

Total and partial amplitude death in networks of diffusively coupled oscillators

Fatihcan M. Atay*

Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22, Leipzig 04103, Germany

Received 19 September 2002; received in revised form 5 May 2003; accepted 13 May 2003

Communicated by Y. Kuramoto

Abstract

Networks of weakly nonlinear oscillators are considered with diffusive and time-delayed coupling. Averaging theory is used to determine parameter ranges for which the network experiences amplitude death, whereby oscillations are quenched and the equilibrium solution has a large domain of attraction. The amplitude death is shown to be a common phenomenon, which can be observed regardless of the precise nature of the nonlinearities and under very general coupling conditions. In addition, when the network consists of dissimilar oscillators, there exist parameter values for which only parts of the network are suppressed. Sufficient conditions are derived for total and partial amplitude death in arbitrary network topologies with general nonlinearities, coupling coefficients, and connection delays.

© 2003 Elsevier B.V. All rights reserved.

PACS: 05.45.-a; 05.45.Xt; 02.30.Ks; 84.35.+i; 87.10.+e

Keywords: Coupled oscillators; Time delay; Stability; Neural networks

1. Introduction

The interaction of oscillatory systems arise naturally in many areas of science and has intrigued investigators for a long time. The earlier studies go back to the 17th century when the Dutch scientist Christiaan Huygens observed that two clocks mounted on a common basis tend to synchronize regardless of the initial conditions [1,2]. In the last several decades, models based on coupled oscillators have found diverse applications in physics and biology. Among many examples are arrays of Josephson junctions [3,4], semiconductor lasers [5,6], relativistic magnetrons [7], chemical reactions [8–11], circadian pacemakers [12,13], intestinal electrical rhythms [14], and a variety of biological processes [15–17]. The investigation of these systems has provided striking examples of different types of dynamical behavior that can be induced by the presence of coupling. Synchronization is probably the most widely studied among these [18–20]. Another interesting dynamics is the so-called amplitude (or oscillator) death, whereby individual oscillators cease to oscillate when coupled and go to an equilibrium solution instead. One of the first

* Tel.: +49-341-9959-518; fax: +49-341-9959-555.

E-mail address: atay@member.ams.org (F.M. Atay).

accounts of this phenomenon was given in the 19th century by Lord Rayleigh, who noted that when two organ pipes stand side by side they “... may almost reduce one another to silence” [21, vol. 2, Chapter XVI]. Later, amplitude death was observed in electronic systems [22,23] and chemical oscillators [24], causing renewed interest in this area. A theoretical analysis was given in [25], which considered a pair of linearly coupled oscillators each of which is near a Hopf bifurcation. In particular, it was established that a fairly general class of such systems can exhibit amplitude death in case the individual oscillator frequencies are sufficiently different and the coupling is diffusive. In a similar vein, it was shown that a network of limit cycle oscillators can experience amplitude death if there is a distribution in frequencies [26–28].

A series of more recent studies have demonstrated that time delays in the coupling can induce amplitude death even in the case of identical frequencies [29,30]. In these works, a pair of identical Stuart–Landau equations are considered under scalar and diffusive coupling. A linear stability analysis and numerical simulations are used to show that the presence of delay can stabilize the equilibrium solution in this case. A similar conclusion is drawn for an all-to-all coupled network of identical Stuart–Landau systems, where the coupling strengths and time delays between pairs are also identical. From a different perspective, other studies have shown the role of feedback delays in suppressing or modifying oscillations in a single oscillator [31–33]. Time delays are thus significant in the oscillations and stability of many systems, which justifies the efforts devoted to their analysis.

The purpose of the present paper is to obtain rigorous conditions for amplitude death when weakly nonlinear planar systems (such as those near a Hopf bifurcation) are diffusively coupled and the coupling involves time delays. We consider arbitrary nonlinearities and network structure to prove the generality of amplitude death for the interconnection of such systems. Furthermore, we show the existence of parameter ranges for which only part of the network are quenched while the others continue to oscillate, even though they remain connected. This phenomenon, which may be termed partial amplitude death, is shown to be possible in case when the individual units are sufficiently different.

This study extends the previous results in several directions. Firstly, we treat a general class of systems and derive conditions for delay-induced amplitude death irrespective of the particular nonlinearities that may be present. In the absence of delays, general conclusions can be drawn by considering normal forms that represent the dynamics on a center manifold when each oscillator is near a Hopf bifurcation [25]. However, the analysis is more difficult when delays are introduced, and as a result, the study of amplitude death in the delayed case so far has been confined to some special equations. For instance, the system studied in [29,30] is obtained by adding delays to the center manifold equations considered in [25], and as such it does not represent general center manifold dynamics for delay-coupled oscillators. Hence, an interesting question is what conclusions can be drawn in the presence of delays when nonlinearities other than the Stuart–Landau system are considered.

Secondly, we consider very general coupling conditions. We do not restrict to scalar coupling and present results for arbitrary diffusion matrices between oscillators. Both the diffusion matrices and the delays are allowed to vary from pair to pair within the network. In addition, there are no assumptions on either the size or the topology of the network, and the results apply to any network layout, including all-to-all, nearest-neighbor, and sparse coupling types.

Finally, instead of a local stability analysis we obtain global results using averaging theory. Among other things, this allows the results to be applicable to more general systems such as bistable oscillators, where the origin is stable but there are also limit cycle solutions which may be annihilated by the effect of coupling. In fact, in such cases the local stability analysis of the origin cannot yield parameter ranges for amplitude death since the origin always remains stable. On the other hand, the averaging approach enables one to study the (averaged) trajectories inside a large sphere and derive conditions that ensure that they all decay to zero. Furthermore, this nonlocal view is essential for investigating the phenomenon partial amplitude death, since here also, the stability type of the origin for the overall system remains unchanged as parts of the network collapse to approach the zero solution.

The paper is organized as follows. In the next section a pair of coupled oscillators is considered. After a brief discussion of averaging theory for delay-differential equations, the averaged equations are obtained, which indicate that qualitatively there are two cases to consider. Section 3 treats the first case where the frequencies of individual oscillators are close to each other. Here, sufficient conditions are derived for the occurrence of amplitude death, and the possibility of death for identical oscillators for almost all positive values of the delay is proved. The case of large frequency differences is discussed in Section 4, where it is shown that it is possible for only one of the oscillators to be suppressed while the other one continues to oscillate. Section 5 extends the results to an arbitrary size network with varying coupling coefficients and delays between the oscillators. A general condition is derived under which the network experiences amplitude death as a whole or only in parts. Results are illustrated on numerical examples.

For notation, $\|\cdot\|$ is used for the Euclidean norm. If K is a matrix, then K^T denotes its transpose, $\text{tr } K$ its trace, and $K_s = (1/2)(K^T + K)$ its symmetric part. An identity matrix of appropriate size is denoted by I , while J denotes the skew-symmetric matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Furthermore, the derivative of a function f is written as Df .

2. A pair of coupled oscillators

We consider oscillators that can be modeled by equations of the form

$$\dot{x}_i(t) = A_i x_i(t) + \varepsilon f_i(x_i(t)), \quad (1)$$

where $x_i \in \mathbf{R}^2$, the matrix $A_i \in \mathbf{R}^{2 \times 2}$ has a pair of purely imaginary eigenvalues $\pm i\omega_i \neq 0$, and ε is a nonnegative parameter. The function f_i includes any nonlinear terms and is assumed to satisfy the following hypothesis.

(H) The function $f_i : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is C^2 and $f_i(0) = 0$.

It is further assumed that A_i is put into the real Jordan form

$$A_i = \omega_i J = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \quad (2)$$

with $\omega_i > 0$, possibly after a linear coordinate transformation. Eq. (1) typically arises from a center manifold reduction of higher order systems near a Hopf bifurcation. Alternatively, it may be viewed as a perturbation of the harmonic oscillator, which includes the classical pendulum, van der Pol, and Duffing equations as special cases.

Denoting $\Phi_i(t) = \exp(tA_i)$, a transformation of variables given by $x_i = \Phi_i(t)u_i$ puts (1) into the form

$$\dot{u}_i = \varepsilon \Phi_i^{-1}(t) f_i(\Phi_i(t)u_i). \quad (3)$$

This time-dependent equation can be made autonomous by averaging theory. The averaged equation corresponding to (3) is defined as

$$\dot{u}_i = \varepsilon \bar{f}_i(u_i), \quad (4)$$

where

$$\bar{f}_i(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi_i^{-1}(t) f_i(\Phi_i(t)u) dt. \quad (5)$$

By the averaging theorem [34], nonzero and hyperbolic equilibria of (4) correspond to periodic solutions of (1) for all sufficiently small ε . If one considers limit cycle oscillators, (4) will usually have one or more asymptotically stable equilibrium points and the origin will be unstable; however, these assumptions are not essential for the purposes of the present paper.

