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Integrability by separation of variables

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ABSTRACT

We study the integrability in the Jacobi sense (integrability by separation of variables), of the Hamiltonian differential systems using the Levi-Civita Theorem. In particular we solve the Stark problem for $N > 3$.

1. Introduction

The study of the integrability by separation of variables of the Hamilton–Jacobi equations is a classical problem in Mechanics, dating back to the foundational works of Jacobi, Stäckel, Levi-Civita and others.

In 1904 and in a letter addressed to P. Stäckel and published in the Matematische Annalen [1], Levi-Civita deals with the problem of the integration by separation of variables. In the introduction of this letter he writtes: *Ho notato che si possono facilmente assegnare (sotto forma esplicita di equazioni a derivate parziali ...) le condizioni necessarie e sufficienti cui deve soddisfare una H affinch'e l'equazione*

$$H\left(z_1,\dots,z_M,\frac{\partial W}{\partial z_1},\dots,\frac{\partial W}{\partial z_M}\right)=h, \tag{1}$$

ammetta un integrale completo della forma

$$W=W_1(z_1,\alpha_1,\dots,\alpha_M)+\cdots+W_M(z_M,\alpha_1,\dots,\alpha_M), \tag{2}$$

dove $\alpha_1,\dots,\alpha_M$ and h sono le costanti arbitrarie. Da queste condizioni scaturiscono alcune conseguenze di indole generale, che mi sembrano abbastanza interessanti, per quanto il dedurre da esse la completa risoluzione del problema apparisca ancora laborioso, e non vi sia nemmeno - oserei affermare - grande speranza di trovare tipi essenzialmente nuovi, oltre a quelli da Lei Stấckel scoperti. Indeed, Levi-Civita shows that

Theorem 1 (*Levi-Civita Theorem*). *Hamilton–Jacobi equation (1) has a first integral of the form (2), if and only if the Hamiltonian H satisfy the $M(M-1)/2$ second-order partial differential equations*

$$\begin{aligned}L_{jk}(z,P):=\;&\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_j\partial z_k}+\frac{\partial H}{\partial z_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_j\partial P_k}\\&-\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial z_j\partial P_k}-\frac{\partial H}{\partial P_k}\frac{\partial H}{\partial z_j}\frac{\partial^2 H}{\partial P_j\partial z_k}=0,\end{aligned} \tag{3}$$

for $j,k=1,\dots,M$ with $j\neq k$. Take into account that $L_{jk}(z,P)=L_{kj}(z,P)$.

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The application of this criterion to the investigation of the integrability of the Hamiltonian systems is a non-trivial problem. Levi-Civita's result provides a criterion for deciding when a given Hamiltonian H independent of the time is separable or not, it does not give an effective method for finding the separable coordinates for a given Hamiltonian. To find such coordinates it is in general a difficult problem which has only been solved for particular Hamiltonians.

The integrability by separation of variables of the Hamilton–Jacobi equations has recently received a big attention due to its applications to the theory of integrable partial differential equations of Korteweg de Vries type and to the theory of quantum integrable systems (see for instance [2–4]). In the review *Separation of Variables. New Trends* see [5] Sklyanin argued that separation of variables could be the most universal tool to solve integrable models of the classical and quantum mechanics.

The question of separation of variables for Hamiltonian systems was studied intensively in the second half of the last century (see for instance [3,6–13]). For an outline of the theory of separation of variables we refer to the book of Kalnins [14].

There exists an equivalent definition of separability, originally due to Jacobi and recently widely used by Sklyanin and his collaborators (see for instance [3]).

Consider a Hamiltonian mechanical system defined by the Hamiltonian $H = H_1$ with M degrees of freedom and *integrable in the Liouville sense*, i.e. there exists a family of $M-1$ first integrals $H_2, \dots, H_M$ such that

(a) they are in involution with respect to the Poisson bracket, that is

$$\{H_j, H_k\} = \sum_{j=1}^{M} \left(\frac{\partial H_j}{\partial P_j}\frac{\partial H_k}{\partial z_j} - \frac{\partial H_j}{\partial z_j}\frac{\partial H_k}{\partial P_j} \right) = 0,$$

for $j, k = 1, \dots, M$,

(b) they are independent, i.e. the rank of the matrix formed by the gradients of $H_1, \dots, H_M$ is M except (perhaps) in a set of zero Lebesgue measure.

An integrable Liouville Hamiltonian system with M degree of freedom and M first integrals $H_1 = H, H_2, \dots, H_M$ is *separable* in the canonical coordinates $(z_1, \dots, z_M, P_1, \dots, P_M)$ if there exist n non-trivial relations

$$\Phi_j(z_j, P_j, H_1, \dots, H_M) = 0, \quad \text{for} \quad j = 1, \dots, M, \tag{4}$$

connecting single pairs (z_j, P_j) of canonical coordinates with the M first integrals $H_1, \dots, H_M$. Note that the knowledge of the separation relations (4) allows to reduce the problem of finding a separated solution of the Hamilton–Jacobi equation to quadratures. Indeed, one can solve the relations (4) with respect to P_j, then we get that $P_j = f_j(z_j, H_1, \dots, H_M)$ and then we can define the generatrix function

$$S(z_1, \dots, z_M, \alpha_1, \dots, \alpha_M) = \sum_{j=1}^{M} \int_{z_{j0}}^{z_j} f_j(u_j, H_1, \dots, H_M)\big|_{H_1=\alpha_1,\dots,H_M=\alpha_M} du_j,$$

of the Hamilton–Jacobi equation (see for instance [5,15]).

The aim of this paper is to study the problem of separation of variables by using the classical approach, i.e. the Levi-Civita approach. In particular we give new properties of the Levi-Civita conditions (see Section 3), and we establish the relations between the integrability in Jacobi and Frobenius sense (see Theorem 10). We determine a new equivalent expression for the Levi-Civita conditions (see Theorem 13). We obtain all the Hamiltonian vector fields admitting a two or three dimensional Lie algebra (see Theorem 18 and Propositions 24 and 25). Finally we prove the integrability of some new Hamiltonian vector fields (see Propositions 26 and 30).

2. Preliminary results. On the Hamilton–Jacobi equation

Let $H = H(z_1, \dots, z_M, P_1, \dots, P_M)$ be a Hamiltonian. We study the Hamiltonian vector field

$$\Gamma_H := \sum_{j=1}^{M} \Gamma_j = \sum_{j=1}^{M} \left(\frac{\partial H}{\partial P_j}\frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial}{\partial P_j} \right), \tag{5}$$

associated to the Hamiltonian system

$$\frac{dz_k}{dt} = \{H, z_k\}, \qquad \frac{dP_k}{dt} = \{H, P_k\}, \quad \text{for} \quad k = 1, \dots, M. \tag{6}$$

The transformation of the $\mathbb{R}^{2M}$ space

$$(z_1, \dots, z_M, P_1, \dots, P_M) \longrightarrow (z_1^*, \dots, z_M^*, P_1^*, \dots, P_m^*) \tag{7}$$

under the condition

$$\det\left(\frac{\partial\left(z_1^*, \dots, z_M^*, P_1^*, \dots, P_M^*\right)}{\partial\left(z_1, \dots, z_M, P_1, \dots, P_M\right)} \right) \neq 0, \tag{8}$$

is called *canonical transformation* if the Hamiltonian system (6) is transformed into the Hamiltonian system

$$\frac{dz_k^*}{dt} = \{H^*, z_k^*\}, \qquad \frac{dP_k^*}{dt} = \{H^*, P_k^*\}, \quad \text{for} \quad k = 1, \dots, M. \tag{9}$$

The Hamilton–Jacobi theory wants to find the canonical transformations which writes system (6) in its simplest form.

The following theorem is well known (see for instance [16,17]).

Theorem 2. *Transformation (7) satisfying (8) is canonical if and only if there exists a function F and a constant c such that*

$$\sum_{k=1}^{M}(P_k^* dz_k^* - P_k dz_k) - (H^* - cH)dt = -dF(t, z_1, \ldots, z_M, P_1, \ldots, P_M), \tag{10}$$

where F is such that $\det\left(\dfrac{\partial^2 F}{\partial z_j \partial P_k}\right) \neq 0.$

Among the group of canonical transformations there exists a subgroup which is determined by the condition

$$\det\left(\frac{\partial\left(z_1^*, \ldots, z_M^*\right)}{\partial\left(P_1, \ldots, P_M\right)}\right) \neq 0.$$

Under this condition it is possible to choose $(t, z_1, \ldots, z_M, z_1^*, \ldots, z_M^*)$ as Hamiltonian variables. In these variables the function F is usually denoted by S, i.e.

$$S(t, z_1, \ldots, z_M, z_1^*, \ldots, z_M^*) = F(t, z_1, \ldots, z_M, P_1, \ldots, P_M)|_{P \longrightarrow z^*}.$$

Under these conditions from (10) we get that

$$\frac{\partial S}{\partial z_k} = cP_k, \qquad \frac{\partial S}{\partial z_k^*} = -P_k^*, \qquad \frac{\partial S}{\partial t} = H^* - cH.$$

The most interesting subcase of canonical transformations is when $H^* = 0$. Clearly in this case from (9) it follows that

$$z_k^* = \alpha_k, \qquad P_k^* = \beta_k.$$

Hence to construct a canonical transformation it is necessary and sufficient to determine S as a solution of the so called *Hamilton–Jacobi equation*

$$\frac{\partial S}{\partial t} + cH\left(t, z_1, \ldots, z_M, \frac{\partial S}{\partial z_1}, \ldots, \frac{\partial S}{\partial z_M}\right) = 0, \tag{11}$$

with $\det\left(\dfrac{\partial^2 S}{\partial z_j \partial \alpha_k}\right) \neq 0.$

Theorem 3 (*Jacobi Theorem*)**.** *The integration of (6) is equivalent to solve (11).*

For more details of Jacobi Theorem see for instance [16,17].

A solution S of the Hamilton–Jacobi equation, contains $M+1$ undetermined constants, the first M of them denoted as $\alpha_1, \alpha_2, \ldots, \alpha_M$, and the last one coming from the integration of $\dfrac{\partial S}{\partial t}$. If the Hamiltonian does not depend on the time explicitly, then the derivative $\dfrac{\partial S}{\partial t}$ in the Hamilton–Jacobi equation must be constant, usually denoted by $-h$, consequently

$$S = W(z_1, z_2, \ldots, z_M) - ht. \tag{12}$$

If the function W can be separated completely into M functions of the form $W_m(z_m, \alpha_1, \alpha_2 \ldots \alpha_M)$ for $m = 1, \ldots, M$, i.e.

$$S = W_1(z_1, \alpha_1, \alpha_2 \ldots \alpha_M) + \cdots + W_M(z_M, \alpha_1, \alpha_2 \ldots \alpha_M) - ht,$$

then we say that the Hamiltonian system is *integrable by separation of variables* or *integrable in the Jacobi sense*.

When the Hamiltonian H does not depend on t from (11) and (12) the Hamilton–Jacobi equation reduces to Eq. (1).

3. Properties of the Hamiltonian systems satisfying Levi-Civita conditions

The aim of this section is to prove some properties of the Hamiltonian systems which admit a separation of variables. Differential system (3) has the following properties.

