#)]. The streamfunction (Hamiltonian) of this system is given by

$$s(x, y, t) = -cy + A \sin(x - vt) \sin(y). \quad (23)$$

We set $c = 0.5$, $A = 1$, and the phase speed to $v = 0.25$. The velocity field is 2π -periodic in the x -direction, which allow us to study the flow on a cylinder. The velocity fields in a comoving frame for these parameters are shown in [Fig. 10](#). The closed recirculation regions adjacent to the walls ($y = 0$ and $y = \pi$) move in the positive x -direction and are separated from the jet flowing regime by the heteroclinic loops of fixed points, which are given below.

This model can be simplified to an autonomous system with a steady streamfunction in the comoving frame by setting $X = x - vt$ and $Y = y$. The steady streamfunction is then given by $S(X, Y, t) = -(c - v)Y + A \sin(X) \sin(Y)$. Let $X_s = \sin^{-1}((c - v)/A)$ and $Y_s = \cos^{-1}((c - v)/A)$. In the comoving frame, the recirculation region at the wall $Y = 0$ contains an elliptic point

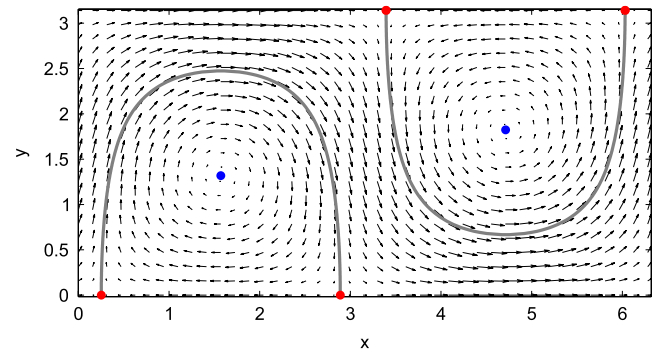


Fig. 10. Vector fields in the comoving frame for the travelling wave flow (23), for $A = 1.0$ and $c = 0.5$. The red dots are the hyperbolic fixed points that are connected by the heteroclinic loops. The blue dots are elliptic points in the centre of recirculation regions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$q_1 = (\pi/2, Y_s)$ and is bounded by the heteroclinic loop of the hyperbolic fixed points $p_1 = (X_s, 0)$ and $p_2 = (\pi - X_s, 0)$. Similarly, those elliptic and hyperbolic points at the wall $Y = \pi$ are $q_2 = (3\pi/2, \pi - Y_s)$, $p_3 = (\pi + X_s, \pi)$, and $p_4 = (2\pi - X_s, \pi)$, respectively, see [Fig. 10](#). One may observe that there is a continuous family of invariant sets in the comoving frame as any fixed level set of the streamfunction bounds an invariant set. In a stationary frame these elliptic and hyperbolic points (and their heteroclinic loops) are just translated in the x -direction. That is, any fixed level set of the time-dependent streamfunction (23) is a (time-dependent) invariant manifold. We note, however, that the recirculation regions are distinguished from the remainder of the cylinder as they are separated from the jet flowing region, which has a different dynamical fate. In the subsequent sections we will perturb this somewhat “degenerate” system to destroy the continuum of invariant sets in the comoving frame and produce a small number of almost-invariant sets (see [Definition 5](#)) in the comoving frame, or coherent sets in the stationary frame).

5.5.1. A coherent family: mixing case

We modify the travelling wave model in the previous section to allow mixing in the jet flowing region. We add a perturbation to the system in the following way:

$$\begin{aligned} \dot{x} &= c - A(\tilde{z}(t)) \sin(x - v\tilde{z}(t)) \cos(y) \\ &\quad + \varepsilon G(g(x, y, \tilde{z}(t))) \sin(\tilde{z}(t)/2) \\ \dot{y} &= A(\tilde{z}(t)) \cos(x - v\tilde{z}(t)) \sin(y). \end{aligned} \quad (24)$$

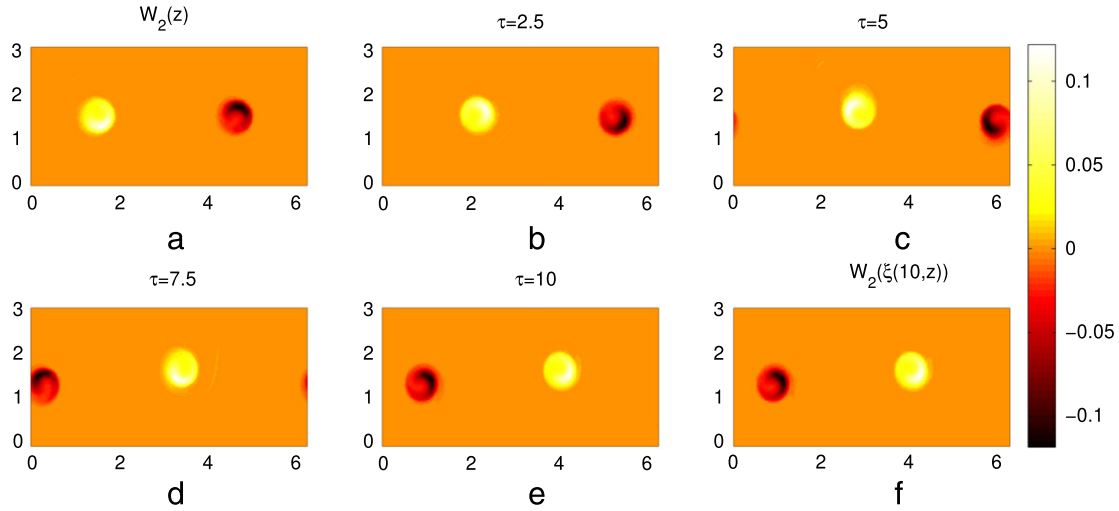


Fig. 11. (a) Graph of approximate Oseledets function $W_2^{(80,40)}(z)$ produced by Algorithm 4. (b)–(e) Push forwards of $W_2^{(80,40)}(z)$ via multiplication by $P^{(\tau)}(z)$ for $\tau = 2.5, 5, 7.5, 10$. (f) $W_2^{(80,40)}(\xi(10, z))$ produced independently by Algorithm 4; compare with (e).

