Physica D 254 (2013) 18–23

Contents lists available at SciVerse ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Turbulence properties and global regularity of a modified Navier–Stokes equation

Tobias Grafke [a], Rainer Grauer [a,*], Thomas C. Sideris [b]

[a] Institut für Theoretische Physik I, Ruhr-Universität Bochum, Germany

[b] Department of Mathematics, University of California, Santa Barbara, USA

HIGHLIGHTS

- We introduce a new nonlocal equation similar to the Navier–Stokes equation.
- The inviscid version of this new equation possesses an infinite number of conserved quantities.
- We prove global regularity for this new equation.
- Turbulence properties lie between those of the Burgers and Naver–Stokes equations.

ARTICLE INFO

Article history:
Received 11 May 2012
Received in revised form
12 December 2012
Accepted 18 March 2013
Available online 26 March 2013
Communicated by Edriss S. Titi

Keywords:
PDEs
Turbulence
Navier–Stokes equation
Burgers equation
Regularity

ABSTRACT

We introduce a modification of the Navier–Stokes equation that has the remarkable property of possessing an infinite number of conserved quantities in the inviscid limit. This new equation is studied numerically and turbulence properties are analyzed concerning energy spectra and scaling of structure functions. The dissipative structures arising in this new equation are curled vortex sheets instead of the vortex tubes arising in Navier–Stokes turbulence. The numerically calculated scaling of structure functions is compared with a phenomenological model based on the She–Lévêque approach.

Finally, for this equation we demonstrate global well-posedness for sufficiently smooth initial conditions in the periodic case and in $\mathbb{R}^3$. The key feature is the availability of an additional estimate which shows that the L^4-norm of the velocity field remains finite.

1. Introduction

In this paper, we introduce a new equation which is a hybrid of the Navier–Stokes equation and the Burgers equation. Our goal is to show the existence of multi-dimensional model equations which possess a direct turbulent cascade to small scales, non-trivial intermittency, nonlocal interaction and yet an infinite number of conserved quantities. To our knowledge, there seems to be no model equation in the literature with these desirable properties. Our model could therefore be seen as an interesting starting point for testing numerous methods like e.g. phenomenological approaches or methods from field theory (Martin–Siggia–Rose formalism [1], instantons [2], OPE [3]), in a multi-dimensional setup. Our new equation may play the same role as the 1D-Burgers equation in higher dimensions.

Turbulence properties of this equation are analyzed using numerical simulations. We calculate energy spectra and scaling of higher order structure functions. A key observation from the numerical simulations of this modified Navier–Stokes equation is that the most dissipative structures consist of curled vortex sheets instead of the vortex tubes in conventional Navier–Stokes turbulence. Using this information, a She–Lévêque type model [4] is derived and compared to the numerically obtained scaling of higher order structure functions.

In addition, for this new equation we can show existence and regularity for H^1 initial conditions of arbitrary size. This will be carried out in $\mathbb{R}^3$ and periodic domains in $\mathbb{R}^3$. The simple modification of the nonlinearity makes the proof of global solutions possible, insofar as an additional estimate is available showing that the solution remains finite in L^p, $2 < p < \infty$. With $p = 4$, this is then coupled with standard estimates for the H^1-norm to complete the proof. In contrast to other approaches for regularization of the Navier–Stokes equation using dispersive mollification [5–7] acting on small scales, our modification acts on all spatial scales.

* Corresponding author. Tel.: +49 234 32 23767.
E-mail addresses: grauer@tp1.ruhr-uni-bochum.de, grauer@tp1.rub.de (R. Grauer).

0167-2789/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physd.2013.03.007

The outline of this paper is as follows: in Section 2 we motivate and introduce our new equation. Turbulence statistics and phenomenological modeling are considered in Section 3. Section 4 contains the proof of existence of global solutions. We finish with remarks on possible further consequences of the existence of an infinite number of conserved quantities.

2. Model equation

We consider a three-dimensional domain Ω which shall be either $\mathbb{R}^3$ or a bounded cube in $\mathbb{R}^3$ with periodic boundary conditions. Let $\mathrm{P} = 1 - \Delta^{-1}\nabla \otimes \nabla$ be the Leray–Hopf projection operator (with periodic boundary conditions when Ω is bounded):

$$\mathrm{P}[\mathrm{P}[\mathbf{u}]] = \mathrm{P}[\mathbf{u}], \qquad \nabla \cdot \mathrm{P}[\mathbf{u}] = 0. \tag{1}$$

The usual incompressible Navier–Stokes equation

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f}, \qquad \nabla \cdot \mathbf{v} = 0 \tag{2}$$

can be written with the projection operator P

$$\frac{\partial}{\partial t}\mathbf{v} + \mathrm{P}[\mathbf{v} \cdot \nabla\mathbf{v}] = \nu\Delta\mathbf{v} + \mathrm{P}[\mathbf{f}], \qquad \nabla \cdot \mathbf{v} = 0$$

so that no explicit pressure term is present in the equation.