(I) Partial differential system (3) can be written in the following equivalent forms

(i)

$$\begin{pmatrix} \dfrac{\partial H}{\partial P_k} & \dfrac{\partial H}{\partial P_j} & \dfrac{\partial H}{\partial z_k} & \dfrac{\partial H}{\partial z_j} \end{pmatrix} \begin{pmatrix} 0 & \dfrac{\partial^2 H}{\partial z_j \partial z_k} & 0 & -\dfrac{\partial^2 H}{\partial P_j \partial z_k} \\ 0 & 0 & -\dfrac{\partial^2 H}{\partial z_j \partial P_k} & 0 \\ 0 & 0 & 0 & \dfrac{\partial^2 H}{\partial P_j \partial P_k} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial H}{\partial P_k} \\ \dfrac{\partial H}{\partial P_j} \\ \dfrac{\partial H}{\partial z_k} \\ \dfrac{\partial H}{\partial z_j} \end{pmatrix} = 0,$$

(ii) By considering that (3) can be rewritten as

$$\left(\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_k \partial P_j} - \frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_k \partial P_j}\right)\frac{\partial H}{\partial z_j} - \left(\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_k \partial z_j} - \frac{\partial H}{\partial z_k}\frac{\partial^2 H_1}{\partial P_k \partial z_j}\right)\frac{\partial H}{\partial P_j} = 0,$$

and introducing the vector field $\Gamma_k = \frac{\partial H}{\partial P_k}\frac{\partial}{\partial z_k} - \frac{\partial H}{\partial z_k}\frac{\partial}{\partial P_k}$ (see formula (5)) we easily deduced that Eqs. (3) can be written as

$$\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j} - \Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j} = 0. \tag{13}$$

(iii) Denote by

$$R_j = -\frac{\partial H}{\partial z_j}\Big/\frac{\partial H}{\partial P_j}, \tag{14}$$

then differential system (3) can be written as

$$\Gamma_k R_j = 0, \qquad \text{where} \qquad k, j = 1, \ldots, M, \quad j \neq k. \tag{15}$$

Indeed, after some computations we can show that

$$\begin{aligned}\Gamma_k R_j &= \frac{\partial H}{\partial P_k}\frac{\partial}{\partial z_k}(R_j) - \frac{\partial H}{\partial z_k}\frac{\partial}{\partial P_k}(R_j)\\ &= -\left(\frac{\partial H}{\partial P_j}\right)^{-2}\left(\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j} - \Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}\right) = 0.\end{aligned}$$

(II) **Proposition 4.** *Let $H = H(z_1, \ldots, z_M, P_1, \ldots, P_M)$ be a solution of (3). Then $H = H(P_1, \ldots, P_M, z_1, \ldots, z_M)$ is also a solution of (3).*
Proof. It is evident. □
We observe that this property is important because if we have a Hamiltonian which is integrable in the Jacobi sense, then the same Hamiltonian under the change $z_j \longleftrightarrow P_j$ is integrable in the Jacobi sense.

(III) **Proposition 5.** *Let H be a solution of (3). Then $F(H)$ satisfies (3) for arbitrary C^2 function F.*
Proof. After some computations it is easy to show that

$$\begin{aligned}\frac{\partial F(H)}{\partial P_j} &= \frac{\partial F(H)}{\partial H}\frac{\partial H}{\partial P_j}, \quad \frac{\partial F(H)}{\partial z_j} = \frac{\partial F(H)}{\partial H}\frac{\partial H}{\partial z_j},\\ \frac{\partial^2 F(H)}{\partial P_k \partial P_j} &= \frac{\partial F(H)}{\partial H}\frac{\partial^2 H}{\partial P_k \partial P_j} + \frac{\partial^2 F(H)}{\partial H \partial H}\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial P_k},\\ \frac{\partial^2 F(H)}{\partial P_k \partial z_j} &= \frac{\partial F(H)}{\partial H}\frac{\partial^2 H}{\partial P_k \partial z_j} + \frac{\partial^2 F(H)}{\partial H \partial H}\frac{\partial H}{\partial z_j}\frac{\partial H}{\partial P_k},\\ \frac{\partial^2 F(H)}{\partial z_k \partial z_j} &= \frac{\partial F(H)}{\partial H}\frac{\partial^2 H}{\partial z_k \partial z_j} + \frac{\partial^2 F(H)}{\partial H \partial H}\frac{\partial H}{\partial z_j}\frac{\partial H}{\partial z_k}.\end{aligned}$$

Hence after some computations we get that

$$\begin{aligned}&\frac{\partial F(H)}{\partial P_j}\frac{\partial F(H)}{\partial P_k}\frac{\partial^2 F(H)}{\partial z_j \partial z_k} + \frac{\partial F(H)}{\partial z_j}\frac{\partial F(H)}{\partial z_k}\frac{\partial^2 F(H)}{\partial P_j \partial P_k}\\ &-\frac{\partial F(H)}{\partial P_j}\frac{\partial F(H)}{\partial z_k}\frac{\partial^2 H}{\partial z_j \partial P_k} - \frac{\partial F(H)}{\partial P_k}\frac{\partial F(H)}{\partial z_j}\frac{\partial^2 F(H)}{\partial P_j \partial z_k}\\ &= \left(\frac{\partial F(H)}{\partial H}\right)^3\left(\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_j \partial z_k} + \frac{\partial H}{\partial z_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_j \partial P_k}\right.\\ &\left.-\frac{\partial H}{\partial P_j}\frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial z_j \partial P_k} - \frac{\partial H}{\partial P_k}\frac{\partial H}{\partial z_j}\frac{\partial^2 H}{\partial P_j \partial z_k}\right).\end{aligned}$$

Consequently in view of (3) we obtain that

$$\begin{aligned}&\frac{\partial F(H)}{\partial P_j}\frac{\partial F(H)}{\partial P_k}\frac{\partial^2 F(H)}{\partial z_j \partial z_k} + \frac{\partial F(H)}{\partial z_j}\frac{\partial F(H)}{\partial z_k}\frac{\partial^2 F(H)}{\partial P_j \partial P_k}\\ &-\frac{\partial F(H)}{\partial P_j}\frac{\partial F(H)}{\partial z_k}\frac{\partial^2 F(H)}{\partial z_j \partial P_k} - \frac{\partial F(H)}{\partial P_k}\frac{\partial F(H)}{\partial z_j}\frac{\partial^2 F(H)}{\partial P_j \partial z_k} = 0,\end{aligned}$$

i.e. since the function H satisfies the Levi-Civita conditions, then the new Hamiltonian $F(H)$ is integrable in the Jacobi sense. □

(IV) **Proposition 6.** *Consider the functions*

$$U = \sum_{j=1}^{m}\alpha_j(z_j, P_j), \qquad V = \sum_{j=1}^{m}\beta_j(z_j, P_j),$$

where $\alpha_j(z_j, P_j)$ and $\beta_j(z_j, P_j)$ are C^2 functions which satisfy the conditions

$$\{\alpha_j,\ \beta_j\} := \frac{\partial\alpha_j}{\partial P_j}\frac{\partial\beta_j}{\partial z_j} - \frac{\partial\alpha_j}{\partial z_j}\frac{\partial\beta_j}{\partial P_j} \neq 0,$$

for $j = 1, \ldots, m$. Let F and G be C^2 functions.

(i) *Then the Hamiltonian vector Γ_H with $H = F(U) + G(V)$ is integrable in the Jacobi sense if and only if*

$$F(U) = -\lambda \log|U + a|\,, \qquad G(V) = \lambda \log|V + b|\,, \tag{16}$$

where a, b and λ are arbitrary constants. Consequently

$$H(U, V) = \log\left|\frac{V + b}{U + a}\right|^{\lambda}.$$

(ii) *Then the Hamiltonian vector Γ_H with $H = F(U)G(V)$ is integrable in the Jacobi sense if and only if*

$$F(U) = (U + a)^{\lambda}\,, \qquad G(V) = (V + b)^{-\lambda} \tag{17}$$

where a, b and λ are arbitrary constants. Consequently

$$H(U, V) = \left(\frac{U + a}{V + b}\right)^{\lambda}.$$

Proof. For statement (i), we insert $H = F(U) + G(V)$ into the Levi-Civita conditions (3) and after some computations we get that

$$\{\alpha_j,\ \beta_j\}\{\alpha_k,\ \beta_k\}\left(\left(\frac{\partial F}{\partial U}\right)^2 \frac{\partial^2 G}{\partial V\partial V} + \left(\frac{\partial G}{\partial V}\right)^2 \frac{\partial^2 F}{\partial U\partial U}\right) = 0,$$

for $j, k = 1, \ldots, M$ and $j \neq k$. Hence, from the previous equation we obtain

$$-\frac{\dfrac{\partial^2 G}{\partial V\partial V}}{\left(\dfrac{\partial G}{\partial V}\right)^2} = \frac{\dfrac{\partial^2 F}{\partial U\partial U}}{\left(\dfrac{\partial F}{\partial U}\right)^2} = \frac{1}{\lambda},$$

or equivalently

$$-\frac{\dfrac{\partial^2 G}{\partial V\partial V}}{\left(\dfrac{\partial G}{\partial V}\right)^2} = \frac{\partial}{\partial V}\left(\frac{\partial G}{\partial V}\right)^{-1} = \frac{1}{\lambda},$$

$$\frac{\dfrac{\partial^2 F}{\partial U\partial U}}{\left(\dfrac{\partial F}{\partial U}\right)^2} = -\frac{\partial}{\partial V}\left(\frac{\partial F}{\partial U}\right)^{-1} = \frac{1}{\lambda}.$$

After integration of these partial differential equations we obtain (16). So statement (i) is proved.
For statement (ii), we insert $H = F(U)G(V)$ into the Levi-Civita conditions (3) and after some computations we get that

$$\{\alpha_j,\ \beta_j\}\{\alpha_k,\ \beta_k\}\left(\left(\frac{\partial F}{\partial U}\right)^2\left(G\frac{\partial^2 G}{\partial V\partial V} - \left(\frac{\partial G}{\partial V}\right)^2\right) + \left(\frac{\partial G}{\partial V}\right)^2\left(F\frac{\partial^2 F}{\partial U\partial U} - \left(\frac{\partial F}{\partial U}\right)^2\right)\right) = 0.$$

Thus

$$\begin{aligned} G\frac{\partial^2 G}{\partial V\partial V} - \left(\frac{\partial G}{\partial V}\right)^2 &= \frac{1}{\lambda}\left(\frac{\partial G}{\partial V}\right)^2, \\ F\frac{\partial^2 F}{\partial U\partial U} - \left(\frac{\partial F}{\partial U}\right)^2 &= -\frac{1}{\lambda}\left(\frac{\partial F}{\partial U}\right)^2. \end{aligned}$$

Consequently

$$\frac{\dfrac{\partial^2 G}{\partial V\partial V}}{\dfrac{\partial G}{\partial V}} = \left(1 + \frac{1}{\lambda}\right)\frac{\dfrac{\partial G}{\partial V}}{G}, \qquad \frac{\dfrac{\partial^2 F}{\partial U\partial U}}{\dfrac{\partial F}{\partial U}} = \left(1 - \frac{1}{\lambda}\right)\frac{\dfrac{\partial F}{\partial U}}{F}.$$

Hence after integration we get (17). Thus the proof of the proposition is done. □

By considering that the Hamiltonian system with Hamiltonian $\dfrac{U + a}{V + b}$ is integrable in the Jacobi sense, then from Proposition 5 we get that any function $F\left(\dfrac{U + a}{V + b}\right)$ is integrable in the Jacobi sense, where F is an arbitrary C^2 function.

Problem 7. *Assume that the Hamiltonian systems with Hamiltonian*

$$F = F(z_1, \dots, z_M, P_1, \dots, P_M) \quad \text{and} \quad G = G(z_1, \dots, z_M, P_1, \dots, P_M)$$

are integrable in the Jacobi sense. Establish the conditions on F and G under which F + G and FG are integrable in the Jacobi sense.

(V) The *Lie algebra* is a vector space $\mathfrak{g}$ together with a bilinear map $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$; $(X, Y) \longrightarrow [X, Y] = XY - YX$, called the Lie bracket, satisfying the Jacobi identity

$$[Z, [X, Y]] + [X, [Y, Z]] + [Y, [Z, X]] = 0,$$

for arbitrary $X, Y, Z \in \mathfrak{g}$. In particular the set of vector fields on a manifold $\mathfrak{M}$ is a Lie algebra.