Here, $\tilde{z}(t) = 6.6685z_1(t)$, where $z_1(t)$ is generated by the Lorenz flow in Example 2 with initial point $z(0) = (0, 1, 1.5)$, $A(\tilde{z}(t)) = 1 + 0.125 \sin(\sqrt{5}\tilde{z}(t))$, $G(\psi) := 1/(\psi^2 + 1)^2$ and the parameter function $\psi = g(x, y, \tilde{z}(t)) := \sin(x - \nu\tilde{z}(t)) \sin(y) + y/2 - \pi/4$ vanishes at the level set of the streamfunction of the unperturbed flow at instantaneous time $t = 0$, i.e., $s(x, y, 0) = \pi/4$, which divides the phase space in half. We set $\varepsilon = 1$ as this value is sufficiently large to ensure no KAM tori remain in the jet regime, but sufficiently small to maintain islands originating from the nested periodic orbits around the elliptic points of the unperturbed system.

We applied Algorithm 3 with $n = 28\,800$, $M = 80$, $N = 40$, $z = (0, 1, 1.5)$, and Algorithm 4 for $z = (0, 1, 1.5)$ and $z = \xi(10, (0, 1, 1.5))$. By using a relatively large number of test points per grid box ($n = 400$ points per box B_j) we are able to flow for $M = 80$ units of time and still well represent $\phi(80, \xi(-40, z), B_j)$. Fig. 11 shows that the resulting Oseledets functions highlight the remaining islands in the perturbed time-dependent flow. We calculate the eigenvalues of $(P^{(80)}(\xi(-40, z)))^T \circ P^{(80)}(\xi(-40, z))^{1/2}$, where $P^{(80)}(\xi(-40, z))$ is defined as in (20), and find the top three to be

$$L_1 \approx 1.1100, \quad L_2 \approx 0.9691, \quad L_3 \approx 0.9676.$$

By part 3 of Theorem 1 (bundle invariance of $W_2(z)$) we should have $P^{(10)}(z)W_2^{(80,40)}(z) \approx W_2^{(80,40)}(\xi(10, z))$. This is demonstrated in Fig. 11 by comparing subplots (e) and (f). In Section 7.3 we will see how to extract coherent sets from these Oseledets functions.

6. Invariant sets, almost-invariant sets, and coherent sets

We begin by briefly recounting some of the background relevant to almost-invariant sets. If Φ (resp. ϕ) is autonomous, then Ω (resp. \mathcal{E}) consists of a single point, and we may write $\Phi(-1, \omega, x) = \Phi(-1, x)$ (resp. $\phi(-t, z, x) = \phi(-t, x)$).

Definition 4. In the autonomous setting, we call A an *invariant set* if $\Phi(-1, A) = A$ (resp. $\phi(-t, A) = A$ for all $t \geq 0$).

The following definition generalises invariant sets to *almost-invariant sets*. In the continuous time case we define:

Definition 5. Let μ be preserved by the autonomous flow ϕ . We will say that a set $A \subset M$ is ρ_0 -almost-invariant over the interval $[0, \tau]$ if

1.

$$\rho_{\mu, \tau}(A) := \frac{\mu(A \cap \phi(-s, A))}{\mu(A)} \geq \rho_0 \quad (25)$$

for all $s \in [0, \tau]$,

2. A is connected.

If $A \subset M$ is *almost-invariant* over the interval $[0, \tau]$, then for each $s \in [0, \tau]$, the probability (according to μ) of a trajectory leaving A at some time in $[0, s]$, and not returning to A at time s is relatively small. In the discrete time setting, $\tau = 1$, and the obvious changes are made in Definition 5. By convention we ask that A is connected; if A is not connected, we consider each connected component to be an almost-invariant set for suitable ρ_0 .

We now begin to discuss the nonautonomous setting. The notion of an invariant set is extended to an *invariant family*.

Definition 6.

1. *Discrete time:* We will call a family of sets $\{A_{\sigma^k \omega}\}, A_{\sigma^k \omega} \subset M$, $\omega \in \Omega$, $k \in \mathbb{Z}$ an *invariant family* if $\Phi(-k, \omega, A_{\omega}) = A_{\sigma^{-k} \omega}$ for all $\omega \in \Omega$ and $k \in \mathbb{Z}^+$.
2. *Continuous time:* We will call a family of sets $\{A_{\xi(t, z)}\}, A_{\xi(t, z)} \subset M$, $z \in \mathcal{E}$, $t \in \mathbb{R}$ an *invariant family* if $\phi(-t, z, A_z) = A_{\xi(-t, z)}$ for all $z \in \mathcal{E}$ and $t \in \mathbb{R}^+$.

Motivated by a model of fluid flow, we imagine coherent sets as a family of *connected* sets with the property that the set A_{ω} is *approximately* mapped onto $A_{\sigma^k \omega}$ by k iterations of the cocycle from “time” ω ; that is, $\Phi(k, \omega, A_{\omega}) \approx A_{\sigma^k \omega}$. The definition of coherent sets combines the properties of almost-invariant sets and an invariant family. As we now have a *family* of sets we require one more property beyond those of Definition 5, in addition to modifying the almost-invariance property. In the continuous time case we define:

Definition 7. Let μ be preserved by a flow ϕ and $0 \leq \rho_0 \leq 1$. Fix a $z \in \mathcal{E}$. We will say that a family $\{A_{\xi(t, z)}\}_{t \geq 0}, A_{\xi(t, z)} \subset M$, $t \geq 0$ is a *family of ρ_0 -coherent sets* over the interval $[0, \tau]$ if:

1.

$$\rho_{\mu}(A_{\xi(t, z)}, A_{\xi(t+s, z)}) := \frac{\mu(A_{\xi(t, z)} \cap \phi(-s, \xi(t+s, z), A_{\xi(t+s, z)}))}{\mu(A_{\xi(t, z)})} \geq \rho_0, \quad (26)$$

for all $s \in [0, \tau]$ and $t \geq 0$.

