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Physica D 218 (2006) 105–121

PHYSICA D

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A bi-Hamiltonian structure for the integrable, discrete non-linear Schrödinger system

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Received 21 April 2005; received in revised form 2 April 2006; accepted 6 April 2006
Available online 6 June 2006
Communicated by A. Doelman

Abstract

This paper shows that the AL (Ablowitz–Ladik) hierarchy of (integrable) equations can be explicitly viewed as a hierarchy of commuting flows which: (a) are Hamiltonian with respect to both a standard, local Poisson operator $\mathcal{J}$, and a new non-local, skew, almost Poisson operator $\mathcal{K}$, on the appropriate space; (b) can be recursively generated from a *recursion* operator $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$. In addition, the proof of these facts relies upon two new pivotal resolvent identities which suggest a general method for uncovering bi-Hamiltonian structures for other families of discrete, integrable equations.

Keywords: Discrete integrable equations; Lattice dynamics; Inverse scattering; Poisson geometry; Bi-Hamiltonian structures

1. Introduction

The Ablowitz–Ladik (AL) system is one of the most studied discrete integrable systems of *soliton* type. It can be thought of as an integrable discretization of the Nonlinear Schrödinger (NLS) equation. However, it has recently received a great deal of attention as the background lattice system in a variety of modelling applications including optical fiber arrays [4,5], chaos in dispersive numerical schemes [22–24], and linkage dynamics [10,18].

AL is an infinite integrable system of soliton type by which one generally means that an infinite family of constants of motion for the AL flow can be constructed through the inverse scattering transform (IST) associated to a particular (discrete) eigenvalue problem. In addition, the IST framework provides a mechanism through which large families of solutions, such as multi-solitons can be explicitly found.

From a geometric standpoint, AL is also an integrable Hamiltonian system with respect to a natural *local* Poisson structure. Indeed this structure is the discretization of the natural (local) Poisson structure for NLS. The NLS system, just as many other integrable partial differential equations (PDEs) of soliton type, has been found to be Hamiltonian with respect to two *distinct* Poisson structures [19]. These distinct structures are compatible in a way that enables one to construct from them a recursion operator which generates the complete hierarchy of commuting flows. As stated above, this hierarchy is the signature of a completely integrable Hamiltonian system and implicitly characterizes its Poisson geometry. If a system has two distinct compatible Poisson structures it is referred to as *bi-Hamiltonian* (see Section 2). The explicit knowledge of two such Poisson structures allows one to "organize" the symmetries of the system by singling out *geometric* sub-hierarchies of flows which may be *explicitly exhibited* as bi-Hamiltonian and pair-wise commuting. Moreover, from the two compatible structures one may recursively and explicitly build the flows of the selected sub-hierarchy. By the same token, explicit knowledge of a *geometric* recursion operator (i.e., one known to result from the compatible Poisson operators) facilitates the construction of other new, well-defined geometric objects (such as hierarchies of Poisson structures associated to the given problem [16]). Finally, in many specific instances, a

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0167-2789/$ - see front matter
doi:10.1016/j.physd.2006.04.014

given bi-Hamiltonian structure provides a means to relate the Poisson geometry to the IST via resolvent relations associated to the linear eigenvalue problem.

Specific examples of bi-Hamiltonian structures related to an IST are much rarer for discrete integrable systems than for continuous ones. The principal goal of this paper is to explicitly build one such structure for AL and to exhibit the details of the connection between this structure and the corresponding IST. In the process, the hierarchies of flows and constants of motion known to be associated to AL will be parsed out in geometric terms.

This paper is organized as follows. In Section 2 we review the second Poisson structure for NLS which is due to Magri [19]. Our approach is to explicitly re-derive Magri's structure from a Wronskian construction of commuting flows for soliton PDEs due to Calogero and Degasperis [7]. We review the derivation of the second structure for the continuous case (NLS) because it turns out to be an important guide for identifying the second structure in the discrete case (as demonstrated in Sections 3.3 and 4.2 of this work).

In Section 3 the necessary inverse scattering background for the discrete case is presented including the derivation, from a generalized Wronskian relation, of two basic operators $\mathcal{L}_+$ and $\mathcal{L}_-$ in terms of which the recursion operator $\mathcal{R}$ and its properties are developed. In the last part of this section, two resolvent identities related to these operators, which are fundamental to the main results of this paper, are derived. These resolvent identities represent a novel contribution to the literature on scattering theory for discrete systems, as they do, in effect, provide an explicit link between the Poisson geometry of AL and the associated IST (see also Section 4). Furthermore, we believe that our derivation of these resolvent identities suggests a constructive procedure for establishing similar relationships between the geometry and the IST of other *discrete* integrable systems.

Finally in Section 4 the Poisson-geometric interpretation of the recursion operator $\mathcal{R}$ is made. We prove that AL has a bi-Hamiltonian character, in the sense defined in this same section. This is established by showing that the operator $\mathcal{K}$ gives rise to an almost Poisson structure for AL. It is an interesting open problem to determine whether or not the bracket defined through $\mathcal{K}$ is Poisson. This topic and other potential investigations are discussed in the conclusions.

2. A generalized Wronskian approach to the Poisson structure of NLS

To illustrate the rationale of our approach to the geometry of the AL equations, we first look at the parallel continuous object (the NLS hierarchy), where the geometry is already understood.

The NLS equation (7) is a well-known integrable PDE. As such, it possesses infinite families of linearly independent constants of motion in involution, and families of explicit special solutions such as N-soliton solutions. In the late 1970's, Magri showed that NLS has a bi-Hamiltonian nature [19]. This property (which now characterizes many integrable PDEs), means that the NLS equation can be written as a Hamiltonian system with respect to two different, independent Poisson brackets. By composing the Poisson operators (denoted J and K, for instance) induced by these brackets in the appropriate manner, one obtains a recursion operator, R, capable of generating a commuting family of Hamiltonian flows, which include NLS. Typically, one of the aforementioned Poisson operators, say J, is invertible and then $R = KJ^{-1}$.

In this section we will outline the derivation of Magri's Poisson structure for NLS from two integro-differential operators associated to the AKNS hierarchy, a collection of commuting integrable evolution equations which includes NLS. The full details of what is outlined here may be found in [18].

The first of these operators, which we denote L_+, was constructed by Calogero and Degasperis using a generalized Wronskian technique [7]. The basic idea behind the generalized Wronskian approach to integrable evolution equations is to generate a set of *scattering* relations between the asymptotic behaviors (as $x \to +\infty$ and $x \to -\infty$) of the wave-function solutions of some initial eigenvalue problem, such as the Zakharov–Shabat problem

$$\begin{pmatrix}\psi_{1x}\\ \psi_{2x}\end{pmatrix} = \begin{pmatrix}-\mathrm{i}k & q\\ r & \mathrm{i}k\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}. \tag{1}$$

The second operator, L_-, appearing in Calogero and Degasperis' work and giving rise to the AKNS hierarchy is related to L_+ in a sort of "adjoint" way, as described below.

The specific form of the operators L_+ and L_- (which act on a certain space C of rapidly decaying complex-valued functions $(q(x), r(x))^{\mathrm{T}}$ — the potentials,) suggests the construction of an anti-symmetric operator K arising from L_+, L_-.

It turns out that (see below) one can indeed use L_+ and L_- to define two geometrically meaningful operators K and R which act on C. The first one is Poisson, and gives rise to Magri's bracket for NLS. The second one is a recursion operator for the AKNS hierarchy which may in fact be portrayed as the composition of K with the standard Poison structure for AKNS (given essentially as multiplication by the imaginary unit i).

The specific structure and relationship between L_+ and L_- in the continuous setting will eventually guide the construction of their discrete counterparts $\mathcal{L}_+$ and $\mathcal{L}_-$ and also motivate the definition of $\mathcal{K}$ and $\mathcal{R}$ – discrete analogs of K and R for the AL hierarchy – in Section 3.

2.1. Obtaining Magri's Poisson structure, K, for NLS through L_+ and L_-

In [7], Calogero and Degasperis begin with the eigenvalue problem (1) and use a generalized Wronskian technique to arrive at the following formally defined class of integrable equations:

$$(r_t(x,t), -q_t(x,t))^{\mathrm{T}} = \gamma(L_+)(r(x,t), q(x,t))^{\mathrm{T}}, \tag{2}$$

where γ is an entire function of the integro-differential operator

$$\begin{aligned} L_+ &= \frac{1}{2\mathrm{i}}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_x + 2\begin{pmatrix} rI_+q & -rI_+r\\ qI_+q & -qI_+r\end{pmatrix}\right],\\ I_+ &= \int_x^{+\infty}(\cdot)\,\mathrm{d}y. \end{aligned} \tag{3}$$

Perhaps one of the best-known procedures for generating hierarchies of non-linear integrable evolution equations is due to the work of Ablowitz, Kaup, Newell and Segur (AKNS) [1]. In our context, their method utilizes once again the Zakharov–Shabat eigenvalue problem together with a second linear operator prescribing the time evolution of the wavefunctions. The hierarchy of integrable evolution equations arises then as a series of compatibility conditions associated to the linear problems just described. All the evolution equations in the hierarchy stem from the same eigenvalue problem, yet each corresponds to a different time-evolution operator.

Calogero and Degasperis write the AKNS hierarchy as

$$(r_t(x,t), -q_t(x,t))^{\mathrm{T}} = -2A(L_-)(r(x,t), q(x,t))^{\mathrm{T}}, \tag{4}$$

where

$$L_- = \frac{1}{2\mathrm{i}}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_x - 2\begin{pmatrix} rI_-q & -rI_-r\\ qI_-q & -qI_-r\end{pmatrix}\right],$$
$$I_- = \int_{-\infty}^{x}(\cdot)\,\mathrm{d}y \tag{5}$$

(A is an arbitrary entire function), and show that (2) and (4) are equivalent.

The proof of this fact entails showing that

$$L_+^n((r,q)^{\mathrm{T}}) = L_-^n((r,q)^{\mathrm{T}}), \tag{6}$$

which follows, in turn, from the observation that $L_+^n((r,q)^{\mathrm{T}})$ is in the kernel of the difference operator $L_- - L_+$. We refer to this as the (continuous) "kernel condition". This can be verified directly for $n = 0$ and other low values of n. The proof for general n (which relies on the definition of two auxiliary functions of the complex variable z) can be found in the Appendix of [7].

We remark that, under the identification $r = -\overline{q}$, the system of coupled integrable evolution equations (4) reduces to a system of evolution equations for a single field q, comprising the NLS hierarchy, which contains the well-known NLS equation

$$-\mathrm{i}q_t = q_{xx} + 2|q|^2 q. \tag{7}$$

In other words, in the reduction $r = -\overline{q}$, (4) becomes the NLS family of equations.

Let us now consider the space of complex, vector valued functions of a real variable x given by

$$C = \left\{(q(x), r(x))^{\mathrm{T}} \in \mathbb{C}^2 : |q|, |r| \to 0 \text{ as } x \to \pm\infty\right\}.$$

The tangent bundle to C may be endowed with a non-degenerate bilinear form locally described by the inner product

$$\langle (v_1, v_2)^{\mathrm{T}}, (w_1, w_2)^{\mathrm{T}}\rangle = -\int v_1 w_2 + v_2 w_1\,\mathrm{d}x, \tag{8}$$

where $(v_1, v_2)^{\mathrm{T}}, (w_1, w_2)^{\mathrm{T}}$ are tangent vectors to C at the point $(q, r)^{\mathrm{T}}$, and $\int f\,\mathrm{d}x = \int_{-\infty}^{\infty} f\,\mathrm{d}x$. One may also define a skew-adjoint operator $J = \mathrm{Diag}(-\mathrm{i}, \mathrm{i})$, on the tangent bundle to C which, together with the inner product just defined, gives rise to the complexified standard Poisson bracket on C, that is,

$$\begin{aligned}\{F, G\}_J &= \langle \nabla F, \mathrm{Diag}(-\mathrm{i}, \mathrm{i})\nabla G\rangle\\ &= \langle (-\delta_r F, -\delta_q F)^{\mathrm{T}}, \mathrm{Diag}(-\mathrm{i}, \mathrm{i})(-\delta_r G, -\delta_q G)^{\mathrm{T}}\rangle.\end{aligned}$$

Here, F and G are functionals on C which become real-valued in the reduction $r_k = \overline{q_k}$ defining the standard setting for NLS. This reality condition implies that F and G must be symmetric (or anti-symmetric) in r and q. Furthermore, $(\delta_q(\cdot), \delta_r(\cdot))^{\mathrm{T}}$ denotes the variational derivative, whereas $\nabla(\cdot) = (-\delta_r(\cdot), -\delta_q(\cdot))^{\mathrm{T}}$ gives the functional gradient of each such functional, with respect to the given inner product.

We now re-consider the operators L_+ and L_- acting on the function space C.

The presence of the integral operators I_+ and I_- suggests that L_+ and L_- may be combined so as to yield an *anti-symmetric* operator, K, related to the hierarchy. Indeed, as we will show below, the special properties of $L_- + L_+$ and $L_- - L_+$ allow us to recover both the (second) Poisson structure for NLS discovered by Magri [19] (prescribed by the Poisson operator K), and the recursion operator $R = KJ^{-1}$ associated to the AKNS hierarchy of flows. We define

$$K \doteq \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial_x + \begin{pmatrix} -q(I_- - I_+)r & q(I_- - I_+)q\\ r(I_- - I_+)r & -r(I_- - I_+)q\end{pmatrix}. \tag{9}$$

This operator K, just as J, acts on the tangent space to C at a point and defines a (point-dependent) operator on the tangent bundle of C.

Proposition 1 (*[18]*). *The bracket $\{F, G\}_K = \langle \nabla F, K\nabla G\rangle$ associated to K is skew-symmetric and satisfies the Jacobi identity. Hence, K is Poisson.* □

To establish that the bracket associated to K is indeed Magri's Poisson structure for NLS check directly that K and J form a compatible pair of Poisson structures and that the recursion operator KJ^{-1} does indeed generate the AKNS hierarchy of flows. This approach has the advantage of giving an explicit description of the bi-Hamiltonian nature of the evolution equations in the AKNS hierarchy.

Recall that the compatibility of two Poisson operators such as K and J amounts to proving that their sum is also a Poisson operator on the manifold in question. Two compatible Poisson operators give rise to a bi-Hamiltonian system when there exists a vector field that is Hamiltonian with respect to both Poisson structures [19]. In this case, one can generate a hierarchy of bi-Hamiltonian vector fields, by recursively applying the composition of one of the Poisson operators with the inverse of the other to the original bi-Hamiltonian field (assuming, of course, that one of the Poisson operators is indeed invertible). We define

$$R = KJ^{-1} = \left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial_x + \begin{pmatrix} -q(I_- - I_+)r & q(I_- - I_+)q\\ r(I_- - I_+)r & -r(I_- - I_+)q\end{pmatrix}\right]$$

$$\times \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (L_- + L_+) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{10}$$

and consider the functional $H = -\int rq$. Then, $J\nabla H = \mathrm{Diag}(-\mathrm{i}, \mathrm{i})(q, r)^{\mathrm{T}} = (-\mathrm{i}q, \mathrm{i}r)^{\mathrm{T}}$, and $X_n = R^n J\nabla H$ defines a hierarchy of bi-Hamiltonian vector fields. Hamilton's equations associated to these fields are

$$((q, r)^{\mathrm{T}})_t = R^n J\nabla H. \tag{11}$$

Notice that $X_0 = J\nabla H = (-\mathrm{i}q, \mathrm{i}r)^{\mathrm{T}}$ and $X_1 = K\nabla H = (q_x, r_x)^{\mathrm{T}}$. The standard NLS flow is prescribed by

$$X_3 = \mathrm{i}\left[(q_{xx}, -r_{xx})^{\mathrm{T}} + (-2rq^2, 2qr^2)^{\mathrm{T}}\right]. \tag{12}$$

The hierarchy (11) is the NLS Hierarchy. (Note: the symbol $\times$ is used in Eq. (10) to denote matrix multiplication. Throughout this paper, it will also be used to denote an action of a matrix operator on vector, and the usual cross product. The precise meaning should be inferred from the context.)