Proposition 8. *Assume that the Hamiltonian vector field $\Gamma_H = \sum_{j=1}^{M} \Gamma_j$, given in (5), is integrable in the Jacobi sense, then $[\Gamma_j, \Gamma_n](H) = 0$. In other words if the Hamiltonian vector field Γ_H is integrable in the Jacobi sense then Γ_j, Γ_k and $[\Gamma_j, \Gamma_k]$ are tangents to the hypersurface $H = h$. Here j, k and n vary in $\{1, \dots, M\}$.*

Proof. Consider

$$\begin{aligned}
[\Gamma_j, \Gamma_k](f) = & \left(\frac{\partial H}{\partial P_j}\frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial}{\partial P_j}\right)\left(\frac{\partial H}{\partial P_k}\frac{\partial}{\partial z_k} - \frac{\partial H}{\partial z_k}\frac{\partial}{\partial P_k}\right)(f) \\
& - \left(\frac{\partial H}{\partial P_k}\frac{\partial}{\partial z_k} - \frac{\partial H}{\partial z_k}\frac{\partial}{\partial P_k}\right)\left(\frac{\partial H}{\partial P_j}\frac{\partial}{\partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial}{\partial P_j}\right)(f) \\
= & \left(\frac{\partial H}{\partial P_j}\frac{\partial^2 H}{\partial P_k \partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial^2 H}{\partial P_k \partial P_j}\right)\frac{\partial f}{\partial z_k} - \left(\frac{\partial H}{\partial P_j}\frac{\partial^2 H}{\partial z_k \partial z_j} - \frac{\partial H}{\partial z_j}\frac{\partial^2 H}{\partial P_j \partial z_k}\right)\frac{\partial f}{\partial P_k} \\
& - \left(\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_k \partial P_j} - \frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_j \partial P_k}\right)\frac{\partial f}{\partial z_j} - \left(\frac{\partial H}{\partial P_k}\frac{\partial^2 H}{\partial z_k \partial z_j} - \frac{\partial H}{\partial z_k}\frac{\partial^2 H}{\partial P_k \partial z_j}\right)\frac{\partial f}{\partial P_j},
\end{aligned}$$

where $f = f(z_1, \dots, z_m, P_1, \dots, P_M)$ is an arbitrary C^2 function, or equivalently

$$[\Gamma_j, \Gamma_k](f) = \left(\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)\frac{\partial f}{\partial z_k} - \Gamma_k\left(\frac{\partial H}{\partial z_k}\right)\frac{\partial f}{\partial P_k}\right) - \left(\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial f}{\partial z_j} - \Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial f}{\partial P_j}\right). \tag{18}$$

Hence taking $f = H$ from (13) we get that

$$[\Gamma_j, \Gamma_k](H) = \left(\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)\frac{\partial H}{\partial z_k} - \Gamma_k\left(\frac{\partial H}{\partial z_k}\right)\frac{\partial H}{\partial P_k}\right) - \left(\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j} - \Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}\right) = 0.$$

Since clearly $\Gamma_j(H) = \Gamma_k(H) = 0$, from the definitions of Γ_j and Γ_k the proposition is proved. □

4. Integrability in the Jacobi and Frobenius sense

Let $\mathfrak{D}$ be a differential system in a manifold $\mathfrak{M}$ and $V(\mathfrak{D})$ be the set of vector fields X such that $X(y) \in \mathfrak{D}$ for all $y \in \mathfrak{M}$ (see for instance [18]).

Theorem 9 (*Frobenius Theorem*)**.** *Differential system $\mathfrak{D}$ is completely integrable if and only if $V(\mathfrak{D})$ is a Lie algebra, i.e. $[X, Y] \in V(\mathfrak{D})$ for all vector fields $X, Y \in V(\mathfrak{D})$, or what is equivalent if $X_1, \dots, X_M$ generate $V(\mathfrak{D})$. Then there exist functions $C_{jn}^k = -C_{nj}^k$ such that $[X_j, X_n] = \sum_{k=1}^{M} C_{jn}^k X_k$, for $j, n = 1, \dots, M$.*

Let Γ_j be the Hamiltonian vector fields given in (5). We say that the Hamiltonian vector field $\Gamma_H = \sum_{j=1}^{M} \Gamma_j$ is *integrable in the Frobenius sense* if and only the vector fields Γ_j for $j, n = 1, \dots, M$ are such that $[\Gamma_j, \Gamma_n] = \sum_{k=1}^{M} C_{jn}^k \Gamma_k$. The Hamiltonian vector field $\Gamma_H = \sum_{j=1}^{M} \Gamma_j$ is integrable in the Jacobi sense if and only the vector fields Γ_j for $j = 1, \dots, M$ satisfy (15).

The next result show that a Hamiltonian is integrable in the Jacobi sense if and only if is integrable in the Frobenius sense.

Theorem 10. *The Hamiltonian vector field $\Gamma_H = \sum_{j=1}^{M} \Gamma_j$ is integrable in the Jacobi sense if and only if it is integrable in the Frobenius sense.*

Proof. From the relation $\Gamma_j = \dfrac{\partial H}{\partial P_j}\dfrac{\partial}{\partial z_j} - \dfrac{\partial H}{\partial z_j}\dfrac{\partial}{\partial P_j}$ it follows that

$$\frac{\partial}{\partial z_j} = \frac{\Gamma_j + \dfrac{\partial H}{\partial z_j}\dfrac{\partial}{\partial P_j}}{\dfrac{\partial H}{\partial P_j}}, \tag{19}$$

Inserting $\dfrac{\partial}{\partial z_j}$ into (18) we obtain

$$\begin{aligned}[\Gamma_j,\ \Gamma_k]= &\left(\Gamma_k\log\left|\frac{\partial H}{\partial P_j}\right|\right)\Gamma_j-\left(\Gamma_j\log\left|\frac{\partial H}{\partial P_k}\right|\right)\Gamma_k\\ &+\left(\left(\Gamma_k\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}-\left(\Gamma_k\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}\right)\frac{\partial}{\partial P_j}\\ &-\left(\left(\Gamma_j\frac{\partial H}{\partial P_k}\right)\frac{\partial H}{\partial z_k}-\left(\Gamma_j\frac{\partial H}{\partial z_k}\right)\frac{\partial H}{\partial P_k}\right)\frac{\partial}{\partial P_k}.\end{aligned}\tag{20}$$

Thus assuming that Γ_H is integrable in the Frobenius sense, i.e. $[\Gamma_j,\ \Gamma_k]=\sum_{n=1}^{M}C_{jk}^n\Gamma_n$, and consequently the coefficients of $\dfrac{\partial H}{\partial P_k}$ and $\dfrac{\partial H}{\partial P_k}$ must be zero, i.e. $\Gamma_j\left(\dfrac{\partial H}{\partial P_k}\right)\dfrac{\partial H}{\partial z_k}-\Gamma_j\left(\dfrac{\partial H}{\partial z_k}\right)\dfrac{\partial H}{\partial P_k}=0$. Hence in view of property (I) (see formula (13)) we get that Γ_H is integrable in the Jacobi sense.

The reciprocity it is easy to obtain, indeed from (20) it follows that if Γ_H is integrable in Jacobi sense then (13) holds, consequently we obtain that

$$[\Gamma_j,\ \Gamma_k]=\Gamma_k\left(\log\left|\frac{\partial H}{\partial P_j}\right|\right)\Gamma_j-\Gamma_j\left(\log\left|\frac{\partial H}{\partial P_k}\right|\right)\Gamma_k=\sum_{n=1}^{M}C_{jk}^n\Gamma_n.\tag{21}$$

Hence we get that the functions C_{jk}^n are such that $C_{jk}^n=0$ if $n\neq j$ or $n\neq k$ and

$$C_{jk}^j=\Gamma_k\left(\log\left|\frac{\partial H}{\partial P_j}\right|\right),\qquad C_{jk}^k:=-\Gamma_j\left(\log\left|\frac{\partial H}{\partial P_k}\right|\right).\tag{22}$$

In short the proposition is proved. □

Remark 11. The proof of Theorem 10 also can be done using

$$\frac{\partial}{\partial P_j}=\frac{-\Gamma_j+\dfrac{\partial H}{\partial P_j}\dfrac{\partial}{\partial z_j}}{\dfrac{\partial H}{\partial z_j}}$$

instead of (19). Indeed inserting $\dfrac{\partial}{\partial P_j}$ into (18) we obtain

$$\begin{aligned}[\Gamma_j,\ \Gamma_k]= &\ \Gamma_k\left(\log|\frac{\partial H}{\partial z_j}|\right)\Gamma_j-\Gamma_j\left(\log|\frac{\partial H}{\partial z_k}|\right)\Gamma_k\\ &+\left(\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}-\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}\right)\frac{\partial}{\partial z_j}\\ &-\left(\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)\frac{\partial H}{\partial z_k}-\Gamma_j\left(\frac{\partial H}{\partial z_k}\right)\frac{\partial H}{\partial P_k}\right)\frac{\partial}{\partial z_k}.\end{aligned}$$

Thus if Γ_H is integrable in the Frobenius sense, then $[\Gamma_j,\ \Gamma_k]=C_{jk}^n\Gamma_n$ and consequently $\Gamma_j\left(\dfrac{\partial H}{\partial P_k}\right)\dfrac{\partial H}{\partial z_k}-\Gamma_j\left(\dfrac{\partial H}{\partial z_k}\right)\dfrac{\partial H}{\partial P_k}=0$, i.e. Γ_H is integrable in the Jacobi sense. Hence we obtain that

$$[\Gamma_j,\ \Gamma_k]=\Gamma_k\left(\log\left|\frac{\partial H}{\partial z_j}\right|\right)\Gamma_j-\Gamma_j\left(\log\left|\frac{\partial H}{\partial z_k}\right|\right)\Gamma_k,\tag{23}$$

Consequently the functions C_{jk}^n are such that $C_{jk}^n=0$ if $n\neq j$ or $n\neq k$ and

$$C_{jk}^j=\Gamma_k\left(\log\left|\frac{\partial H}{\partial z_j}\right|\right),\qquad C_{jk}^k=-\Gamma_j\left(\log\left|\frac{\partial H}{\partial z_k}\right|\right).$$

Proposition 12. *The system*

$$\begin{aligned}\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)&=-C_{jk}^k\frac{\partial H}{\partial P_k}, &\Gamma_j\left(\frac{\partial H}{\partial z_k}\right)&=-C_{jk}^k\frac{\partial H}{\partial z_k},\\ \Gamma_k\left(\frac{\partial H}{\partial P_j}\right)&=C_{jk}^j\frac{\partial H}{\partial P_j}, &\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)&=C_{jk}^j\frac{\partial H}{\partial z_j},\end{aligned}\tag{24}$$

can be written in the matrix form

$$W\,\xi=\mathbf{0},\tag{25}$$

where

$$W=\begin{pmatrix}\dfrac{\partial^2 H}{\partial P_k\partial z_j} & -\dfrac{\partial^2 H}{\partial P_k\partial P_j} & C^j_{jk} & 0\\ \dfrac{\partial^2 H}{\partial z_k\partial z_j} & -\dfrac{\partial^2 H}{\partial P_j\partial z_k} & 0 & C^j_{jk}\\ -C^k_{jk} & 0 & \dfrac{\partial^2 H}{\partial P_j\partial z_k} & -\dfrac{\partial^2 H}{\partial P_k\partial P_j}\\ 0 & -C^k_{jk} & \dfrac{\partial^2 H}{\partial z_k\partial z_j} & -\dfrac{\partial^2 H}{\partial P_k\partial z_j}\end{pmatrix}$$

and $\xi=\left(\dfrac{\partial H}{\partial P_j},\dfrac{\partial H}{\partial z_j},\dfrac{\partial H}{\partial P_k},\dfrac{\partial H}{\partial z_k},\right)^T$. *Moreover the differential system* (25) *has non-trivial solutions if and only if*

$$\det(W)=\left(-C^k_{jk}C^j_{jk}+\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}\right)^2=0.$$

Proof. From (25) by considering that the vector ξ is a non-zero vector, then in view of the relation

$$\det W=\left(-C^k_{jk}C^j_{jk}+\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}\right)^2,$$

we obtain the proof of the proposition. □

Theorem 13. *Assume that*

$$\left(\frac{\partial^2 H}{\partial P_j\partial P_k}\right)^2+\left(\frac{\partial^2 H}{\partial z_k\partial z_j}\right)^2\neq 0,\quad \textit{for}\quad j\neq k, \tag{26}$$

then the Levi-Civita second order partial differential equations (3) *are equivalent to the equations*

$$-C^k_{jk}C^j_{jk}+\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}=0, \tag{27}$$

for $j,k=1,\dots,M$ *and* $j\neq k$.

Proof. The following identities hold

$$\begin{aligned}&\frac{\partial^2 H}{\partial P_k\partial P_j}\left(\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}-\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}\right)\\ &\quad=\frac{\partial H}{\partial P_k}\frac{\partial H}{\partial P_j}\left(-C^k_{jk}C^j_{jk}+\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}\right),\\ &\frac{\partial^2 H}{\partial z_k\partial z_j}\left(\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}-\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}\right)\\ &\quad=-\frac{\partial H}{\partial z_k}\frac{\partial H}{\partial z_j}\left(-C^k_{jk}C^j_{jk}+\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}\right),\end{aligned}$$

for $j\neq k$. Indeed, from the relations

$$\begin{aligned}&\frac{\partial^2 H}{\partial P_k\partial P_j}\left(\Gamma_k\left(\frac{\partial H}{\partial z_j}\right)\frac{\partial H}{\partial P_j}-\Gamma_k\left(\frac{\partial H}{\partial P_j}\right)\frac{\partial H}{\partial z_j}\right)\\ &=\frac{\partial H}{\partial P_k}\frac{\partial H}{\partial P_j}\left(\frac{\partial^2 H}{\partial P_j\partial P_k}\frac{\partial^2 H}{\partial z_k\partial z_j}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}\right)+\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)\Gamma_k\left(\frac{\partial H}{\partial P_j}\right),\end{aligned}$$

and in view of (22) we get that

$$\Gamma_j\left(\frac{\partial H}{\partial P_k}\right)\left(\Gamma_k\frac{\partial H}{\partial P_j}\right)=-C^k_{jk}C^j_{jk}\frac{\partial H}{\partial P_k}\frac{\partial H}{\partial P_j},$$

where Γ_j are the vector fields given in (5).