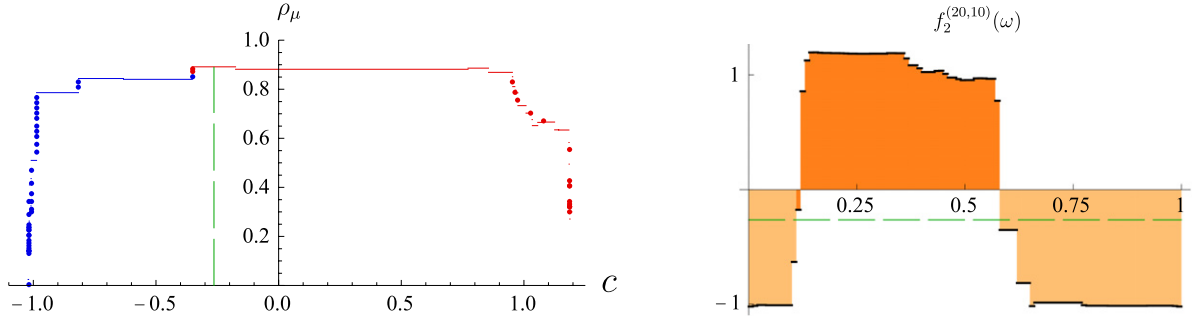


Fig. 12. (left): The function $\rho_\mu(\hat{A}_\omega^+(c), \hat{A}_{\sigma\omega}^+(\eta(c)))$ takes its maximum on the interval $(-0.352, -0.176)$ and so we take the midpoint $c^* = -0.264$ as the optimal threshold. (right): Taking this optimal threshold (shown in dashed green) for the eigenvector $f_2^{(20,10)}(\omega)$ identifies the coherent set $A_\omega = [0.11, 0.58]$ (shown in dark orange). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

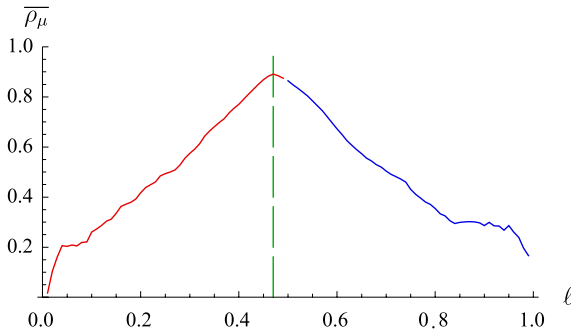


Fig. 13. The graph of $\bar{\rho}_\mu := \frac{1}{6} \sum_{k=0}^5 \rho_\mu(\hat{A}_{\sigma^k\omega}^\pm(c_k(\ell)), \hat{A}_{\sigma^{k+1}\omega}^\pm(c_{k+1}(\ell)))$ against ℓ , where we take $\hat{A}_{\sigma^k\omega}^+(c_k(\ell))$ for $\ell \leq 0.5$ and $\hat{A}_{\sigma^k\omega}^-(c_k(\ell))$ otherwise. The maximum 0.891 occurs at $\ell^* = 0.47$. The red section of the curve corresponds to $\hat{A}_{\sigma^k\omega}^+(c_k(\ell))$ and the blue section to $\hat{A}_{\sigma^k\omega}^-(c_k(\ell))$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

2. Each $A_{\xi(t,z)}$, $t \geq 0$ is connected.
 3. $\mu(A_{\xi(t,z)}) = \mu(A_{\xi(t',z)})$ for all $t, t' \geq 0$.
- In discrete time, we replace (26) with

$$\rho_\mu(A_\omega) := \frac{\mu(A_\omega \cap \Phi(-1, \sigma\omega, A_{\sigma\omega}))}{\mu(A_\omega)} \geq \rho_0, \quad (27)$$

τ necessarily becomes 1, and we make the obvious changes to the other items in Definition 7.

We remark that by selecting some $A \subset M$ of positive μ measure and defining $A_{\xi(t,z)} := \phi(t, z, A)$, $t \geq 0$, the family $\{A_{\xi(t,z)}\}_{t \geq 0}$ is a family of 1-coherent sets. Such a family is not of much dynamical interest, as there is nothing distinguishing this family from one constructed with another connected subset $A' \subset M$. We are not interested in these constructions of coherent sets, and in practice the numerical algorithm we present in the next section is unlikely to find such sets for chaotic systems.

7. Coherent sets from Oseledets functions

We wish to find a family of sets $\{A_z\}$ so that

$$\rho_\mu(A_z, A_{\xi(s,z)}) := \frac{\mu(A_z \cap \phi(-s, \xi(s,z), A_{\xi(s,z)}))}{\mu(A_z)} \quad (28)$$

is large for $s \in [0, \tau]$. We may rewrite the RHS of (28) as

$$\begin{aligned} & \left(\int \chi_{A_z} \cdot \chi_{\phi(-s, \xi(s,z), A_{\xi(s,z)})} d\mu \right) / \mu(A_z) \\ &= \left(\int \mathcal{P}_z^{(s)} \chi_{A_z} \cdot \chi_{A_{\xi(s,z)}} d\mu \right) / \mu(A_z). \end{aligned} \quad (29)$$

For (29) to be large we require $\mathcal{P}_z^{(s)} \chi_{A_z} \approx \chi_{A_{\xi(s,z)}}$.