Now suppose a pair of systems of the form (1) are coupled diffusively,

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + \varepsilon f_1(x_1(t)) + \varepsilon K(x_2(t - \tau) - x_1(t)), \\ \dot{x}_2(t) &= A_2 x_2(t) + \varepsilon f_2(x_2(t)) + \varepsilon K(x_1(t - \tau) - x_2(t)).\end{aligned}\quad (6)$$

Here $K \in \mathbf{R}^{2 \times 2}$ is a matrix of coupling coefficients and τ is a nonnegative number representing the transmission delay. The matrix K is scaled by the parameter ε to indicate that the coupling strength is assumed to be of the same order of magnitude as the attraction to the limit cycles of individual oscillators. Following the change of variables $x_i = \Phi_i(t)u_i$, (6) becomes

$$\begin{aligned}\dot{u}_1(t) &= \varepsilon \Phi_1^{-1}(t) f_1(\Phi_1(t)u_1(t)) + \varepsilon \Phi_1^{-1}(t) K(\Phi_2(t - \tau)u_2(t - \tau) - \Phi_1(t)u_1(t)), \\ \dot{u}_2(t) &= \varepsilon \Phi_2^{-1}(t) f_2(\Phi_2(t)u_2(t)) + \varepsilon \Phi_2^{-1}(t) K(\Phi_1(t - \tau)u_1(t - \tau) - \Phi_2(t)u_2(t)).\end{aligned}\quad (7)$$

For small ε , u_1 and u_2 are slowly changing variables and the method of averaging is applicable.

Averaging theory for functional differential equations such as (7) is in principle similar to the theory for ordinary differential equations. To introduce some notation, let y denote the pair $(u_1, u_2) \in \mathbf{R}^n$, with $n = 4$ in the present case. Then (7) is a relation between the derivative $\dot{y}(t)$ and the present and past values of $y(t)$. A solution $y(t)$ of (7) describes a trajectory on the state space $\mathcal{C} = C([- \tau, 0], \mathbf{R}^n)$, the space of continuous functions mapping the interval $[- \tau, 0]$ to \mathbf{R}^n . The trajectory consists of points y_t which are defined by $y_t(\theta) = y(t + \theta)$, $\theta \in [- \tau, 0]$. Hence each point $y_t \in \mathcal{C}$ on the trajectory corresponds to a window of the solution $y(t)$ over an interval of length τ . The system (7) is a particular case of the more general functional differential equation

$$\dot{y}(t) = \varepsilon g(t, y_t), \quad (8)$$

where $y \in \mathbf{R}^n$ and $g : (\mathbf{R} \times \mathcal{C}) \rightarrow \mathbf{R}^n$. For small values ε , (8) can be viewed as a perturbation of a trivial ordinary differential equation. In fact, if the following limit exists

$$\bar{g}(\varphi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(t, \varphi) dt,$$

then the averaged equation corresponding to (8) is defined to be

$$\dot{y}(t) = \varepsilon \bar{g}(y_t)$$

in which y_t is interpreted as a constant element of \mathcal{C} . In this way, averaging reduces the infinite-dimensional system (8) to an ordinary differential equation on \mathbf{R}^n [35]. In the following, we shall exploit this fact for the analysis of (7). The main averaging result is contained in next lemma.

Lemma 1. Let $\Phi_i = \exp(tA_i)$, where A_i is given by (2), and let $K \in \mathbf{R}^{2 \times 2}$. Then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi_i^{-1}(t) K \Phi_j(t - \tau) dt = \begin{cases} \bar{K} \Phi_j(-\tau) & \text{if } \omega_i = \omega_j \\ 0 & \text{if } \omega_i \neq \omega_j \end{cases} \quad (9)$$

where

$$\bar{K} = \frac{1}{2}(\text{tr } K)I - \frac{1}{2}\text{tr}(JK)J. \quad (10)$$

Proof. By (2) one has

$$\Phi_i(t) = \exp(tA_i) = \begin{bmatrix} \cos \omega_i t & \sin \omega_i t \\ -\sin \omega_i t & \cos \omega_i t \end{bmatrix}. \quad (11)$$

Clearly, $\Phi_i^{-1}(t) = \Phi_i^T(t)$ and $\Phi_i(t - \tau) = \Phi_i(t)\Phi_i(-\tau)$. Thus, if L denotes the left-hand side of (9),

$$L = \left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi_i^T(t) K \Phi_j(t) dt \right] \Phi_j(-\tau). \quad (12)$$

When $\omega_i = \omega_j = \omega$, the integrand above consists of second-order homogenous polynomials in the variables $\sin \omega t$ and $\cos \omega t$. Let $[k_{mn}]$ denote the elements of K . Then, using the elementary facts that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin^2 \omega t dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \cos^2 \omega t dt = \frac{1}{2}$$

and

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sin \omega t \cos \omega t dt = 0,$$

one can calculate

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi_i^T(t) K \Phi_j(t) dt = \frac{1}{2} \begin{bmatrix} k_{11} + k_{22} & k_{12} - k_{21} \\ k_{21} - k_{12} & k_{11} + k_{22} \end{bmatrix} = \frac{1}{2} (\text{tr } K) I - \frac{1}{2} \text{tr}(JK) J.$$

So, $L = \bar{K} \Phi_j(-\tau)$ for the case of equal frequencies. On the other hand if $\omega_i \neq \omega_j$, then the entries of the matrix $\Phi_i^T(t) K \Phi_j(t)$ are linear combinations of terms of the form $p(\omega_i t) q(\omega_j t)$, where p, q can be the sine or the cosine functions. It follows by the theory of almost periodic functions [36] that the average in the above sense of such terms is zero whenever $\omega_i \neq \omega_j$. \square

As the lemma implies, it is necessary to distinguish between two qualitatively different cases, determined by the difference $|\omega_1 - \omega_2|$ between the frequencies. These cases shall be discussed in the next two sections.

3. Similar oscillators and amplitude death

We first consider the coupling of two oscillators with similar native frequencies. More precisely, suppose that $|\omega_1 - \omega_2| = \mathcal{O}(\varepsilon)$. This includes the particular case of identical frequencies. In fact, writing $A_1 = A_2 + \mathcal{O}(\varepsilon)$, and subsuming the $\mathcal{O}(\varepsilon)$ difference into the definition of the function f_1 or f_2 in (6), it is clear that there is no loss of generality in taking $\omega_1 = \omega_2$, which will be assumed for the rest of this section.

With $\omega_1 = \omega_2 = \omega$, applying Lemma 1 to the coupled system (17) gives the averaged equations

$$\dot{u}_1 = \varepsilon(\bar{f}_1(u_1) - \bar{K}u_1 + \bar{K}\Phi(-\tau)u_2), \quad \dot{u}_2 = \varepsilon(\bar{f}_2(u_2) - \bar{K}u_2 + \bar{K}\Phi(-\tau)u_1), \quad (13)$$

where $\Phi = \Phi_1 = \Phi_2$ and the \bar{f}_i are defined by (5). We shall show the existence of parameter values such that inside an arbitrary ball centered at the origin all solutions $(u_1(t), u_2(t))$ of (13) approach zero as $t \rightarrow \infty$. This will imply the amplitude death in the original equations (6) for sufficiently small ε .

Theorem 1. Suppose $K \in \mathbf{R}^{2 \times 2}$ and $R > 0$, and let the numbers q_i , $i = 1, 2$, be defined by

$$q_i = \sup \left\{ \frac{u^T \bar{f}_i(u)}{\|u\|^2} : \|u\| \leq R, \ u \neq 0 \right\}. \quad (14)$$

If

$$\operatorname{tr} K - |(\operatorname{tr} K) \cos \omega \tau - \operatorname{tr}(JK) \sin \omega \tau| > 2 \max\{q_1, q_2\} \quad (15)$$

then all solutions of (13) satisfy $\lim_{t \rightarrow \infty} (u_1(t), u_2(t)) = 0$ whenever $\|(u_1(0), u_2(0))\| \leq R$. On the other hand, if

$$\operatorname{tr} K < \frac{1}{2} \operatorname{tr}(Df_1(0) + Df_2(0)) \quad (16)$$

then the zero solution of (13) is unstable.