Hence we obtain the proof of the first identity. The proof of the second identity is obtained in analogous way.

From these two identities under condition (26) we get that the Levi-Civita conditions or what is equivalent the conditions (13) are equivalent to (27). Thus the proof of theorem is done. □

Proposition 14. *The following equations hold*

$$\Gamma_m C^j_{jk} + \Gamma_k C^j_{mj} + C^j_{mj} C^j_{kj} + C^j_{jk} C^j_{mj} + C^k_{km} C^j_{jk} + C^m_{km} C^j_{jm} = 0 \tag{28}$$

for $j, n = 1, \ldots, M$ with $n \neq j$. Moreover if C^j_{mk} =constants for $j, k, m = 1, \ldots, N$, then $C^k_{jk} C^j_{jk} = 0$, for $j \neq k$.

Proof. From the Jacobi identity

$$[\Gamma_m, [\Gamma_j, \Gamma_k]] + [\Gamma_j, [\Gamma_k, \Gamma_m]] + [\Gamma_k, [\Gamma_m, \Gamma_j]] = 0,$$

and after some computations we get that

$$\begin{aligned}
[\Gamma_m, [\Gamma_j, \Gamma_k]] = {} & \Gamma_m \left(C^j_{jk}\Gamma_j + C^k_{jk}\Gamma_k\right) - \left(C^j_{jk}\Gamma_j + C^k_{jk}\Gamma_k\right)\Gamma_m \\
= {} & \left(\Gamma_m C^j_{jk}\right)\Gamma_j + \left(\Gamma_m C^k_{jk}\right)\Gamma_k + C^j_{jk}[\Gamma_m, \Gamma_j] + C^k_{jk}[\Gamma_m, \Gamma_k] \\
= {} & \left(\Gamma_m C^j_{jk}\right)\Gamma_j + \left(\Gamma_m C^k_{jk}\right)\Gamma_k + C^j_{jk}(C^m_{mj}\Gamma_m + C^j_{mj}\Gamma_j) \\
& + C^k_{jk}(C^m_{mk}\Gamma_m + C^k_{mk}\Gamma_k) \\
= {} & \left(\Gamma_m C^j_{jk} + C^j_{jk} C^j_{mj}\right)\Gamma_j + \left(\Gamma_m C^k_{jk} + C^k_{jk} C^k_{mk}\right)\Gamma_k \\
& + \left(C^j_{jk} C^m_{mj} + C^k_{jk} C^m_{mk}\right)\Gamma_m .
\end{aligned}$$

Consequently

$$\begin{aligned}
[\Gamma_k, [\Gamma_m, \Gamma_j]] = {} & \left(\Gamma_k C^m_{mj} + C^m_{mj} C^k_{kj}\right)\Gamma_m + \left(\Gamma_k C^j_{mj} + C^j_{mj} C^j_{kj}\right)\Gamma_j \\
& + \left(C^m_{mj} C^k_{kj} + C^j_{jm} C^k_{kj}\right)\Gamma_k, \\
[\Gamma_j, [\Gamma_k, \Gamma_m]] = {} & \left(\Gamma_j C^k_{km} + C^k_{km} C^j_{jk}\right)\Gamma_k + \left(\Gamma_j C^m_{km} + C^m_{km} C^m_{jm}\right)\Gamma_m \\
& + \left(C^k_{km} C^j_{jk} + C^m_{km} C^j_{jm}\right)\Gamma_j,
\end{aligned}$$

Hence, inserting these expression into the Jacobi identity we get

$$\begin{aligned}
0 = {} & [\Gamma_m, [\Gamma_j, \Gamma_k]] + [\Gamma_j, [\Gamma_k, \Gamma_m]] + [\Gamma_k, [\Gamma_m, \Gamma_j]] \\
= {} & \left(\Gamma_m C^j_{jk} + \Gamma_k C^j_{mj} + C^j_{mj} C^j_{kj} + C^j_{jk} C^j_{mj} + C^k_{km} C^j_{jk} + C^m_{km} C^j_{jm}\right)\Gamma_j + \cdots
\end{aligned}$$

Hence we get (28). The proof of the second statement we obtain by putting $m = j$ in (28). In short the proposition is proved. □

Corollary 15. *Under the assumptions of Proposition 14 if C^j_{nk} is a constant for $j, n, k = 1, \ldots, N$ then*

$$C^j_{jk} C^k_{jk} = 0 \quad \text{for all} \quad j \neq k.$$

Proof. It follows from (28) by considering that C^j_{nk} = constant for all j, n, k and by putting $n = j$. □

Proposition 16. *Relations (24) are equivalent to the equations*

$$\begin{aligned}
\left(\frac{\partial H}{\partial P_j}\right)^2 \frac{\partial R_j}{\partial P_k} = {} & C^k_{jk}\frac{\partial H}{\partial P_k}, \quad \left(\frac{\partial H}{\partial P_j}\right)^2 \frac{\partial R_j}{\partial z_k} = C^k_{jk}\frac{\partial H}{\partial z_k}, \\
\left(\frac{\partial H}{\partial P_k}\right)^2 \frac{\partial R_k}{\partial P_j} = {} & C^j_{jk}\frac{\partial H}{\partial P_j}, \quad \left(\frac{\partial H}{\partial P_k}\right)^2 \frac{\partial R_k}{\partial z_j} = C^j_{jk}\frac{\partial H}{\partial z_j},
\end{aligned} \tag{29}$$

with $j \neq k$, where the functions R_j for $j = 1, \ldots, M$, are defined in (14) ,

Proof. From (24) we have

$$\left(\frac{\partial H}{\partial P_j}\right)^2 \frac{\partial}{\partial P_k}\left(R_j\right) = \left(\frac{\partial H}{\partial P_j}\right)^2 \frac{\partial}{\partial P_k}\left(-\frac{\dfrac{\partial H}{\partial z_j}}{\dfrac{\partial H}{\partial P_j}}\right) = -\Gamma_j\left(\frac{\partial H}{\partial P_k}\right) = C^k_{jk}\frac{\partial H}{\partial P_k}.$$

Hence after some similar computations we obtain the proof of the proposition. □

5. Integrability in the Jacobi sense in some particular cases

Assume that the Hamiltonian H satisfies the conditions

$$\frac{\partial^2 H}{\partial P_j \partial P_k} \neq 0, \quad \text{for all} \quad j \neq k. \tag{30}$$

If conditions (30) hold then the Levi-Civita conditions (3) can be written as (27). In this section we shall study conditions (27) when

$$C^k_{jk} C^j_{jk} = 0, \qquad \text{for} \quad \text{for all} \quad j \neq k.$$

Therefore the Levi-Civita conditions (27) become

$$\frac{\partial^2 H}{\partial P_j \partial P_k} \frac{\partial^2 H}{\partial z_k \partial z_j} - \frac{\partial^2 H}{\partial P_k \partial z_j} \frac{\partial^2 H}{\partial z_k \partial P_j} = 0. \tag{31}$$

Theorem 17. *Assume that a Hamiltonian vector field Γ_H (see formula (5)) is integrable in the Jacobi sense. Then Γ_H admits a Lie algebra with the basis $\Gamma_1, \ldots, \Gamma_M$ such that*

$$[\Gamma_j, \Gamma_k] = 0, \quad \textit{for} \quad j, k = 1, \ldots, M, \quad j \neq k,$$

(i.e admits an Abelian Lie algebra) if and only if

$$H = H(H_1(z_1, P_1), H_2(z_2, P_2), \ldots, H_M(z_M, P_M)). \tag{32}$$

Proof. If H is given by the formula (32) then

$$\begin{aligned}
\frac{\partial H}{\partial P_j} &= \frac{\partial H}{\partial H_j}\frac{\partial H_j}{\partial P_j}, \quad \frac{\partial H}{\partial z_j} = \frac{\partial H}{\partial H_j}\frac{\partial H_j}{\partial z_j},\\
\frac{\partial^2 H}{\partial P_k \partial P_j} &= \frac{\partial^2 H}{\partial H_j \partial H_k}\frac{\partial H_j}{\partial P_j}\frac{\partial H_k}{\partial P_k}, \quad \frac{\partial^2 H}{\partial P_k \partial z_j} = \frac{\partial^2 H}{\partial H_j \partial H_k}\frac{\partial H_j}{\partial z_j}\frac{\partial H_k}{\partial P_k},\\
\frac{\partial^2 H}{\partial z_j \partial z_k} &= \frac{\partial^2 H}{\partial H_j \partial H_k}\frac{\partial H_j}{\partial z_j}\frac{\partial H_k}{\partial z_k},\\
\frac{\partial^2 H}{\partial z_j \partial P_k} &= \frac{\partial^2 H}{\partial H_j \partial H_k}\frac{\partial H_j}{\partial z_j}\frac{\partial H_k}{\partial P_k},
\end{aligned} \tag{33}$$

Consequently

$$\Gamma_k\left(\log\left|\frac{\partial H}{\partial P_j}\right|\right) = C^j_{jk} = 0, \quad \Gamma_j\left(\log\left|\frac{\partial H}{\partial P_k}\right|\right) = -C^k_{jk} = 0.$$

Hence by (23) we get that $[\Gamma_j, \Gamma_k] = 0$. Thus the Lie algebra is Abelian. The reciprocity can be obtained as follows. By considering that the vector fields $\Gamma_1, \ldots, \Gamma_M$ generate an Abelian Lie algebra, then $[\Gamma_k, \Gamma_j] = 0$, consequently from (23) we get $C^j_{jk} = C^k_{jk} = 0$. Therefore from (29) we obtain that

$$\frac{\partial R_j}{\partial P_k} = \frac{\partial R_k}{\partial P_j} = 0, \qquad \frac{\partial R_j}{\partial z_k} = \frac{\partial R_k}{\partial z_j} = 0, \quad \text{for all} \quad j \neq k,$$

Consequently $R_j = R(z_j, P_j)$ for $j = 1, \ldots, M$. Hence from (14) H must be such that

$$\frac{\partial H}{\partial z_j} + R_j(z_j, P_j)\frac{\partial H}{\partial P_j} = 0, \quad \text{for} \quad j = 1, \ldots, M.$$

The function H given by the formula (32) satisfies the previous equations. Indeed

$$R_j(z_j, P_j) = -\frac{\dfrac{\partial H}{\partial z_j}}{\dfrac{\partial H}{\partial P_j}} = -\frac{\dfrac{\partial H}{\partial H_j}\dfrac{\partial H_j}{\partial z_j}}{\dfrac{\partial H}{\partial H_j}\dfrac{\partial H_j}{\partial P_j}} = -\frac{\dfrac{\partial H_j}{\partial z_j}}{\dfrac{\partial H_j}{\partial P_j}},$$

where $H_j = H_j(z_j, P_j)$. Clearly that in view of (33) the Levi-Civita conditions (31) hold. In short the proposition is proved. □

Theorem 18. *Assume that a Hamiltonian vector field Γ_H is integrable in the Jacobi sense. Then Γ_H admits a Lie algebra satisfying*

$$\begin{aligned}
[\Gamma_j, \Gamma_k] = &\quad C^j_{jk}\Gamma_j, \quad \textit{for} \quad j, k = 1, \ldots, M, \quad j > k, \quad C^k_{jk} = 0,\\
[\Gamma_j, \Gamma_k] = &\quad C^k_{jk}\Gamma_j, \quad \textit{if} \quad C^j_{jk} = 0,
\end{aligned} \tag{34}$$

for $j, k = 1, \ldots, M$ and $j \neq k$, if and only if the Hamiltonian H is of the form

$$H = H_M(z_M, P_M, H_{M-1}(z_{M-1}, P_{M-1}, H_{M-2}(z_{M-2}, P_{M-2}, H_{M-3}(\ldots, H_1(z_1, P_1)\ldots)))). \tag{35}$$

Proof. We shall study only the first case. The second case is proved in analogous form.