Let us now make a connection with the Oseledets functions $f_2(z) = \sum_{i=1}^n w_{2,i}(z) \chi_{B_i}$ where $w_2(z) \in W_2^{(M,N)}(z)$ obtained in Algorithm 4. In the following discussion, we scale $f_2(z)$ so that $\|f_2(z)\|_1 = 1$ for all $z \in \mathcal{E}$. To convert the family of Oseledets functions into a family of coherent sets, we modify a heuristic due to [18] that has been successfully used in the autonomous setting. The heuristic is to set $A_z = \{f_2(z) > 0\}$, $z \in \mathcal{E}$. We show that $\mathcal{P}_z^{(s)} f_2^+(z) - f_2^+(\xi(s,z))$ is small for moderate s and large λ_2 ; we then heuristically infer that $\mathcal{P}_z^{(s)} \chi_{A_z} \approx \chi_{A_{\xi(s,z)}}$.

Proposition 3. Let $\lambda_2 = \lim_{s \rightarrow \infty} (1/s) \log \|\mathcal{P}_z^{(s)} f_2(z)\| < 0$ be the second largest Lyapunov exponent from Theorem 1 and $f_2(z) \in W_2(z)$ a corresponding Oseledets function, normalised so that $\|f_2(z)\|_1 = 1$. Given an $\epsilon > 0$ there is an $S \geq 0$ so that $s \geq S$ implies $\|\mathcal{P}_z^{(s)} f_2^+(z) - f_2^+(\xi(s,z))\|_1 \leq (1 - e^{(\lambda_2 - \epsilon)s})/2$.

Proof. Given $\epsilon > 0$ we know that there exists $S \geq 0$ such that for all $s \geq S$ one has $e^{\lambda_2 - \epsilon} \leq \|\mathcal{P}_z^{(s)} f_2(z)\|^{1/s} \leq 1$. Since $\mathcal{P}_z^{(s)} f_2(z) = (\mathcal{P}_z^{(s)} f_2(z))^+ - (\mathcal{P}_z^{(s)} f_2(z))^-$ and $\int \mathcal{P}_z^{(s)} f_2(z) dm = 0$, one has $\|\mathcal{P}_z^{(s)} f_2(z)\|_1 = \int (\mathcal{P}_z^{(s)} f_2(z))^+ + (\mathcal{P}_z^{(s)} f_2(z))^- dm = 2 \int (\mathcal{P}_z^{(s)} f_2(z))^+ dm$. Thus $\int (\mathcal{P}_z^{(s)} f_2(z))^+ dm \geq e^{(\lambda_2 - \epsilon)s}/2$. Since $(\mathcal{P}_z^{(s)} f_2(z))^+ \leq \mathcal{P}_z^{(s)} f_2^+(z)$, one has $\|\mathcal{P}_z^{(s)} f_2^+(z) - (\mathcal{P}_z^{(s)} f_2(z))^+\| = \int \mathcal{P}_z^{(s)} f_2^+(z) - (\mathcal{P}_z^{(s)} f_2(z))^+ dm$. As $\|f_2(z)\| = 1$ and $\int f_2(z) dm = 0$, we have $\int f_2^+(z) dm = 1/2$ and since $\mathcal{P}_z^{(s)}$ preserves integrals, $\int \mathcal{P}_z^{(s)} f_2^+(z) dm = 1/2$. Thus, $\int \mathcal{P}_z^{(s)} f_2^+(z) - (\mathcal{P}_z^{(s)} f_2^+(z)) dm \leq (1 - e^{(\lambda_2 - \epsilon)s})/2$. \square

The preceding discussion heuristically addresses item 1 of Definition 7. Regarding item 2 of Definition 7, as we are extracting the sets A_z from the Oseledets functions $f_2(z)$, the connectivity of the sets will depend on the regularity of the Oseledets functions. This is a delicate question and relatively little can be said formally at present. In the autonomous case, roughly speaking, one expects smooth eigenfunctions for Perron–Frobenius operators of smooth expanding systems [37,49], and eigendistributions (smooth in expanding directions, distributions in contracting directions) in uniformly hyperbolic settings [33]. These properties may carry over to the nonautonomous setting; recent results in the bounded variation setting show they do [39]. If a small amount of noise is added by postmultiplying the Perron–Frobenius operator by a smoothing (e.g. diffusion) operator, then the Oseledets functions must be smooth. This physical addition of a small amount of noise is one way to guarantee regularity of the Oseledets functions and connectivity of the associated coherent sets.

Finally we note that if $\mu = m$ one has $\int f_2(z)(x) d\mu(x) = 0$ and so we must have $\mu(A_z) = 1/2$ for all $z \in \mathcal{E}$. Thus, item 3. of Definition 7 is satisfied by the choice $A_z = \{f_2(z) > 0\}$. If $\mu \neq m$,

then it may be necessary to further tweak the choice of the A_z to ensure that item 3. of [Definition 7](#) is satisfied. This additional tweak is described in [Algorithm 5](#).

7.1. A numerical algorithm

For a fixed time $z \in \mathcal{E}$, we seek to approximate a pair of sets A_z and $A_{\xi(\tau, z)}$ for which

$$\rho_\mu(A_z, A_{\xi(\tau, z)}) := \frac{\mu(A_z \cap \phi(-\tau, \xi(\tau, z), A_{\xi(\tau, z)}))}{\mu(A_z)} \quad (30)$$

is maximal. The quantity $\rho_\mu(A_z, A_{\xi(\tau, z)})$ is simply the fraction of μ -measure of A_z that is covered by a pullback of the set $A_{\xi(\tau, z)}$ over a duration of τ . For maximal coherence, we wish to find pairs $A_z, A_{\xi(\tau, z)}$ that maximise $\rho_\mu(A_z, A_{\xi(\tau, z)})$. We present a heuristic to find such a pair of sets based upon the vectors $W^{(M, N)}(z)$ and $W^{(M, N)}(\xi(\tau, z))$ corresponding to some Lyapunov spectral value λ close to 0. This heuristic is a modification of heuristics to determine maximal almost-invariant sets, see [\[20,21,38\]](#). In the terminology of the prior discussion in [Section 7](#), rather than setting $A_z := \{f(z) > 0\}$, we allow $A_z := \{f(z) > c\}$ or $A_z := \{f(z) < c\}$ for some $c \in \mathbb{R}$ in the hope of finding $A_z, A_{\xi(\tau, z)}$ with an even greater value of $\rho_\mu(A_z, A_{\xi(\tau, z)})$. This additional flexibility also permits a matching of $\mu(A_z)$ and $\mu(A_{\xi(\tau, z)})$.