Proof. Under the conditions (H), it is easy to see that the supremum in (14) is finite. In fact, since the integrand in (5) is periodic in t ,

$$\bar{f}_i(u) = \frac{1}{T'} \int_0^{T'} \Phi_i^{-1}(t) f_i(\Phi_i(t)u) dt, \quad (17)$$

where T' is the period. Therefore, \bar{f}_i are differentiable and vanish at the origin, and one has the finite Taylor series expansion

$$\bar{f}_i(u) = D\bar{f}_i(0)u + \mathcal{R}_i(u), \quad (18)$$

where the remainder term \mathcal{R}_i is continuous and $\lim_{u \rightarrow 0} \|\mathcal{R}_i(u)\|/\|u\| = 0$. Hence $u^T \bar{f}_i(u)/\|u\|^2$ is bounded inside any ball with the origin removed, so the q_i defined by (14) are finite. Now taking the inner product of the first equation in (13) by u_1 and the second one by u_2 , and adding, one obtains

$$\frac{1}{2} \frac{d}{dt} (\|u_1\|^2 + \|u_2\|^2) = \varepsilon(u_1^T \bar{f}_1(u_1) + u_2^T \bar{f}_2(u_2) - u_1^T \bar{K}u_1 - u_2^T \bar{K}u_2) + \varepsilon(u_1^T \bar{K}\Phi(-\tau)u_2 + u_2^T \bar{K}\Phi(-\tau)u_1). \quad (19)$$

By (14) $u_i^T \bar{f}_i(u_i) \leq q_i \|u_i\|^2$. Hence, if $v = (u_1, u_2)$ denotes the vector formed by concatenating u_1 and u_2 , then (19) yields

$$\frac{d}{dt} \|v\|^2 \leq 2\varepsilon v^T P v, \quad (20)$$

where

$$P = \begin{bmatrix} q_1 I - \bar{K} & \bar{K}\Phi(-\tau) \\ \bar{K}\Phi(-\tau) & q_2 I - \bar{K} \end{bmatrix}. \quad (21)$$

Clearly, $v^T P v = v^T P_s v$, where P_s denotes the symmetric part of P . We claim that P_s is negative definite if the condition (15) is satisfied. Indeed, it is immediate from (10) that $\bar{K}_s = (1/2)(\operatorname{tr} K)I$, and a calculation using (11) shows that

$$(\bar{K}\Phi(-\tau))_s = \frac{1}{2}[(\operatorname{tr} K) \cos \omega \tau - \operatorname{tr}(JK) \sin \omega \tau]I. \quad (22)$$

Hence, P_s has the form

$$P_s = \begin{bmatrix} q_1 I - \bar{K}_s & (\bar{K}\Phi(-\tau))_s \\ (\bar{K}\Phi(-\tau))_s & q_2 I - \bar{K}_s \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \eta_1 I & \alpha I \\ \alpha I & \eta_2 I \end{bmatrix} \quad (23)$$

with

$$\eta_i = 2q_i - \operatorname{tr} K, \quad i = 1, 2$$

and

$$\alpha = (\operatorname{tr} K) \cos \omega \tau - \operatorname{tr}(JK) \sin \omega \tau.$$

If (15) holds, then η_1 and η_2 are both negative and $\eta_1 \eta_2 > \alpha^2$. Thus, by the well-known tests for definiteness, the matrix P_s in (23) is negative definite, and the claim is proved. It now follows under these conditions that

$$v^T P v = v^T P_s v \leq -\lambda \|v\|^2 \quad \text{for all } v \in \mathbf{R}^4,$$

where $-\lambda < 0$ is the largest eigenvalue of P_s . Substituting into (20) gives

$$\|v(t)\|^2 \leq \|v(0)\|^2 \exp(-2\varepsilon \lambda t) \quad \text{for } t \geq 0$$

which establishes the first statement of the theorem.

To prove the second statement, consider the linearization of (13) about zero:

$$\dot{v} = \varepsilon \begin{bmatrix} D\bar{f}_1(0) - \bar{K} & \bar{K}\Phi(-\tau) \\ \bar{K}\Phi(-\tau) & D\bar{f}_2(0) - \bar{K} \end{bmatrix} v. \quad (24)$$

By (17) and Lemma 1,

$$D\bar{f}_i(0) = \frac{1}{T'} \int_0^{T'} \Phi_i^{-1}(t) Df_i(0) \Phi_i(t) dt = \frac{1}{2} \operatorname{tr}[Df_i(0)]I - \frac{1}{2} \operatorname{tr}(J[Df_i(0)]J). \quad (25)$$

Since J has zero trace,

$$\operatorname{tr} D\bar{f}_i(0) = \operatorname{tr} Df_i(0). \quad (26)$$

Also, $\operatorname{tr} K = \operatorname{tr} \bar{K}$ by (10). Thus the matrix in (24) has trace equal to

$$\operatorname{tr}(Df_1(0) + Df_2(0) - 2K)$$

which is positive if (16) holds, in which case at least one of its eigenvalues has a positive real part. \square

Remark 1. The quantities q_i defined in (14) satisfy the lower bounds

$$q_i \geq \frac{1}{2} \operatorname{tr}[Df_i(0)], \quad (27)$$

which provides a connection between the stability condition (15) and the instability condition (16). The inequality (27) follows by taking limits in (18) and using (25) to obtain

$$\lim_{u \rightarrow 0} \frac{u^T \bar{f}_i(u)}{\|u\|^2} = \lim_{u \rightarrow 0} \frac{u^T [D\bar{f}_i(0)]u}{\|u\|^2} = \frac{1}{2} \operatorname{tr}[Df_i(0)].$$

Thus if q_i satisfies (14) for any $R > 0$ then necessarily $q_i \geq (1/2) \operatorname{tr}[Df_i(0)]$. This relates q_i to the local stability of the origin for the individual uncoupled oscillator. For instance, if the linear part of the oscillator (1) has the eigenvalues $\alpha \pm i\omega$, then $q_i \geq \alpha$ by (27). In particular, $q_i > 0$ in case the oscillations arise from a supercritical Hopf bifurcation. On the other hand, if the origin is asymptotically stable then $\operatorname{tr} Df_i(0) < 0$, and q_i may be negative in a sufficiently small neighborhood of the origin, but not necessarily in larger neighborhoods (see Example 1). To

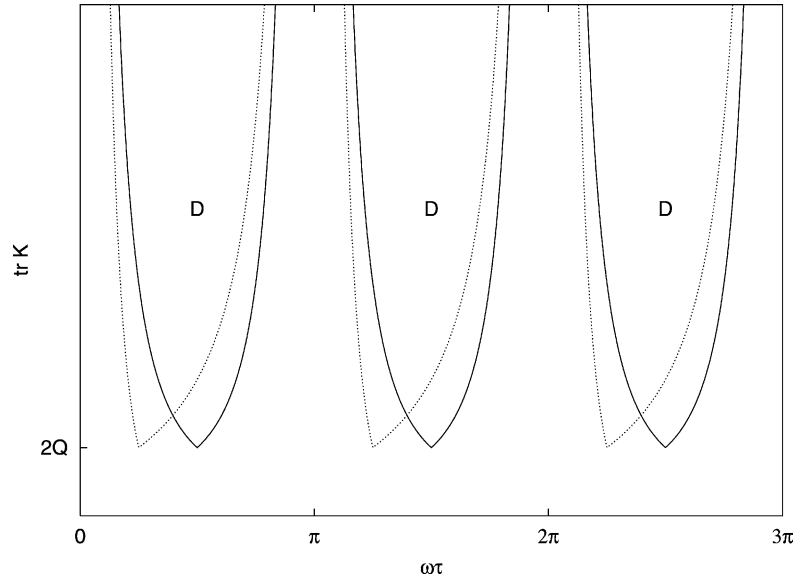


Fig. 1. Regions marked D are parameter values for which amplitude death occurs, and $Q = \max\{q_1, q_2\}$. The solid lines correspond to the case when the matrix K is symmetric. For a nonsymmetric K , the death regions are warped to the side, as shown by dotted lines.

avoid trivial cases from the point of study of the death of limit cycles, it will be assumed in this paper that the q_i are positive in the region of interest.