From (23) and (34) we get that $C_{jk}^k = 0$ for $j > k$. Consequently from (29) it follows that

$$\begin{aligned}
\frac{\partial R_1}{\partial P_2} = \frac{\partial R_1}{\partial z_2} = \frac{\partial R_1}{\partial P_3} = \frac{\partial R_1}{\partial z_3} = \cdots = \frac{\partial R_1}{\partial P_M} = \frac{\partial R_1}{\partial z_M} &= 0,\\
\frac{\partial R_2}{\partial P_3} = \frac{\partial R_2}{\partial z_3} = \cdots = \frac{\partial R_2}{\partial P_M} = \frac{\partial R_2}{\partial z_M} &= 0,\\
\vdots \qquad \vdots \qquad &\vdots\\
\frac{\partial R_{M-1}}{\partial P_M} = \frac{\partial R_{M-1}}{\partial z_M} &= 0,
\end{aligned}$$

Thus

$$R_1 = R_1(z_1,\ P_1), \quad R_2 = R_2(z_2, P_2, z_1, P_1), \ldots, R_{M-1} = R_{M-1}(z_{M-1}, P_{M-1}, \ldots, z_1, P_1).$$

Hence from (14) we obtain

$$\begin{aligned}
\frac{\partial H}{\partial z_1} + R_1(z_1, P_1)\frac{\partial H}{\partial P_1} &= 0,\\
\frac{\partial H}{\partial z_2} + R_2(z_2, P_2, z_1, P_1)\frac{\partial H}{\partial P_2} &= 0,\\
\vdots \qquad \vdots \qquad &\vdots\\
\frac{\partial H}{\partial z_{M-1}} + R_{M-1}(z_{M-1}, P_{M-1}, \ldots, z_1, P_1)\frac{\partial H}{\partial P_{M-1}} &= 0,\\
\frac{\partial H}{\partial z_M} + R_M(z_M, P_M, z_{M-1}, P_{M-1}, \ldots, z_1, P_1)\frac{\partial H}{\partial P_M} &= 0,
\end{aligned}$$

The solution of this system is the function (35).

The reciprocity is obtained as follows. Assume that H is given by the formula (35) then the following relations hold

$$\begin{aligned}
\frac{\partial H}{\partial z_1} &= \frac{\partial H_1}{\partial z_1}\prod_{j=2}^{M}\frac{\partial H_j}{\partial H_{j-1}}, \qquad \frac{\partial H}{\partial P_1} = \frac{\partial H_1}{\partial P_1}\prod_{j=2}^{M}\frac{\partial H_j}{\partial H_{j-1}},\\
\frac{\partial H}{\partial z_2} &= \frac{\partial H_2}{\partial z_2}\prod_{j=3}^{M}\frac{\partial H_j}{\partial H_{j-1}}, \qquad \frac{\partial H}{\partial P_2} = \frac{\partial H_2}{\partial P_2}\prod_{j=3}^{M}\frac{\partial H_j}{\partial H_{j-1}},\\
\vdots \qquad &\vdots \qquad \vdots\\
\frac{\partial H}{\partial z_{M-1}} &= \frac{\partial H_{M-1}}{\partial z_{M-1}}\frac{\partial H_M}{\partial H_{M-1}}, \qquad \frac{\partial H}{\partial P_{M-1}} = \frac{\partial H_{M-1}}{\partial z_{P-1}}\frac{\partial H_M}{\partial H_{M-1}},\\
\frac{\partial H}{\partial z_M} &= \frac{\partial H_M}{\partial z_M}, \qquad \frac{\partial H}{\partial P_M} = \frac{\partial H_M}{\partial z_{P_M}}.
\end{aligned} \tag{36}$$

Thus again from (14) we get that

$$\begin{aligned}
R_1(z_1, P_1) &= -\frac{\dfrac{\partial H_1(z_1,P_1)}{\partial z_1}\prod_{j=2}^{M}\dfrac{\partial H_j}{\partial H_{j-1}}}{\dfrac{\partial H_1(z_1,P_1)}{\partial P_1}\prod_{j=2}^{M}\dfrac{\partial H_j}{\partial H_{j-1}}}\\
&= -\frac{\dfrac{\partial H_1(z_1,P_1)}{\partial z_1}}{\dfrac{\partial H_1(z_1,P_1)}{\partial P_1}},\\
R_2(z_2, P_2, z_1, P_1) &= -\frac{\dfrac{\partial H_2(z_2,P_2,H_1(z_1,P_1))}{\partial z_2}\prod_{j=3}^{M}\dfrac{\partial H_j}{\partial H_{j-1}}}{\dfrac{\partial H_2(z_2,P_2,H_1(z_1,P_1))}{\partial P_2}\prod_{j=3}^{M}\dfrac{\partial H_j}{\partial H_{j-1}}}\\
&= -\frac{\dfrac{\partial H_2(z_2,P_2,H_1(z_1,P_1))}{\partial z_2}}{\dfrac{\partial H_2(z_2,P_2,H_1(z_1,P_1))}{\partial P_2}},\\
\vdots \qquad &\vdots \qquad \vdots
\end{aligned}$$

$$R_{M-1}(z_{M-1},P_{M-1},\dots,z_1,P_1) = -\frac{\dfrac{\partial H_{M-1}}{\partial z_{M-1}}\dfrac{\partial H_M}{\partial H_{M-1}}}{\dfrac{\partial H_{M-1}}{\partial P_{M-1}}\dfrac{\partial H_M}{\partial H_{M-1}}} = -\frac{\dfrac{\partial}{\partial z_{M-1}}H_{M-1}(z_{M-1},P_{M-1},H_{M-2}(\dots,H_1(z_1,P_1)\dots))}{\dfrac{\partial}{\partial P_{M-1}}H_{M-1}(z_{M-1},P_{M-1},H_{M-2}(\dots,H_1(z_1,P_1)\dots))},$$

$$R_M(z_M,P_M,\dots,z_1,P_1) = -\frac{\dfrac{\partial}{\partial z_M}H_M(z_M,P_M,H_{M-1}(\dots,H_1(z_1,P_1)\dots))}{\dfrac{\partial}{\partial P_M}H_M(z_M,P_M,H_{M-1}(\dots,H_1(z_1,P_1)\dots))}.$$

Hence

$$\frac{\partial R_k}{\partial P_j} = \frac{\partial R_k}{\partial z_j} = 0, \quad \text{for} \quad j > k.$$

From (29) we obtain that $C^j_{jk} = 0$ for $j > k$. Thus from (24) and (23) we get (34).

In view of (36) the function (35) satisfies the Levi-Civita conditions (31). Indeed, from (36) for $k > j$ it follows that

$$\begin{aligned}
\frac{\partial H}{\partial z_j} &= \frac{\partial H_j}{\partial z_j}\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}, \quad \frac{\partial H}{\partial P_j} = \frac{\partial H_j}{\partial P_j}\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}},\\
\frac{\partial^2 H}{\partial P_j \partial P_k} &= \frac{\partial H_j}{\partial P_j}\frac{\partial}{\partial P_k}\left(\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}\right), \quad \frac{\partial^2 H}{\partial z_j \partial z_k} = \frac{\partial H_j}{\partial z_j}\frac{\partial}{\partial z_k}\left(\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}\right),\\
\frac{\partial H}{\partial z_j} &= \frac{\partial H_j}{\partial z_j}\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}, \quad \frac{\partial H}{\partial P_j} = \frac{\partial H_j}{\partial P_j}\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}},\\
\frac{\partial^2 H}{\partial P_j \partial z_k} &= \frac{\partial H_j}{\partial P_j}\frac{\partial}{\partial z_k}\left(\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}\right), \quad \frac{\partial^2 H}{\partial z_j \partial P_k} = \frac{\partial H_j}{\partial z_j}\frac{\partial}{\partial P_k}\left(\prod_{n=j+1}^{M}\frac{\partial H_n}{\partial H_{n-1}}\right),
\end{aligned}$$

Consequently

$$\frac{\partial^2 H}{\partial P_j \partial P_k}\frac{\partial^2 H}{\partial z_k \partial z_j} - \frac{\partial^2 H}{\partial P_k \partial z_j}\frac{\partial^2 H}{\partial z_k \partial P_j} \equiv 0, \qquad \text{for} \quad k > j.$$

In short the proposition is proved. □

Remark 19. The functions (32) and (35) are well-known Hamiltonians (see for instance [17]). The originality of Theorems 17 and 18 consists in providing the relation with the Lie algebra of the Jacobi integrable vector field Γ_H.

5.1. Example. The stark problem in arbitrary dimensions

The main integrable problem of Celestial Mechanics is the Kepler problem, the Euler problem (two center Newtonian gravitational motion), and the Stark problem. This small number of integrable problems explains why the Stark problem, corresponding to the motion in a Newtonian gravitational field subjected to an additional uniform force of constant magnitude and direction, has received special attention over almost two and one-half centuries (see for instance [19–22]). The integrability in the Jacobi sense of two and three dimensional Stark problem was first established in [19,23]. Here we extend this problem to arbitrary dimension and show that it continues being integrable in the Jacobi sense.

The equations of motions of two-dimensional Stark problem are

$$\ddot{x} = -\frac{\mu\, x}{r^3}, \qquad \ddot{y} = -\frac{\mu\, y}{r^3} + \varepsilon,$$

where μ and ε are constants and $r = \sqrt{x^2+y^2}$, and for the three-dimensional case the equations of motions are

$$\ddot{x} = -\frac{\mu\, x}{r^3}, \qquad \ddot{y} = -\frac{\mu\, y}{r^3} \qquad \ddot{z} = -\frac{\mu\, z}{r^3} + \varepsilon, \tag{37}$$

where $r = \sqrt{x^2+y^2+z^2}$.

In [19] the integrals of motion are determined, and the resulting quadratures are analytically given. A complete list of exact, closed-form solutions is deduced in terms of elliptic functions.

Now we shall illustrate the integrability in the Jacobi sense for the Stark problem in dimension $N > 3$ whose kinetic energy for unit of mass is $T = \frac{1}{2}\sum_{j=1}^N \dot{x}_j^2$ and its potential function is $U = -\dfrac{\mu}{r} - \varepsilon\, x_N$, where $r = \sqrt{\sum_{j=1}^N x_j^2}$. The equations of motion are

$$\ddot{x}_j = -\frac{\mu\, x_j}{r^3}, \qquad \ddot{x}_N = -\frac{\mu\, x_N}{r^3} + \varepsilon,$$

for $j = 1, \ldots, N-1$. Clearly the Hamiltonian vector field Γ_H with $H = T + U$ is not integrable in Jacobi sense in cartesian coordinates. However the authors of [21,22] and others have found that it becomes separable in parabolic coordinates for $N = 2$ and $N = 3$. We developed these results and proved that Γ_H is integrable in the Jacobi sense for $N > 3$.

Proposition 20. *The Hamiltonian vector field Γ_H with*

$$H = \frac{1}{2}\sum_{j=1}^{N}\dot{x}_j^2 - \frac{\mu}{r} - \varepsilon\, x_N,$$

in coordinates

$$x = \xi\eta, \qquad y = \frac{\xi^2 - \eta^2}{2},$$

for $N = 2$ becomes

$$H = \frac{P_\xi^2 + P_\eta^2 - 4\mu - \varepsilon(\xi^4 - \eta^4)}{2(\xi^2 + \eta^2)}$$

and for $N > 2$ in coordinates

$$\begin{aligned}
x_1 = &\ \ \xi\eta\cos\alpha_1 \prod_{j=2}^{N-2}\sin\alpha_j,\\
x_2 = &\ \ \xi\eta\sin\alpha_1 \prod_{j=2}^{N-2}\sin\alpha_j,\\
x_3 = &\ \ \xi\eta\cos\alpha_2 \prod_{j=3}^{N-2}\sin\alpha_j,\\
x_4 = &\ \ \xi\eta\sin\alpha_2 \prod_{j=3}^{N-2}\sin\alpha_j,\\
\vdots \quad & \quad \vdots \quad \quad \vdots \quad \quad \vdots\\
x_{N-2} = &\ \ \xi\eta\cos\alpha_1\cos\alpha_{N-3}\sin\alpha_{N-2},\\
x_{N-1} = &\ \ \xi\eta\cos\alpha_{N-2},\\
x_N = &\ \ \frac{\xi^2 - \eta^2}{2},
\end{aligned}$$

becomes

$$H = \frac{P_\xi^2 + P_\eta^2 - 4\mu - \varepsilon(\xi^4 - \eta^4)}{2(\xi^2 + \eta^2)} + \frac{C_{N-2}}{\xi^2\eta^2}, \tag{38}$$

where

$$\begin{aligned}
C_{N-2} = &\ \ H_{N-2}\left(P_{N-2}, \alpha_{N-2}, H_{N-3}(P_{N-3}, \alpha_{N-3}, \ldots, H_1(P_1, \alpha_1)\ldots)\right),\\
H_1 = &\ \ P_1^2, \qquad H_j = P_j^2 + \frac{H_{j-1}}{\sin^2\alpha_j}, \quad \textit{for} \quad j = 2, \ldots, N-2.
\end{aligned}$$