Algorithm 5 (To Determine a Pair of Maximally Coherent Sets at Times $z, \xi(\tau, z)$).

1. Determine $W^{(M, N)}(z)$ and $W^{(M, N)}(\xi(\tau, z))$ for some $\tau > 0$ according to [Algorithm 4](#).
2. Set $\hat{A}_z^+(c) = \bigcup_{i: W^{(M, N)}(z) > c} B_i$ and $\hat{A}_{\xi(\tau, z)}^+(c) = \bigcup_{i: W^{(M, N)}(\xi(\tau, z)) > c} B_i$, restricting the values of c so that $\mu(\hat{A}_z^+(c)), \mu(\hat{A}_{\xi(\tau, z)}^+(c)) \leq 1/2$. These are sets constructed from grid boxes whose corresponding entry in the $W^{(M, N)}$ vectors is above a certain value.
3. Define $\eta(c) = \operatorname{argmin}_{c' \in \mathbb{R}} |\mu(\hat{A}_z^+(c)) - \mu(\hat{A}_{\xi(\tau, z)}^+(c'))|$. Given a value of c , $\eta(c)$ determines the set $\hat{A}_{\xi(\tau, z)}^+(\eta(c))$ that best matches the μ -measure of $\hat{A}_z^+(c)$, as required by item 3 of [Definition 7](#).
4. Set $c^* = \operatorname{argmax}_{c \in \mathbb{R}} \rho_\mu(\hat{A}_z^+(c), \hat{A}_{\xi(\tau, z)}^+(\eta(c)))$. The value of c^* is selected to maximise the coherence.
5. Define $A_z := \hat{A}_z^+(c^*)$ and $A_{\xi(\tau, z)} := \hat{A}_{\xi(\tau, z)}^+(\eta(c^*))$.

Remark 6.

1. One can repeat [Algorithm 5](#), replacing $\hat{A}_z^+(c)$ and $\hat{A}_{\xi(\tau, z)}^+(c)$ with $\hat{A}_z^-(c) = \bigcup_{i: W^{(M, N)}(z) < c} B_i$ and $\hat{A}_{\xi(\tau, z)}^-(c) = \bigcup_{i: W^{(M, N)}(\xi(\tau, z)) < c} B_i$ respectively. See [\[38\]](#) for further details.
2. Care should be taken regarding the sign of $W^{(M, N)}(z)$ and $W^{(M, N)}(\xi(\tau, z))$. Visual inspection may be required in order to check that the vectors have the same “parity”.

7.2. Coherent sets for a 1D discrete time nonautonomous system

We return to the map cocycle Φ and Perron–Frobenius cocycle described in [Example 1](#) and identify coherent sets. We use two methods: firstly, inspection of the composition of maps as perturbations of maps with invariant sets, and secondly using the general method of [Algorithm 5](#).

The map cocycle Φ is defined in terms of a map H_a which has an almost-invariant set, and this gives rise to a family of coherent sets in the following way. Recall the definitions of the maps T_i , $i = 1, \dots, 4$ and shift space Ω determined by the adjacency matrix B . The maps T_i have the property that if $B_{i,j} = 1$, then any inner R

factors cancel in $T_j \circ T_i$. More generally, for any $\omega \in \Omega$, we have cancellation of all intermediate R factors:

$$\Phi(k, \omega, \cdot) = R^s \circ H_{a_{(\sigma^{k-1}\omega)_0}} \circ \dots \circ H_{a_{\omega_0}} \circ R^{-t}, \quad (31)$$

where $s, t \in \{0, 1\}$ are given by

$$s(\omega, k) = \begin{cases} 0, & \omega_{k-1} \text{ odd,} \\ 1, & \omega_{k-1} \text{ even,} \end{cases} \quad \text{and} \quad t(\omega, k) = \begin{cases} 0, & \omega_0 \leq 2 \\ 1, & \omega_0 > 2. \end{cases}$$

For the map H_0 , the interval $[0, 0.5]$ is invariant. Moreover, $[0, 0.5]$ is almost-invariant for H_a with $\rho_\mu([0, 0.5]) = 1 - 2a$. By [\(31\)](#), we if we set

$$\tilde{A}_{\sigma^k \omega} = R^{s(\omega, k)}([0, 0.5]), \quad \text{for each } k \in \mathbb{N}, \quad (32)$$

then

$$\rho_\mu(\tilde{A}_{\sigma^k \omega}, \tilde{A}_{\sigma^{k+1} \omega}) = 1 - 2a_{\omega_k}.$$

Thus $\{\tilde{A}_{\sigma^k \omega}\}_{k \in \mathbb{N}}$ is a family of ρ_0 -coherent sets with $\rho_0 = 1 - 2 \max\{a_1, \dots, a_4\} = 0.843$. In the same way, the invariant set $[0.5, 1]$ of H_0 leads to a family $\{R^{s(\omega, k)}([0.5, 1])\}_{k \in \mathbb{N}}$ of ρ_0 -coherent sets with the same ρ_0 .