In essence, [Theorem 1](#) states that in case of equal frequencies, amplitude death occurs whenever the trace of the matrix K is sufficiently large, provided that $\sin \omega\tau \neq 0$. By the averaging theory, for a set of parameter values satisfying (15) there exists $\varepsilon_0 > 0$ such that the original coupled system (6) undergoes amplitude death if $\varepsilon \in (0, \varepsilon_0)$. Condition (15) is graphically depicted on the parameter plane in [Fig. 1](#). It should be noted that the graph is not uniform in ε . For instance, it may be necessary to decrease ε as $\text{tr } K$ increases in order to observe amplitude death. This should be expected since the foregoing analysis assumes that the coupling coefficients εK are $\mathcal{O}(\varepsilon)$ terms. Nevertheless, the value of ε_0 can be chosen to be constant over any compact subset of the parameter space. The extent of the death region in the figure suggests that amplitude death is quite a robust dynamical behavior for diffusively coupled systems.

Some general qualitative characteristics of the region of amplitude death can be deduced from [Fig. 1](#). For instance, the death region is a disconnected set, formed of “islands of death” as mentioned in [30]. Furthermore, it is bounded away from the horizontal axis, implying that there is a threshold of coupling strength below which amplitude death is not possible. Strictly speaking, these properties do not follow from (15), which is only a sufficient condition for amplitude death. Nevertheless, they can be established directly by a linear stability analysis of the origin. Thus, the statement that $\text{tr } K$ should be sufficiently large for amplitude death follows from the instability condition (16). Similarly, it can be shown that amplitude does not occur if $\omega\tau = n\pi$ for some integer n . Indeed, in this case $\Phi(-\tau) = (-1)^n I$ by (11), and the linear variational [equation \(24\)](#) takes the form

$$\dot{v} = \varepsilon \begin{bmatrix} D\bar{f}(0) - \bar{K} & (-1)^n \bar{K} \\ (-1)^n \bar{K} & D\bar{f}(0) - \bar{K} \end{bmatrix} v,$$

where for simplicity it is assumed that the oscillators are identical, and $f = f_1 = f_2$. Letting C denote the matrix on the right side, we note that

$$C \begin{bmatrix} I \\ (-1)^n I \end{bmatrix} = \begin{bmatrix} I \\ (-1)^n I \end{bmatrix} D\bar{f}(0).$$

If λ is an eigenvalue of $D\bar{f}(0)$ corresponding to an eigenvector p , then

$$C \begin{bmatrix} I \\ (-1)^n I \end{bmatrix} p = \begin{bmatrix} I \\ (-1)^n I \end{bmatrix} [D\bar{f}(0)]p = \lambda \begin{bmatrix} I \\ (-1)^n I \end{bmatrix} p,$$

i.e., λ is also an eigenvalue of C . From (25) it is seen that $(1/2)\text{tr}[Df(0)] \pm i(1/2)\text{tr}(J[Df(0)])$ are the eigenvalues of $D\bar{f}(0)$, and thus of C . Consequently, the origin is unstable if $\text{tr}[Df(0)] > 0$, e.g. when the oscillations result from a supercritical Hopf bifurcation, as noted in Remark 1. This proves that amplitude death is not possible for identical oscillators if $\omega\tau = n\pi$, $n \in \mathbf{Z}$; in particular death does not occur for $\tau = 0$, which agrees with earlier results obtained for the undelayed case [25]. It should be pointed out that a disconnected death region in the parameter space arises under the assumptions that ε is small and the delays are discrete. For large values of ε single or multiple death islands (or none) may be possible, as was shown for the delayed Stuart–Landau equations in [30]. On the other hand, recent results indicate that regions of amplitude death typically grow and merge if the delays are distributed over an interval rather than concentrated at a single point [37].

The global nature of Theorem 1 makes it applicable also to cases where a linear stability analysis may be inconclusive—for instance, to bistable oscillators, as the next example illustrates.

Example 1. Consider the following variants of van der Pol oscillators which are coupled through their velocities:

$$\begin{aligned} \ddot{z}_1 + \varepsilon \dot{z}_1 \left(1 - (1 + \rho^2)z_1^2 + \frac{1}{2}\rho^2 z_1^4 \right) + z_1 &= \varepsilon \beta \left(\dot{z}_2 \left(t - \frac{\pi}{2} \right) - \dot{z}_1(t) \right), \\ \ddot{z}_2 + \varepsilon \dot{z}_2 \left(1 - (1 + \rho^2)z_2^2 + \frac{1}{2}\rho^2 z_2^4 \right) + z_2 &= \varepsilon \beta \left(\dot{z}_1 \left(t - \frac{\pi}{2} \right) - \dot{z}_2(t) \right). \end{aligned} \quad (28)$$

For $\rho = 0$ and $\varepsilon < 0$ the left-hand sides of (28) describe the usual van der Pol oscillators. The scalar β denotes the coupling strength. Putting (28) into the vector form (6) by letting $x_i = (z_i, \dot{z}_i)$, it is seen that

$$K = \begin{bmatrix} 0 & 0 \\ 0 & \beta \end{bmatrix},$$

thus $\text{tr } K = \beta$. In the absence of coupling, the averaged equations have the form (4) with

$$\bar{f}_i(u) = -\frac{1}{2} \left(1 - \frac{1 + \rho^2}{4} \|u\|^2 + \frac{\rho^2}{16} \|u\|^4 \right) u, \quad i = 1, 2. \quad (29)$$

The roots u of \bar{f}_i satisfy $\|u\| = 0, 2$, or $2/\rho$. For definiteness we take $\rho = 2$. Then for $\beta = 0$ and ε small and positive, each (uncoupled) oscillator has two attractors: the zero solution and a periodic solution with amplitude near 2. In addition, there is an unstable periodic solution whose amplitude is near 1. Now by (29),

$$u^T \bar{f}_i(u) = -\frac{1}{2} \left(1 - \frac{5}{4} \|u\|^2 + \frac{1}{4} \|u\|^4 \right) \|u\|^2 \quad (30)$$

from which, by simple calculus on the quantity in parenthesis, it follows that

$$u^T \bar{f}_i(u) \leq \frac{9}{32} \|u\|^2, \quad u \in \mathbf{R}^2, \quad i = 1, 2$$

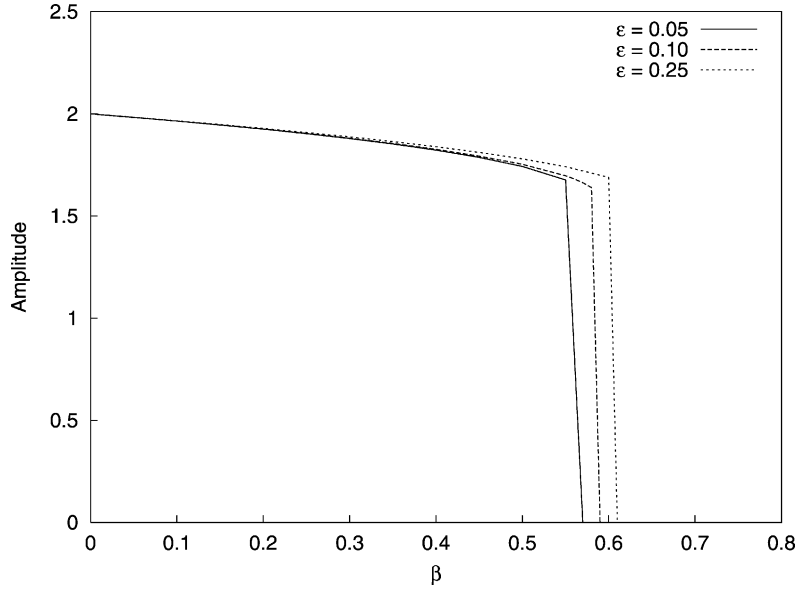


Fig. 2. Death of the periodic solutions of coupled bistable oscillators as the coupling strength β is varied. The zero solution remains asymptotically stable throughout the parameter scale.