We shall next show that the NLS hierarchy (11) is equivalent to a sub-hierarchy of the family of Calogero–Degasperis flows (2). In fact, we will demonstrate that the entire Calogero–Degasperis hierarchy can be generated from polynomials in R. The argument's basic idea is to write the recursion operator R in terms of $L_- + L_+$ and exploit the relationship between L_+ and L_- given in (6).

Proposition 2. *The NLS hierarchy, obtained by recursively applying the recursion operator R to the (Hamiltonian) field $(-\mathrm{i}q, \mathrm{i}r)^{\mathrm{T}}$, is equivalent to a sub-hierarchy of the Calogero–Degasperis family (2).*

Proof. We re-write Hamilton's equations (11), using the form of the recursion operator given in (10), to obtain

$$((q, r)^{\mathrm{T}})_t = \sigma_2 (L_- + L_+)^n (r, q)^{\mathrm{T}}, \tag{13}$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{14}$$

is the standard Pauli matrix.

Using induction on n together with formula (6) one sees that

$$(L_- + L_+)^n (r, q)^{\mathrm{T}} = 2^n (L_+)^n (r, q)^{\mathrm{T}}, \tag{15}$$

so that (13) becomes

$$((q, r)^{\mathrm{T}})_t = 2^n \sigma_2 L_+^n (r, q)^{\mathrm{T}}. \tag{16}$$

Multiplying both sides of (16) by $-\mathrm{i}\sigma_2$, we obtain

$$((-r, q)^{\mathrm{T}})_t = -2^n \mathrm{i} L_+^n (r, q)^{\mathrm{T}}, \tag{17}$$

which is clearly a sub-hierarchy of the Calogero–Degasperis family of flows. □

Using Weierstrass approximation one can replace the monomial in (17) by an arbitrary analytic function, A, of L_+ and finally re-write the equation as $((r, -q)^{\mathrm{T}})_t + A(L_+)(r, q)^{\mathrm{T}} = 0$, which is the form of the flow derived by Calogero and Degasperis. Hence,

Corollary 1. *The entire Calogero–Degasperis family of flows may be formally generated from polynomials in R.* □

3. Inverse scattering preliminaries

This section begins with the direct scattering formulation for the Zakharov–Shabat eigenvalue problem (18) originally studied by Ablowitz and Ladik in connection with the integrable equations bearing their name.

After identifying (and characterizing) two pairs of eigenfunctions $\phi_k(z)$, $\hat{\phi}_k(z)$ and $\psi_k(z)$, $\hat{\psi}_k(z)$ (which behave nicely at either $-\infty$ or ∞), we focus on a Wronskian relation introduced by Ladik and Chiu as a means to study a family of integrable evolution equations arising from the linear problem (18) (which includes AL).

The generalized Wronskian identity presented by Ladik and Chiu provides, once again, a way to relate the evolution of the potentials r_k and q_k (defining (18)) to the evolution of the scattering coefficients (i.e., the parameters specifying the asymptotic behavior of the eigenfunctions mentioned above). By so doing, it also singles out two sum-difference operators L and L^{-1}, which together generate the collection of (integrable) equations prescribing the potentials' evolution.

As we will soon see, the specific form and relationship between L and L^{-1} will guide the construction of two new operators $\mathcal{L}_+$ and $\mathcal{L}_-$, associated to a sub-hierarchy of the Chiu–Ladik equations which encompasses the AL system. The latter operators can be viewed as discrete analogs of L_+ and L_- in more than one way. In particular, they will play a key role in unveiling the Poisson-geometric picture behind AL, as we begin to indicate below.

Exploiting the time-independence of one of the scattering coefficients (an essential feature of these kinds of integrable equations), one can obtain both a sequence of constants of motion, and a generating function for the associated hierarchy of variation (gradient) fields. It turns out that it is also possible to explicitly write down (resolvent) identities giving the infinite hierarchy of gradient fields in terms of powers of L and L^{-1}. These identities will be instrumental for exhibiting the bi-Hamiltonian nature of the AL hierarchy in Section 4.

3.1. A generalized Wronskian leading to AL flows

In [8], Chiu and Ladik developed a generalized Wronskian identity based on the discrete version of the Zakharov–Shabat eigenvalue problem:

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_{k+1} = \begin{pmatrix} z & q_k \\ r_k & 1/z \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_k = \mathcal{E}_k \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}_k, \tag{18}$$

which generated a large family of integrable evolution equations associated to AL. Here, $z \in \mathbb{C}$ is the eigenvalue parameter, and the complex-valued potentials q_k and r_k are assumed to vanish rapidly for $k \to \pm\infty$.

Let $\Phi_k(z)$ and $\Phi'_k(z)$ respectively denote matrix solutions to (18) and to a *second* eigenvalue problem of the same form but for a new set of potentials q'_k, r'_k. The initial Wronskian identity proposed by Chiu and Ladik takes the form

$$\sum_{k=-\infty}^{\infty} P(k+1)\ {\Phi'_k}^{\mathrm{T}}\ M(k,z)\ \Phi_{k+1} = {\Phi'_k}^{\mathrm{T}} P(k) F(k) \Phi_k\ |_{-\infty}^{\infty},$$

where

$$M(k,z) = \left[\begin{pmatrix} z & r'_k \\ q'_k & 1/z \end{pmatrix} F(k+1) - F(k) \begin{pmatrix} 1/z & -q_k \\ -r_k & z \end{pmatrix}\right]. \tag{19}$$

Above, $P(k) = \prod_{j=k}^{\infty}(1-r_k q_k)$ and $F(k,z)$ is a matrix that can be chosen arbitrarily, independent of q, q', r, r' and of the particular choice of fundamental matrices, Φ, Φ'. In this paper we will be using a particular choice of F that is presented in (28). The validity of (19) results from the telescoping of the terms in the specified series, which can be checked directly. This identity serves to relate correlations between components of Φ and Φ' to the asymptotic behavior, in k, of the scattering coefficients $a, \hat{a}, b, \hat{b}$ for Φ (defined below) and $a', \hat{a}', b', \hat{b}'$ for Φ'.

Due to the asymptotic decay of the potentials, the right-hand side of (19) makes sense and may be written in terms of the scattering parameters (for the "primed" and "unprimed" linear problems). Its precise form depends both on the choice of matrix $F(k,z)$ and on the particular eigenfunctions making up the columns of $\Phi_k(z)$.

Remark 1. We will use the notation f_k and $f(k)$ interchangeably, depending on emphasis, to denote functions of the discrete variable k. The z dependence of objects such as Φ_k, $F(k)$ and the scattering parameters $a, \hat{a}, b, \hat{b}$ may or may not be explicitly noted, depending on the context.

Following the work of Chiu and Ladik, we now specify the precise form of the right-hand side of (19) by choosing a particular fundamental matrix $\Phi_k(z)$ and determining its asymptotic form. We begin by considering the four special solutions

$$\phi_k(z) \sim z^k(1,0)^{\mathrm{T}}, \quad \hat{\phi}_k(z) \sim z^{-k}(0,1)^{\mathrm{T}}, \quad \text{for } k \to -\infty \tag{20}$$

$$\psi_k(z) \sim z^{-k}(0,1)^{\mathrm{T}}, \quad \hat{\psi}_k(z) \sim z^{k}(1,0)^{\mathrm{T}}, \quad \text{for } k \to \infty \tag{21}$$

of (18) and the scattering relations prescribed by

$$\begin{pmatrix} \phi_k(z) \\ \hat{\phi}_k(z) \end{pmatrix} = \begin{pmatrix} b(z) & a(z) \\ \hat{a}(z) & \hat{b}(z) \end{pmatrix} \begin{pmatrix} \psi_k(z) \\ \hat{\psi}_k(z) \end{pmatrix} = S(z) \begin{pmatrix} \psi_k(z) \\ \hat{\psi}_k(z) \end{pmatrix}. \tag{22}$$

We then set $\Phi_k(z) = (\phi_k(z), \psi_k(z))$ and determine that

$$\Phi_k \sim \begin{pmatrix} z^k & \dfrac{-\hat{b}}{C_0} z^k \\ 0 & \dfrac{a}{C_0} z^{-k} \end{pmatrix}, \quad \text{for } k \to -\infty, \tag{23}$$
$$\Phi_k \sim \begin{pmatrix} a z^k & 0 \\ b z^{-k} & z^{-k} \end{pmatrix}, \quad \text{for } k \to \infty,$$

where $\det S(z) = a\hat{a} - b\hat{b} = C_0$, and $C_0 = \lim_{k\to-\infty} P(k) = \prod_{-\infty}^{\infty}(1-r_k q_k)$, as shown by Ablowitz et al. [3].

We now adapt the results established by Chiu and Ladik to our choice of $\Phi(z)$ and arrive at two Wronskian-type identities based on (19). They are:

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} P(k+1)\, {\Phi'_k}^{\mathrm{T}} \left[\mathcal{I}\Lambda^l \begin{pmatrix} H_1^{(0)}(k) \\ H_4^{(0)}(k) \end{pmatrix}\right] \Phi_{k+1} \\ &\quad = z^{2l} \begin{pmatrix} a'b & a'-a \\ 0 & \dfrac{a\hat{b}'}{C'_0} \end{pmatrix} \\ &\qquad + \sum_{j=1}^{l} z^{2(l-j)} ({\Phi'_k}^{\mathrm{T}} P(k) F^{(j)}(k) \Phi_k)\ |_{-\infty}^{\infty}, \end{aligned} \tag{24}$$

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} P(k+1)\, {\Phi'_k}^{\mathrm{T}} \left[\mathcal{I}(\Lambda^{-1})^l \begin{pmatrix} h_1^{(0)}(k) \\ h_4^{(0)}(k) \end{pmatrix}\right] \Phi_{k+1} \\ &\quad = \frac{1}{z^{2l}} \begin{pmatrix} ab' & 0 \\ a - \dfrac{a'C_0}{C'_0} & \dfrac{a'\hat{b}}{C'_0} \end{pmatrix} \\ &\qquad + \sum_{j=1}^{l} \frac{1}{z^{2(l-j)}} ({\Phi'_k}^{\mathrm{T}} P(k) F^{(j)}(k) \Phi_k)\ |_{-\infty}^{\infty}, \end{aligned} \tag{25}$$

where the integro-differential operators Λ, Λ^{-1} are given by

$$\begin{aligned} \Lambda = &\begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} + \begin{pmatrix} -r_k \mathcal{J}_k(q_j E^-) & r_k \mathcal{J}_k(r'_j E^+) \\ -q'_k \mathcal{J}_k(q_j E^-) & q'_k \mathcal{J}_k(r'_j E^+) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ q'_k q_k E^- & -q'_k r'_k E^+ \end{pmatrix} \\ &+ (1-r_k q_k) \begin{pmatrix} -E^-(r'_k)\hat{\mathcal{J}}_k(q'_j) & E^-(r'_k)\hat{\mathcal{J}}_k(r_j) \\ 0 & 0 \end{pmatrix} \\ &+ (1-r'_k q'_k) \begin{pmatrix} 0 & 0 \\ -E^+(q_k \hat{\mathcal{J}}_k(q'_j)) & E^+(q_k \hat{\mathcal{J}}_k(r_j)) \end{pmatrix}, \end{aligned} \tag{26}$$

$$\begin{aligned} \Lambda^{-1} = &\begin{pmatrix} E^+ & 0 \\ 0 & E^- \end{pmatrix} + \begin{pmatrix} r'_k \mathcal{J}_k(q'_j E^+) & -r'_k \mathcal{J}_k(r_j E^-) \\ q_k \mathcal{J}_k(q'_j E^+) & -q_k \mathcal{J}_k(r_j E^-) \end{pmatrix} \\ &+ \begin{pmatrix} -r'_k q'_k E^+ & r'_k r_k E^- \\ 0 & 0 \end{pmatrix} \\ &+ (1-r_k q_k) \begin{pmatrix} 0 & 0 \\ E^-(q'_k)\hat{\mathcal{J}}_k(q_j) & -E^-(q'_k)\hat{\mathcal{J}}_k(r'_j) \end{pmatrix} \\ &+ (1-r'_k q'_k) \begin{pmatrix} E^+(r_k \hat{\mathcal{J}}_k(q_j)) & -E^+(r_k \hat{\mathcal{J}}_k(r'_j)) \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{27}$$

as demonstrated in the original Chiu–Ladik work [8]. The shift operators $E^\pm$ act in the standard fashion, $E^\pm v_k = v_{k\pm 1}$, whereas $\mathcal{J}_k(v(j))\ u \doteq \sum_{j=k}^{\infty} v(j)u(j)$, and $\hat{\mathcal{J}}_k(v(j))\ u \doteq p(k)/2 + \sum_{j=k}^{\infty} [(\prod_{i=k}^{j-1}(1-r'_i q'_i)/(1-r_i q_i))(v(j)u(j))/(1-r_j q_j)]$, for $p(k) \doteq p\prod_{j=k}^{\infty}(1-r'_k q'_k)/(1-r_k q_k)$ and p an arbitrary constant.

As explained both in [18] and in Chiu and Ladik's paper, Eq. (24) arises from an iteration in powers of z^2 prescribed essentially through a suitably chosen ansatz for the $M(k,z)$

factor in the summand of (19). Specifically, we can choose a sequence of M to be of the form

$$M^{(l)}(k) = \begin{pmatrix} -z^2 H_1^{(l-1)}(k) + H_1^{(l)}(k) & 0 \\ 0 & -z^2 H_4^{(l-1)}(k) + H_4^{(l)}(k) \end{pmatrix}, \quad l \geq 0 \tag{28}$$

where

$$M^{(0)}(k) = \begin{pmatrix} r_k & 0 \\ 0 & q_k' \end{pmatrix} = \begin{pmatrix} H_1^{(0)}(k) & 0 \\ 0 & H_4^{(0)}(k) \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} H_1^{(l+1)}(k) \\ H_4^{(l+1)}(k) \end{pmatrix} = \Lambda \begin{pmatrix} H_1^{(l)}(k) \\ H_4^{(l)}(k) \end{pmatrix}. \tag{29}$$

Such an ansatz completely determines the structure of the corresponding $F(k, z)$ [8]. The second equation emerges in a similar manner from a sequence M^{-l} related to an iteration in powers of $1/z^2$.