Proof. For $N \geq 2$ we get that $r^2 = \sum_{j=1}^{N} x_j^2 = \left(\dfrac{\xi^2 + \eta^2}{2}\right)^2$. The kinetic energy T and potential energy U become

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2), \quad U = -\frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}\left(\xi^2 - \eta^2\right),$$

respectively for $N = 2$ and

$$\begin{aligned}
T = &\ \ \frac{1}{2}\sum_{j=1}^{N}\dot{x}_j^2\\
= &\ \ \frac{\xi^2\eta^2}{2}\left(\dot{\alpha}_1^2 \prod_{j=2}^{N-2}\sin^2\alpha_j + \dot{\alpha}_2^2 \prod_{j=3}^{N-2}\sin^2\alpha_j + \cdots + \dot{\alpha}_{N-2}^2\right) + \frac{(\xi^2 + \eta^2)}{2}(\dot{\xi}^2 + \dot{\eta}^2),\\
U = &\ \ -\frac{2\mu}{\xi^2 + \eta^2} - \frac{\varepsilon}{2}\left(\xi^2 - \eta^2\right),
\end{aligned}$$

for $N>2$, respectively. Consequently the Hamiltonian H in coordinates $(\xi,\eta,\alpha_1,\dots,\alpha_{N-2})$ for $N>2$ becomes

$$\begin{aligned} H = &\ \frac{1}{2\xi^2\eta^2}\left(\frac{P_1^2}{\prod_{j=2}^{N-2}\sin^2\alpha_j}+\frac{P_2^2}{\prod_{j=3}^{N-2}\sin^2\alpha_j}+\cdots+\frac{P_{N-3}^2}{\sin^2\alpha_{N-2}}+P_{N-2}^2\right)\\ &+\frac{P_\xi^2+P_\eta^2}{2(\xi^2+\eta^2)}-\frac{2\mu}{\xi^2+\eta^2}-\frac{\varepsilon}{2}\left(\xi^2-\eta^2\right), \end{aligned}$$

or equivalently

$$\begin{aligned} H = &\ \frac{1}{2\xi^2\eta^2}\left(P_{N-2}^2+\frac{1}{\sin^2\alpha_{N-2}}\left(P_{N-3}^2+\frac{1}{\sin^2\alpha_{N-3}}\left(\dots\right.\right.\right.\\ &\left.\left.\left.\left(\left(P_3^2+\frac{1}{\sin^2\alpha_3}\left(P_2^2+\frac{P_1^2}{\sin^2\alpha_2}\right)\right)\right)\dots\right)\right)\right)\\ &+\frac{P_\xi^2+P_\eta^2}{2(\xi^2+\eta^2)}-\frac{2\mu}{\xi^2+\eta^2}-\frac{\varepsilon}{2}\left(\xi^2-\eta^2\right). \end{aligned} \tag{39}$$

By introducing the Hamiltonians

$$H_1=P_1^2,\qquad H_j=P_j^2+\frac{H_{j-1}}{\sin^2\alpha_j},\qquad \text{for}\quad j=2,\dots,N-2,$$

we get that (39) can be written as

$$\begin{aligned} H = &\ \frac{1}{2\xi^2\eta^2}\left(H_{N-2}\left(P_{N-2},\alpha_{N-2},H_{N-3}(P_{N-3},\alpha_{N-3},\dots,H_1(P_1,\alpha_1)\dots\right)\right))\\ &+\frac{P_\xi^2+P_\eta^2}{2(\xi^2+\eta^2)}+\frac{2\mu}{\xi^2+\eta^2}+\frac{\varepsilon}{2}\left(\xi^2-\eta^2\right)\\ = &\ \frac{C_{N-2}}{2\xi^2\eta^2}+\frac{P_\xi^2+P_\eta^2}{2(\xi^2+\eta^2)}-\frac{2\mu}{\xi^2+\eta^2}-\frac{\varepsilon}{2}\left(\xi^2-\eta^2\right) \end{aligned}$$

where C_{N-2} is the level of the function

$$H_{N-2}\left(P_{N-2},\alpha_{N-2},H_{N-3}(P_{N-3},\alpha_{N-3},\dots,H_1(P_1,\alpha_1)\dots)\right).$$

In short the proposition is proved. □

Proposition 21. *The Hamiltonian vector field Γ_H with Hamiltonian (38) is integrable in the Jacobi sense. Moreover the N dimensional Stark problem has $N-1$ independent first integrals*

$$\begin{aligned} H_1 = &\ P_1^2,\qquad H_j=P_j^2+\frac{H_{j-1}}{\sin^2\varphi_j},\quad \textit{for}\quad j=2,\dots,N-2,\\ H_{N-1} := &\ \int\frac{\eta\,d\eta}{\sqrt{R_6(\eta)}}-\int\frac{\xi\,d\xi}{\sqrt{S_6(\xi)}}, \end{aligned} \tag{40}$$

where

$$\begin{aligned} R_6(\eta) = &\ -\varepsilon\,\eta^6+2h\eta^4+(2\mu+\lambda)\eta^2+2C_{N-2},\\ S_6(\xi) = &\ \varepsilon\xi^6+2h\xi^4+(2\mu-\lambda)\xi^2+2C_{N-2}, \end{aligned}$$

where C_{N-2} and λ are arbitrary constants.

Proof. Now we prove the integrability in the Jacobi sense of Γ_H. Indeed, from (38) it follows that

$$\begin{aligned} H = &\ \frac{P_\xi^2+P_\eta^2-4\mu-\varepsilon(\xi^4-\eta^4)}{2(\xi^2+\eta^2)}+\frac{C_{N-2}}{\xi^2\eta^2}\\ = &\ \frac{1}{2(\xi^2+\eta^2)\xi^2\eta^2}\left(\xi^2\eta^2(P_\xi^2+P_\eta^2-4\mu-\varepsilon(\xi^4-\eta^4))+2C_{N-2}(\xi^2+\eta^2)\right). \end{aligned}$$

Thus

$$\xi^2\eta^2(P_\xi^2+P_\eta^2-4\mu-\varepsilon(\xi^4-\eta^4))+2C_{N-2}(\xi^2+\eta^2)-2(\xi^2+\eta^2)\xi^2\eta^2h=0.$$

Consequently

$$\xi^2\left(\eta^2(P_\eta^2-2\mu+\varepsilon\eta^4-2h\eta^2)+2C_{N-2}\right)+\eta^2\left(\xi^2(P_\xi^2-2\mu-\varepsilon\xi^4-2h\xi^2)+2C_{N-2}\right)=0.$$

which is equivalent to

$$\begin{aligned} \eta^2(P_\eta^2-2\mu+\varepsilon\eta^4-2h\eta^2)+2C_{N-2} = &\ \lambda\eta^2,\\ \xi^2(P_\xi^2-2\mu-\varepsilon\xi^4-2h\xi^2)+2C_{N-2} = &\ -\lambda\xi^2, \end{aligned}$$

where λ is an arbitrary constant. Hence we get that

$$\eta^2(P_\eta^2-(2\mu+\lambda)+\varepsilon\eta^4-2h\eta^2)+2C_{N-2}=0,\qquad \xi^2(P_\xi^2-(2\mu-\lambda)-\varepsilon\xi^4-2h\xi^2)+2C_{N-2}=0.$$

Thus the variable is separated. Hence

$$\begin{aligned}\eta P_\eta=&\ \pm\sqrt{-\varepsilon\,\eta^6+2h\eta^4+(2\mu+\lambda)\eta^2+2C_{N-2}}:=\sqrt{R_6(\eta)},\\ \xi P_\xi=&\ \pm\sqrt{\varepsilon\xi^6+2h\xi^4+(2\mu-\lambda)\xi^2+2C_{N-2}}:=\sqrt{S_6(\xi)},\end{aligned}$$

or equivalently

$$\begin{aligned}\eta(\xi^2+\eta^2)\dot\eta=&\ \pm\sqrt{-\varepsilon\,\eta^6+2h\eta^4+(2\mu+\lambda)\eta^2+2C_{N-2}}:=\sqrt{R_6(\eta)},\\ \xi(\xi^2+\eta^2)\dot\xi=&\ \pm\sqrt{\varepsilon\xi^6+2h\xi^4+(2\mu-\lambda)\xi^2+2C_{N-2}}:=\sqrt{S_6(\xi)}.\end{aligned}$$

By introducing the new time τ such that $dt=(\xi^2+\eta^2)d\tau$ we get after the integration

$$\int\frac{\eta\,d\eta}{\sqrt{R_6(\eta)}}=\tau+\tau_0,\qquad \int\frac{\xi\,d\xi}{\sqrt{S_6(\xi)}}=\tau+\tau_1, \tag{41}$$

where τ_0 and τ_1 are arbitrary constants. From these relations eliminating the new time variable τ we get the first integral

$$F:=\int\frac{\eta\,d\eta}{\sqrt{R_6(\eta)}}-\int\frac{\xi\,d\xi}{\sqrt{S_6(\xi)}}=C$$

where $C=\tau_0-\tau_1$. Hence the Hamiltonian vector field Γ_H is integrable in the Jacobi sense, it has $N-1$ independent first integrals (40). □

Proposition 22. *The parametric expressions of the solutions of the N dimensional Stark problem are the following*

$$\xi^2=-\frac{2h}{\varepsilon}+\mathfrak{P}(\sqrt{\varepsilon}(\tau+\tau_0)),\qquad \eta^2=\frac{2h}{\varepsilon}+\mathfrak{P}(\sqrt{-\varepsilon}(\tau+\tau_1)),$$

where $\mathfrak{P}$ is the Weierstrass function.

Proof. After the change $\xi^2=X-\dfrac{2h}{\varepsilon}$ and $\eta^2=Y+\dfrac{2h}{\varepsilon}$ we get that the polynomials $4R_6(\xi)/\varepsilon$ and $4S_6(\eta)/(-\varepsilon)$ become

$$\begin{aligned}P_3(X)=&\ 4X^2-\frac{4}{3\varepsilon^2}\left((3\lambda-6\mu)\varepsilon+4h^2\right)X+\frac{8}{27\varepsilon^3}\left(27C_{N-2}\varepsilon^2+(9h\lambda-18h\mu)\varepsilon+8h^3\right),\\ Q_3(Y)=&\ 4Y^2-\frac{4}{3\varepsilon^2}\left((3\lambda+6\mu)\varepsilon+4h^2\right)X-\frac{8}{27\varepsilon^3}\left(27C_{N-2}\varepsilon^2+(9h\lambda+18h\mu)\varepsilon+8h^3\right)\end{aligned}$$

Consequently (41) can be rewritten as

$$\int\frac{dX}{\sqrt{P_3(X)}}=\sqrt{\varepsilon}(\tau+\tau_0),\qquad \int\frac{dY}{\sqrt{Q_3(Y)}}=\sqrt{-\varepsilon}(\tau+\tau_1). \tag{42}$$

The problem is therefore reduced to quadratures, more specifically to elliptic integrals. The key is now to invert those integrals to find parametric expressions of the variables X and Y in functions of new time τ. By inverting integrals (42) we get that

$$X=\mathfrak{P}(\sqrt{\varepsilon}(\tau+\tau_0)),\qquad Y=\mathfrak{P}(\sqrt{-\varepsilon}(\tau+\tau_1)),$$

where $\mathfrak{P}$ is the Weierstrass function. In short the proposition is proved. □

Remark 23. For $N=3$ the Stark problem does not conserve the generalized Laplace–Lenz vector $F=(f_1,f_2,f_3)$ defined by

$$\begin{aligned}f_1=&\ \dot y(x\dot y-y\dot x)-\dot z(z\dot x-x\dot z)-\frac{\mu x}{\sqrt{x^2+y^2+z^2}},\\ f_2=&\ \dot z(y\dot z-z\dot y)-\dot x(x\dot y-y\dot x)-\frac{\mu y}{\sqrt{x^2+y^2+z^2}},\\ f_3=&\ \dot x(z\dot x-x\dot z)-\dot y(y\dot z-z\dot y)-\frac{\mu z}{\sqrt{x^2+y^2+z^2}}.\end{aligned}$$

But since along the solutions of (37) we get that

$$\begin{aligned}\dot f_1=&\ \varepsilon\,(2x\dot z-z\dot x)\,,\\ \dot f_2=&\ \varepsilon\,(2y\dot z-z\dot y)\,,\\ \dot f_3=&\ -\varepsilon\,(x\dot x+y\dot y)\,,\end{aligned}$$

we get the first integral

$$\dot x(z\dot x-x\dot z)-\dot y(y\dot z-z\dot y)-\frac{\mu z}{\sqrt{x^2+y^2+z^2}}+\frac{\varepsilon}{2}(x^2+y^2)=C. \tag{43}$$

In particular if we put in the previous equations $y = \dot{y} = 0$ then we obtain that

$$\begin{aligned}\dot{f}_1 &= \varepsilon\,(2x\dot{z} - z\dot{x})\,,\\ \dot{f}_3 &= -\varepsilon x\dot{x}.\end{aligned}$$

Hence we get that the first integral (43) becomes

$$\dot{x}(z\dot{x} - x\dot{z}) - \frac{\mu z}{\sqrt{x^2+z^2}} + \frac{\varepsilon}{2}x^2 = C_0.$$

This first integral was given in [19].