so $q_1 = q_2 = 9/32$. [Theorem 1](#) then implies that for $\beta > 9/16$ and sufficiently small positive ε the coupled system has no periodic solutions and the origin is the only attractor. The agreement with the numerical solutions of (28) is shown in [Fig. 2](#) for several values of ε . Here the vertical scale is the amplitude (in the variables z_1 and z_2) of the attracting periodic solution, which exists for the uncoupled system, and thus also for small values of β . As the coupling strength β increases past a critical value, the periodic solutions are annihilated. The critical coupling strength approaches $9/16 \approx 0.56$ as ε is made smaller, as predicted by averaging theory. A closer inspection of the averaged equations reveals the mechanism through which amplitude death occurs. With $\tau = \pi/2$, [Lemma 1](#) gives $\bar{K} = (1/2)\beta I$ and $\bar{K}\Phi(-\tau) = -(1/2)\beta J$. The latter being an antisymmetric matrix, the last two terms in (19) vanish,

$$\frac{1}{2} \frac{d}{dt} (\|u_1\|^2 + \|u_2\|^2) = \varepsilon \left(u_1^T \bar{f}_1(u_1) + u_2^T \bar{f}_2(u_2) - \frac{1}{2} \beta (\|u_1\|^2 + \|u_2\|^2) \right).$$

Since the oscillators are identical and the coupling is symmetric, one could look for equal amplitude solutions, say $\|u_1\|^2 = \|u_2\|^2 = r^2$. Using (30) in the above equation then gives

$$\dot{r} = -\frac{\varepsilon}{2} \left(\beta + 1 - \frac{5}{4}r^2 + \frac{1}{4}r^4 \right) r \stackrel{\text{def}}{=} \frac{\varepsilon}{2} h(r). \quad (31)$$

The equilibrium solutions of (31) are $r = 0$ and $\sqrt{5/2 \pm (1/2)\sqrt{9 - 16\beta}}$, whenever the radicands are nonnegative. Thus for $-1 < \beta < 9/16$ there are two positive equilibria corresponding to the unstable and stable periodic solutions of the oscillators. As $\beta \rightarrow 9/16$, these two equilibria approach each other and disappear in a saddle-node bifurcation. Correspondingly, for $\beta > 9/16$ there are no periodic solutions. The situation is graphically shown in [Fig. 3](#). It is seen that the sufficient condition $\beta > 9/16$ given by [Theorem 1](#) is also necessary for amplitude death in the present case. Furthermore, throughout the whole range of β values the origin remains asymptotically stable. Hence, the local stability analysis of the origin reveals no information about the death of the periodic solutions in this example.

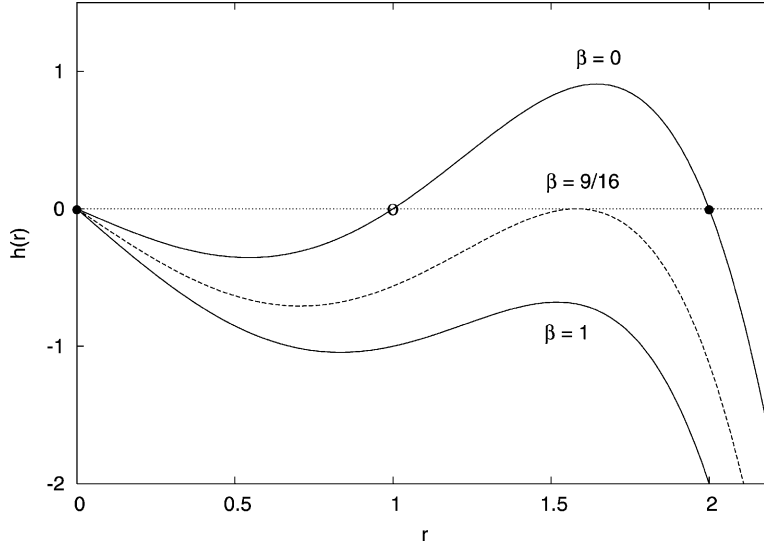


Fig. 3. The shape of the right-hand side of (31) for various values of β . The roots correspond to the amplitudes of the stable (●) and unstable (○) periodic solutions. As β is increased the two positive roots coalesce and disappear. The critical value for this saddle-node bifurcation is $\beta = 9/16$.

4. Dissimilar oscillators and partial amplitude death

Suppose now that $|\omega_1 - \omega_2| = \mathcal{O}(1)$, i.e., the frequencies differ by a nonzero constant which does not depend on the parameter ε . Applying Lemma 1 to (7), the averaged equations are found to be

$$\dot{u}_1 = \varepsilon(\bar{f}_1(u_1) - \bar{K}u_1), \quad \dot{u}_2 = \varepsilon(\bar{f}_2(u_2) - \bar{K}u_2). \quad (32)$$

Note that the equations are decoupled and do not depend on the delay τ . In analogy with Theorem 1, the following result can be stated.

Theorem 2. Let $R > 0$, and q_i be defined by (14). If

$$\text{tr } K > 2q_i \quad (33)$$

then all solutions of the i th equation in (32) satisfy $\lim_{t \rightarrow \infty} u_i(t) = 0$ whenever $\|u_i(0)\| \leq R$. On the other hand, if

$$\text{tr } K < \text{tr } Df_i(0) \quad (34)$$

then the zero solution of i th equation in (32) is unstable.

Proof. Taking the inner product of u_i and the i th equation in (32), and using (10) and (14), one obtains

$$\frac{1}{2} \frac{d}{dt} \|u_i\|^2 \leq \varepsilon u_i^T (q_1 I - \bar{K}) u_i = \varepsilon u_i^T (q_1 I - \bar{K})_s u_i = \varepsilon \left(q_1 - \frac{1}{2} \text{tr } K \right) \|u_i\|^2$$

from which the first statement of the theorem follows. On the other hand, the linearized equation

$$\dot{u}_1 = \varepsilon(D\bar{f}_i(0) - \bar{K})u_i$$

is unstable if $\text{tr}(D\bar{f}_i(0) - \bar{K})$ is positive. But the latter quantity is equal to $\text{tr}(Df_i(0) - K)$ as observed in the proof of [Theorem 1](#), which establishes the second statement. \square

Similar to [Theorem 1](#), the above result implies that the pair of oscillators undergo amplitude death if $\text{tr } K$ is sufficiently large, but this time regardless of the value of the delay. Comparison of the conditions (15) and (33) shows that amplitude death becomes more likely when the coupled oscillators have a frequency difference, which agrees with previous studies of amplitude death caused by frequency distributions [27,28]. More importantly, in this case the conditions for amplitude death are separate for each equation. Thus, it is possible that one of the oscillators is suppressed independently of the other one although they remain connected, leading to a partial amplitude death in the coupled system. The following example illustrates this behavior.

Example 2. Consider a pair of coupled van der Pol oscillators

$$\begin{aligned}\ddot{z}_1 + \varepsilon(z_1^2 - \alpha_1)\dot{z}_1 + \omega_1^2 z_1 &= \varepsilon\beta(\dot{z}_2(t-1) - \dot{z}_1(t)), \\ \ddot{z}_2 + \varepsilon(z_2^2 - \alpha_2)\dot{z}_2 + \omega_2^2 z_2 &= \varepsilon\beta(\dot{z}_1(t-1) - \dot{z}_2(t))\end{aligned}\quad (35)$$

where $\alpha_i > 0$ and $\beta \in \mathbf{R}$. As in the previous example, letting $x_i = (\omega_i z_i, \dot{z}_i)$ shows that β is equal to the trace of the matrix K in (6). In the averaged equations (32) one has

$$\bar{f}_i(u) = \frac{1}{2}(\alpha_i - \frac{1}{4}\|u\|^2)u, \quad i = 1, 2 \quad (36)$$

which confirms that in the absence of coupling the i th oscillator has an attracting limit cycle solution $z_i = 2\sqrt{\alpha_i} \cos \omega_i t + \mathcal{O}(\varepsilon)$. Furthermore, (36) yields

$$u^T \bar{f}_i(u) \leq \frac{\alpha_i}{2} \|u\|^2.$$

Hence, with $q_i = \alpha_i/2$ an application of [Theorem 2](#) gives the condition $\beta > \alpha_i$ for the death of the i th oscillator in the coupled system. In particular, if $\alpha_2 < \beta < \alpha_1$ then only the second oscillator will be suppressed. The results of the numerical solution of (35) is summarized in [Fig. 4](#), where the parameter values are $\alpha_1 = 1$, $\alpha_2 = 0.5$, $\omega_1 = 0.1$, $\omega_2 = 1$, and $\varepsilon = 0.1$. It is seen that amplitude death occurs for $\beta > 1$, where both oscillators are quenched. On the other hand, a partial death occurs for $0.5 < \beta < 1$, where only the second oscillator is damped while the first one continues to exhibit high-amplitude oscillations. For this parameter range the amplitude of the second oscillator is almost (but not exactly) zero; in fact, the amplitude approaches zero as $\varepsilon \rightarrow 0^+$. Indeed, the averaged equations (32) in this case are