We can rewrite the Navier–Stokes equation without the incompressibility constraint in the form

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}] \cdot \nabla\mathrm{P}[\mathbf{u}] = \nu\Delta\mathbf{u} + \mathbf{f}, \tag{3}$$

where the solution of the Navier–Stokes equation can be recovered by taking $\mathbf{v} = \mathrm{P}[\mathbf{u}]$.

Eq. (3) can be compared with the Burgers equation whose structure is formally similar:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}. \tag{4}$$

For Eq. (4) the nonlinearity is purely local, whereas for Eq. (3) the nonlinear interaction involves the nonlocal projection.

A natural hybrid of these two equations leads a new model equation involving a compressible velocity field $\mathbf{u}$ that is convected by its solenoidal part $\mathrm{P}[\mathbf{u}]$:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}] \cdot \nabla\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}. \tag{5}$$

More accurately this means that the convection of the velocity field $\mathbf{u}$ is local in position space, but the projection operator is local in Fourier space and thus shares this mixture of local and nonlocal interactions with the original Navier–Stokes equation.

Writing this equation in the more conventional form of a solenoidal velocity field $\mathbf{v}$,

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{v} + \nabla\left(\frac{\partial}{\partial t}\varphi - \nu\Delta\varphi\right) + \mathbf{v} \cdot \nabla\nabla\varphi = \nu\Delta\mathbf{v}, \tag{6}$$

where the compressible vector $\mathbf{u}$ is decomposed as $\mathbf{u} = \mathbf{v} + \nabla\varphi$, the similarity as well as the difference to the original Navier–Stokes equation (2) is stressed: the gradient term $\nabla\big(\frac{\partial}{\partial t}\varphi - \nu\Delta\varphi\big)$ corresponds to the pressure contribution ∇p whereas the additional term $\mathbf{v} \cdot \nabla\nabla\varphi$ forms the difference to the Navier–Stokes equation.

3. Turbulence statistics

By construction the presented model equation is an intermediate step between the Navier–Stokes and Burgers equation, which in turn differ significantly in their dynamical evolution and turbulent behavior. In Navier–Stokes turbulence, on the one hand, the most dissipative structures are vortex filaments, while for Burgers equation shocks dominate the turbulent flow. It is of obvious interest in how far our model equation bridges between those, which structures are the most dominant for turbulent flows and how these structures influence the turbulence statistics. We therefore extend the She–Lévêque reasoning, which describes Navier–Stokes and Burgers turbulence well, to our model equation and test it against numerical simulations by comparing the scaling exponents of the structure functions.

Numerical simulations are carried out with a second-order in space finite difference scheme with a strongly stable third-order Runge–Kutta time integration with resolutions up to 512^3. The initial conditions were chosen as Orszag–Tang-like (see [8]) large-scale perturbations:

$$\begin{aligned} u_x &= A\left(-2\sin(2y) + \sin(z) + 2\cos(2y) + \cos(z)\right) \\ u_y &= A\left(-2\sin(x) + \sin(z) + 2\cos(x) + \cos(z)\right) \\ u_z &= A\left(\sin(x) + \sin(y) - 2\cos(2x) + \cos(y)\right). \end{aligned}$$

For simplicity and comparability both velocity and its solenoidal projection are set to equal values. The physical domain stretches from $-\pi$ to π; the above defined conditions, thus, are both large-scale perturbations and periodic. All hydrodynamical models will be simulated in comparison, using these initial conditions. We consider only decaying turbulence without external forces. For the parameters of all performed runs see Table 1, which shows the numerical value of the quantities at the time of maximum enstrophy $t = t_{\mathcal{E}}$.

Fig. 1 shows the decay of kinetic energy for the considered hydrodynamical models in comparison. The tendency of Burgers turbulence to form shocks and the dissipative nature of these structures lead to a faster energy decay compared to the Navier–Stokes equation. The new model equation exhibits a less violent form of dissipation; its energy decay lies in between the others. The difference in turbulence development is identified in a more precise way when comparing the time $t_{\mathcal{E}}$ of maximum enstrophy $\mathcal{E} = \int_\Omega \boldsymbol{\omega}^2 \mathrm{d}x$. As Fig. 1 (right) indicates, the enstrophy of Burgers turbulence reaches its peak significantly faster than for the Navier–Stokes equation, in which vortex filaments dominate the turbulent flow. The proposed model equation ranges between them. This hints at the development of coherent structures at a timescale slower than shock-formation of Burgers equation but faster than the formation of vortex tubes for the Navier–Stokes equation. Precise values for the timescales are stated in Table 1.