In order to demonstrate the methods of this article, we now show how [Algorithm 5](#) can be used. We may use the Oseledets subspaces computed in [Section 5.4](#) to find a family of coherent sets. First we apply [Algorithm 5](#) to find a coherent set for the time step $k = 0$ to $k = 1$. We calculate $\rho_\mu(\hat{A}_\omega^+(c), \hat{A}_{\sigma \omega}^+(c))$ as c varies over the elements of the vector $f_2^{(20, 10)}(\omega)$; see [Fig. 12](#) (left). The maximum value of $\rho_\mu(\hat{A}_\omega^+(c), \hat{A}_{\sigma \omega}^+(c))$ is 0.890. The set A_ω is found to be the interval $[0.11, 0.58]$ of length $\mu(A_\omega) = 0.47$; see [Fig. 12](#) (right).

We note that the set A_ω found by [Algorithm 5](#) is not the same as the \tilde{A}_ω produced by the intuitive construction [\(32\)](#). In the latter case, $\tilde{A}_\omega = [0, 1/2]$, $\tilde{A}_{\sigma \omega} = [0, 1/2]$, and $\rho_\mu(\tilde{A}_\omega, \tilde{A}_{\sigma \omega}) = 1 - 2a_{\omega_0} = 1 - 2a_1 = 1 - \pi/20 \approx 0.843$, significantly lower than the value of 0.890 found using [Algorithm 5](#).

We may extend [Algorithm 5](#) in order to find a sequence of coherent sets $\{A_{\sigma^i \omega}\}_{i=0}^K$. Since we require the measure of a sequence of coherent sets to be constant, we seek to maximise the mean value of ρ_μ over a given time range as we vary the measure of the sets.

Algorithm 6 (To Determine a Sequence of Maximally Coherent Sets Over a Range of Times $\omega, \dots, \sigma^K \omega$).

1. Follow steps 1–3 of [Algorithm 5](#) for each $k = 0, \dots, K-1$ using $\tau = 1$ to obtain sets $\hat{A}_{\sigma^k \omega}^+(c)$.
2. Let $c_k(\ell) := \operatorname{argmin}_{c \in \mathbb{R}} |\mu(\hat{A}_{\sigma^k \omega}^+(c)) - \ell|$.
3. Compute $\ell^* := \operatorname{argmax}_{\ell \in (0, 0.5)} \frac{1}{K} \sum_{k=0}^{K-1} \rho_\mu(\hat{A}_{\sigma^k \omega}^+(c_k(\ell)), \hat{A}_{\sigma^{k+1} \omega}^+(c_{k+1}(\ell)))$.
4. For $k = 0, \dots, K-1$, define $A_{\sigma^k \omega} := \hat{A}_{\sigma^k \omega}^+(c_k(\ell^*))$.

To demonstrate [Algorithm 6](#), we use the approximate Oseledets functions $f_2(\sigma^k \omega)$, $k = 0, \dots, 5$, to find a sequence of six coherent sets $\{A_{\sigma^k \omega}\}_{k=0}^5$ for the map cocycle Φ . Plotting $\frac{1}{6} \sum_{k=0}^5 \rho_\mu(\hat{A}_{\sigma^k \omega}^+(c_k(\ell)), \hat{A}_{\sigma^{k+1} \omega}^+(c_{k+1}(\ell)))$ against ℓ (see [Fig. 13](#)), we find a unique maximum of 0.891, which occurs at $\ell^* = 0.47$.

[Fig. 8](#) shows the graph of $f_2^{(20, 10)}(\sigma^k \omega)$ with the threshold $c_k(\ell^*)$ for $k = 0, \dots, 5$, and in each case the set $\hat{A}_{\sigma^k \omega}^+(c_k(\ell^*))$ is indicated by shading. Since coherent sets are required to be connected, we must find the interval closest to each $\hat{A}_{\sigma^k \omega}^+(c_k(\ell^*))$. For $k = 0, 2, 3, 4, 5$ the set $\hat{A}_{\sigma^k \omega}^+(c_k(\ell^*))$ is itself an interval and we set $A_{\sigma^k \omega} = \hat{A}_{\sigma^k \omega}^+(c_k(\ell^*))$. The set $\hat{A}_{\sigma^1 \omega}^+(c_1(\ell^*))$ has two components, $[0.12, 0.58]$ and $[0.60, 0.61]$, and so we set $A_{\sigma \omega} =$

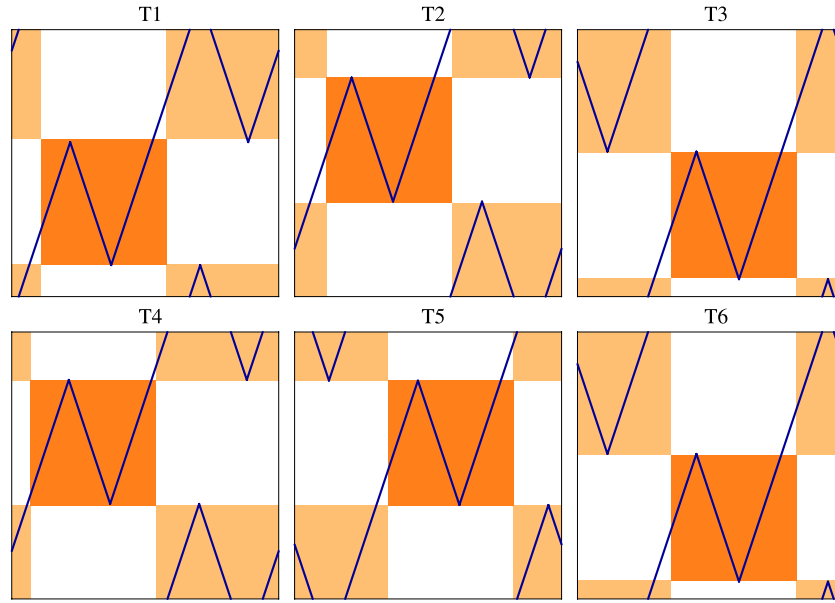


Fig. 14. Graphs of $T_{\sigma^k \omega}$ showing $A_{\sigma^k \omega} \times A_{\sigma^{k+1} \omega}$ for $k = 0, \dots, 5$.