$$\dot{u}_i = \frac{1}{2}(\alpha_i - \frac{1}{4}\|u_i\|^2 - \beta)u_i, \quad i = 1, 2 \quad (37)$$

which have an attracting fixed point of the form $(u_1, 0)$ with $\|u_1\| = 2\sqrt{\alpha_1 - \beta}$ when $\alpha_2 < \beta < \alpha_1$. Thus for small ε the amplitudes of z_1 and z_2 in the original system (35) are $2\sqrt{1 - \beta} + \mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon)$, respectively. It should be remarked that, for this range of β , although the zero solution is asymptotically stable in the equation for u_2 it is not stable for the full system of equations (37). As the origin remains unstable throughout the range $0 < \beta < 1$, a linear stability analysis alone is insufficient to reveal the occurrence of the partial amplitude death of the coupled system. Also note that the actual value of the delay is not significant in this behavior, and one obtains pictures similar to [Fig. 4](#) even for zero delay.

It is rather interesting to observe partial death under diffusive coupling, since one intuitively expects this type of connection to act towards reducing the differences between the oscillators, when, for instance, K is a positive scalar. For the undelayed case, results related to this behavior was given in [38], which essentially follow by the observation

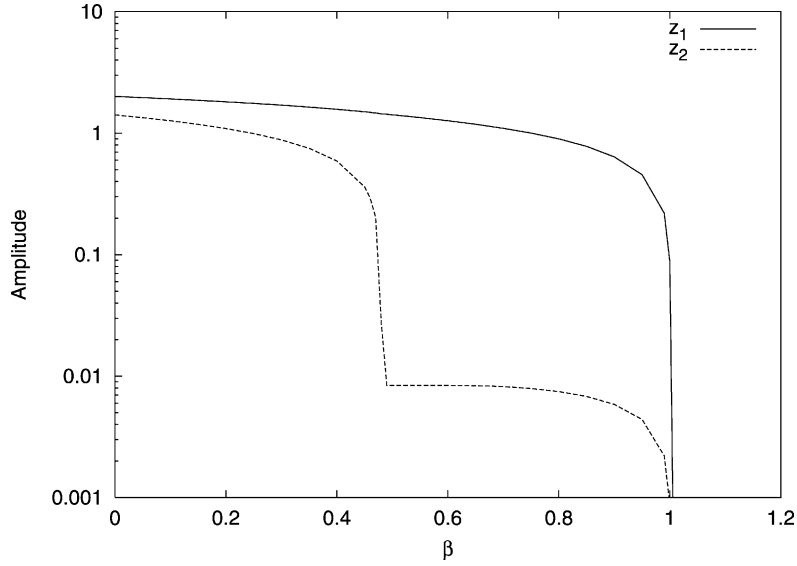


Fig. 4. Behavior of coupled van der Pol oscillators whose limit cycles have different amplitudes and frequencies, as the coupling strength β is varied. Partial amplitude death occurs for $0.5 < \beta < 1$, where only one of the oscillators is quenched while the other one continues to exhibit high-amplitude oscillations. The region $\beta > 1$ corresponds to the usual amplitude death of the whole system. Note that the vertical scale is logarithmic.

that weakly connected systems near a Hopf bifurcation can be decoupled in the $\mathcal{O}(\varepsilon)$ terms if the frequencies are different, a fact which has been applied to the study of neural networks [39,40]. Here we see that the same conclusion also holds in the presence of arbitrary coupling delays. However, it should be noted that higher-order terms in ε may still be coupled and dependent on τ . This can be significant at large values of ε ; for instance, the results of [30] indicate that in such a case amplitude death may depend on the value of τ , especially when the frequency difference between the oscillators is small. In closing we mention that a form of partial death was also observed numerically in undelayed networks of relaxation oscillators [41]. Since Theorem 2 holds under quite general conditions, partial amplitude death can be expected to be among the common behaviors of many practical networks formed by the interaction of dissimilar units.

5. Network of oscillators

The results of the previous sections will now be extended to a general network of N coupled oscillators modeled in the form

$$\dot{x}_i(t) = A_i x_i(t) + \varepsilon f_i(x_i(t)) + \varepsilon \sum_{j=1}^N K_{ij}(x_j(t - \tau_{ij}) - x_i(t)), \quad i = 1, 2, \dots, N. \quad (38)$$

Here $x_i \in \mathbf{R}^2$, $A_i = \omega_i J$, f_i satisfies the hypothesis (H), and $\varepsilon \geq 0$. The nonnegative quantity τ_{ij} is the transmission delay and $K_{ij} \in \mathbf{R}^{2 \times 2}$ are the coupling or diffusion coefficients between the i th and the j th oscillator. To prevent self-coupling K_{ii} is set to zero for all i . It is assumed that the coupling between pairs is symmetric, that is, $K_{ij} = K_{ji}$ and $\tau_{ij} = \tau_{ji}$ for all i, j . On the other hand, both the coupling coefficients and the delays are allowed to vary from pair to pair. This is an especially realistic assumption if the delays arise as a result of spatial separation of the oscillators.

With the change of variables $x_i = \Phi_i(t)u_i = \exp(tA_i)u_i$, (38) is transformed to

$$\dot{u}_i(t) = \varepsilon \Phi_i^{-1}(t) f_i(\Phi_i(t)u_i(t)) - \varepsilon \sum_{j=1}^N \Phi_i^{-1}(t) K_{ij} \Phi_i(t)u_i + \varepsilon \sum_{j=1}^N \Phi_i^{-1}(t) K_{ij} \Phi_j(t - \tau_{ij})u_j(t - \tau_{ij}) \quad (39)$$

for $i = 1, 2, \dots, N$. If $\tau = \max\{\tau_{ij} | i, j = 1, \dots, N\}$, then (39) has the form (8) where the right-hand side is a function on $\mathbf{R} \times C([- \tau, 0], \mathbf{R}^{2N})$, and can be averaged as before. To this end, suppose there are M distinct frequencies $\omega_1, \dots, \omega_M$ among the oscillators, and let \mathcal{I}_n denote the set of indices of those oscillators having frequency ω_n , for $n = 1, \dots, M$. More precisely, $j \in \mathcal{I}_n$ if and only if the matrix A_j has eigenvalues $\pm i\omega_n$. It follows by Lemma 1 that the average of the terms of the form $\Phi_i^{-1}(t) K_{ij} \Phi_j(t - \tau_{ij})$ in (39) will be zero whenever $i \in \mathcal{I}_n$ and $j \notin \mathcal{I}_n$. Application of the lemma gives the averaged equations

$$\dot{u}_i = \varepsilon \left(\bar{f}_i(u_i) - \sum_{j=1}^N \bar{K}_{ij} u_j + \sum_{j \in \mathcal{I}_n} \bar{K}_{ij} \Phi_j(-\tau_{ij}) u_j \right), \quad i \in \mathcal{I}_n \quad (40)$$

for $n = 1, \dots, M$, where

$$\bar{K}_{ij} = \frac{1}{2}(\text{tr } K_{ij})I - \frac{1}{2}\text{tr}(JK_{ij})J. \quad (41)$$

The averaged equations consist of M decoupled sets of equations. The results of the previous sections suggest that each set can undergo amplitude death independently of the others. This observation is made precise by the following theorem.

Theorem 3. Suppose $K \in \mathbf{R}^{2 \times 2}$ and $R > 0$, and let $q_i, i \in \mathcal{I}_n$, be defined by (14). Let v_n denote the vector formed by the concatenation of u_i for $i \in \mathcal{I}_n$. If

$$\sum_{j=1}^N \text{tr } K_{ij} - \sum_{j \in \mathcal{I}_n} |(\text{tr } K_{ij}) \cos \omega_n \tau_{ij} - \text{tr}(JK_{ij}) \sin \omega_n \tau_{ij}| > 2q_i, \quad \text{for every } i \in \mathcal{I}_n \quad (42)$$

then all solutions of the n th set of equations (40) satisfy $\lim_{t \rightarrow \infty} v_n(t) = 0$ whenever $\|v_n(0)\| \leq R$. On the other hand, if

$$\sum_{i \in \mathcal{I}_n} \left(\sum_{j=1}^N \text{tr } K_{ij} - \text{tr } Df_i(0) \right) < 0 \quad (43)$$

then the zero solution of (40) is unstable.