5.2. Hamiltonian vector field admitting a finite dimensional Lie algebra

We shall study the Levi-Civita integrable Hamiltonian systems which admit a finite dimensional Lie algebra formed by the vector fields $\Gamma_1, \ldots, \Gamma_M$ such that

$$[\Gamma_j,\ \Gamma_k] = C^j_{jk}\Gamma_j + C^k_{jk}\Gamma_k, \tag{44}$$

where C^j_{jk} and C^k_{jk} are constants which in view of Corollary 15 satisfy the conditions

$$C^j_{jk}C^k_{jk} = 0, \quad \text{for} \quad j,k = 1,\ldots,M, \quad \text{and} \quad j \neq k.$$

Consequently (44) becomes

$$[\Gamma_j,\ \Gamma_k] = C^k_{jk}\Gamma_k \quad \text{when} \quad C^j_{jk} = 0, \quad \text{or} \quad [\Gamma_j,\ \Gamma_k] = C^j_{jk}\Gamma_j \quad \text{when} \quad C^k_{jk} = 0,$$

or $[\Gamma_j,\ \Gamma_k] = 0$ when $C^j_{jk} = 0$, $C^k_{jk} = 0$, which is a particular case of (34).

Proposition 24. *A Hamiltonian vector field Γ_H integrable in the Jacobi sense, admits a two dimensional Lie algebra with basis Γ_1 and Γ_2 satisfying*

$$[\Gamma_1,\ \Gamma_2] = C^1_{12}\Gamma_1 + C^2_{12}\Gamma_2, \quad C^1_{12}C^2_{12} = 0,$$

if and only if

(i) $H = H(H_1(z_1,P_1),\ H_2(z_2,P_2))$ *when* $C^1_{12} = C^2_{12} = 0$,
(ii) $H = H(z_2,P_2,H_1(z_1,P_1))$ *when* $C^2_{12} = 0$,
(iii) $H = H(z_1,P_1,H_2(z_2,P_2))$ *when* $C^1_{12} = 0$,

Proof. Is a consequence of Theorems 17 and 18. □

Proposition 25. *The Jacobi integrable Hamiltonian vector field Γ_H admits a three dimensional Lie algebra with basis $\Gamma_1,\ \Gamma_2$ and Γ_3 satisfying*

$$[\Gamma_1,\ \Gamma_2] = a\Gamma_2, \qquad [\Gamma_2,\ \Gamma_3] = 0, \qquad [\Gamma_3,\ \Gamma_1] = -a\Gamma_3, \tag{45}$$

if and only if

$$H = H(z_1,P_1,H_2(z_2,P_2),H_3(z_3,P_3)). \tag{46}$$

Proof. For $N = 3$ we get that (see (21))

$$\begin{aligned}
&[\Gamma_1,\Gamma_2] = C^1_{12}\Gamma_1 + C^2_{12}\Gamma_2, \quad \text{with} \quad C^1_{12}C^2_{12} = 0,\\
&[\Gamma_2,\Gamma_3] = C^2_{23}\Gamma_2 + C^3_{23}\Gamma_3, \quad \text{with} \quad C^2_{23}C^3_{23} = 0,\\
&[\Gamma_3,\Gamma_1] = C^1_{31}\Gamma_1 + C^3_{31}\Gamma_3, \quad \text{with} \quad C^1_{31}C^3_{31} = 0,
\end{aligned} \tag{47}$$

On the other hand from the Bianchi classification of three dimensional Lie-algebra we get that the Lie algebra with basis $e_1,\ e_2$ and e_3 has the following representation

$$[e_1,e_2] = ae_2 + b_3e_3, \qquad [e_2,e_3] = b_1e_1, \qquad [e_3,e_1] = b_2e_2 - ae_3.$$

By comparing with (47) we obtain that $C^1_{12} = C^3_{23} = C^2_{23} = C^1_{31} = 0$. Hence in view of (29) we get that

$$\begin{aligned}
\frac{\partial R_2}{\partial z_1} &= \frac{\partial R_2}{\partial P_1} = \frac{\partial R_2}{\partial z_3} = \frac{\partial R_2}{\partial P_3} = 0,\\
\frac{\partial R_3}{\partial z_1} &= \frac{\partial R_3}{\partial P_1} = \frac{\partial R_3}{\partial z_2} = \frac{\partial R_3}{\partial P_2} = 0,\\
C^3_{31} &= -C^2_{12} = -a.
\end{aligned}$$

Hence from (29) we obtain

$$\begin{aligned}\frac{\partial H}{\partial z_2}+R_2(z_2,P_2)\frac{\partial H}{\partial P_2} &= 0,\\ \frac{\partial H}{\partial z_3}+R_3(z_3,P_3)\frac{\partial H}{\partial P_3} &= 0.\end{aligned}$$

Clearly the function (46) is a solution of these equations. In short the proposition is proved. □

5.3. *Integrability in the Jacobi sense for the Hamiltonians such that* $\dfrac{\partial^2 H}{\partial P_j\partial P_k}=0$ *and* $\dfrac{\partial^2 H}{\partial z_j\partial z_k}\neq 0$ *with* $j\neq k$

Clearly under these conditions the function H can be written as

$$H=\sum_{j=1}^{M}H_j(z_1,z_2,\ldots,z_M,P_j).$$

For these Hamiltonians the partial differential equations (3) become

$$\frac{\partial^2 H}{\partial z_j\partial z_k}-\frac{\partial H}{\partial z_k}\frac{\partial}{\partial z_j}\left(\log\left|\frac{\partial H}{\partial P_k}\right|\right)-\frac{\partial H}{\partial z_j}\frac{\partial}{\partial z_k}\left(\log\left|\frac{\partial H}{\partial P_j}\right|\right)=0, \tag{48}$$

for $j\neq k$, or equivalently from Theorem 13 we have

$$C^j_{jk}C^k_{jk}-\frac{\partial^2 H}{\partial P_k\partial z_j}\frac{\partial^2 H}{\partial z_k\partial P_j}=0, \tag{49}$$

where

$$C^k_{jk}=\begin{cases}-\Gamma_j\left(\log\left|\dfrac{\partial H}{\partial P_k}\right|\right)=-\dfrac{\dfrac{\partial H}{\partial P_j}}{\dfrac{\partial H}{\partial P_k}}\dfrac{\partial^2 H}{\partial P_k\partial z_j},\\[3ex] -\Gamma_j\left(\log\left|\dfrac{\partial H}{\partial z_k}\right|\right)=-\dfrac{\dfrac{\partial H}{\partial P_k}}{\dfrac{\partial H}{\partial P_j}}\dfrac{\partial^2 H}{\partial P_j\partial z_k},\end{cases}.$$

Let $|\Delta|$ be the determinant of the matrix

$$\Delta=\begin{pmatrix}\varphi_{11}(z_1) & \ldots & \varphi_{1M}(z_1)\\ \vdots & \ldots & \vdots\\ \varphi_{M1}z_M & \ldots & \varphi_{MM}(z_M)\end{pmatrix},$$

where $\varphi_{jk}(z_j)$ is a set of M^2 arbitrary functions. Let $\psi_j(z_j)$ be a set of M functions. Then the vector field Γ_H with

$$H=\frac{1}{2}\sum_{j=1}^{M}A_j(z_1,\ldots,z_M)\left(P_j^2-\psi_j(z_j)\right),$$

where $A_j=\dfrac{1}{|\Delta|}\left(\dfrac{\partial|\Delta|}{\partial\varphi_{j1}}\right)$ is called the *Stäckel vector field* (see for instance [24,25]).

Proposition 26. *The Hamiltonian vector field Γ_H with*

$$H=\sum_{j=1}^{M}A_j(z_1,z_2,\ldots,z_M)H_j(z_j,P_j),$$

where $H_j(z_j,P_j)$ is an arbitrary function for $j=1,\ldots,M$, is integrable in the Jacobi sense if and only if the functions A_j for $j=1,2,\ldots,M$ satisfy the partial differential equations

$$A_jA_k\frac{\partial^2 A_n}{\partial z_j\partial z_k}-A_j\frac{\partial A_k}{\partial z_j}\frac{\partial A_n}{\partial z_k}-A_k\frac{\partial A_j}{\partial z_k}\frac{\partial A_n}{\partial z_j}=0 \tag{50}$$

for $j,k,n=1,\ldots,M$ and $j\neq k$.

Moreover if $H_j(z_j,P_j)=P_j^2-\psi_j(z_j)$ where $\psi_j(z_j)$ is an arbitrary function, for $j=1,\ldots,M$, then we obtain an Stäckel system (see for instance [24,25]).

Proof. By considering that

$$\begin{aligned}
\frac{\partial H}{\partial z_n} &= \sum_{j=1}^{M} \frac{\partial A_j}{\partial z_n} H_j(z_j, P_j), \quad \text{for} \quad n \neq j,\\
\frac{\partial H}{\partial z_j} &= \sum_{j=1}^{M} \left(\frac{\partial A_j}{\partial z_j} H_j(z_j, P_j) + A_j \frac{\partial H_j(z_j, P_j)}{\partial z_j} \right),\\
\frac{\partial H}{\partial P_j} &= \sum_{j=1}^{M} \left(A_j \frac{\partial H_j(z_j, P_j)}{\partial P_j} \right),\\
\frac{\partial^2 H}{\partial z_j \partial z_n} &= \sum_{j=1}^{M} \frac{\partial^2 A_j}{\partial z_j \partial z_n} H_j(z_j, P_j) + \frac{\partial A_j}{\partial\, z_n} \frac{\partial H_j(z_j, P_j)}{\partial z_j}, \quad \text{for} \quad n \neq j,\\
\frac{\partial^2 H}{\partial P_j \partial P_n} &= 0 \quad \text{for} \quad n \neq j,
\end{aligned}$$

Inserting these relations into (48) and after some computations we get

$$A_j A_k \frac{\partial H_k}{\partial z_j} \frac{\partial H_j}{\partial z_k} \sum_{n=1}^{M} H_n(z_n, P_n) \left(A_j A_k \frac{\partial^2 A_n}{\partial z_j \partial z_k} - A_j \frac{\partial A_k}{\partial z_j} \frac{\partial A_n}{\partial z_k} - A_k \frac{\partial A_j}{\partial z_k} \frac{\partial A_n}{\partial z_j} \right) = 0,$$

for $j, k, = 1, \dots, M$ and $j \neq k$. Hence we obtain (50).

It is interesting to observe that if we apply (49) then we obtain that

$$\frac{\partial^2 H}{\partial P_k \partial z_j} \frac{\partial^2 H}{\partial z_k \partial P_j} = \frac{\partial A_k}{\partial z_j} \frac{\partial A_j}{\partial z_k} \frac{\partial H_k}{\partial P_k} \frac{\partial H_j}{\partial P_j}.$$

On the other hand by considering that

$$C_{jk}^k = -\Gamma_j \left(\log \left| \frac{\partial H}{\partial P_k} \right| \right) = -\frac{A_j}{A_k} \frac{\partial H_j}{\partial P_j} \frac{\partial A_k}{\partial z_j},$$

we obtain

$$C_{jk}^k C_{jk}^j - \frac{\partial^2 H}{\partial P_k \partial z_j} \frac{\partial^2 H}{\partial z_k \partial P_j} = \frac{A_j}{A_k} \frac{\partial H_j}{\partial P_j} \frac{\partial A_k}{\partial z_j} \frac{A_k}{A_j} \frac{\partial H_k}{\partial P_k} \frac{\partial A_j}{\partial z_k} - \frac{\partial A_k}{\partial z_j} \frac{\partial A_j}{\partial z_k} \frac{\partial H_k}{\partial P_k} \frac{\partial H_j}{\partial P_j} \equiv 0.$$

Thus the condition (49) holds identically. On the other hand, since

$$C_{jk}^k = -\frac{A_j}{A_k} \frac{\partial H_j}{\partial P_j} \frac{\partial A_k}{\partial z_j} = -\Gamma_j \left(\log \left| \frac{\partial H}{\partial z_k} \right| \right) \quad \text{for} \quad j \neq k,$$

therefore after some computations we obtain that the functions $A_j(z_1, \dots, z_m)$ are a solution of (50). In short the proposition is proved. □

Finally we observe that the integrability in the Jacobi sense of the Hamiltonian $H = \sum_{j=1}^{M} A_j(z_1, z_2, \dots, z_M) \left(P_j^2 - \psi_j(z_j) \right)$ was established by Stäckel (see for instance [24,25]).