Table 1

Coherent sets $A_{\sigma^k \omega}$ and the values of $\rho_\mu(A_{\sigma^k \omega}, A_{\sigma^{k+1} \omega})$ for $k = 0, \dots, 5$.

k	0	1	2	3	4	5
ω_k	1	2	3	2	4	3
$A_{\sigma^k \omega}$	[0.11, 0.58]	[0.12, 0.59]	[0.35, 0.82]	[0.07, 0.54]	[0.35, 0.82]	[0.35, 0.82]
ρ_k	0.89	0.87	0.87	0.96	0.90	0.87

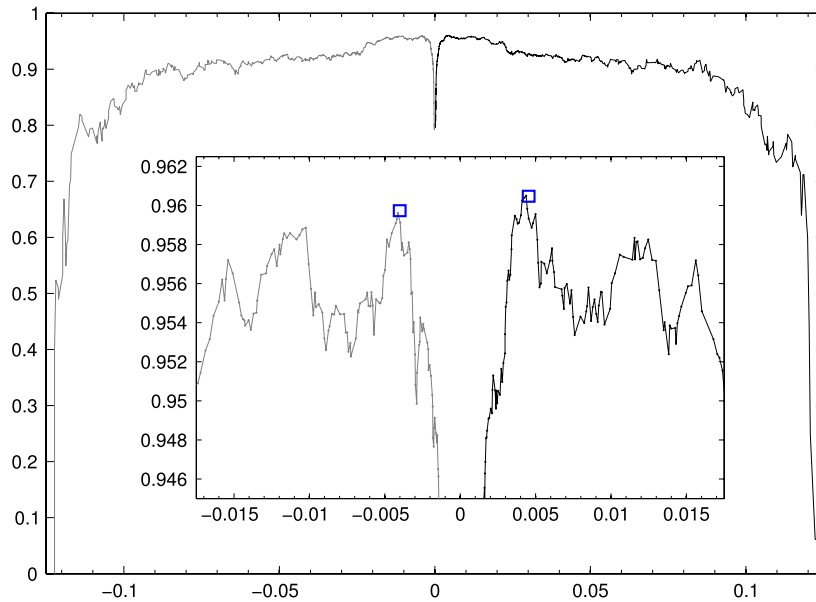


Fig. 15. Thresholding curves $\rho_\mu(\hat{A}_z^-(c), \hat{A}_{\xi(10,z)}^-(\eta(c)))$ and $\rho_\mu(\hat{A}_z^+(c), \hat{A}_{\xi(10,z)}^+(\eta(c)))$ are plotted in grey and black on the negative c axis and positive c axis, respectively. The optimal thresholds are marked with a rectangle.

[0.12, 0.59]. Table 1 lists the coherent sets $A_{\sigma^k \omega}$ and the values of $\rho_\mu(A_{\sigma^k \omega}, A_{\sigma^{k+1} \omega})$ for $k = 0, \dots, 5$.

It is interesting to compare the locations of the coherent sets with their corresponding maps in the mapping cocycle, see Fig. 14. As for the previously constructed family $\{R^{(\omega,k)}([0.5, 1])\}_{k \in \mathbb{N}}$, the coherent sets alternate between two positions separated by a rotation of approximately 0.25. However, the mean value of ρ_μ is greater for the sequence $A_{\sigma^k \omega}$ constructed from Algorithm 6 since in each case the coherent set matches up well with local maxima and minima of the preceding map.

7.3. Coherent sets in a 2D continuous time nonautonomous system

We apply Algorithm 5 to the Oseledets subspaces $W^{(80,40)}(z)$ and $W^{(80,40)}(\xi(10, z))$ calculated in Section 5.5.1 and displayed in Fig. 11. The optimal coherent sets \hat{A}_z^+ and $\hat{A}_{\xi(10,z)}^+$ are obtained at the threshold values $c^* = 0.0043$ and $\eta(c^*) = 0.0052$, which gives $\rho_\mu(\hat{A}_z^+(c^*), \hat{A}_{\xi(10,z)}^+(\eta(c^*))) = 0.9605$, see Fig. 15. For \hat{A}_z^- and $\hat{A}_{\xi(10,z)}^-$ the optimal threshold is at $c^* = -0.0040$ and $\eta(c^*) =$