Proof. Taking the inner product of the i th equation in (40) with u_i and adding over $i \in \mathcal{I}_n$ gives

$$\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathcal{I}_n} \|u_i\|^2 = \varepsilon \sum_{i \in \mathcal{I}_n} \left(u_i^T \bar{f}_i(u_i) - \sum_{j=1}^N u_i^T \bar{K}_{ij} u_j + u_i^T \sum_{j \in \mathcal{I}_n} \bar{K}_{ij} \Phi_j(-\tau_{ij}) u_j \right).$$

Let $R > 0$. By (10) and (14),

$$\frac{d}{dt} \sum_{i \in \mathcal{I}_n} \|u_i\|^2 \leq 2\varepsilon \sum_{i \in \mathcal{I}_n} \left(q_i \|u_i\|^2 - \frac{1}{2} \left(\sum_{j=1}^N \text{tr } K_{ij} \right) \|u_i\|^2 + u_i^T \sum_{j \in \mathcal{I}_n} \bar{K}_{ij} \Phi_j(-\tau_{ij}) u_j \right) \quad (44)$$

provided $\max_{i \in \mathcal{I}_n} \|u_i\| \leq R$. Let S denote the last summation in (44), i.e.,

$$\begin{aligned} S &= \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} u_i^T \bar{K}_{ij} \Phi_j(-\tau_{ij}) u_j = \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} \frac{1}{2} u_i^T [\bar{K}_{ij} \Phi_j(-\tau_{ij}) + \Phi_i^T(-\tau_{ji}) \bar{K}_{ji}^T] u_j \\ &= \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} \frac{1}{2} u_i^T [\bar{K}_{ij} \Phi_j(-\tau_{ij}) + \Phi_j^T(-\tau_{ij}) \bar{K}_{ij}^T] u_j, \end{aligned}$$

where the last line follows since $\bar{K}_{ij} = \bar{K}_{ji}$ and $\tau_{ij} = \tau_{ji}$ for all i, j , and $\Phi_i \equiv \Phi_j$ for $i, j \in \mathcal{I}_n$ by our assumptions. Hence from (22),

$$S = \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} u_i^T [\bar{K}_{ij} \Phi_j(-\tau_{ij})] u_j = \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} \frac{1}{2} \alpha_{ij} u_i^T u_j \quad (45)$$

where

$$\alpha_{ij} = (\text{tr } K_{ij}) \cos \omega_n \tau_{ij} - \text{tr}(JK_{ij}) \sin \omega_n \tau_{ij}. \quad (46)$$

From the identity

$$0 \leq \|u_i \pm u_j\|^2 = \|u_i\|^2 + \|u_j\|^2 \pm 2u_i^T u_j$$

follows the estimate

$$|u_i^T u_j| \leq \frac{\|u_i\|^2 + \|u_j\|^2}{2}.$$

Using in (45) and noting that $\alpha_{ij} = \alpha_{ji}$ gives

$$S \leq \frac{1}{4} \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} |\alpha_{ij}| (\|u_i\|^2 + \|u_j\|^2) = \frac{1}{2} \sum_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} |\alpha_{ij}| \|u_i\|^2.$$

Substitution into (44) yields

$$\frac{d}{dt} \sum_{i \in \mathcal{I}_n} \|u_i\|^2 \leq 2\varepsilon \sum_{i \in \mathcal{I}_n} \left(q_i - \frac{1}{2} \sum_{j=1}^N \text{tr } K_{ij} + \frac{1}{2} \sum_{j \in \mathcal{I}_n} |\alpha_{ij}| \right) \|u_i\|^2 \leq \varepsilon \lambda \sum_{i \in \mathcal{I}_n} \|u_i\|^2,$$

where we have defined

$$\lambda = \max_{i \in \mathcal{I}_n} \left(2q_i - \sum_{j=1}^N \text{tr } K_{ij} + \sum_{j \in \mathcal{I}_n} |\alpha_{ij}| \right).$$

Thus

$$\|v_n(t)\|^2 \leq \|v_n(0)\|^2 \exp(\varepsilon \lambda t).$$

If condition (42) holds, then λ is negative, and the first statement of the theorem is proved.

To prove the second statement, consider the linearization of (40) about zero:

$$\dot{u}_i = \varepsilon \left(D\bar{f}_i(0)u_i - \sum_{j=1}^N \bar{K}_{ij}u_i + \sum_{j \in \mathcal{I}_n} \bar{K}_{ij}\Phi_j(-\tau_{ij})u_j \right), \quad i \in \mathcal{I}_n.$$

Recalling that $K_{ii} = 0$, and using (26) and (41), the divergence of the vector field defined by the right-hand side is calculated as

$$\sum_{i \in \mathcal{I}_n} \text{tr} \left(D \bar{f}_i(0) - \sum_{j=1}^N \bar{K}_{ij} \right) = \sum_{i \in \mathcal{I}_n} \left(\text{tr} D f_i(0) - \sum_{j=1}^N \text{tr} K_{ij} \right)$$

which is positive if (43) holds. In this case, any sufficiently small neighborhood of the origin expands in volume under the evolution of the differential equations (40), which proves the instability of the zero solution. \square

In view of the above theorem it is worthwhile to examine the death of a given network as additional units are connected to it. Suppose that the network consists of N oscillators with identical (or $\mathcal{O}(\varepsilon)$ close) frequencies, and consider the condition (42) for amplitude death. When an additional oscillator with a different frequency is connected, the first sum in (42) changes by $\text{tr}(K_{i,N+1})$ while the other terms remain the same. Thus if $\text{tr}(K_{i,N+1})$ is sufficiently large for $i = 1, \dots, N$, then the original set of N oscillators will experience amplitude death after the new connection. Conversely, if $\text{tr}(K_{i,N+1})$ is negative for some i , then it is possible for the new connection to “awaken” the network from a previous death state, by satisfying the instability condition (43).

When the newly added oscillator has the same frequency as the rest, the coupling delay becomes important. For simplicity let us assume that all oscillators and the coupling conditions between them are identical, so the indices can be dropped. The death condition (42) then has the form

$$\text{tr} K - |(\text{tr} K) \cos \omega \tau - \text{tr}(JK) \sin \omega \tau| > \frac{2q}{N}. \quad (47)$$

Assuming $q > 0$, this condition cannot be satisfied for any N if either $\sin \omega \tau = 0$ or $\text{tr} K \leq 0$. However, if $\sin \omega \tau \neq 0$ and K is symmetric with positive trace, the network will undergo amplitude death for all sufficiently large network sizes N . For nonsymmetric matrices the same conclusion holds whenever the left-hand side of (47) is positive, i.e., if $\text{tr} K$ is sufficiently larger than $|\text{tr}(JK)|$. However, for highly nonsymmetric matrices the presence of the connection delay may make amplitude death impossible, as the left-hand side of (47) becomes negative.

Although physically meaningful in most models, the assumption that there are no self-connections in the network (38) is not an essential requirement for Theorem 3. In fact, an investigation of the first part of the proof shows that (42) remains a sufficient condition for amplitude death even when the matrices K_{ii} are not zero. This observation points to an interesting connection between the present paper and previous works on feedback control of oscillations. For if one considers the system

$$\dot{x}(t) - Ax(t) - \varepsilon f(x(t)) = \varepsilon K(x(t - \tau) - x(t)) \quad (48)$$

as a special case of the network (38) consisting of a single oscillator, Theorem 3 gives the sufficient condition

$$\text{tr} K - |(\text{tr} K) \cos \omega \tau - \text{tr}(JK) \sin \omega \tau| > 2q$$

for its stability. Since the latter inequality can be satisfied if $\text{tr} K$ is large and $\sin \omega \tau \neq 0$, one concludes that for almost all positive values of the delay τ and sufficiently small $\varepsilon > 0$, the oscillator given by the left-hand side of (48) can be stabilized by a linear delayed feedback of the form $K(x(t - \tau) - x(t))$, regardless of a precise knowledge about the nonlinearity f . This is in agreement with related results which were obtained for similar systems in the context of delayed-feedback control of oscillatory behavior [31–33,42].