The pivotal identity for obtaining the precise time-evolution of the vector potential (r_k, q_k) and the scattering parameters associated to (18) arises by taking linear combinations of (24) and (25) for different values of $l > 0$. In particular, the family of Wronskian identities can be used to generate families of evolution equations by taking the primed spectral ingredients (eigenfunctions and scattering coefficients) to be the time-evolved counter-parts of the initial (unprimed) eigenfunctions and scattering coefficients:

$$\begin{aligned} &a' \doteq a(t), \quad \hat{a}' \doteq \hat{a}(t), \quad b' \doteq b(t), \quad \hat{b}' \doteq \hat{b}(t), \\ &r_k' \doteq r_k(t), \quad q_k' \doteq q_k(t), \quad \Phi_k' \doteq \Phi_k(t),\ C_0' \doteq C_0(t). \end{aligned} \tag{30}$$

As indicated in the original Chiu and Ladik work, by considering linear combinations of Eqs. (24) and (25) in light of the substitutions (30), one arrives at the operators

$$\begin{aligned} L = {} & \begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} + (1 - r_k q_k) \begin{pmatrix} -pE^-(r_k) \\ -\check{p}E^+(q_k) \end{pmatrix} + (1 - r_k q_k) \\ & \times \begin{pmatrix} -E^-\left(r_k J_k^+\left(\frac{q_j}{1 - r_j q_j}\right)\right) & E^-\left(r_k J_k^+\left(\frac{r_j}{1 - r_j q_j}\right)\right) \\ -E^+\left(q_k J_k^+\left(\frac{q_j}{1 - r_j q_j}\right)\right) & E^+\left(q_k J_k^+\left(\frac{r_j}{1 - r_j q_j}\right)\right) \end{pmatrix} \\ & + (1 - r_k q_k) \begin{pmatrix} E^-\left(\frac{r_k q_k}{1 - r_k q_k}\right) & -E^-\left(\frac{r_k^2}{1 - r_k q_k}\right) \\ 0 & 0 \end{pmatrix} \\ & + \begin{pmatrix} -r_k J_k^+(q_j E^-) & r_k J_k^+(r_j E^+) \\ -q_k J_k^+(q_j E^-) & q_k J_k^+(r_j E^+) \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 \\ q_k^2 E^- & -q_k r_k E^+ \end{pmatrix}, \end{aligned} \tag{31}$$

$$\begin{aligned} L^{-1} = {} & \begin{pmatrix} E^+ & 0 \\ 0 & E^- \end{pmatrix} + (1 - r_k q_k) \begin{pmatrix} pE^+(r_k) \\ \check{p}E^-(q_k) \end{pmatrix} + (1 - r_k q_k) \\ & \times \begin{pmatrix} E^+\left(r_k J_k^+\left(\frac{q_j}{1 - r_j q_j}\right)\right) & -E^+\left(r_k J_k^+\left(\frac{r_j}{1 - r_j q_j}\right)\right) \\ E^-\left(q_k J_k^+\left(\frac{q_j}{1 - r_j q_j}\right)\right) & -E^-\left(q_k J_k^+\left(\frac{r_j}{1 - r_j q_j}\right)\right) \end{pmatrix} \\ & + (1 - r_k q_k) \begin{pmatrix} 0 & 0 \\ -E^-\left(\frac{q_k^2}{1 - r_k q_k}\right) & E^-\left(\frac{q_k r_k}{1 - r_k q_k}\right) \end{pmatrix} \\ & + \begin{pmatrix} r_k J_k^+(q_j E^+) & -r_k J_k^+(r_j E^-) \\ q_k J_k^+(q_j E^+) & -q_k J_k^+(r_j E^-) \end{pmatrix} \\ & + \begin{pmatrix} -r_k q_k E^+ & r_k^2 E^- \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{32}$$

(arising from Λ and Λ^{-1} respectively), and to the following (simple!) time-evolution equations for the scattering parameters:

$$\begin{aligned} &((a(z), \hat{a}(z))^{\mathrm{T}})_t = 0, \\ &((-b(z), \hat{b}(z))^{\mathrm{T}})_t = \omega(z^2)(b(z), \hat{b}(z))^{\mathrm{T}}, \quad (C_0)_t = 0. \end{aligned} \tag{33}$$

(The interested reader may refer to either [8] or [18] for the actual calculations leading to these identities.)

Remark 2. The operator J_k^+ present in (31) and (32) is defined by $J_k^+(u_j) \doteq \sum_{j=k}^{\infty} u_j$ and is the discrete analog of the integral operator I_+, appearing in the expression for the continuous integro-differential operator L_+ given in (3). The symbols p, $\check{p}$ denote (discrete) integration constants.

Obtaining evolution equations for the potentials $r_k(t)$, $q_k(t)$ requires more careful arguments which exploit the analytic properties of the Jost functions associated to the linear problem (18) and its four special solutions (20) and (21). This derivation is carried out systematically in [18] (Theorem 1), and yields the time-flows which also appear in [8]:

$$((-r_k, q_k)^{\mathrm{T}})_t = \omega(L)(r_k, q_k)^{\mathrm{T}}, \tag{34}$$

where $\omega(x) \doteq \omega_1(x) + \omega_2(x^{-1})$ and L and L^{-1} are the sum-difference operators (31) and (32).

The family of discrete integrable evolution equations (34) arising from the Chiu–Ladik Wronskian approach includes several discrete versions of mKdV equations as well as the well-known discretization of NLS constructed by Ablowitz and Ladik, namely,

$$\mathrm{i}(q_k)_t = (1 + |q_k|^2)(q_{k+1} + q_{k-1}), \tag{35}$$

(up to the linear term $-2q_k$.) The latter – which we will also call the AL equation – can essentially be obtained from one of the simplest symmetric $\omega(L)$, namely $L + L^{-1}$. Indeed, as one can directly check,

$$\begin{aligned} ((-r_k, q_k)^{\mathrm{T}})_t &= -\mathrm{i}(L + L^{-1})(r_k, q_k)^{\mathrm{T}} \\ &= -\mathrm{i}(1 - r_k q_k)(E^+ + E^-)(r_k, q_k)^{\mathrm{T}}. \end{aligned} \tag{36}$$

In the reduction $r_k = -\overline{q}_k$, Eq. (36) becomes (35).

3.2. *The constant of motion* log *a and associated gradient fields*

Recall that the scattering parameter a is an analytic function of the complex variable z on the complement of the unit disc, and can therefore be represented by its Laurent series expansion [3]. Since a is time-independent – as made evident through the evolution equations (33) – so are the coefficients of its Laurent series. Such coefficients then constitute an infinite family of constants of motion associated with the flows (34). The same reasoning shows that the coefficients of the Taylor series for $\hat{a}$ (on $|z| < 1$) also give an infinite family of integrals.

Ablowitz and Ladik [2] utilized a recursive technique to arrive at the collection of constants of motion arising from $\log \hat{a}$:

$$\begin{aligned}\hat{C}_1 &= \sum q_k r_{k+1},\\ \hat{C}_2 &= \sum r_{k+1}q_{k-1}(1-r_kq_k) - \frac{1}{2}q_k^2 r_{k+1}^2, \ldots .\end{aligned} \tag{37}$$

The same formal method may be used to compute the hierarchy of integrals associated to $\log a$, namely,

$$\begin{aligned}C_1 &= \sum r_k q_{k+1},\\ C_2 &= \sum q_{k+1}r_{k-1}(1-r_kq_k) - \frac{1}{2}r_k^2 q_{k+1}^2, \ldots .\end{aligned} \tag{38}$$

One may also obtain a hierarchy of "symmetric" constants of motion, by combining the above as $C_0, C_1+\hat{C}_1, C_2+\hat{C}_2, \ldots$, where $C_0 = \prod_{k=-\infty}^{\infty}(1-r_kq_k)$. (See (23) and (33).)

We now determine a hierarchy of gradient fields associated to the conserved quantities arising from the series expansion of $\log a$ and $\log \hat{a}$ just discussed.

The argument is based upon considering a variation $(\dot{q}_k, \dot{r}_k)^{\mathrm{T}} = (a_k, b_k)^{\mathrm{T}}$ of $(q, r)^{\mathrm{T}}$ along the space of rapidly vanishing (vector) potentials, while keeping the eigenvalue z fixed. By finding the expression for the induced variation on the special eigenfunctions $\phi_k, \hat{\phi}_k, \psi_k, \hat{\psi}_k$ (the special solutions given in (20), (21) to the linear problem (18)), one is able to precisely calculate the effect of the potentials' perturbation on the scattering parameters $a, \hat{a}, b, \hat{b}$. For instance,

$$\dot{\psi}_n = -\Phi(n)\sum_{k=n}^{\infty}\Phi(k+1)^{-1}\begin{pmatrix}0 & \dot{q}_k\\ \dot{r}_k & 0\end{pmatrix}\psi_k, \tag{39}$$

leads to

$$\begin{aligned}\frac{C_0}{a}\left(\frac{\dot{a}}{C_0}\right) &= \left[\log a - \sum \dot{\log}(1-r_kq_k)\right]\\ &= \sum \frac{P(k+1)}{a}\left(-\phi_{k+1}^{(1)}\psi_k^{(1)}\,\dot{r}_k + \phi_{k+1}^{(2)}\psi_k^{(2)}\,\dot{q}_k\right),\end{aligned} \tag{40}$$

in the manner outlined in Appendix A of this paper. This, in turn, allows us to deduce an explicit expression for the variational derivative $\delta_k \log a \doteq (\delta \log a/\delta q_k, \delta \log a/\delta r_k)$ of $\log a$, namely,

$$\begin{aligned}&\delta_k\left(\log a - \sum \log(1-r_kq_k)\right)\\ &\quad= \frac{P(k+1)}{a}\left(\phi_{k+1}^{(2)}\psi_k^{(2)}, -\phi_{k+1}^{(1)}\psi_k^{(1)}\right)^{\mathrm{T}}.\end{aligned} \tag{41}$$

By Laurent-expanding the expression for this generating function one can then obtain the desired hierarchy of "gradient" fields, each given by the variational derivative of the corresponding coefficient in the expansion of $\log a$.

The results just outlined, together with the supporting arguments presented in Appendix A, prove the following theorem.

Theorem 1. *The (z-dependent) gradients of $\log a(z)$ and $\log \hat{a}(z)$ are generating functions for the hierarchy of variation fields $\delta_k C_n$ and $\delta_k \hat{C}_n$ associated to the constants of motion (37) and (38) respectively. Their explicit expressions are*

$$\delta_k \log a + \delta_k H_0 = \frac{P(k+1)}{a}\left(\phi_{k+1}^{(2)}\psi_k^{(2)},\ -\phi_{k+1}^{(1)}\psi_k^{(1)}\right)^{\mathrm{T}}, \tag{42}$$

$$\delta_k \log \hat{a} + \delta_k H_0 = \frac{P(k+1)}{\hat{a}}\left(-\hat{\phi}_{k+1}^{(2)}\hat{\psi}_k^{(2)},\ \hat{\phi}_{k+1}^{(1)}\hat{\psi}_k^{(1)}\right)^{\mathrm{T}}, \tag{43}$$

where $H_0 = -\sum_{k=-\infty}^{\infty}\log(1-r_kq_k) = -\log C_0$ and $P(k+1) = \prod_{j=k+1}^{\infty}(1-r_kq_k)$. □

Notice that, on the unit circle $|z| = 1$, both Eqs. (42) and (43) make sense (as the scattering coefficients $a(z), \hat{a}(z)$ are holomorphic on the regions $|z| > 1$ and $|z| < 1$ of $\mathbb{C}$ respectively). One could then combine them to obtain a "symmetric" representation of the variational derivatives. This line of reasoning leads to the symmetric constants of motion stemming from the combination of (37) and (38).

3.3. Constructing the generating operators: $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{R}$

Prima facie, one observes several qualitative analogies between the operators L_+^2 (defined in Eq. (3)) associated to continuous NLS and $L + L^{-1}$. Both are second order operators, and both may be obtained via parallel (generalized Wronskian) techniques (in the discrete and continuous settings respectively). Furthermore, they each give rise to (continuous and discrete) integrable versions of NLS in a parallel manner:

$$\begin{aligned}\mathrm{i}((-r,q)^{\mathrm{T}})_t &= L_+^2(r,q)^{\mathrm{T}}\\ &= -1/2\left((1/2)r_{xx} - r^2q, (1/2)q_{xx} - q^2r\right)^{\mathrm{T}},\\ \mathrm{i}((-r_k,q_k)^{\mathrm{T}})_t &= (L+L^{-1})(r_k,q_k)^{\mathrm{T}}\\ &= (1-r_kq_k)(E^+ + E^-)(r_k,q_k)^{\mathrm{T}}.\end{aligned}$$

We recall that the later equation is the differential-difference version of NLS discussed earlier (36) and which we also refer to as the AL equation.

More precisely, setting $\mathcal{L}_+ \doteq L + L^{-1}$, we note that, based on Eqs. (31) and (32),

$$\begin{aligned}\mathcal{L}_+ &= (E^+ + E^-) + (1-r_kq_k)\begin{pmatrix}p(E^+ - E^-)r_k\\ -\check{p}(E^+ - E^-)q_k\end{pmatrix}\\ &\quad + (1-r_kq_k)(E^+ - E^-)\\ &\quad\times\begin{pmatrix} r_kJ_k^+\left(\dfrac{q_j}{1-r_jq_j}\right) & -r_kJ_k^+\left(\dfrac{r_j}{1-r_jq_j}\right)\\ -q_kJ_k^+\left(\dfrac{q_j}{1-r_jq_j}\right) & q_kJ_k^+\left(\dfrac{r_j}{1-r_jq_j}\right)\end{pmatrix}\\ &\quad + (1-r_kq_k)E^-\begin{pmatrix}\dfrac{r_kq_k}{1-r_kq_k} & \dfrac{-r_k^2}{1-r_kq_k}\\ \dfrac{-q_k^2}{1-r_kq_k} & \dfrac{q_kr_k}{1-r_kq_k}\end{pmatrix}\end{aligned}$$

$$+\begin{pmatrix} r_k J_k^+(q_j) & r_k J_k^+(r_j) \\ q_k J_k^+(q_j) & q_k J_k^+(r_j) \end{pmatrix}(E^+ - E^-) + \begin{pmatrix} -r_k q_k E^+ & r_k^2 E^- \\ q_k^2 E^- & -q_k r_k E^+ \end{pmatrix}. \tag{44}$$

Remark 3. From this point on, we will assume that the integration constants $p, \check{p}$ vanish, unless explicitly noted otherwise.

Proposition 3. *The operator $\mathcal{L}_+$ generates a sub-hierarchy of the family of evolution equations (34) given by*

$$((-r_k, q_k)^{\mathrm{T}})_t = \mathcal{P}(\mathcal{L}_+)(r_k, q_k)^{\mathrm{T}}, \tag{45}$$

where $\mathcal{P}$ denotes an arbitrary polynomial in $\mathcal{L}_+$.

Proof. Since L and L^{-1} commute, all powers of $\mathcal{L}_+ = L + L^{-1}$ can also be written as linear combinations of powers of L and L^{-1}, and hence $\mathcal{P}(\mathcal{L}_+) = \omega_1(L) + \omega_2(L^{-1})$. □

Note that the hierarchy (45) encompasses various versions of the AL equations. In the reduction $r = -\overline{q}$, these include (35), and also the more standard

$$\mathrm{i}(q_k)_t = (1 + |q_k|^2)(q_{k+1} + q_{k-1}) - 2q_k, \tag{46}$$

obtained from $\mathcal{P}(\mathcal{L}_+) = \mathcal{L}_+ - 2$. Observe also the parallel with the Calogero–Degasperis hierarchy (2) associated with NLS.