Fig. 2 depicts a volume render of the fully developed turbulence, the snapshot in each case taken at $t = t_{\mathcal{E}}$. As expected, the Burgers flow (left) is dominated by shocks and the Navier–Stokes flow (right) consists of vortex filaments. For the proposed model equation (middle), the most dominant structures are two-dimensional folded vortex-sheets.

A phenomenological description, which takes into account the most dissipative structures of the turbulent flow, is the model of She and Lévêque [4] connected to log-Poisson statistics of the local energy dissipation [9]. They state that the scaling exponent ζ_p of the p-th structure function behaves like

$$\zeta_p = \frac{(1-k)p}{3} + C_0\left(1 - \left(\frac{C_0 - k}{C_0}\right)^{\frac{p}{3}}\right), \tag{7}$$

where C_0 is the co-dimension of the most dissipative structures in the evolved flow and k is the time-scaling exponent. This formula will be referred to as the *She–Lévêque model* even though in [4] it is applied exclusively to the Navier–Stokes equation (where $C_0 = 2,\ k = 2/3$).

Table 1
Parameters of the numerical simulations. Number of collocation points N^3; grid spacing dx; time of fully developed turbulence $t_\mathcal{E}$; viscosity ν; root-mean-square velocity $v_{rms} = \sqrt{2/3E_{kin}}$; mean energy dissipation rate ε; integral scale $L = (2/3E_{kin})^{3/2}/\varepsilon$; dissipation length scale $\eta = (\nu^3/\varepsilon)^{1/4}$; Taylor–Reynolds number $R_\lambda = \sqrt{15 v_{rms} L/\nu}$; all taken at the time of maximum enstrophy $t = t_\mathcal{E}$.

PDE	N	Δx	$t_\mathcal{E}$	ν	v_{rms}	ε	L	η	R_λ
Burgers	128	0.0491	0.461	0.0241	1.761	3.564	1.533	0.0445	40.99
	256	0.0245	0.603	0.015	1.637	3.320	1.320	0.0318	46.49
	512	0.0123	0.634	0.009	1.656	3.361	1.352	0.0216	61.10
Model equation	128	0.0491	1.438	0.01	1.316	0.983	2.318	0.0318	67.65
	256	0.0245	1.459	0.006	1.369	1.044	2.459	0.0213	91.75
	512	0.0123	1.624	0.0036	1.370	1.096	2.344	0.0144	115.7
Navier–Stokes	128	0.0491	2.502	0.007	1.482	0.918	3.549	0.0247	106.2
	256	0.0245	2.616	0.00278	1.518	1.268	2.761	0.0114	150.4
	512	0.0123	2.545	0.00110	1.551	1.572	2.376	0.00539	224.2

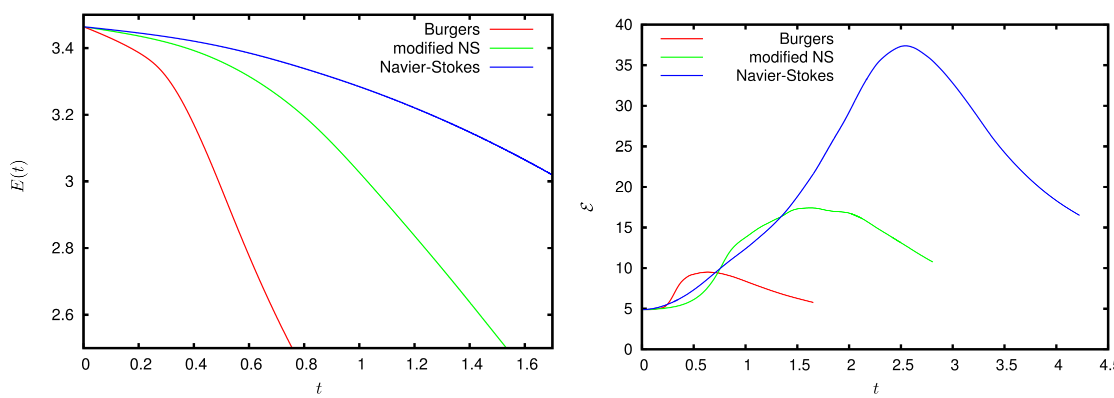

Fig. 1. *Left:* comparison of the kinetic energies for the Navier–Stokes, Burgers and the proposed model equation. The steep discontinuities of a Burgers flow explain the fast energy dissipation. *Right:* Evolution of free-falling turbulence for the Burgers equation, the proposed model equation and Navier–Stokes equation. The turbulent flow is fully developed at $t = t_\mathcal{E}$, when the enstrophy $\mathcal{E}$ reaches its maximum.