Now we state the following problem.

Problem 27.

(i) Let $A_j = A_j(z_1, \dots, z_M)$ and $\alpha_j = \alpha_j(H_1(z_1, P_1), \dots, H_M(z_M, P_M))$ where $H_j = H_j(z_j, P_j)$ for $j = 1, \dots, M$. Determine the conditions on the functions A_j and α_j in such a way that the Hamiltonian vector field Γ_H with

$$H = \sum_{j=1}^{M} A_j(z_1, \dots, z_M) \alpha_j(H_1, \dots, H_M),$$

is integrable in the Jacobi sense.

(ii) Let $A_j = A_j(z_1, \dots, z_M)$ and

$$\alpha_j = \alpha_j(z_M, P_M, H_{M-1}\,(z_{M-1}, P_{M-1}, H_{M-2}\,(z_{M-2}, P_{M-2}, \dots, H_1\,(z_1, P_1) \dots))).$$

Determine the conditions on the functions A_j and α_j in such away that the Hamiltonian vector field Γ_H with H equal to

$$\sum_{j=1}^{M} A_n(z_1, \dots, z_M) \alpha_n(z_M, P_M, H_{M-1})\,(z_{M-1}, P_{M-1}, H_{M-2}\,(z_{M-2}, P_{M-2}, \dots, H_1\,(z_1, P_1) \dots)),$$

is integrable in the Jacobi sense.

The integrability in the Jacobi sense for the Hamiltonian vector field Γ_H with such that

$$\frac{\partial^2 H}{\partial P_j \partial P_k} \neq 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial z_j \partial z_k} = 0 \quad \text{with} \quad j \neq k, \tag{51}$$

can be obtained by applying Proposition 4 on the integrable Hamiltonian vector fields in the Jacobi sense.

Clearly under the condition (51) the function H can be written as

$$H = \sum_{j=1}^{M} H_j(z_j, P_1, \ldots, P_M). \tag{52}$$

For these Hamiltonians the partial differential equations (3) become

$$\frac{\partial^2 H}{\partial P_j \partial P_k} - \frac{\partial H}{\partial P_k}\frac{\partial}{\partial P_j}\left(\log\left|\frac{\partial H}{\partial z_k}\right|\right) - \frac{\partial H}{\partial P_j}\frac{\partial}{\partial P_k}\left(\log\left|\frac{\partial H}{\partial z_j}\right|\right) = 0 \quad \text{for} \quad j \neq k,$$

or equivalently from Theorem 13 we have

$$C^k_{jk} C^j_{jk} - \frac{\partial^2 H}{\partial P_k \partial z_j}\frac{\partial^2 H}{\partial z_k \partial P_j} = 0,$$

where

$$C^k_{jk} = \begin{cases} -\Gamma_j\left(\log\left|\dfrac{\partial H}{\partial P_k}\right|\right), \\ -\Gamma_j\left(\log\left|\dfrac{\partial H}{\partial z_k}\right|\right), \end{cases} .$$

Proposition 28. *The Hamiltonian vector field Γ_H with*

$$H = \sum_{j=1}^{M} A_j(P_1, P_2, \ldots, P_M) H_j(z_j, P_j)$$

is integrable in Jacobi sense if and only if the functions A_j for $j = 1, 2, \ldots, M$ satisfy the partial differential equations

$$A_j A_k \frac{\partial^2 A_n}{\partial P_j \partial P_k} - A_j \frac{\partial A_k}{\partial P_j}\frac{\partial A_n}{\partial P_k} - A_k \frac{\partial A_j}{\partial P_k}\frac{\partial A_n}{\partial P_j} = 0$$

for $j, k, n = 1, \ldots, M$ and $j \neq k$.

Proof. This proof is analogous to the proof of Proposition 26 after the change $P \longleftrightarrow z$, by using the properties of solutions of the Levi-Civita conditions (see Proposition 4). In short the proposition is proved. □

In view of Proposition 4 Problem 27 can be stated for the Hamiltonians (52) after the change $P_j \longleftrightarrow z_j$.

5.4. Integrability in the Jacobi sense and Lax pair

One of the best methods for determining the involution set of first integrals for an integrable Liouville Hamiltonian vector field Γ_H associated to a Hamiltonian differential system (6) is the Lax pair method, for more details see the ones given in what follows, see [26]

A *Laix pair* for a Hamiltonian vector field is a pair of smooth quadratic matrices $A = (a_{jk})$ and $B = (b_{jk})$, satisfying the equation called the *Lax equation*

$$\frac{dA}{dt} = [B, A] = BA - AB, \tag{53}$$

where the derivative $\dfrac{dA}{dt}$ is calculated along the solutions of (5).

If the Hamiltonian system admits a Lax pair then it has the involution set of first integrals

$$F_j = \text{trace}(A^j), \quad \text{for} \quad j = 1, \ldots, M.$$

For the particular case of the Hamiltonian vector field Γ_H with

$$\begin{aligned} H = \; & \frac{1}{2}\sum_{j=1}^{M} P_j^2 + \sum_{k,j=1,\, k<j}^{M} a_{kj}(z_k - z_j) a_{jk}(z_j - z_k) \\ = \; & \frac{1}{2}\sum_{j=1}^{M} P_j^2 + \sum_{j=1}^{M} V_j(z_j - z_{j+1}) = T + V, \end{aligned} \tag{54}$$

we shall study the Lax equation for the case when the matrices A and B are such that

$$A=\begin{pmatrix} P_1 & a_{12} & a_{13} & \dots & a_{1M}\\ a_{21} & P_2 & a_{23} & \dots & a_{2M}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ a_{M-11} & \dots & \dots & P_{M-1} & a_{M-1M}\\ a_{M1} & \dots & \dots & a_{MM-1} & P_M \end{pmatrix},$$

$$B=\begin{pmatrix} \Psi_1 & a'_{12} & a'_{13} & \dots & a'_{1M}\\ a'_{21} & \Psi_2 & a'_{23} & \dots & a'_{2M}\\ \vdots & \vdots & \vdots & \dots & \vdots\\ a'_{M-11} & \dots & \dots & \Psi_{M-1} & a'_{M-1M}\\ a'_{M1} & \dots & \dots & a'_{MM-1} & \Psi_M \end{pmatrix},$$

where $\Psi_j=\Psi_j(P_1,\dots,P_M,z_1,\dots,z_M)$ are convenient functions, $a_{jk}=a_{jk}(z_j-z_k)$ for $j\neq k$, and $a'_{jk}=\dfrac{d}{d\xi}\left(a_{jk}(\xi)\right)$ with $\xi=z_j-z_k$, respectively. Clearly in this case $F_2=2H=\text{trace}(A^2)$.

The relation between the integrability in Jacobi and Liouville sense, as we observed in the introduction, was studied in particular in [3].

Problem 29. Determine the functions $\Psi_1,\dots,\Psi_M$ and $a_{jk}=a_{jk}(z_j-z_k)$ for $j,k=1,\dots,M$ for which the Hamiltonian vector field Γ_H with Hamiltonian (54) with $M-1$ independent first integrals $F_j=\text{trace}(L^j)$ for $j=1,\dots,M-1$ is integrable in the Jacobi sense.

To illustrate the solution of this problem we study the following particular case.

Proposition 30. *Let A and B be the matrix such that*

(i)

$$A=\begin{pmatrix} P_1 & a_{12} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & a_{1M}\\ -a_{12} & P_2 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \dots & \vdots\\ 0 & \ddots & P_k & a_{kk+1} & \dots & \dots & \dots & \dots & \dots & 0\\ 0 & 0 & -a_{kk+1} & P_{k+1} & \dots & \dots & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & & \dots & \vdots\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & P_{M-2} & a_{M-3,M-2} & 0\\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -a_{M-3,M-2} & P_{M-1} & 0\\ -a_{1M} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & P_M \end{pmatrix}$$

if M is odd, and

$$A=\begin{pmatrix} P_1 & a_{12} & 0 & 0 & \dots & \dots & \dots & \dots & 0\\ a_{12} & P_2 & 0 & 0 & \dots & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots\\ 0 & \ddots & P_k & a_{kk+1} & \dots & \dots & \dots & \dots & 0\\ 0 & 0 & -a_{kk+1} & P_{k+1} & \dots & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 0 & P_{M-3} & a_{M-3,M-2} & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & -a_{M-3,M-2} & P_{M-2} & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & P_{M-1} & a_{M-1,M}\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{M-1,M} & P_M \end{pmatrix},$$

if M is even, and set

$$B=\begin{pmatrix} \Psi_2 & a'_{12} & 0 & 0 & \dots & \dots & \dots & 0\\ a'_{12} & \Psi_2 & 0 & 0 & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots\ \vdots\\ 0 & \ddots & \Psi_{2k} & a'_{2k+1k} & \dots & \dots & \dots & 0\\ 0 & 0 & a'_{2k+1k} & \Psi_{2k} & \dots & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 0 & \Psi_{M-1} & a'_{M-2,M-1} & 0\\ 0 & 0 & 0 & \dots & 0 & a'_{M-2,M-1} & \Psi_{M-1} & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \Psi_2 \end{pmatrix},$$

if M *is odd and* $k = 1, \dots, \dfrac{M-1}{2}$, *and set*

$$B = \begin{pmatrix} \Psi_2 & a'_{12} & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ a'_{12} & \Psi_2 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & \ddots & \Psi_{2k} & a'_{2k+1k} & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & a'_{2k+1k} & \Psi_{2k} & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \Psi_{M-2} & a'_{M-3,M-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & a'_{M-3,M-2} & \Psi_{M-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \Psi_2 & a_{M-1,M} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & a_{M-1,M} & \Psi_2 \end{pmatrix},$$

if $M > 2$ *is even and* $k = 1, \dots, \dfrac{M-2}{2}$.

Assuming that the Hamiltonian (54) *is*

$$H = \frac{1}{2}\sum_{j=1}^{2N} P_j^2 - \sum_{k=1}^{N} a_{2k-1,2k}^2 = T + V, \tag{55}$$

if $M = 2N$, *and*

$$H = \frac{1}{2}\sum_{j=1}^{2N+1} P_j^2 - \sum_{k=1}^{N} a_{2k-1,2k}^2 = T + V, \tag{56}$$

if $M = 2N+1$, *where* $a_{2k-1,2k} = a_{2k-1,2k}(z_{2k-1} - z_{2k})$ *are arbitrary functions, then the Lax equation* (53) *becomes* $\dot{P}_j = \dfrac{\partial V}{\partial z_j}$ *for* $j = 1, \dots, M$.

Proof. After some computations it is easy to show that independently on the parity of M the Lax equation (53) becomes $\dot{P}_j = \dfrac{\partial V}{\partial z_j}$ for $j = 1, \dots, M$. We note that the Hamiltonian vector field with Hamiltonian Γ_H with H given in (55) and (56) is integrable in the Liouville sense. □

Proposition 31. *The Hamiltonian vector field* Γ_H *with* H *given in* (55) and (56) *is integrable in the Jacobi sense.*

Proof. Indeed, after the change of variables

$$u_{2k-1} = \frac{z_{2k-1} - z_{2k}}{2}, \qquad v_{2k-1} = \frac{z_{2k-1} + z_{2k}}{2},$$

for $k = 1, \dots, N$, we get

$$P_{2k-1} = \dot{z}_{2k-1} = \dot{u}_{2k-1} + \dot{v}_{2k-1}, \qquad P_{2k} = \dot{z}_{2k} = \dot{u}_{2k-1} - \dot{v}_{2k-1}.$$

Consequently the Hamiltonians (55) and (56) become

$$H = \sum_{j=2}^{2N} \left(\dot{u}_{2j-1}^2 + \dot{v}_{2j-1}^2\right) - \sum_{j=1}^{N} a_{2j-1,\,2j}^2(2u_{2j-1})$$

and

$$H = \sum_{j=2}^{2N+1} \left(\dot{u}_{2j-1}^2 + \dot{v}_{2j-1}^2\right) - \sum_{j=1}^{N} a_{2j-1,2j}^2(2u_{2j-1})$$

respectively. Hence the Hamiltonian system is integrable by variable separation. □

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