In the continuous setting, we explored a second recursive-type operator giving rise to the NLS hierarchy generated by L_+. This operator, denoted by R, was essentially constructed as the sum of L_+ and L_- (see (3) and (5)) and was later seen to carry important geometric information concerning the Poisson geometry of NLS.

In the remaining part of this section we will construct the adjoint of $\mathcal{L}_+$ and use it to define $\mathcal{L}_-$, the discrete counterpart of L_-. We then show that a certain conjugate of the sum $\mathcal{L}_+ + \mathcal{L}_-$ defines a (discrete) recursion operator $\mathcal{R}$ for AL which is the analog of the "continuous" R associated to NLS.

Let $\langle \cdot, \cdot \rangle$ denote the (non-Hermitean) inner product on the complex space $\cdots \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \cdots$ (indexed by k in $\mathbb{Z}$) defined by

$$\langle (a,b)^{\mathrm{T}}, (c,d)^{\mathrm{T}} \rangle = -\sum (a_k d_k + b_k c_k)(1 - q_k r_k)^{-1}, \tag{47}$$

where $(a,b)^{\mathrm{T}} \doteq (a_k, b_k)^{\mathrm{T}}$, $(c,d)^{\mathrm{T}} \doteq (c_k, d_k)^{\mathrm{T}}$, $k \in \mathbb{Z}$.

Inspired by an analogous relation between L_+ and L_- observed in the continuous case [18], we define

$$\mathcal{L}_- \doteq \sigma_3 \mathcal{L}_+^* \sigma_3^{-1} = \sigma_3 (L^* + (L^{-1})^*) \sigma_3^{-1}, \tag{48}$$

where $*$ denotes the adjoint of an operator acting on $\cdots \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \cdots$ with respect to the inner product (47) just defined and $\sigma_3 = \mathrm{Diag}(1, -1)$. We then have,

$$\begin{aligned} L^* &= (1 - r_k q_k) \\ &\times \begin{pmatrix} E^-\left(\dfrac{1}{1 - r_k q_k}\right) & 0 \\ 0 & E^+\left(\dfrac{1}{1 - r_k q_k}\right) \end{pmatrix} + (1 - r_k q_k) \\ &\times \begin{pmatrix} E^-\left(r_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right)\right) & E^-\left(r_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right)\right) \\ -E^+\left(q_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right)\right) & -E^+\left(q_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right)\right) \end{pmatrix} \\ &+ (1 - r_k q_k) \begin{pmatrix} -E^-\left(\dfrac{r_k q_k}{1 - r_k q_k}\right) & 0 \\ E^+\left(\dfrac{q_k^2}{1 - r_k q_k}\right) & 0 \end{pmatrix} \\ &+ \begin{pmatrix} r_k J_k^-(q_j E^-) & r_k J_k^-(r_j E^+) \\ -q_k J_k^-(q_j E^-) & -q_k J_k^-(r_j E^+) \end{pmatrix} \\ &+ \begin{pmatrix} 0 & -r_k^2 E^+ \\ 0 & q_k r_k E^+ \end{pmatrix}, \end{aligned} \tag{49}$$

$$\begin{aligned} (L^{-1})^* &= (1 - r_k q_k) \\ &\times \begin{pmatrix} E^+\left(\dfrac{1}{1 - r_k q_k}\right) & 0 \\ 0 & E^-\left(\dfrac{1}{1 - r_k q_k}\right) \end{pmatrix} + (1 - r_k q_k) \\ &\times \begin{pmatrix} -E^+\left(r_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right)\right) & -E^+\left(r_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right)\right) \\ E^-\left(q_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right)\right) & E^-\left(q_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right)\right) \end{pmatrix} \\ &+ (1 - r_k q_k) \begin{pmatrix} 0 & E^+\left(\dfrac{r_k^2}{1 - r_k q_k}\right) \\ 0 & -E^-\left(\dfrac{q_k r_k}{1 - r_k q_k}\right) \end{pmatrix} \\ &+ \begin{pmatrix} -r_k J_k^-(q_j E^+) & -r_k J_k^-(r_j E^-) \\ q_k J_k^-(q_j E^+) & q_k J_k^-(r_j E^-) \end{pmatrix} \\ &+ \begin{pmatrix} r_k q_k E^+ & 0 \\ -q_k^2 E^+ & 0 \end{pmatrix}, \end{aligned} \tag{50}$$

where J_k^- denotes the equivalent of the integral operator I_- in the discrete context, that is, $J_k^-(u_j) \doteq \sum_{j=-\infty}^{k} u_j$. Summing the two expressions above and conjugating by σ_3, yields

$$\begin{aligned} \mathcal{L}_- &= (E^+ + E^-) - (1 - r_k q_k)(E^+ - E^-) \\ &\times \begin{pmatrix} r_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right) & -r_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right) \\ -q_k J_k^-\left(\dfrac{q_j}{1 - r_j q_j}\right) & q_k J_k^-\left(\dfrac{r_j}{1 - r_j q_j}\right) \end{pmatrix} \\ &- (1 - r_k q_k) E^+ \begin{pmatrix} \dfrac{-r_k q_k}{1 - r_k q_k} & \dfrac{r_k^2}{1 - r_k q_k} \\ \dfrac{q_k^2}{1 - r_k q_k} & \dfrac{-q_k r_k}{1 - r_k q_k} \end{pmatrix} \\ &- \begin{pmatrix} r_k J_k^-(q_j) & r_k J_k^-(r_j) \\ q_k J_k^-(q_j) & q_k J_k^-(r_j) \end{pmatrix}(E^+ - E^-) \\ &- \begin{pmatrix} r_k q_k E^- & -r_k^2 E^+ \\ -q_k^2 E^+ & q_k r_k E^- \end{pmatrix}. \end{aligned} \tag{51}$$

The (lengthy but straight-forward) calculations yielding formulas (49) for L^* and (50) for $(L^{-1})^*$ may be found in Appendix B of [18].

Let us now introduce the operator

$$\begin{aligned}
\mathcal{R} &\doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= 2(E^+ + E^-) + (1 - r_k q_k)(E^+ - E^-) \\
&\quad \times \left(\begin{matrix} q_k (J_k^+ - J_k^-) \left(\frac{r_j}{1 - r_j q_j} \right) & q_k (J_k^+ - J_k^-) \left(\frac{q_j}{1 - r_j q_j} \right) \\ r_k (J_k^+ - J_k^-) \left(\frac{r_j}{1 - r_j q_j} \right) & r_k (J_k^+ - J_k^-) \left(\frac{q_j}{1 - r_j q_j} \right) \end{matrix} \right) \\
&\quad + (1 - r_k q_k)(E^+ + E^-) \begin{pmatrix} \dfrac{q_k r_k}{1 - r_k q_k} & \dfrac{q_k^2}{1 - r_k q_k} \\ \dfrac{r_k^2}{1 - r_k q_k} & \dfrac{r_k q_k}{1 - r_k q_k} \end{pmatrix} \\
&\quad + \begin{pmatrix} q_k (J_k^+ - J_k^-)(r_j) & -q_k (J_k^+ - J_k^-)(q_j) \\ -r_k (J_k^+ - J_k^-)(r_j) & r_k (J_k^+ - J_k^-)(q_j) \end{pmatrix} \\
&\quad \times (E^+ - E^-) + \begin{pmatrix} -q_k r_k & -q_k^2 \\ -r_k^2 & -r_k q_k \end{pmatrix} (E^+ + E^-). \qquad (52)
\end{aligned}$$

Based solely on its form and method of construction, $\mathcal{R}$ may be regarded as a discrete analog of R. The analogy is strengthened by the fact that recursive applications of $\mathcal{R}$ to the (Hamiltonian) field $(\mathrm{i}q_k, -\mathrm{i}r_k)^{\mathrm{T}}$ yield the same hierarchy of evolution equations generated by $\mathcal{L}_+$. This can be verified by direct calculation for the first few iterations; it will be proven in the general case in Theorem 4. We note, once again, the remarkable parallel with the continuous setting.

In Section 4 we will also prove that $\mathcal{R}$ is in fact the true *geometric* analog of the recursion operator R associated to the NLS hierarchy. In particular, we will realize $\mathcal{R}$ as the composition of two skew-symmetric operators $\mathcal{J}^{-1}$ and $\mathcal{K}$ on the appropriate space of complex, vector valued functions of a discrete real variable k.

The next section contains a pivotal result for rigorously proving the geometric character of $\mathcal{R}$; thus the framework for unveiling the Poisson geometry of the AL equations will begin to emerge.

3.4. Resolvent identities for L and L^{-1}

In this section we establish two resolvent identities. The first one links the hierarchy of evolution equations generated by powers of L to the hierarchy of gradient fields (42) stemming from the generating function $\delta_k \log a + \delta_k H_0$. The second one relates the flows given in terms of powers of L^{-1} to the hierarchy of gradient fields (43) stemming from the generating function $\delta_k \log \hat{a} + \delta_k H_0$.

The fact that the operators L and L^{-1} satisfy these identities will be key for postulating and proving several fundamental results in Section 4. Through them, we will be able to show: (1) that a discrete analog of the *kernel condition* holds (Theorem 4); (2) that the hierarchy of flows (65) generated by the recursion operator $\mathcal{R}$ is equivalent to the hierarchy arising from powers of $\mathcal{L}_+ = L + L^{-1}$ (Corollary 3); that this hierarchy is a sub-hierarchy of the Chiu–Ladik flows; and that the $\mathcal{R}$-generated flows are bi-Hamiltonian (Theorem 5).

Theorem 2. *Let L, L^{-1} be as in (31), (32) respectively. Then L and L^{-1} satisfy the following resolvent identities:*

$$(I - z^{-2} L)^{-1} (r_k, q_k)^{\mathrm{T}} = (1 - r_k q_k)(\delta_k \log a + \delta_k H_0), \qquad (53)$$

$$(I - z^{2} L^{-1})^{-1} (r_k, q_k)^{\mathrm{T}} = (1 - r_k q_k)(\delta_k \log \hat{a} + \delta_k H_0), \qquad (54)$$

where $H_0 = -\sum_{k=-\infty}^{\infty} \log(1 - r_k q_k)$.

Proof. Setting $(\beta_k, -\alpha_k)^{\mathrm{T}} = P(k+1)/a\ (\psi_k^{(2)} \phi_{k+1}^{(2)}, -\phi_k^{(1)} \psi_{k+1}^{(1)})^{\mathrm{T}}$, in accordance with (95) in Appendix B, we recall from Theorem 1 that the non-standard squared eigenfunctions given by $-\alpha_k$ and β_k encode the hierarchy of gradient fields stemming from the generating function $\delta_k \log a + \delta_k H_0$. The resolvent formula (53) we seek to establish explicitly portrays $(-\alpha_k, \beta_k)^{\mathrm{T}}$ as a generating function for the hierarchy of fields arising from powers of L.

Guided by the form of the leading term of L we select the first component of (101) and the second component of (100) (re-writing the $\delta\gamma$-terms in each using (104)) of Appendix B and combine them so as to get

$$\begin{aligned}
&\begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} \left[(1 - r_k q_k) \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix} \right] \\
&= z^2 \left[(1 - r_k q_k) \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix} - \begin{pmatrix} r_k \\ q_k \end{pmatrix} \right] \\
&\quad + \begin{pmatrix} -(r_{k-1} + z^2 r_k) & 0 \\ 0 & q_{k+1} + z^2 q_k \end{pmatrix} \begin{pmatrix} E^- & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} -J_{k+1}^+(q_j) & J_{k+1}^+(r_j) \\ J_{k+1}^+(q_j) & -J_{k+1}^+(r_j) \end{pmatrix} \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix}. \qquad (55)
\end{aligned}$$

Re-grouping terms and dividing through by $-z^2$, the last equation is equivalent to

$$\begin{aligned}
&\left(I - z^{-2} \begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} \right) \left[(1 - r_k q_k) \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix} \right] + z^{-2} \mathcal{B} \\
&= \begin{pmatrix} r_k \\ q_k \end{pmatrix}, \qquad (56)
\end{aligned}$$

where $\mathcal{B}$ is the second term on the right-hand side of (55).

Based on the expected form of the resolvent identity and the formula for L, Eq. (56) suggests that $\mathcal{B}$ ought to simply amount to all but the first term of the operator $-L$ acting on the vector $[(1 - r_k q_k)(\beta_k, -\alpha_k)^{\mathrm{T}}]$. In other words,

$$\begin{aligned}
\mathcal{B} &\overset{?}{=} \begin{pmatrix} -r_k & 0 \\ 0 & q_k \end{pmatrix} \Upsilon(k) \begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} \left[(1 - r_k q_k) \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix} \right] \\
&\quad + \begin{pmatrix} -r_{k-1} & 0 \\ 0 & q_{k+1} \end{pmatrix} (1 - r_k q_k) \Upsilon(k) \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix}, \qquad (57) \\
\Upsilon(k) &\doteq \begin{pmatrix} -J_k^+(q_j) & J_k^+(r_j) \\ J_{k+1}^+(q_j) & -J_{k+1}^+(r_j) \end{pmatrix},
\end{aligned}$$

(where the symbol $\overset{?}{=}$ signifies a claim yet-to-be-established). Expressing $\mathcal{B}$ (in (55)) in terms of $\Upsilon(k)$, one notices that (57) is equivalent to $\mathrm{Diag}(-r_k, q_k)\mathcal{A} \overset{?}{=} 0$, where

$$\mathcal{A} = \left[\begin{pmatrix} r_{k-1} q_k + z^2 & 0 \\ 0 & q_{k+1} r_k + z^2 \end{pmatrix} \Upsilon(k) \right.$$

$$- \Upsilon(k) \begin{pmatrix} E^- & 0 \\ 0 & E^+ \end{pmatrix} (1 - r_k q_k) \Bigg] \begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix}, \qquad (58)$$

which in turn holds *if* $\mathcal{A} = 0$. Iterating once, we re-write the factor $\mathrm{Diag}(E^-, E^+)[(1-r_k q_k)(\beta_k, -\alpha_k)^{\mathrm{T}}]$ in the second term of $\mathcal{A}$ with its equivalent form given in Eq. (55). Grouping terms in z^2, we see that $\mathcal{A}$ amounts to the following quadratic equation in $\Upsilon(k)$:

$$\left[\begin{pmatrix} r_{k-1}q_k & 0 \\ 0 & q_{k+1}r_k \end{pmatrix} - \Upsilon(k)\begin{pmatrix} -r_{j-1} & 0 \\ 0 & q_{j+1} \end{pmatrix}\right] \times \left[\Upsilon(k)\begin{pmatrix} \beta_k \\ -\alpha_k \end{pmatrix}\right] + z^2\,\Upsilon(k) \times \left[r_j q_j \begin{pmatrix} -\beta_j \\ \alpha_j \end{pmatrix} - \begin{pmatrix} -r_j & 0 \\ 0 & q_j \end{pmatrix}\begin{pmatrix} q_j\beta_j + r_j\alpha_j \\ 0 \end{pmatrix}\right]. \qquad (59)$$

Now, due to the form of the matrix $\Upsilon(k)$, a simple calculation shows that the coefficient of the z^2 term in Eq. (59) vanishes. The quadratic term within the first line of (59) telescopes down to its boundary term, $J_k^+(q_j\beta_j + r_j\alpha_j)(q_k r_{k-1}, -q_{k+1}r_k)^{\mathrm{T}}$, which is evidently the negative of the linear term in the first line of (59). Hence $\mathcal{A} = 0$ and the resolvent formula follows. □

Remark 4. Resolvent identities similar to those we have just proven in Theorem 2, but for integrable PDEs such as KdV or NLS, have been known for a long time (see for instance [21] or [7]). As pointed out in our introduction, in the continuous setting, such identities have proven useful for relating scattering data to the geometric picture underlying a particular integrable equation. In our discrete scenario, Eqs. (53) and (54) also give a direct means for linking the geometry of the AL equations (depicted through L and L^{-1} — the key building blocks for $\mathcal{R}$ and $\mathcal{K}$) to the corresponding IST (through the scattering coefficient a). The methods presented here should extend naturally to other discrete integrable equations with second order scattering transforms.