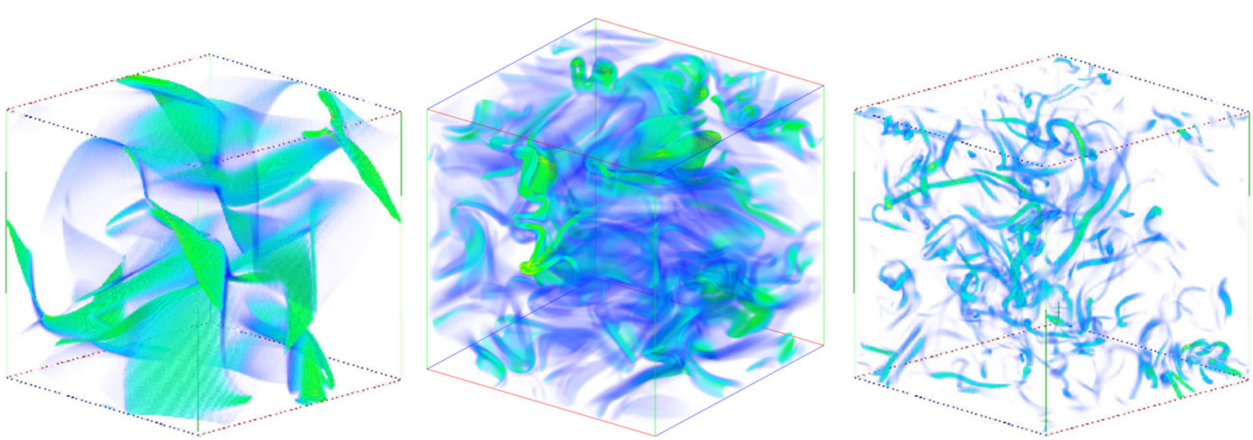

Fig. 2. Dissipative structures in fully developed turbulence for different hydrodynamical models. *Left:* 2-dimensional shock fronts in Burgers turbulence. *Middle:* Folded vortex sheets in the new model equation. *Right:* Vortex filaments in Navier–Stokes turbulence.

The typical time t_l for the evolution of discontinuities for turbulent Burgers flows scales linearly with l, which accounts for $k = 1$. As the shocks traveling through the domain are two-dimensional we furthermore obtain $C_0 = 1$. Inserting this into Eq. (7) leads to $\zeta_p = 1$. Since parts of the velocity field are continuous, for $p < 1$ the smoother regions of the velocity field are pronounced. This would equal a scaling exponent of $\zeta_p = p$ for these orders. Since this result is smaller than $\zeta_p = 1$, it is dominant for $p < 1$. Therefore, we get

$$\zeta_p = \begin{cases} p & \text{for } p < 1 \\ 1 & \text{for } p \geq 1 \end{cases} \tag{8}$$

for the Burgers equation.

Table 2
Scaling exponents ζ_p for the Burgers equation. The α_p measured with ESS are compared to the ζ_p predicted by the She–Lévêque model.

p	1	2	4	5	6	7
α_p	0.70	0.95	1.00	1.00	0.98	1.00
ζ_p	1.00	1.00	1.00	1.00	1.00	1.00

Table 2 shows the measurements of a direct numerical simulation with 512^3 grid points. Here, α_p are the data obtained via ESS and ζ_p is the prediction of Eq. (8). The visualization of this result is shown in Fig. 3 on the left. Especially for high order of p ($p > 3$) the scaling exponent agrees with the prediction, yet smears out for smaller p (see also [10]).

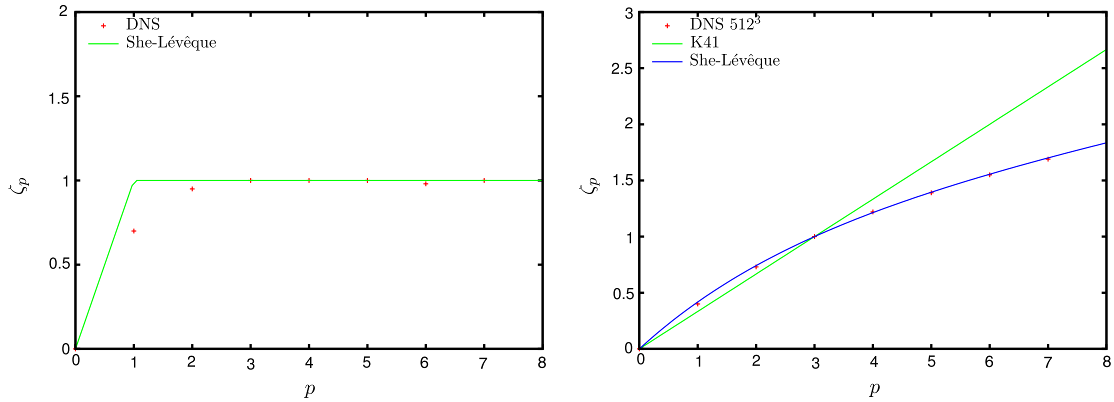

Fig. 3. Scaling exponents for fully developed turbulence, comparison between theory and numerical simulation. *Left:* Burgers equation. *Right:* Presented model equation.