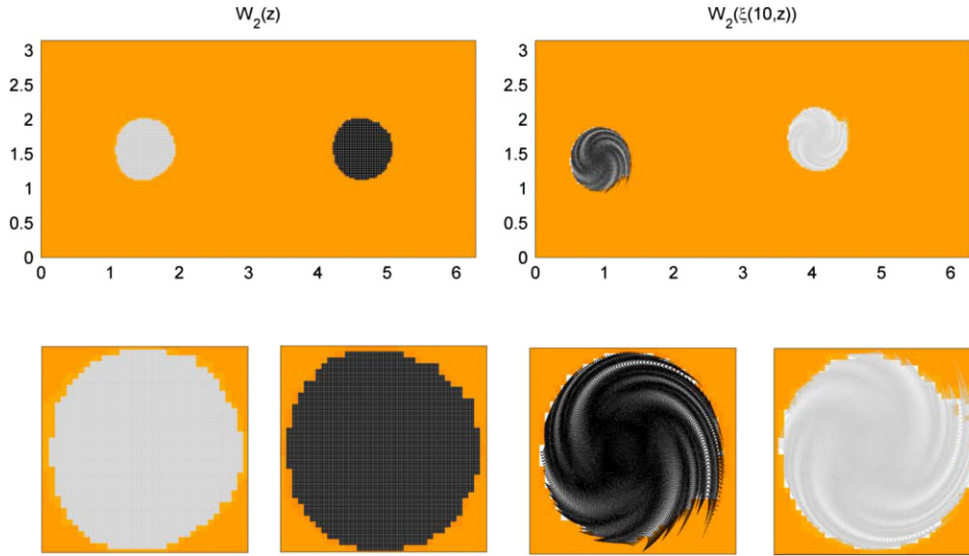


Fig. 16. [Top left] The coherent sets A_z^+ (grey) and A_z^- (black) and [Top right] $A_{\xi(10,z)}^+$ (grey) and $A_{\xi(10,z)}^-$ (black) are identified by thresholding the Oseledets functions. Overlays of $\phi(10, z, A_z^+)$ (grey) and $\phi(10, z, A_z^-)$ (black) are also shown. [Bottom left] Zooms of A_z^+ and A_z^- . [Bottom right] Overlays of $\phi(10, z, A_z^\pm)$ (grey/black dots) on $A_{\xi(10,z)}^\pm$ (white), displaying the loss of mass over 10 time units duration from $z = (0, 1, 1.5)$.

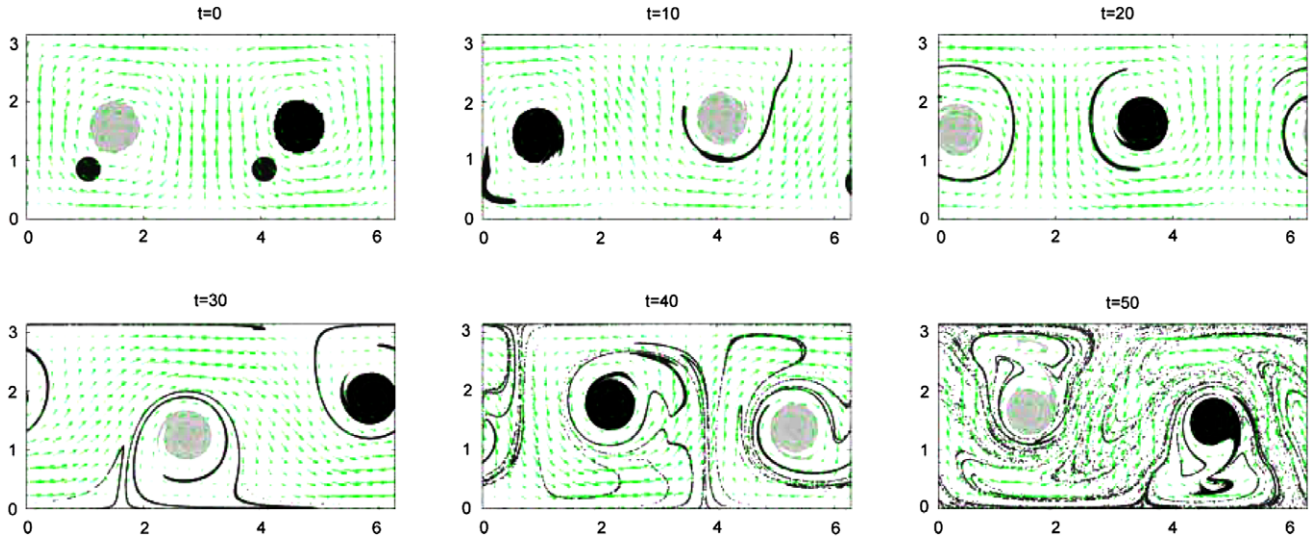


Fig. 17. Trajectories of the perturbed system (24) for $\varepsilon = 1$. The large light grey ($A_{\xi(t,z)}^+$) and black ($A_{\xi(t,z)}^-$) blobs are the coherent sets identified by our approach. The other (medium grey) blobs are chosen nearby the coherent sets to show strong mixing away from the coherent regions.

-0.0051 and $\rho_\mu(\hat{A}_z^-(c^*), \hat{A}_{\xi(10,z)}^-(\eta(c^*))) = 0.9599$. The coherent sets at \hat{A}_z^\pm and $\hat{A}_{\xi(10,z)}^\pm$ and the images of sample points in \hat{A}_z^\pm are shown in Fig. 16.

In Fig. 16 we note that the grey set A_z^+ on the left at time $z = (0, 1, 1.5)$ flows approximately to the light grey set $A_{\xi(10,z)}^+$ on the right at time $\xi(10, z)$. Similarly for the black sets A_z^- and $A_{\xi(10,z)}^-$. This carrying of the time z coherent sets to the time $\xi(10, z)$ coherent sets by the aperiodic flow is only approximate, as $\rho_\mu(\hat{A}_z^+(c^*), \hat{A}_{\xi(10,z)}^+(\eta(c^*))) = 0.9605$ and $\rho_\mu(\hat{A}_z^-(c^*), \hat{A}_{\xi(10,z)}^-(\eta(c^*))) = 0.9599$. Thus, we expect a loss of about 4% under the advection of the flow. Fig. 16 also zooms onto A_z^+ and A_z^- to demonstrate this loss of mass. To make this loss even more apparent, we continue to flow forward for 50 time units. Fig. 17 shows that the (black) coherent sets A_z^+ and A_z^- do indeed disperse over time, however at a much slower rate than the arbitrarily chosen

(grey) sets. The coherent sets A_z^+, A_z^- are just single elements of a time parameterised family $\{A_{\xi(t,z)}^+, A_{\xi(t,z)}^-\}_{t \geq 0}$ of coherent sets that at any given initial time describe those sets that will disperse most slowly over a duration of 10 time units.

8. Final remarks

We have formulated a new mathematical and algorithmic approach for identifying and tracking coherent sets in nonautonomous systems. Our new approach generalises existing successful transfer operator methodologies that have been used in the autonomous setting. Our constructions address the question raised by [6] of how to study strange eigenmodes and persistent patterns observed in forced fluid flows in the general time-dependent situation. Future work will include applying these techniques to detect and track mobile coherent regions in oceanic and atmospheric flows, extending significantly the flow times studied in [8,9,50].

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