4. Geometry: Almost Poisson structure of AL

In this section we exhibit the bi-Hamiltonian character of the AL hierarchy of equations, as defined in Eq. (65).

This postulated characterization is attained through the construction of two geometrically meaningful operators on the phase space $\mathcal{C}$ of these equations: an almost-Poisson operator $\mathcal{K}$, and a recursion operator $\mathcal{R}$ which can be realized either in terms of $\mathcal{L}_+$ and $\mathcal{L}_-$ (see (44) and (51)), or as $\mathcal{K}\mathcal{J}^{-1}$ (where $\mathcal{J} = \mathrm{Diag}(-\mathrm{i}, \mathrm{i})$ is the standard Poisson operator on $\mathcal{C}$).

The former depiction of $\mathcal{R}$ allows us to view the $\mathcal{R}$-generated hierarchy (65) as a sub-hierarchy of the Chiu–Ladik flows. The latter realization of this recursion operator attests to its true geometric character, and it is responsible for elucidating the bi-Hamiltonian character of the hierarchy.

The skew operator $\mathcal{K}$ can be portrayed as the discrete counterpart of Magri's Poisson operator for NLS, based both on its mode of construction and on its geometric properties. For example, we will see that, once their bi-Hamiltonian nature is established, the skewness of $\mathcal{K}$ serves to prove the commutativity of the AL flows generated by $\mathcal{R}$.

4.1. The geometric context: AL as a Hamiltonian system

We begin by defining the phase space $\mathcal{C}$ of the AL equations as the space of complex, vector valued functions of a discrete real variable $k \in \mathbb{Z}$ given by

$$\mathcal{C} = \{(q_k, r_k)^{\mathrm{T}} \in \cdots \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \cdots : \\ |q_k|, |\, r_k\, | \to 0 \text{ as } k \to \pm\infty\}.$$

A point in $\mathcal{C}$ will either be denoted by $(q, r)^{\mathrm{T}} \doteq [\ldots, (q_k, r_k)^{\mathrm{T}}, (q_{k+1}, r_{k+1})^{\mathrm{T}}, \ldots]$, or just by $(q_k, r_k)^{\mathrm{T}}$, depending on the context.

As in the continuous case, we may endow the tangent-bundle to $\mathcal{C}$, with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ arising from a point-dependent inner product defined by

$$\langle (a,b)^{\mathrm{T}}, (c,d)^{\mathrm{T}}\rangle_{(q,r)^{\mathrm{T}}} = -\sum (a_k d_k + b_k c_k)(1 - q_k r_k)^{-1}, \qquad (60)$$

on the tangent space to $\mathcal{C}$ at $(q, r)^{\mathrm{T}}$; that is, $(a, b)^{\mathrm{T}}$, $(c, d)^{\mathrm{T}}$ lie in $T_{(q,r)^{\mathrm{T}}}\mathcal{C} \simeq \mathcal{C}$. Under the identification $r_k = -\overline{q}_k$, (60) defines a positive-definite, real inner product.

Given a functional F over $\mathcal{C}$ we may define the (discrete) functional gradient of F, ∇F, by $\nabla F((q,r)^{\mathrm{T}}) = [\ldots, \nabla_k F((q,r)^{\mathrm{T}}), \nabla_{k+1} F((q,r)^{\mathrm{T}}), \ldots]$, where for k in $\mathbb{Z}$,

$$\nabla_k F((q,r)^{\mathrm{T}}) = -(1 - q_k r_k)\sigma_1(\delta F/\delta q_k, \delta F/\delta r_k)^{\mathrm{T}} \\ \doteq \nabla F((q_k, r_k)^{\mathrm{T}}), \qquad (61)$$

where σ_1 is the usual Pauli matrix with *ones* in the off-diagonal.

Now, let $\mathcal{J}_{(q,r)^{\mathrm{T}}} \doteq \mathrm{Diag}(\ldots, \mathcal{J}_k, \mathcal{J}_{k+1}, \ldots)$, $\mathcal{J}_k = \mathrm{Diag}(-\mathrm{i}, \mathrm{i})$ for all k, define the (standard, point-independent) skew-symmetric operator on $T_{(q,r)^{\mathrm{T}}}\mathcal{C}$, acting as multiplication by $\mathcal{J}_k$ on each $\mathbb{C}^2$ component. The operator $\mathcal{J}$ (which may also be regarded as an operator on the tangent bundle to $\mathcal{C}$) may be used to define the following Poisson bracket on $\mathcal{C}$:

$$\{F, G\}_{\mathcal{J}}((q,r)^{\mathrm{T}}) = \langle \nabla F, \mathcal{J}\nabla G\rangle((q,r)^{\mathrm{T}}) \\ = \langle (1 - q_k r_k)(\delta F/\delta r_k, \delta F/\delta q_k)^{\mathrm{T}}, \\ \mathrm{Diag}(-\mathrm{i}, \mathrm{i})(1 - q_k r_k)(\delta F/\delta r_k, \delta F/\delta q_k)^{\mathrm{T}}\rangle_{(q,r)^{\mathrm{T}}}, \qquad (62)$$

where F and G are smooth functionals over $\mathcal{C}$ which become real-valued under the identification $r_k = -\overline{q}_k$. Note that as in the continuous setting, this reality requirement implies that F and G must be either symmetric or anti-symmetric in r_k, q_k (for all k).

Remark 5. Since the operator $\mathcal{J}_k$ defined above does not actually depend on k, we will often use the notation $\mathcal{J}$ to refer to the 2×2 complex matrix $\mathrm{Diag}(-\mathrm{i}, \mathrm{i})$ originally defined as $\mathcal{J}_k$. By the same token, from now on we may omit noting explicitly the point dependence of operators which are clearly characterized as such by their definitions.

Let $H((q_k, r_k)^{\mathrm{T}}) = -\sum_{-\infty}^{\infty}(q_k r_{k+1} + r_k q_{k+1})$. Then the AL equations (36) can be obtained as a Hamiltonian system on $\mathcal{C}$ with respect to the $\mathcal{J}$-bracket (62) for the Hamiltonian functional H. In this same manner, the Hamiltonian $H((q_k, r_k)^{\mathrm{T}}) = -\sum_{-\infty}^{\infty}(q_k r_{k+1} + r_k q_{k+1} + 2\log(1 - q_k r_k))$ produces the standard (vector) AL flow, which becomes (46) in the reduction $r_k = -\overline{q}_k$. (See also [11].)

4.2. The operators $\mathcal{K}$ and $\mathcal{R}$ and the AL hierarchy

Guided by the parallel with the continuous NLS setting and the form of the operators $\mathcal{L}_+$ and $\mathcal{L}_-$, we re-consider the operator $\mathcal{R}$ defined in (52), and define a new operator

$$\mathcal{K}((q_k,r_k)^{\mathrm{T}}) \doteq \mathcal{R}((q,r)^{\mathrm{T}})\mathcal{J}$$
$$= -\mathrm{i}\left\{ 2(E^+ + E^-) + (1-r_kq_k)(E^+ - E^-) \right.$$
$$\times \begin{pmatrix} q_k(J_k^+ - J_k^-)\left(\frac{r_j}{1-r_jq_j}\right) & -q_k(J_k^+ - J_k^-)\left(\frac{q_j}{1-r_jq_j}\right) \\ r_k(J_k^+ - J_k^-)\left(\frac{r_j}{1-r_jq_j}\right) & -r_k(J_k^+ - J_k^-)\left(\frac{q_j}{1-r_jq_j}\right) \end{pmatrix}$$
$$+ (1-r_kq_k)(E^+ + E^-)\begin{pmatrix} \frac{q_kr_k}{1-r_kq_k} & \frac{-q_k^2}{1-r_kq_k} \\ \frac{r_k^2}{1-r_kq_k} & \frac{-r_kq_k}{1-r_kq_k} \end{pmatrix}$$
$$+ \begin{pmatrix} q_k(J_k^+ - J_k^-)(r_j) & q_k(J_k^+ - J_k^-)(q_j) \\ -r_k(J_k^+ - J_k^-)(r_j) & -r_k(J_k^+ - J_k^-)(q_j) \end{pmatrix}$$
$$\left. \times (E^+ - E^-) + \begin{pmatrix} -q_kr_k & q_k^2 \\ -r_k^2 & r_kq_k \end{pmatrix}(E^+ + E^-) \right\}. \tag{63}$$

Just as with $\mathcal{R}$, $\mathcal{K}$ is point-dependent and non-local; its definition at site k involves functions of the vector potential at neighboring sites, for example, functions of $q_{k\pm1}, r_{k\pm1}$.

As $\mathcal{K}$ acts on the tangent bundle of $\mathcal{C}$, it may be used to define the bracket

$$\{F,G\}_{\mathcal{K}}((q_k,r_k)^{\mathrm{T}}) = \langle \nabla F, \mathcal{K}\nabla G\rangle((q,r)^{\mathrm{T}}), \tag{64}$$

on the class of functionals over $\mathcal{C}$ just discussed.

Theorem 3. *The operator $\mathcal{K}$ is skew-symmetric.*

Proof. One first checks that $\mathcal{R}^* = \sigma_3\mathcal{R}\sigma_3$, where, as before, * denotes the adjoint of an operator with respect to the inner product (60), and $\sigma_3 = \mathrm{Diag}(1,-1)$. This calculation uses the fact that both $\pm\mathrm{i}\sigma_2$ and σ_3 are self-adjoint, while $\mathcal{L}_- \doteq \sigma_3\mathcal{L}_+^*\sigma_3^{-1}$.

The result now follows from the definition of $\mathcal{K}$ in terms of $\mathcal{R}$ and $\mathcal{J} = -\mathrm{i}\sigma_3$. Indeed,

$$\begin{aligned}\{F,G\}_{\mathcal{K}} &= \langle \nabla F, \mathcal{K}\nabla G\rangle = \langle \nabla F, \mathcal{R}\mathcal{J}\nabla G\rangle \\ &= \langle \mathcal{R}^*\nabla F, \mathcal{J}\nabla G\rangle = \langle -\mathcal{J}(\sigma_3\mathcal{R}\sigma_3)\nabla F, \nabla G\rangle \\ &= \langle \mathcal{R}(\mathrm{i}\sigma_3)\nabla F, \nabla G\rangle = -\langle \mathcal{R}\mathcal{J}\nabla F, \nabla G\rangle \\ &= -\langle \nabla G, \mathcal{K}\nabla F\rangle = -\{G,F\}_{\mathcal{K}}. \quad \square\end{aligned}$$

As discussed in Section 4.3.1 of [18], the $\mathcal{K}$-bracket can be seen to satisfy Leibnitz's identity, in addition to being skew. This leads to the following corollary.

Corollary 2. *The $\mathcal{K}$-bracket (64) defines an almost Poisson structure on $\mathcal{C}$.* $\square$

4.3. The bi-Hamiltonian character of the AL equations

Let us now consider the functional $H_0((q_k,r_k)^{\mathrm{T}}) = -\sum_{-\infty}^{\infty}\log(1-q_kr_k)$, on $\mathcal{C}$, and observe that $\mathcal{J}\nabla H_0((q_k,r_k)^{\mathrm{T}}) = \mathrm{i}(q_k,-r_k)^{\mathrm{T}}$. If we now define $X_{(n)} \doteq \mathcal{R}^n\mathcal{J}\nabla H_0$, then $X_{(n)}$ defines a hierarchy of fields on the tangent bundle to $\mathcal{C}$ which, we claim, has the following properties:

(i) The fields $X_{(n)}$ mutually commute;
(ii) The hierarchy comprises the AL equations (36), which occur as $X_{(1)}$;
(iii) The evolution equations
$$((q_k,r_k)^{\mathrm{T}})_t = \mathcal{R}^n\mathcal{J}\nabla_k H_0 = X_{(n)}, \tag{65}$$
are equivalent to a sub-hierarchy of the family (45) and are therefore integrable;
(iv) The fields $X_{(n)}$ in the hierarchy are Hamiltonian with respect to $\mathcal{J}$ and also with respect to $\mathcal{K}$, i.e., bi-Hamiltonian.

Remark 6. The term *bi-Hamiltonian* here refers to sequences of fields $X_{(n)}$ which satisfy the *Lenard relations* $X_{(n)} = \mathcal{K}\nabla H_{n-1} = \mathcal{J}\nabla H_n$ [9]. The clarification is pertinent due to the fact that the Poisson nature of the skew operator $\mathcal{K}$ has not yet been established.

It is a good exercise to verify properties (iii) and (iv) explicitly for the first two iterations (in the process, property (ii) becomes apparent). As such calculations indicate, being able to show that the fields $\mathcal{L}_+^n((r_k,q_k)^{\mathrm{T}})$ are in the kernel of $\mathcal{L}_+ - \mathcal{L}_-$ for all n is of central importance in proving (iii). This is precisely the content of Theorem 4. In it, we use once again the identities derived in Appendix B to extend the Calogero–Degasperis "kernel-condition" result (see (6)) to the discrete setting. Corollary 3, the discrete analog of Proposition 2, will establish property (iii) in full generality. Theorem 5 will then use the resolvent identities (53) and (54) together with Corollary 3 to establish (iv). The proof of property (i) is rather immediate given (iv). It is the content of Corollary 5.