Table 3
Scaling exponents ζ_p for the new model equation. The α_p for P**u** measured with ESS are compared to the ζ_p predicted by the She–Lévêque model.

p	1	2	4	5	6	7
$\alpha_p(\mathrm{P}\mathbf{u})$	0.4	0.73	1.22	1.39	1.55	1.69
ζ_p	0.418	0.751	1.213	1.395	1.556	1.701

Similar to the Burgers equation the introduced model equation allows for shocks to develop, since no incompressibility condition is stated. Nevertheless, energy decays solely via the dissipative term. Because of this, the hypotheses of Kolmogorov may be adapted to the Euler–Burgers equation. The Richardson cascade as well as the properties of the energy spectrum and the scaling behavior of structure functions should agree with the conclusions of Kolmogorov and the She–Lévêque model. On the other hand, the structures that evolve seem to be significantly different from the vortex filaments known from Navier–Stokes.

Fig. 2 (middle) shows the most dissipative structures of fully developed turbulent flow. The norm of the vorticity visualizes two-dimensional folded vortex sheets as the structures that correspond to the vortex tubes of Navier–Stokes. This suggest $C_0 = 1$. The time-scaling exponent t_l for the introduced equation is estimated as $k = 2/3$, with the same reasoning as for Navier–Stokes. Thus,

$$\zeta_p = \frac{p}{9} + \left(1 - \left(\frac{1}{3}\right)^{\frac{p}{3}}\right) \tag{9}$$

is the prediction for the scaling exponents proposed by the She–Lévêque model.

Table 3 features the results measured via ESS for the scaling exponents of P**u**. As can be seen in Fig. 3 (right) the numerical data agree very well with the prediction of the She–Lévêque model. The solenoidal field is consistent with theory from low orders of p up to the highest order that was measured.

4. Global solutions

In this section, we show global regularity for Eq. (5) for suitable initial data without any size restrictions. For this, we prove the remarkable property that this equation possesses an infinite number of conserved quantities in the inviscid limit. Especially, the finiteness of the L^4-norm of the velocity field coupled with standard estimates for the H^1-norm enables us to show global regularity.

The problem of whether the three-dimensional incompressible Navier–Stokes equations can develop a finite time singularity from smooth initial conditions or if it has global solutions remains unresolved (see [11–15] and the references therein). The answer to this important question is recognized as one of the Millennium prize problems [16,17].

Despite the complexity of the topic, a lot of progress has been made on this field in the past. For the two-dimensional case, global-in-time existence of unique weak and strong solutions is well-known (see [11,12]). In three dimensions weak solutions are known to exist globally in time. For strong solutions, existence and uniqueness is known for a short interval of time which depends continuously on the initial data [18]. Many results published in the past, starting with [19], provide criteria for the global regularity of solutions via conditions applied to the velocity field [20,21] or components thereof [22], the vorticity [23], its direction [24] or to the pressure field [25,26].

The theory for the compressible Navier–Stokes equation is less well developed, and we will not attempt a summary here. The multi-dimensional Burgers equation [27] can be regarded as a crude simplification of this model. Global existence and uniqueness of strong solutions can be established in two and three-dimensions for suitably small initial conditions, much as with the Navier–Stokes system. Irrotational flows do possess global solutions for large data in arbitrary dimension, thanks to the Cole–Hopf transformation [28,29]. However, there is no multi-dimensional weak theory because of the absence of a mechanism for energy dissipation, unlike Navier–Stokes.

The situation for this new modified Navier–Stokes like equation is rather different. In this section, the existence of global solutions is proven for the model equation (5) in a domain Ω which shall either be $\mathbb{R}^3$ or a periodic cube in $\mathbb{R}^3$.

Theorem 1. *Let* $\mathbf{u}_0 \in H^1(\Omega)$. *Let* $\mathbf{f} \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega)) \cap L^1_{loc}(\mathbb{R}^+, L^4(\Omega))$. *Then the initial value problem for the model equation (5) has a unique global solution*

$$u \in C(\mathbb{R}^+, H^1(\Omega)) \cap L^2_{loc}(\mathbb{R}^+, H^2(\Omega)).$$

The aim is to show that the solution remains *a priori* bounded in $L^\infty([0,T), H^1(\Omega)) \cap L^2([0,T), H^2(\Omega))$ for any $T > 0$, which implies its existence and uniqueness with standard arguments comparable to e.g. [11,30]. Throughout the argument, we denote the Euclidean norm of the vector $\mathbf{u} = \sum_i u_i \mathbf{e}_i$ by $u = (\sum_i u_i^2)^{1/2}$. We first prove the following lemma.

Lemma 1. *Let* $\mathbf{u}_0$, $\mathbf{f}$, Ω *be defined as above. Then the quantity* $\|\mathbf{u}(t)\|_{L^p}$ *remains finite for* $2 \le p \in \mathbb{R}$.