The possibility that the network may be divided into clusters which separately experience amplitude death entails the formation of spatial patterns consisting of oscillating and suppressed units. Such pattern formation is best studied under more specific assumptions on the network structure; therefore, it is not treated further in this paper.

6. Conclusion

We have presented a mathematical analysis of the phenomenon of amplitude death for a general class of weakly nonlinear systems which are diffusively coupled with time delays. Sufficient conditions are derived for the coupled network to exhibit amplitude death. It is shown that when the individual oscillators are sufficiently different, it is possible for only parts of the network to be suppressed, while the rest continues to oscillate. The size of the network may enhance or inhibit amplitude death depending on the nature of the coupling matrices, where both the magnitude and the sign of the trace of the matrix plays an important role. The quantitative conditions presented are rigorously justified for weakly nonlinear oscillators. In particular, they are significant when considering networks where each individual unit is near a Hopf bifurcation. On the other hand, numerical simulations suggest that in many cases the qualitative aspects of the results remain valid for stronger nonlinearities as well. In view of the few assumptions required on the network structure and the individual units, the generality of the results imply that the death of the network, either in part or as a whole, may be among the common and robust dynamical behaviors in many practical systems. This should be significant in studies of interacting periodic processes, such as physical, biological, and financial cycles.

References

- [1] C. Huygens. Oeuvres complètes de Christiaan Huygens, Martinus Nijhoff, The Hague, 1888–1950.
- [2] M. Bennett, M.F. Schatz, H. Rockwood, K. Wiesenfeld, Huygens' clocks, *Proc. R. Soc. Lond. A* 458 (2002) 563–579.
- [3] P. Hadley, M.R. Beasley, K. Wiesenfeld, Phase locking of Josephson-junction series arrays, *Phys. Rev. B* 38 (1988) 8712–8719.
- [4] P. Hadley, M.R. Beasley, K. Wiesenfeld, Phase locking of Josephson-junction arrays, *Appl. Phys. Lett.* 52 (1988) 1619–1621.
- [5] P.M. Varangis, A. Gavrielides, T. Erneux, V. Kovanis, L.F. Lester, Frequency entrainment in optically injected semiconductor lasers, *Phys. Rev. Lett.* 78 (1997) 2353–2356.
- [6] A. Hohl, A. Gavrielides, T. Erneux, V. Kovanis, Localized synchronization in two coupled nonidentical semiconductor lasers, *Phys. Rev. Lett.* 78 (1997) 4745–4748.
- [7] J. Benford, H. Sze, W. Woo, R.R. Smith, B. Harteneck, Phase locking of relativistic magnetrons, *Phys. Rev. Lett.* 62 (1989) 969–971.
- [8] I. Schreiber, M. Marek, Strange attractors in coupled reaction–diffusion cells, *Physica D* 5 (2–3) (1982) 258–272.
- [9] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer, Berlin, 1984.
- [10] M.F. Crowley, I.R. Epstein, Experimental and theoretical studies of a coupled chemical oscillator: phase death, multistability, and in-phase and out-of-phase entrainment, *J. Phys. Chem.* 93 (1989) 2496–2502.
- [11] M. Dolnik, I.R. Epstein, Coupled chaotic chemical oscillators, *Phys. Rev. E* 54 (1996) 3361–3368.
- [12] S. Daan, C. Berde, Two coupled oscillators, simulations of the circadian pacemaker in mammalian activity rhythms, *J. Theor. Biol.* 70 (1978) 297–314.
- [13] M. Kawato, R. Suzuki, Two coupled neural oscillators as a model of the circadian pacemaker, *J. Theor. Biol.* 86 (3) (1980) 547–575.
- [14] D.A. Linkens, The stability of entrainment conditions for RLC coupled van der Pol oscillators used as a model for intestinal electrical rhythms, *Bull. Math. Biol.* 39 (1977) 359–372.
- [15] A.T. Winfree, *The Geometry of Biological Time*, Springer, New York, 1980.
- [16] N. Holstein-Rathlou, Synchronization of proximal intratubular pressure oscillations—Evidence for interaction between nephrons, *Pflügers Archiv-Eur. J. Physiol.* 408 (5) (1987) 438–443.
- [17] J.D. Murray, *Mathematical Biology*, 2nd ed., Springer, Berlin, 1993.
- [18] S. Shinomoto, Y. Kuramoto, Cooperative phenomena in two-dimensional active rotator systems, *Prog. Theor. Phys.* 75 (6) (1986) 1319–1327.
- [19] L. Bonilla, J. Casado, M. Morillo, Self-synchronization of populations of nonlinear oscillators in the thermodynamic limit, *J. Stat. Phys.* 48 (3–4) (1987) 571–591.
- [20] A. Pikovsky, M. Rosenblum, J. Kurths, *Synchronization—A Universal Concept in Nonlinear Science*, Cambridge University Press, Cambridge, 2001.
- [21] J.W.S. Rayleigh, *The Theory of Sound*, 2nd ed., Dover, New York, 1945 (Originally published in 1877–1878).
- [22] J.A. Copeland, A new mode of operation for bulk negative resistance oscillators, *Proc. Lett. IEEE* 54 (10) (1966) 1479.
- [23] J.A. Copeland, LSA oscillator-diode theory, *J. Appl. Phys.* 38 (8) (1967) 3096.
- [24] K. Bar-Eli, On the stability of coupled chemical oscillators, *Physica D* 14 (1985) 242–252.
- [25] D.G. Aronson, G.B. Ermentrout, N. Kopell, Amplitude response of coupled oscillators, *Physica D* 41 (1990) 403–449.

- [26] M. Shiino, M. Frankowicz, Synchronization of infinitely many coupled limit-cycle type oscillators, *Phys. Lett. A* 136 (3) (1989) 103–108.
- [27] G.B. Ermentrout, Oscillator death in populations of “all to all” coupled nonlinear oscillators, *Physica D* 41 (1990) 219–231.
- [28] R.E. Mirollo, S.H. Strogatz, Amplitude death in an array of limit-cycle oscillators, *J. Stat. Phys.* 60 (1–2) (1990) 245–262.
- [29] D.V. Ramana Reddy, A. Sen, G.L. Johnston, Time delay induced death in coupled limit cycle oscillators, *Phys. Rev. Lett.* 80 (23) (1998) 5109–5112.
- [30] D.V. Ramana Reddy, A. Sen, G.L. Johnston, Time delay effects on coupled limit cycle oscillators at Hopf bifurcation, *Physica D* 129 (1–2) (1999) 15–34.
- [31] F.M. Atay, Van der Pol’s oscillator under delayed feedback, *J. Sound Vib.* 218 (2) (1998) 333–339.
- [32] F.M. Atay, Delayed-feedback control of oscillations in non-linear planar systems, *Internat. J. Contr.* 75 (5) (2002) 297–304.
- [33] F.M. Atay, Oscillation control in delayed feedback systems, *Lecture Notes in Control and Information Sciences*, vol. 273, Springer, Berlin, 2002, pp. 103–116.
- [34] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.
- [35] J.K. Hale, Averaging methods for differential equations with retarded arguments and a small parameter, *J. Diff. Eqns.* 2 (1966) 57–73.
- [36] H. Bohr, *Fastperiodische Funktionen*, Springer, Berlin, 1932.
- [37] F.M. Atay, Distributed delays facilitate amplitude death of coupled oscillators. Max-Planck-Institute for Mathematics in the Sciences Preprint Series Nr. 23, 2003.
- [38] F.M. Atay, Partial amplitude death in diffusively coupled oscillators, in: *Proceedings of the 10th Workshop on Nonlinear Dynamics of Electronic Systems (NDES 2002)*, Izmir, Turkey, 2002, Section 4, pp. 5–8.
- [39] F.C. Hoppensteadt, E.M. Izhikevich, Synaptic organizations and dynamic properties of weakly connected neural oscillators, *Biol. Cybernet.* 75 (1996) 117–127.
- [40] F.C. Hoppensteadt, E.M. Izhikevich, *Weakly Connected Neural Networks*, Springer, Berlin, 1997.
- [41] E.I. Volkov, D.V. Volkov, Multirhythmicity generated by slow variable diffusion in a ring of relaxation oscillators and noise-induced abnormal interspike variability, *Phys. Rev. E* 65 (2002) 046232-1–046232-13.
- [42] F.M. Atay, Balancing the inverted pendulum using position feedback, *Appl. Math. Lett.* 12 (5) (1999) 51–56.