Throughout the remainder of this paper, $\sum f_j = \sum_{j=-\infty}^{\infty} f_j$; also the difference operator $\mathcal{L}_+ - \mathcal{L}_-$ will often be written as $\mathcal{L}_+ - \mathcal{L}_- = \mathcal{D}_1 + \mathcal{D}_2$, where

$$\mathcal{D}_1 \doteq (1-r_kq_k)(E^+ - E^-) \times \begin{pmatrix} r_k\sum q_j/(1-r_jq_j) & -r_k\sum r_j/(1-r_jq_j) \\ -q_k\sum q_j/(1-r_jq_j) & q_k\sum r_j/(1-r_jq_j) \end{pmatrix}, \tag{66}$$

and

$$\mathcal{D}_2 \doteq \begin{pmatrix} r_k\sum q_j & r_k\sum r_j \\ q_k\sum q_j & q_k\sum r_j \end{pmatrix}(E^+ - E^-). \tag{67}$$

Lemma 1. *Let $j = 1,2$. If $\mathcal{D}_j\left(L^m(r_k,q_k)^{\mathrm{T}}\right) = \mathcal{D}_j((L^{-1})^m(r_k,q_k)^{\mathrm{T}}) = 0$ for all m, then $(\mathcal{L}_+ - \mathcal{L}_-)\left(\mathcal{L}_+^n(r_k,q_k)^{\mathrm{T}}\right) = 0$ for all n (m,n non-negative integers).*

Proof. This follows immediately from the definition of $\mathcal{L}_+$ (see (44) above), the inverse relation between L and L^{-1} and the binomial theorem. $\square$

Proposition 4. *For $\mathcal{D}_1$ as in (66), $\mathcal{D}_1\left(L^m(r_k,q_k)^{\mathrm{T}}\right) = \mathcal{D}_1\left((L^{-1})^m(r_k,q_k)^{\mathrm{T}}\right) = 0$.*

Proof. Recall that the vector of squared eigenfunctions $(\beta_k,-\alpha_k)^{\mathrm{T}}$ defined in Appendix B may be explicitly viewed a generating function for the hierarchy of fields arising from powers of L (through the resolvent identities in Theorem 2). The same is true of $(-\hat\beta_k,\alpha_k)^{\mathrm{T}}$ and the fields given by $L^{-1}((r_k,q_k)^{\mathrm{T}})$. Using the inner product (60) one observes certain relations between the aforementioned fields, which can be compactly expressed as

$$\begin{aligned}&\langle (1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}, -\mathrm{i}\sigma_3(1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}\rangle\\ &\quad = \mathrm{i}\sum(1-r_kq_k)(\beta_k\alpha_k-\alpha_k\beta_k)=0,\end{aligned} \tag{68}$$

$$\langle (1-r_kq_k)(-\hat\beta_k,\alpha_k)^{\mathrm{T}}, -\mathrm{i}\sigma_3(1-r_kq_k)(-\hat\beta_k,\alpha_k)^{\mathrm{T}}\rangle = 0. \tag{69}$$

Let us now focus attention on the particular identities

$$\begin{aligned}0 &= \langle (r_k,q_k)^{\mathrm{T}}, -\mathrm{i}\sigma_3(1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}\rangle\\ &= \mathrm{i}\sum(q_k\beta_k+r_k\alpha_k),\end{aligned} \tag{70}$$

$$\begin{aligned}0 &= \langle (r_k,q_k)^{\mathrm{T}}, -\mathrm{i}\sigma_3(1-r_kq_k)(-\hat\beta_k,\alpha_k)^{\mathrm{T}}\rangle\\ &= -\mathrm{i}\sum(q_k\hat\beta_k+r_k\hat\alpha_k),\end{aligned} \tag{71}$$

obtained from expanding the first entry on the left-hand side of (68) (respectively (69)) in powers of z^{-2} (respectively z^2) as suggested by the resolvent formula (53) (respectively (54)).

Eqs. (70) and (66) together with (71) and (66) immediately show that

$$\begin{aligned}&\mathcal{D}_1\left((1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}\right)=0, \quad \text{and}\\ &\mathcal{D}_1\left((1-r_kq_k)(-\hat\beta_k,\alpha_k)^{\mathrm{T}}\right)=0.\end{aligned} \tag{72}$$

Using now (53) and the first part of (72), we see that

$$\begin{aligned}0 &= \mathcal{D}_1(r_k,q_k)^{\mathrm{T}} + z^{-2}\mathcal{D}_1\left(L(r_k,q_k)^{\mathrm{T}}\right)+\cdots\\ &\quad + z^{-2m}\mathcal{D}_1\left(L^m(r_k,q_k)^{\mathrm{T}}\right)+\cdots\end{aligned}$$

or, equivalently, $\mathcal{D}_1\left(L^m(r_k,q_k)^{\mathrm{T}}\right)=0$, for all m. The identity $\mathcal{D}_1\left((L^{-1})^m(r_k,q_k)^{\mathrm{T}}\right)=0$ stems from Eq. (54) and the second piece of (72) in a parallel manner. □

Remark 7. It is interesting to notice that identities (68) and (69) directly translate into commutativity relations for certain special Hamiltonian functions, H_n^a and $H_n^{\hat a}$, defined implicitly through Eqs. (83) and (85) in Theorem 5 below. (Specifically, the aforementioned identities imply that $\{H_m^a, H_n^a\} = \{H_m^{\hat a}, H_n^{\hat a}\} = 0$). As we will see in Corollary 5, obtaining this commutativity result for any two Hamiltonians in the AL hierarchy requires the full strength of Corollary 3.

Theorem 4. *For any non-negative integer n, $(\mathcal{L}_+ - \mathcal{L}_-)(\mathcal{L}_+^n(r_k,q_k)^{\mathrm{T}})=0$.*

Proof. Due to Lemma 1 and Proposition 4 it suffices to prove that

$$\mathcal{D}_2((L^{-1})^m(r_k,q_k)^{\mathrm{T}}) \overset{(a)}{=} \mathcal{D}_2(L^m(r_k,q_k)^{\mathrm{T}}) \overset{(b)}{=} 0. \tag{73}$$

Below, we present the argument showing that equality (b) in (73) holds. An analogous argument (based on the identities given at the end of Appendix B) shows that identity (a) holds as well, validating the theorem.

The key step in the argument closely resembles the recurrence strategy employed in the derivation of the resolvent identities (53) and (54), as described in Appendix B. In the current scenario, the technique entails re-writing the vector $(E^+-E^-)\left[(1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}\right]$ as

$$(z^{-2}-z^2)\mathrm{Diag}(1,-1)\left[(1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}}-(r_k,q_k)^{\mathrm{T}}\right] \tag{74}$$

$$\begin{aligned}&+\mathrm{Diag}(r_{k+1}+z^{-2}r_k,\ q_{k+1}+z^2q_k)\\ &\qquad\times(q_k\beta_k+z\gamma_k+1,\ -(r_k\alpha_k+z^{-1}\delta_k-1))^{\mathrm{T}}\end{aligned} \tag{75}$$

$$\begin{aligned}&-\mathrm{Diag}(-(r_{k-1}+z^2r_k),q_{k-1}+z^{-2}q_k)\\ &\qquad\times E^-\ (q_k\beta_k+z\gamma_k,r_k\alpha_k+z^{-1}\delta_k)^{\mathrm{T}},\end{aligned} \tag{76}$$

by considering the difference between Eqs. (100) and (101) in Appendix B.

The remainder of the proof amounts to a series of calculations. First we utilize Eqs. (74)–(76) together with (98) and (104) in Appendix B, to obtain a simplified three-term expression for $\mathcal{D}_2((1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}})$. We then exploit the particular form of the resulting expressions to show that their combined sum vanishes. Finally, the same argument used at the end of the proof of Proposition 4 shows that the vanishing of the vector $\mathcal{D}_2((1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}})$ implies the vanishing of $\mathcal{D}_2(L^m(r_k,q_k)^{\mathrm{T}})$ for all m.

Due to (70), the matrix pre-factor of the operator $\mathcal{D}_2$ applied to expression (74), becomes

$$(z^{-2}-z^2)\begin{pmatrix} r_k & 0\\ 0 & q_k\end{pmatrix}\begin{pmatrix}\sum q_j & -\sum r_j\\ \sum q_j & -\sum r_j\end{pmatrix}\left[-r_kq_k\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix}\right]. \tag{77}$$

Expression (77) is the first term in the sought-after three-term expression for $\mathcal{D}_2((1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}})$. The second and third terms (appearing in (78) and (79)) are essentially obtained by re-writing (75) and (76) by means of (104), in Appendix B. Specifically, the matrix pre-factor of $\mathcal{D}_2$ applied to expression (75) yields the new version of the second term in the expression for $\mathcal{D}_2((1-r_kq_k)(\beta_k,-\alpha_k)^{\mathrm{T}})$, namely,

$$\begin{aligned}&\begin{pmatrix} r_k & 0\\ 0 & q_k\end{pmatrix}\begin{pmatrix}\sum q_j & \sum r_j\\ \sum q_j & \sum r_j\end{pmatrix}\\ &\quad\times\begin{pmatrix} r_{k+1}+z^{-2}r_k & 0\\ 0 & q_{k+1}+z^2q_k\end{pmatrix}\\ &\quad\times\left[\begin{pmatrix}-J_{k+1}^+(q_j) & J_{k+1}^+(r_j)\\ J_{k+1}^+(q_j) & -J_{k+1}^+(r_j)\end{pmatrix}\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\right]\\ &\quad=\mathrm{Diag}(r_k,q_k)\Big[\sum q_j(r_{j+1}+z^{-2}r_j)(1,1)^{\mathrm{T}}\\ &\qquad+\sum[-q_j(r_{j+1}+z^{-2}r_j)+r_j(q_{j+1}+z^2q_j)]\\ &\qquad\times\ J_{j+1}^+(q_l\beta_l+r_l\alpha_l)(1,1)^{\mathrm{T}}\Big]\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Diag}(r_k, q_k) \sum [q_j(r_{j+1} + z^{-2} r_j) \\
&\quad + (-q_{j-1}(r_j + z^{-2} r_{j-1}) + r_{j-1}(q_j + z^2 q_{j-1})) \\
&\quad \times J_j^+(q_l \beta_l + r_l \alpha_l)](1,1)^{\mathrm{T}} \\
&= \mathrm{Diag}(r_k, q_k) \sum [q_j(r_{j+1} + z^{-2} r_j) \\
&\quad + (q_j \beta_j + r_j \alpha_j) J_j^-(-q_{l-1} r_l + r_{l-1} q_l \\
&\quad - (z^{-2} - z^2) q_{l-1} r_{l-1})](1,1)^{\mathrm{T}},
\end{aligned}
\tag{78}
$$

where, recall, $J_k^-(u_j) \doteq \sum_{j=-\infty}^{k} u_j$. The last equality in the above string of identities results from a direct application of the summation by parts formula

$$
\sum_{k=-\infty}^{\infty} a_k \left[\left(\sum_{j=k}^{\infty} - \sum_{j=-\infty}^{k} \right) b_j \right]
= - \sum_{k=-\infty}^{\infty} b_k \left[\left(\sum_{j=k}^{\infty} - \sum_{j=-\infty}^{k} \right) a_j \right].
$$

The previous one amounts simply to a convenient shift in the summation index for the bi-infinite sum. This same series of calculations leads to the desired third and final term. That is, using (104), and summation by parts, the action of the matrix pre-factor of $\mathcal{D}_2$ on (76) leads to the vector

$$
\begin{aligned}
&\mathrm{Diag}(r_k, q_k) \sum [-r_j(q_{j-1} + z^{-2} q_j) + (q_j \beta_j + r_j \alpha_j) \\
&\quad \times J_j^-(-r_{l-1} q_l + q_{l-1} r_l + (z^{-2} - z^2) q_l r_l)](1,1)^{\mathrm{T}}.
\end{aligned}
\tag{79}
$$

Consider now the sum of expressions (78) and (79). The J_j^--independent terms all disappear: the two z-dependent ones cancel out; the remaining two telescope away, leaving no boundary terms. Focusing now on the J_j^--dependent terms, one observes that the z-independent ones cancel off identically. The four z-dependent terms are pairwise-telescoping, and hence reduce to $\mathrm{Diag}(r_k, q_k) \sum [(q_j \beta_j + r_j \alpha_j) q_j r_j (z^{-2} - z^2)](1,1)^{\mathrm{T}}$, which is exactly the negative of (77). It follows that, for all (non-negative) m,

$$
\begin{aligned}
&\mathcal{D}_2((1 - r_k q_k)(\beta_k, -\alpha_k)^{\mathrm{T}}) = 0, \quad \text{and hence,} \\
&\mathcal{D}_2 \left(L^m (r_k, q_k)^{\mathrm{T}} \right) = 0.
\end{aligned}
\tag{80}
$$

A parallel argument, based on identities (106)–(108) in Appendix B, yields

$$
\begin{aligned}
&\mathcal{D}_2((1 - r_k q_k)(-\hat{\beta}_k, \alpha_k)^{\mathrm{T}}) = 0, \quad \text{and hence,} \\
&\mathcal{D}_2((L^{-1})^m (r_k, q_k)^{\mathrm{T}}) = 0.
\end{aligned}
\tag{81}
$$

The theorem follows. □

Corollary 3. *If* $(\mathcal{L}_+ - \mathcal{L}_-)\left(\mathcal{L}_+^m (r_k, q_k)^{\mathrm{T}}\right) = 0$ *for* $m \geq 0$, *then*

$$
\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{J} \nabla_k H_0 \\
&= 2^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{L}_+^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{J} \nabla_k H_0,
\end{aligned}
\tag{82}
$$

for $n \geq 0$ *and* $H_0 = -\sum_{k=-\infty}^{\infty} \log(1 - r_k q_k)$.

Proof. The statement may be verified by direct computation for $n = 1$. One then proceeds by using induction on n. Assuming that identity (82) holds for $n - 1$, we show below that it also holds for n. Theorem 4 is used in the last line below. Indeed,

$$
\begin{aligned}
&\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{J} \nabla_k H_0 \\
&= \mathrm{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-)(\mathcal{L}_+ + \mathcal{L}_-)^{n-1} \begin{pmatrix} r_k \\ q_k \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-) \left[\mathrm{i} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right. \\
&\quad \left. \times\ (\mathcal{L}_+ + \mathcal{L}_-)^{n-1} \begin{pmatrix} r_k \\ q_k \end{pmatrix} \right] \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&\quad \times \left[\mathrm{i} 2^{n-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{L}_+^{n-1} \begin{pmatrix} r_k \\ q_k \end{pmatrix} \right] \\
&= \mathrm{i} 2^{n-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-) \mathcal{L}_+^{n-1} \begin{pmatrix} r_k \\ q_k \end{pmatrix} \\
&= \mathrm{i} 2^{n-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[\mathcal{L}_+^n \begin{pmatrix} r_k \\ q_k \end{pmatrix} + \mathcal{L}_- \left(\mathcal{L}_+^{n-1} \begin{pmatrix} r_k \\ q_k \end{pmatrix} \right) \right] \\
&= \mathrm{i} 2^{n-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[\mathcal{L}_+^n \begin{pmatrix} r_k \\ q_k \end{pmatrix} + \mathcal{L}_+^n \begin{pmatrix} r_k \\ q_k \end{pmatrix} \right] \\
&= 2^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{L}_+^n \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{J} \nabla_k H_0 \right]. \quad \square
\end{aligned}
$$

Corollary 4 (*Validity of Property (iii)*). *The evolution equations (65) are equivalent to a sub-hierarchy of the Chiu–Ladik family (45) and, thus, are integrable.*

Proof. Working from Eq. (65) and using Corollary 3, we have

$$
\begin{aligned}
\begin{pmatrix} q_k \\ r_k \end{pmatrix}_t &= \mathcal{R}^n \mathcal{J} \nabla_k H_0 \\
&= \mathrm{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\mathcal{L}_+ + \mathcal{L}_-)^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_k \\ -r_k \end{pmatrix} \\
&= \mathrm{i} 2^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{L}_+^n \begin{pmatrix} r_k \\ q_k \end{pmatrix};
\end{aligned}
$$

or, equivalently, $((-r_k, q_k)^{\mathrm{T}})_t = \mathrm{i} 2^n \mathcal{L}_+^n (r_k, q_k)^{\mathrm{T}}$, which is of form (45), as claimed. □