Proof. Taking the Euclidean inner product of (5) with $\mathbf{u}$ yields the identity

$$\frac{1}{2}\left(\frac{\partial}{\partial t}u^2 + \mathrm{P}\mathbf{u} \cdot \nabla u^2\right) = \frac{\nu}{2}\Delta u^2 - \nu|\nabla \mathbf{u}|^2 + \mathbf{f} \cdot \mathbf{u}. \tag{10}$$

Integrate (10) over Ω, use the fact that P$\mathbf{u}$ is divergence free, and then apply the Cauchy–Schwarz inequality to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\mathbf{u}\|_{L^2}^2 + \nu\|\nabla\mathbf{u}\|_{L^2}^2 \le \|\mathbf{f}\|_{L^2}\|\mathbf{u}\|_{L^2}. \tag{11}$$

Defining $x(t) = \frac{1}{2}\left(\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t \nu\|\nabla\mathbf{u}(s)\|_{L^2}^2 ds\right)$, we have that

$x'(t) \le \|\mathbf{f}(t)\|_{L^2}(2x(t))^{1/2}$.

Upon integration, this gives the inequality

$$\|\mathbf{u}(t)\|_{L^2}^2 + \int_0^t \nu\|\nabla\mathbf{u}(s)\|_{L^2}^2 ds \le \left(\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\mathbf{f}(s)\|_{L^2} ds\right)^2. \tag{12}$$

With this estimate the lemma is shown for the case $p = 2$.

Let $2 \le n \in \mathbb{R}$ and multiply the identity (10) by $u^{2(n-1)}$:

$$\begin{aligned}\frac{1}{2n}&\left(\frac{\partial}{\partial t}u^{2n} + \mathrm{P}\mathbf{u}\cdot\nabla u^{2n}\right)\\ &= \frac{\nu}{2}\left(\frac{\partial}{\partial x_j}\left(u^{2(n-1)}\frac{\partial}{\partial x_j}u^2\right) - \frac{4(n-1)}{n^2}|\nabla u^n|^2\right)\\ &\quad - \nu u^{2(n-1)}|\nabla\mathbf{u}|^2 + u^{2(n-1)}\mathbf{f}\cdot\mathbf{u}.\end{aligned}$$

Integrate this over Ω and apply Hölder's inequality:

$$\begin{aligned}\frac{1}{2n}\frac{\partial}{\partial t}\|\mathbf{u}\|_{L^{2n}}^{2n} &+ \int_\Omega\left(\frac{2\nu(n-1)}{n^2}\left|\nabla u^n\right|^2 + \nu u^{2(n-1)}|\nabla\mathbf{u}|^2\right)d\mathbf{x}\\ &\le \|\mathbf{f}\|_{L^{2n}}\|\mathbf{u}\|_{L^{2n}}^{2n-1}.\end{aligned}$$

If we let

$$\begin{aligned}y(t) = \frac{1}{2n}\|\mathbf{u}(t)\|_{L^{2n}}^{2n} &+ \int_0^t\int_\Omega\left(\frac{2\nu(n-1)}{n^2}\left|\nabla u^n(s)\right|^2\right.\\ &\left. + \nu u^{2(n-1)}(s)|\nabla\mathbf{u}(s)|^2\right)d\mathbf{x}ds,\end{aligned}$$

then we obtain

$y'(t) \le \|\mathbf{f}(t)\|_{L^{2n}}(2n\, y(t))^{\frac{2n-1}{2n}}$.

This leads to the estimate

$$\begin{aligned}\|\mathbf{u}(t)\|_{L^{2n}}^{2n} &+ 2n\int_0^t\int_\Omega\left(\frac{2\nu(n-1)}{n^2}\left|\nabla u^n(s)\right|^2\right.\\ &\left. + \nu u^{2(n-1)}(s)|\nabla\mathbf{u}(s)|^2\right)d\mathbf{x}ds\\ &\le \left(\|\mathbf{u}_0\|_{L^{2n}} + \int_0^t\|\mathbf{f}(s)\|_{L^{2n}}ds\right)^{2n}.\end{aligned} \tag{13}$$

Since $n \ge 2$, this proves the result for $p \ge 4$. The cases $2 < p < 4$ follow by interpolation. □

Remark. This key argument fails for the case of the Navier–Stokes equation. At the same time, this estimate establishes an infinite number of conserved quantities in the unforced inviscid case.

Proof of Theorem 1. Take the L^2-inner product of (5) with $\Delta\mathbf{u}$ and integrate by parts:

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\Delta\mathbf{u}\|_{L^2}^2 = \underbrace{\int_\Omega(\mathrm{P}\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,d\mathbf{x}}_{\text{(i)}} + \underbrace{\int_\Omega\mathbf{f}\cdot\Delta\mathbf{u}\,d\mathbf{x}}_{\text{(ii)}}.$$

The forcing term (ii) has the bound

$$\int_\Omega\mathbf{f}\cdot\Delta\mathbf{u}d\mathbf{x} \le \|\mathbf{f}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2} \le \frac{\nu}{4}\|\Delta\mathbf{u}\|_{L^2}^2 + \frac{1}{\nu}\|\mathbf{f}\|_{L^2}^2.$$

The nonlinear term (i) is estimated as follows:

$$\begin{aligned}\int_\Omega(\mathrm{P}\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\,d\mathbf{x} &= -\int\frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_k}\left((\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}u_i\right)d\mathbf{x}\\ &= -\int\frac{\partial}{\partial x_k}u_i\left((\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i + \frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}u_i\right)d\mathbf{x}\\ &= -\int\left(\frac{1}{2}(\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}|\nabla\mathbf{u}|^2 + \frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j u_i\right)\right)d\mathbf{x}\\ &= \int\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j u_i\,d\mathbf{x}\\ &\le \|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathrm{P}\mathbf{u}\|_{L^4}\|\mathbf{u}\|_{L^4}.\end{aligned}$$

The second norm above is handled by interpolation. We first note that

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^4} \le \|\nabla\mathrm{P}\mathbf{u}\|_{L^6}^{3/4}\|\nabla\mathrm{P}\mathbf{u}\|_{L^2}^{1/4}.$$

Now when $\Omega = \mathbb{R}^3$, the Sobolev embedding theorem gives

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^6} \le C\|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2}. \tag{14}$$

When Ω is a periodic domain, the norm on the right must be replaced by $\|\nabla\mathrm{P}\mathbf{u}\|_{H^1}$. However, since $\nabla\mathrm{P}\mathbf{u}$ has zero mean, this is bounded again by $C\|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2}$, by the Poincaré inequality. Therefore, (14) holds in both cases. Using the facts that the operator P commutes with derivatives and that it is a projection in L^2, we have that

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^2} \le \|\nabla\mathbf{u}\|_{L^2} \quad\text{and}\quad \|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2} \le \|\nabla^2\mathbf{u}\|_{L^2}.$$

Next, we use integration by parts to obtain the simple ellipticity identity

$$\begin{aligned}\|\nabla^2\mathbf{u}\|_{L^2}^2 &= \int_\Omega\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i\,d\mathbf{x}\\ &= \int_\Omega\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}u_i\,d\mathbf{x}\\ &= \|\Delta\mathbf{u}\|_{L^2}^2.\end{aligned} \tag{15}$$

Combining these observations with Young's inequality, we conclude that the nonlinear term (i) is bounded by

$$C\|\Delta\mathbf{u}\|_{L^2}^{7/4}\|\nabla\mathbf{u}\|_{L^2}^{1/4}\|\mathbf{u}\|_{L^4} \le \frac{\nu}{4}\|\Delta\mathbf{u}\|_{L^2}^2 + \frac{C}{\nu^7}\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^4}^8.$$

Altogether, we get the inequality

$$\frac{\partial}{\partial t}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\Delta\mathbf{u}\|_{L^2}^2 \le \frac{C}{\nu^7}\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^4}^8 + \frac{C}{\nu}\|\mathbf{f}\|_{L^2}^2.$$

Using Gronwall's inequality, we find that

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^2}^2 &+ \nu\int_0^t\|\Delta\mathbf{u}(s)\|_{L^2}^2\,ds\\ &\le \|\nabla\mathbf{u}_0\|_{L^2}^2\exp\frac{C}{\nu^7}\int_0^t\|\mathbf{u}(s)\|_{L^4}^8\,ds\\ &\quad + \frac{C}{\nu}\int_0^t\left(\exp\frac{C}{\nu^7}\int_s^t\|\mathbf{u}(\sigma)\|_{L^4}^8\,d\sigma\right)\|\mathbf{f}(s)\|_{L^2}^2\,ds.\end{aligned} \tag{16}$$

Combining (12) and Lemma 1 with $p = 4$, and (16), we see that the quantity

$$\|\mathbf{u}(t)\|_{H^1}^2 + \int_0^t\nu[\|\nabla\mathbf{u}(s)\|_{L^2}^2 + \|\Delta\mathbf{u}(s)\|_{L^2}^2]\,ds$$

remains finite. However, by (15) and the fact that $L^\infty_{loc}(\mathbb{R}^+, L^2(\Omega)) \subset L^2_{loc}(\mathbb{R}^+, L^2(\Omega))$ we have that

$$\|\mathbf{u}(t)\|^2_{H^1} + \int_0^t \nu \ \|\mathbf{u}(s)\|^2_{H^2(\Omega)} \ ds$$

also remains finite. □

5. Final remarks

In this paper, a modified Navier–Stokes equation is presented. Its dynamics and turbulent behavior are studied in terms of the scaling properties of its structure functions. The most dissipative structures are identified as vortex sheets of co-dimension 1, which allows us to compare the numerically measured scaling exponents to a modified phenomenologically based She–Lévêque approach.