Theorem 5 (*Validity of Property (iv)*). *The fields* $X_{(n)}$ *in the hierarchy (65) are Hamiltonian with respect to both,* $\mathcal{J}$ *and* $\mathcal{K}$, *i.e., bi-Hamiltonian in the sense of Remark 6.*

Proof. Let us once again consider the resolvent formulas for L and L^{-1}. Starting from (53) and matching inverse powers of z^2, we see that

$$
\begin{aligned}
&(1 - r_k q_k) \delta_k H_0 = (r_k, q_k)^{\mathrm{T}}, \quad \text{and also,} \\
&(1 - r_k q_k) \delta_k H_j^a = L^j (r_k, q_k)^{\mathrm{T}}.
\end{aligned}
\tag{83}
$$

Using the definition of ∇_k given in (61), we observe

$$\begin{aligned}\mathcal{J}\nabla_k H_j^a &= \mathrm{Diag}(-\mathrm{i},\mathrm{i})\left[-\begin{pmatrix}0&1\\1&0\end{pmatrix}L^j\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}q_k\\r_k\end{pmatrix}\right]\\ &= \mathrm{i}\begin{pmatrix}0&1\\-1&0\end{pmatrix}L^j\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}q_k\\-r_k\end{pmatrix}\\ &= \begin{pmatrix}0&1\\-1&0\end{pmatrix}L^j\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{J}\nabla_k H_0. \end{aligned}\tag{84}$$

Reasoning in a parallel fashion for the L^{-1} resolvent, (54), we get

$$\begin{aligned}(1-r_kq_k)\delta_k H_j^{\hat{a}} &= (L^{-1})^j\begin{pmatrix}r_k\\q_k\end{pmatrix},\\ \mathcal{J}\nabla_k H_j^{\hat{a}} &= \begin{pmatrix}0&1\\-1&0\end{pmatrix}(L^{-1})^j\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{J}\nabla_k H_0.\end{aligned}\tag{85}$$

(Notice that, based on Eqs. (38) and (37), one has $\delta_k H_j^a = \delta_k C_j$, $\delta_k H_j^{\hat{a}} = \delta_k \hat{C}_j$, for $j > 0$.) Based in Eqs. (84) and (85), we have

$$\begin{aligned}\begin{pmatrix}q_k\\r_k\end{pmatrix}_t &= \mathcal{R}^n\mathcal{J}\nabla_k H_0\\ &\overset{\text{Corollary 3}}{=} 2^n\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}(\mathcal{L}_+)^n\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{J}\nabla_k H_0\right]\\ &= 2^n\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}(L+L^{-1})^n\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{J}\nabla_k H_0\right]\\ &= 2^n\begin{pmatrix}0&1\\-1&0\end{pmatrix}\left[\sum_{n-k\geq k}\binom{n}{k}L^{(n-k)-k}\right.\\ &\quad\left.+\sum_{n-k<k}\binom{n}{k}(L^{-1})^{k-(n-k)}\right]\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathcal{J}\nabla_k H_0\\ &= 2^n\mathcal{J}\nabla_k\\ &\quad\times\left(\sum_{n-k\geq k}\binom{n}{k}H^a_{n-2k}+\sum_{n-k<k}\binom{n}{k}H^{\hat{a}}_{-(n-2k)}\right)\\ &= \mathcal{J}\nabla_k\tilde{H}_n,\end{aligned}$$

and so the last equality explicitly depicts the alluded hierarchy of flows as $\mathcal{J}$-Hamiltonian. Given this, one uses the fact that $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$ in a recursive fashion to show that these flows are also $\mathcal{K}$-Hamiltonian (in the sense of Remark 6). Indeed, for $n = 1$ and $n = 2$,

$$\begin{aligned}&\mathcal{R}\mathcal{J}\nabla_k H_0 = \mathcal{K}\nabla_k H_0 = \mathcal{J}\nabla_k\tilde{H}_1, \quad \text{and}\\ &\mathcal{R}^2\mathcal{J}\nabla_k H_0 = \mathcal{K}\mathcal{J}^{-1}\mathcal{K}\nabla_k H_0 = \mathcal{K}\nabla_k\tilde{H}_1,\end{aligned}$$

respectively. It then follows via an inductive argument, that $\mathcal{R}^n\mathcal{J}\nabla_k H_0 = \mathcal{K}\nabla_k\tilde{H}_{n-1} = \mathcal{J}\nabla_k\tilde{H}_n$, for $n > 0$, where $\tilde{H}_0 = H_0$. □

Corollary 5 (*Validity of Property (i)*). *Let $\mathcal{J}$ and $\mathcal{K}$ be skew operators on the tangent bundle to $\mathcal{C}$, and let $\{H_j\}$ be a sequence of functionals on $\mathcal{C}$ indexed by j in $\mathbb{Z}$. If $\mathcal{J}\nabla H_{k+1} = \mathcal{K}\nabla H_k$, then the H_j pairwise commute, that is $\{H_j, H_k\}_{\mathcal{J}} = 0$.*

Proof. Simply using the skew-symmetry of $\mathcal{K}$ and $\mathcal{J}$, we have

$$\begin{aligned}\{H_j,H_k\}_{\mathcal{J}} &= \langle\nabla H_j,\mathcal{J}\nabla H_k\rangle = \langle\nabla H_j,\mathcal{K}\nabla H_{k-1}\rangle\\ &= -\langle\mathcal{K}\nabla H_j,\nabla H_{k-1}\rangle\\ &= -\langle\mathcal{J}\nabla H_{j+1},\nabla H_{k-1}\rangle\\ &= \langle\nabla H_{j+1},\mathcal{J}\nabla H_{k-1}\rangle = \{H_{j+1},H_{k-1}\}_{\mathcal{J}}.\end{aligned}\tag{86}$$

Assume now $k > j$. Then after $(k-j)$ iterations of the procedure yielding (86), we obtain $\{H_j, H_k\}_{\mathcal{J}} = \{H_k, H_j\}_{\mathcal{J}}$, which implies $\{H_j, H_k\}_{\mathcal{J}} = 0$, as the $\mathcal{J}$-bracket is skew. □

5. Conclusion

To summarize, this paper shows that the AL hierarchy can be explicitly viewed as a hierarchy of commuting flows which: (a) are Hamiltonian with respect to both the standard, local Poisson operator $\mathcal{J}$, and a new non-local, skew, almost Poisson operator $\mathcal{K}$, on the appropriate space; (b) can be recursively generated from the *recursion* operator $\mathcal{R} = \mathcal{K}\mathcal{J}^{-1}$. In addition, the proof of these facts relies upon two new pivotal resolvent identities which suggest a general method for uncovering explicit bi-Hamiltonian structures for other families of discrete, integrable equations.

Another result stemming from the current research is the clarification of the geometric framework that underlies a certain class of geodesic linkages evolving on the sphere [10,18]. A linkage on a Riemannian manifold is essentially defined by specifying a sequence of points connected by geodesic arcs. A closed linkage is usually called a polygon. Such linkages are related to the AL hierarchy via the evolution for their “discrete” geodesic curvature [10]. In this regard, Lozano’s preliminary results include a geometric interpretation of a compatibility condition associated to a Lax pair for AL and, consequently, a bijective correspondence between discrete, integrable mKdV flows (also AL flows) and linkage flows. (For details on this, see [18]; also see [10,15,17] for background in terms of continuous-analogs of the linkage models.)

In closing we also want to mention some of the many possible avenues for further research. First, a definite answer to the question of whether or not $\mathcal{K}$ actually defines a Poisson structure would be desirable on several counts. In [20], Maltsev and Novikov give a non-standard set of coordinates for the phase space of NLS in which both the second and third Poisson brackets for this system become local. Given the parallel between our $\mathcal{K}$-bracket and the second Poisson bracket for NLS (i.e., Magri’s), one should explore the possibility that our $\mathcal{K}$-bracket also be local in the appropriate coordinates. If this were the case, the problem of deciding the validity of Jacobi’s identity for the $\mathcal{K}$-bracket may simplify considerably.

Also, some of the explicit calculations in the current paper suggest that the $\mathcal{K}$-bracket (and probably the remaining brackets in its hierarchy) could be seen as a discrete version of what Maltsev and Novikov call weakly non-local brackets [20]. Again, the parallel between $\mathcal{K}$ and Magri’s Poisson bracket for NLS strengthens this hypothesis.

Now, if $\mathcal{K}$ were Poisson, one could also explore possible connections between our $\mathcal{K}$-induced Poisson bracket and (a) the bi-Hamiltonian structure for finite AL described by Faybusovich and Gekhtman [12]; (b) the family of symplectic forms associated to periodic AL encoded in the formula presented by Vaninsky in Section 7 of [25]. The former work considers the AL hierarchy within the larger class of full Toda flows in $sl(n)$ and presents a bi-Hamiltonian formulation for (finite) AL stemming from the bi-Hamiltonian structure of these Toda flows. The latter paper develops some of the Hamiltonian formalism for AL in order to construct an invariant Gibbs' state for NLS.

On the other hand, an obstruction to the Jacobi identity would place the $\mathcal{K}$-bracket in the category of almost Poisson structures and could perhaps steer further investigation in the direction of non-holonomic mechanical systems [6]. Such systems possess an underlying Hamiltonian structure that is (strictly) almost Poisson.

The elucidation of new bi-Hamiltonian structures and their connection with the evolution of non-stretching classes of linkages could also be pursued in the context of the de-focusing Ablowitz–Ladik system (obtained from (36) in the reduction $r_k = \overline{q}_k$) and other discrete integrable equations. This would serve as a test of the robustness of our methods for deriving operators such as $\mathcal{K}$, $\mathcal{R}$, and resolvent identities such as those obtained for L and L^{-1}. It would be good, for instance, to understand how recursion operators (such as $\mathcal{R}$, or even L and L^{-1}) are encoded in the squared eigenfunctions of a linear problem (through resolvent identities of the right kind).

The connection with "physical linkage" spaces could also be pursued further, both for discrete, integrable mKdV, AL and potentially for other discrete integrable equations. In the context of discrete integrable mKdV, AL and non-stretching spherical linkages for instance, one should aim at understanding the linkage recursion schemes proposed in [18] in Poisson-geometric terms. Ideally, a well-defined lift of $\mathcal{R}$ to the space of non-stretching linkages could be defined and then parsed out as the composition of two Poisson (or perhaps one Poisson and one almost Poisson) operators. Connections with results of Langer and Perline (for the case of continuous NLS and the FM model [16]), and Kapovich and Millson (regarding the symplectic geometry of non-stretching polygons [13,14]), could also be explored and addressed in the appropriate context.

Acknowledgments

G.I. Lozano would like to thank Hermann Flaschka for useful discussions and feedback. N.M. Ercolani and G.I. Lozano were supported in part by NSF grant no. 0073087.

Appendix A. Key elements in the proof of Theorem 1

Let $(\dot q_k, \dot r_k)^{\mathrm{T}} = (a_k, b_k)^{\mathrm{T}}$ denote an arbitrary variation of (q_k, r_k), and v_k denote a (vector) solution to the eigenvalue problem (18). The induced variation on v_k satisfies

$$\dot v_{k+1} - \begin{pmatrix} z & q_k \\ r_k & 1/z \end{pmatrix} \dot v_k = \begin{pmatrix} 0 & \dot q_k \\ \dot r_k & 0 \end{pmatrix} v_k. \tag{87}$$

Setting $\dot v_k = \Phi(k) u_k$ for $\Phi(k)$ as in (23), and using the standard variation of constants method to solve (87) for $\dot v_k$ yields

$$\begin{aligned} &\Phi(k+1)u_{k+1} - \begin{pmatrix} z & q_k \\ r_k & 1/z \end{pmatrix} \Phi(k) u_k \\ &\quad = \Phi(k+1)(E^+ - 1)u_k = \begin{pmatrix} 0 & \dot q_k \\ \dot r_k & 0 \end{pmatrix} v_k. \end{aligned} \tag{88}$$

Upon inverting $(E^+ - 1)$, one obtains

$$\begin{aligned} u_n &= -\sum_{k=n}^{\infty} \Phi(k+1)^{-1} \begin{pmatrix} 0 & \dot q_k \\ \dot r_k & 0 \end{pmatrix} v_k + c, \\ \dot v_n &= \Phi(n) u_n. \end{aligned} \tag{89}$$

In particular, choosing $v_n = \psi_n$, we see that as $n \to \infty$, $\psi_n \sim (0, z^{-n})^{\mathrm{T}}$. So $\dot\psi_n$ vanishes and, clearly, so does the sum in (89), which implies $c = 0$. Multiplying the resulting equation through by z^n and focusing on the variation of just the second component of the resulting vector (i.e., $z^n \psi_n^{(2)}$) we see that

$$\begin{aligned} z^n \dot{\psi_n^{(2)}} &= -\sum_{k=n}^{\infty} \left[(z^n \Phi(n)) \Phi(k+1)^{-1} \begin{pmatrix} 0 & \dot q_k \\ \dot r_k & 0 \end{pmatrix} \psi_k \right]^{(2)} \\ &= -\sum_{k=n}^{\infty} \left[\left(z^n \phi_n^{(2)} \right) \left(\beta_{k+1} \dot r_k \psi_k^{(1)} + \alpha_{k+1} \dot q_k \psi_k^{(2)} \right) \right. \\ &\quad \left. + \left(z^n \psi_n^{(2)} \right) \left(\delta_{k+1} \dot r_k \psi_k^{(1)} + \gamma_{k+1} \dot q_k \psi_k^{(2)} \right) \right], \end{aligned} \tag{90}$$

where

$$\begin{aligned} \begin{pmatrix} \alpha_{k+1} & \beta_{k+1} \\ \gamma_{k+1} & \delta_{k+1} \end{pmatrix} &= \Phi(k+1)^{-1} \\ &= \frac{P(k+1)}{a} \begin{pmatrix} \psi_{k+1}^{(2)} & -\psi_{k+1}^{(1)} \\ -\phi_{k+1}^{(2)} & \phi_{k+1}^{(1)} \end{pmatrix}. \end{aligned} \tag{91}$$

(Here, one uses relations (22) to determine $\det \Phi(k) = \phi_k^{(1)} \psi_k^{(2)} - \psi_k^{(1)} \phi_k^{(2)} = a/P(k)$ [18,3].)

Taking the limit of (90) as $n \to -\infty$ and using the asymptotics (23) together with (91) above, we get

$$\begin{aligned} (a \dot{/} C_0) &= \sum_{k=-\infty}^{\infty} (P(k+1)/C_0) \\ &\quad \times \left(-\phi_{k+1}^{(1)} \psi_k^{(1)} \dot r_k + \phi_{k+1}^{(2)} \psi_k^{(2)} \dot q_k \right). \end{aligned} \tag{92}$$

Multiplying (92) by C_0/a, we obtain (40) and, hence, (41).

Focusing on the variation of $z^{-n} \hat\psi_n^{(1)}$ and arguing in a similar manner, one arrives at the identity

$$\begin{aligned} (\hat a \dot{/} C_0) &= \sum_{k=-\infty}^{\infty} (P(k+1)/C_0) \\ &\quad \times \left(\hat\phi_{k+1}^{(1)} \hat\psi_k^{(1)} \dot r_k - \hat\phi_{k+1}^{(2)} \hat\psi_k^{(2)} \dot q_k \right), \end{aligned} \tag{93}$$

Theorem 1 follows.