Furthermore, we prove the existence of global solutions for this equation. A remarkable consequence of this proof is the existence of an infinite number of conserved quantities $\|\mathbf{u}\|_{L^p}$ in the ideal (non-viscous) case without forcing. This property is not only responsible for the existence of global solutions but should show up in the statistics of intermittent turbulent fluctuations. Work in this direction is in progress.

Acknowledgment

This work benefited from support through DFG-GR 967/3-1.

References

[1] P.C. Martin, E.D. Siggia, H.A. Rose, Statistical dynamics of classical systems, Phys. Rev. A 8 (1973) 423.
[2] G. Falkovich, I. Kolokolov, V. Lebedev, A. Migdal, Instantons and intermittency, Phys. Rev. E 54 (1996) 4896.
[3] A.M. Polyakov, Turbulence without pressure, Phys. Rev. E 52 (1995) 6183–6188.
[4] Z.-S. She, E. Lévêque, Universal scaling laws in fully developed turbulence, Phys. Rev. Lett. 72 (1994) 336–339.
[5] S. Chen, C. Foias, D. Holm, E. Olson, E. Titi, S. Wynne, A connection between the Camassa–Holm equations and turbulent flows in channels and pipes, Phys. Fluids 11 (1999) 2343.
[6] C. Foias, D. Holm, E. Titi, The Navier–Stokes-alpha model of fluid turbulence, Phys. D 152 (2001) 505–519.
[7] J.D. Gibbon, D.D. Holm, Estimates for the LANS-α, Leray-α & Bardina models in terms of a Navier–Stokes Reynolds number, Indiana Univ. Math. J. 57 (2008) 2761–2773.
[8] H. Politano, A. Pouquet, P.L. Sulem, Current and vorticity dynamics in three-dimensional magnetohydrodynamic turbulence, Phys. Plasmas 2 (1995) 2931.
[9] B. Dubrulle, Intermittency in fully developed turbulence: log-Poisson statistics and generalized scale covariance, Phys. Rev. Lett. 73 (1994) 959–962.
[10] D. Mitra, J. Bec, R. Pandit, U. Frisch, Is multiscaling an artifact in the stochastically forced Burgers equation? Phys. Rev. Lett. 94 (2005) 194501.
[11] R. Temam, Navier–Stokes Equations, Elsevier, Amsterdam, 1984.
[12] P. Constantin, C. Foias, Navier–Stokes Equations, The University of Chicago Press, Chicago, London, 1988.
[13] C.R. Doering, J. Gibbon, Applied analysis of the Navier–Stokes equations, in: Cambridge Texts in Applied Mathematics, Cambridge, 1995.
[14] C. Foias, O. Manley, R. Rosa, R. Temam, Navier–Stokes Equations and Turbulence, Cambridge University Press, Cambridge, 2001.
[15] A.J. Majda, A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.
[16] C. Fefferman, Existence and smoothness of the Navier–Stokes equation, in: The Millennium Prize Problems, Clay Math. Inst, Cambridge, MA, 2006.
[17] O.A. Ladyzhenskaya, Sixth problem of the millenium: Navier–Stokes equations, existence and smoothness, Russian Math. Surveys 58 (2001) 251–286.
[18] T. Kato, Nonstationary flows of viscous and ideal fluids in R^3, J. Funct. Anal. 9 (1972) 296–305.
[19] J. Serrin, On the interior regularity of weak solutions of the Navier–Stokes equations, Arch. Ration. Mech. Anal 9 (1962).
[20] L. Berselli, On a regularity criterion for the solutions to the 3D Navier–Stokes equations, Differential Integral Equations 15 (2002) 1129–1137.
[21] C. Cao, E. Titi, Regularity criteria for the three-Dimensional Navier–Stokes equations, Indiana Univ. Math. J. 57 (2008) 2643–2662.
[22] I. Kukavica, M. Ziane, One component regularity for the Navier–Stokes equations, Nonlinearity 19 (2006) 453–469.
[23] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, Comm. Math. Phys. 94 (1984) 61–66.
[24] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier–Stokes equations, Indiana Univ. Math. J. 42 (1993) 775–789.
[25] D. Chae, J. Lee, Regularity criterion in terms of pressure for the Navier–Stokes equations, Nonlinear Anal. 46 (2001) 727–735.
[26] L. Berselli, G. Galdi, Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations, Proc. Amer. Math. Soc 130 (2002) 3585–3595.
[27] J.M. Burgers, The Nonlinear Diffusion Equation, Asymptotic Solutions and Statistical Problems, Reidel, Dordrecht, 1974.
[28] J. Cole, On a quasilinear parabolic equation occuring in aerodynamics, Quart. J. Appl. Math. 9 (1951) 225–236.
[29] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201–230.
[30] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer Verlag, New York, 1988.