Appendix B. Key elements in the proofs of Theorems 2 and 4

We begin by defining the following matrices of squared eigenfunctions:

$$\begin{aligned}V_k &= \Phi_k \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix} \Phi_k^{-1}\\ &= \begin{pmatrix}\psi_k^{(1)}\phi_k^{(2)} & -\psi_k^{(1)}\phi_k^{(1)}\\ \psi_k^{(2)}\phi_k^{(2)} & -\psi_k^{(2)}\phi_k^{(1)}\end{pmatrix} \frac{P(k)}{a}\\ &= \begin{pmatrix}C_k & -A_k\\ B_k & -D_k\end{pmatrix},\end{aligned} \tag{94}$$

$$\begin{aligned}W_k &= \Phi_k \begin{pmatrix}0 & 0\\ 0 & -1\end{pmatrix} \Phi_{k+1}^{-1}\\ &= \begin{pmatrix}\psi_k^{(1)}\phi_{k+1}^{(2)} & -\psi_k^{(1)}\phi_{k+1}^{(1)}\\ \psi_k^{(2)}\phi_{k+1}^{(2)} & -\psi_k^{(2)}\phi_{k+1}^{(1)}\end{pmatrix} \frac{P(k+1)}{a}\\ &= \begin{pmatrix}\gamma_k & -\alpha_k\\ \beta_k & -\delta_k\end{pmatrix}.\end{aligned} \tag{95}$$

Note that V_k's entries are standard squared eigenfunctions, whereas W_k is composed of semi-shifted products. Observe also that the constant matrix defining these identities is chosen so that the off-diagonal entries of W_k are precisely the components of the vector of squared eigenfunctions defining the generating gradient $\delta_k \log a + \delta_k H_0$.

One can directly verify the following identities for V_k and W_k:

- $W_{k+1} = \mathcal{E}_k W_k \mathcal{E}_{k+1}^{-1}$; (96)
- $W_k = V_k \Phi_k \Phi_{k+1}^{-1} = V_k \mathcal{E}_k^{-1}$; (97)
- $(r_k\alpha_k + z^{-1}\delta_k) - (q_k\beta_k + z\gamma_k) = 1$, (98)

where $\mathcal{E}_k$ is the coefficient matrix for our eigenvalue problem (18), and the entries of W_k are now written in terms of α_k, β_k, γ_k and δ_k, as in (95).

By writing (96) in terms of the new form of W_k and tracking only the off-diagonal entries of the resulting matrix, one observes that

$$E^+\left[(1-r_kq_k)\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix}\right] = \begin{pmatrix}z^{-2} & 0\\ 0 & z^2\end{pmatrix}\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix} + \begin{pmatrix}r_kr_{k+1}\alpha_k + z^{-1}(r_k\gamma_k + r_{k+1}\delta_k)\\ -q_kq_{k+1}\beta_k - z(q_k\delta_k + q_{k+1}\gamma_k)\end{pmatrix}. \tag{99}$$

Adding and subtracting the expression $\mathrm{Diag}(z^{-2}, z^2)[-r_kq_k(\beta_k, -\alpha_k)^{\mathrm{T}} - (r_k, q_k)^{\mathrm{T}}]$ on the right-hand side of (99), and using identity (98), one may re-write it as

$$\begin{aligned}E^+\left[(1-r_kq_k)\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix}\right] &= \begin{pmatrix}z^{-2} & 0\\ 0 & z^2\end{pmatrix}\\ &\times\left[(1-r_kq_k)\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix} - \begin{pmatrix}r_k\\ q_k\end{pmatrix}\right]\\ &+\begin{pmatrix}r_{k+1} + z^{-2}r_k & 0\\ 0 & q_{k+1} + z^2q_k\end{pmatrix}\\ &\times\begin{pmatrix}q_k\beta_k + z\gamma_k + 1\\ -(r_k\alpha_k + z^{-1}\delta_k - 1)\end{pmatrix}.\end{aligned} \tag{100}$$

By multiplying (100) by $\mathrm{Diag}(z^2, z^{-2})E^-$, we obtain an almost perfectly symmetric equation in terms of the opposite shift:

$$\begin{aligned}E^-\left[(1-r_kq_k)\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix}\right] &= \begin{pmatrix}z^2 & 0\\ 0 & z^{-2}\end{pmatrix}\\ &\times\left[(1-r_kq_k)\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix} - \begin{pmatrix}r_k\\ q_k\end{pmatrix}\right]\\ &+\begin{pmatrix}r_{k-1} + z^2r_k & 0\\ 0 & q_{k-1} + z^{-2}q_k\end{pmatrix}\\ &\times E^-\begin{pmatrix}-(q_k\beta_k + z\gamma_k)\\ r_k\alpha_k + z^{-1}\delta_k\end{pmatrix}.\end{aligned} \tag{101}$$

Spelling out the relationship given in (97) one discovers

$$D_k = z^{-1}\delta_k - q_k\beta_k, \quad \text{and} \quad -C_k = r_k\alpha_k - z\gamma_k, \tag{102}$$

so that (98) may be written as $D_k - C_k = (z^{-1}\delta_k - q_k\beta_k) - (r_k\alpha_k - z\gamma_k) = 1$. Using this fact together with the previous relations (102), we see that the $\delta\gamma$-dependent terms of (100) and (101) may be written as

$$(q_k\beta_k + r_k\alpha_k + C_k)\begin{pmatrix}1\\ -1\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}, \quad \text{and}$$

$$(q_k\beta_k + r_k\alpha_k + C_k)\begin{pmatrix}1\\ 1\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}, \tag{103}$$

respectively. Finally, writing out V_{k+1} in terms of its entries, we find that C_k satisfies a difference equation given in terms of α_k and β_k, namely

$$(E^+ - 1)C_k = q_k\beta_k + r_k\alpha_k.$$

Taking $C_k = -\sum_{j=k}^{\infty} q_j\beta_j + r_j\alpha_j$, and substituting this expression into Eqs. (103), we obtain

- $$\begin{pmatrix}q_k\beta_k + z\gamma_k + 1\\ -(r_k\alpha_k + z^{-1}\delta_k - 1)\end{pmatrix} = \begin{pmatrix}-J_{k+1}^+(q_j) & J_{k+1}^+(r_j)\\ J_{k+1}^+(q_j) & -J_{k+1}^+(r_j)\end{pmatrix} \times \begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix},$$
- $$\begin{pmatrix}q_k\beta_k + z\gamma_k\\ r_k\alpha_k + z^{-1}\delta_k\end{pmatrix} = \begin{pmatrix}-J_{k+1}^+(q_j) & J_{k+1}^+(r_j)\\ -J_{k+1}^+(q_j) & J_{k+1}^+(r_j)\end{pmatrix}\begin{pmatrix}\beta_k\\ -\alpha_k\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}, \tag{104}$$

expressing the $\delta\gamma$-dependent terms in (100) and (101) in terms of the sum operator J_k^+ in the formula for L, and the non-standard squared eigenfunctions α_k, β_k.

Eqs. (100) and (101) used in conjunction with (104) (as well as (106) and (107) together with (108) below) play key roles in the proofs of Theorems 2 and 4, as indicated therein. These last three formulas are established by arguments similar to those just given but applied to the matrix of (semi-shifted) squared eigenfunctions tied to the generating function

$(-\hat{\beta}_k, \alpha_k)^{\mathrm{T}}$ described in (43), namely,

$$\begin{pmatrix} -\hat{\gamma}_k & \hat{\alpha}_k \\ -\hat{\beta}_k & \hat{\delta}_k \end{pmatrix} = \frac{P(k+1)}{\hat{a}} \begin{pmatrix} -\hat{\psi}_k^{(1)}\hat{\phi}_{k+1}^{(2)} & \hat{\psi}_k^{(1)}\hat{\phi}_{k+1}^{(1)} \\ -\hat{\psi}_k^{(2)}\hat{\phi}_{k+1}^{(2)} & \hat{\psi}_k^{(2)}\hat{\phi}_{k+1}^{(1)} \end{pmatrix}, \tag{105}$$

$$(r_k\hat{\alpha}_k + z^{-1}\hat{\delta}_k) - (q_k\hat{\beta}_k + z\hat{\gamma}_k) = 1.$$

Starting with (105), and applying a procedure analogous to the one described in this appendix, we obtain

$$\begin{aligned} E^+\left[(1-r_kq_k)\begin{pmatrix} -\hat{\beta}_k \\ \hat{\alpha}_k \end{pmatrix}\right] &= \begin{pmatrix} z^{-2} & 0 \\ 0 & z^2 \end{pmatrix} \\ &\times \left[(1-r_kq_k)\begin{pmatrix} -\hat{\beta}_k \\ \hat{\alpha}_k \end{pmatrix} + \begin{pmatrix} r_k \\ q_k \end{pmatrix}\right] \\ &- \begin{pmatrix} r_{k+1}+z^{-2}r_k & 0 \\ 0 & q_{k+1}+z^2q_k \end{pmatrix} \\ &\times \begin{pmatrix} q_k\hat{\beta}_k + z\hat{\gamma}_k + 1 \\ -(r_k\hat{\alpha}_k + z^{-1}\hat{\delta}_k - 1) \end{pmatrix}, \end{aligned} \tag{106}$$

$$\begin{aligned} E^-\left[(1-r_kq_k)\begin{pmatrix} -\hat{\beta}_k \\ \hat{\alpha}_k \end{pmatrix}\right] &= \begin{pmatrix} z^{2} & 0 \\ 0 & z^{-2} \end{pmatrix} \\ &\times \left[(1-r_kq_k)\begin{pmatrix} -\hat{\beta}_k \\ \hat{\alpha}_k \end{pmatrix} + \begin{pmatrix} r_k \\ q_k \end{pmatrix}\right] \\ &+ \begin{pmatrix} r_{k-1}+z^{2}r_k & 0 \\ 0 & q_{k-1}+z^{-2}q_k \end{pmatrix} \\ &\times E^-\begin{pmatrix} q_k\hat{\beta}_k + z\hat{\gamma}_k \\ -(r_k\hat{\alpha}_k + z^{-1}\hat{\delta}_k) \end{pmatrix}, \end{aligned} \tag{107}$$

where

$$\begin{aligned} \begin{pmatrix} q_k\hat{\beta}_k + z\hat{\gamma}_k \\ -(r_k\hat{\alpha}_k + z^{-1}\hat{\delta}_k) + 1 \end{pmatrix} &= \begin{pmatrix} J_{k+1}^+(q_j) & -J_{k+1}^+(r_j) \\ -J_{k+1}^+(q_j) & J_{k+1}^+(r_j) \end{pmatrix} \\ &\times \begin{pmatrix} -\hat{\beta}_k \\ \hat{\alpha}_k \end{pmatrix} \end{aligned} \tag{108}$$

as just mentioned above.

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell, H. Segur, The inverse scattering transform-Fourier analysis for nonlinear problems, Stud. Appl. Math. 53 (1974) 249–315.

[2] M.J. Ablowitz, J.F. Ladik, Nonlinear differential-difference equations and Fourier analysis, J. Math. Phys. 17 (1976) 1011–1018.

[3] M.J. Ablowitz, B. Prinari, A.D. Trubatch, Discrete and continuous nonlinear Schrödinger systems, in: London Mathematical Society Lecture Note Series, vol. 302, Cambridge University Press, Cambridge, 2004.

[4] A.B. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Lederer, S. Trillo, S. Wabnitz, Discrete self-trapping, soliton interactions, and beam steering in non-linear waveguide arrays, Phys. Rev. E 53 (1996).

[5] A.B. Aceves, C.M. deSterke, M. Weinstein, Theory of non-linear pulse propagation in periodic structures, in: B. Eggleton, R.E. Slusher (Eds.), Non-linear Photonic Crystals, in: Lecture Notes in Physics, Springer Verlag, 2002 (chapter 2).

[6] A.M. Bloch, Nonholonomic mechanics and control, in: Interdisciplinary Applied Mathematics, vol. 24, Springer-Verlag, New York, ISBN: 0-387-95535-6, 2003.

[7] F. Calogero, A. Degasperis, Nonlinear evolution equations solvable by the inverse spectral transform. II, Nuovo Cimento B (11) 39 (1977) 1–54. (English, with Italian and Russian summaries).

[8] S.C. Chiu, J.F. Ladik, Generating exactly soluble nonlinear discrete evolution equations by a generalized Wronskian technique, J. Math. Phys. 18 (1977) 690–700.

[9] P. Damianou, Nonlinear Poisson brackets, Ph.D. Thesis, University of Arizona, 1989.

[10] A. Doliwa, P.M. Santini, Integrable dynamics of a discrete curve and the Ablowitz–Ladik hierarchy, J. Math. Phys. 36 (1995) 1259–1273.

[11] L.D. Faddeev, L.A. Takhtajan, Hamiltonian methods in the theory of solitons, in: Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, ISBN: 3-540-15579-1, 1987.

[12] L. Faybusovich, M. Gekhtman, Elementary Toda orbits and integrable lattices, J. Math. Phys. 41 (2000) 2905–2921.

[13] M. Kapovich, J.J. Millson, The symplectic geometry of polygons in Euclidean space, J. Differential Geom. 44 (1996) 479–513.

[14] M. Kapovich, J.J. Millson, T. Treloar, The symplectic geometry of polygons in hyperbolic 3-space, Asian J. Math. 4 (2000) 123–164.

[15] J. Langer, Recursion in curve geometry, New York J. Math. 5 (1999) 25–51 (electronic).

[16] J. Langer, R. Perline, Poisson geometry of the filament equation, J. Nonlinear Sci. 1 (1991) 71–93.

[17] J. Langer, R. Perline, Curve motion inducing modified Korteweg–de Vries systems, Phys. Lett. A 239 (1998) 36–40.

[18] G. Lozano, Poisson geometry of the Ablowitz–Ladik equations, Ph.D. Thesis, University of Arizona, 2004 (PDF available at www.math.lsa.umich.edu/~guada/documents/GILdiss_Aug10_04.pdf).

[19] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156–1162.

[20] A.Ya. Maltsev, S.P. Novikov, On the local systems Hamiltonian in the weakly non-local Poisson brackets, Physica D 156 (2001) 53–80.

[21] A.C. Newell, Solitons in mathematics and physics, in: CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 48, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, ISBN: 0-89871-196-7, 1985.

[22] M.J. Ablowitz, B.M. Herbst, C. Schober, On the numerics of integrable discretizations, in: Symmetries and Integrability of Difference Equations (Estérel, PQ, 1994), in: CRM Proc. Lecture Notes, vol. 9, Amer. Math. Soc., Providence, RI, 1996, pp. 1–11.

[23] A. Calini, N.M. Ercolani, D.W. McLaughlin, C.M. Schober, Mel'nikov analysis of numerically induced chaos in the nonlinear Schrödinger equation, Physica D 89 (1996) 227–260.

[24] C.M. Schober, Symplectic integrators for the Ablowitz–Ladik discrete nonlinear Schrödinger equation, Phys. Lett. A 259 (1999) 140–151.

[25] K.L. Vaninsky, An additional Gibbs' state for the cubic Schrödinger equation on the circle, Comm. Pure Appl. Math. 54 (2001) 537–582.