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Shift in the speed of reaction–diffusion equation with a cut-off: Pushed and bistable fronts

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HIGHLIGHTS

- We study the effect of a cut-off on the speed of pushed and bistable fronts.
- Explicit expression for the speed shift of general reaction terms is found.
- Applied to examples with vanishing and non-vanishing derivative at the origin.

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ABSTRACT

We study the change in the speed of pushed and bistable fronts of the reaction–diffusion equation in the presence of a small cut-off. We give explicit formulas for the shift in the speed for arbitrary reaction terms $f(u)$. The dependence of the speed shift on the cut-off parameter is a function of the front speed and profile in the absence of the cut-off. In order to determine the speed shift we solve the leading order approximation to the front profile $u(z)$ in the neighborhood of the leading edge and use a variational principle for the speed. We apply the general formula to the Nagumo equation and recover the results which have been obtained recently by geometric analysis. The formulas given are of general validity and we also apply them to a class of reaction terms which have not been considered elsewhere.

1. Introduction

The one-dimensional reaction–diffusion equation

$$u_t = u_{xx} + f(u) \quad \text{with } f(0) = f(1) = 0, \tag{1}$$

has been extensively studied since the original works of Fisher [1], Kolmogorov, Petrovskii and Piscounov (KPP hereafter) [2] and Zeldovich [3], as it is the simplest equation that describes the propagation of traveling fronts in a variety of problems arising in physics, population dynamics, chemistry and others. The time evolution of localized initial conditions leads to the appearance of monotonic traveling fronts joining the stable $u = 1$ to the unstable $u = 0$ equilibrium point. For bistable reaction terms the traveling front may join two stable equilibria. The convergence of initial conditions to monotonic fronts was first studied rigorously in [2] where it was shown that positive reaction terms with non-vanishing derivative at the origin which in addition satisfy $f'(u) < f'(0)$ evolve into a monotonic front of speed $c = c_{KPP} = 2\sqrt{f'(0)}$. These fronts are also called pulled fronts as its speed depends on properties of the leading edge of the front. In contrast, fronts for which the speed depends on the full reaction term are called pushed. These results were extended and generalized by Aronson and Weinberger [4]. They included general positive reaction functions and bistable reaction functions, which satisfy $f(u) < 0$ for u in $(0, a)$, $f > 0$ on $(a, 1)$ with $\int_0^1 f(u)\,du > 0$. It was proved in [4] that sufficiently localized initial conditions evolve into a monotonic traveling front $u = U(x - c\,t)$ joining the stable state $u = 1$ to the state $u = 0$. For reaction terms which are positive in $(0, 1)$ there is continuum of values of c for which a monotonic front exists and the system evolves into the front of minimal speed. The minimal speed, c, satisfies $2\sqrt{f'(0)} \le c \le 2 \sup \sqrt{f(u)/u}$ thus generalizing the condition given in [2]. For bistable reaction terms there is a single isolated value of the speed for which the monotonic front exists.

The reaction–diffusion equation (1) is the simplest model exhibiting front propagation. In many problems effects such as density-dependent diffusion, memory, convective terms, and others, are important and have been studied as well. A different effect was studied by Brunet and Derrida [5] who wished to model the effect of additive noise and the finiteness in the number of diffusing

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particles on the front. They conjectured that these effects can be modeled by the classical reaction–diffusion equation (1) introducing a small cut-off at the edge of the propagating front. More precisely, it is assumed that the reaction term vanishes in a small region $0 \leq u < \epsilon$ and remains unchanged in the interval $\epsilon < u \leq 1$. This conjecture was validated numerically in [5] for a KPP type reaction function. Using perturbation methods they calculated the leading order correction to the speed,

$$c \approx 2 - \frac{\pi^2}{(\log \epsilon)^2}, \tag{2}$$

result obtained solving the equation for traveling fronts $U(z = x - ct)$ of (1) asymptotically. Recently these results were established rigorously in [6] where a reaction–diffusion equation of KPP type with noise is studied and where it is proved that the speed is effectively (2) to leading order. The speed for a KPP type reaction term with a cut-off was studied rigorously in [7] using geometric singular perturbation theory and a blow-up method and in [8,9] through a variational approach. While the speed shift due to the cut-off equation (2) is valid for all KPP fronts with a cut-off, for pushed and bistable fronts the speed, and the change in the speed due to the cut-off, depends on the full reaction term. The shift in the speed was first found to depend on powers of the cut-off [10] for the Nagumo equation with the exact value depending on the reaction term itself. Small perturbations to the reaction term were shown to have a marked effect on the speed [11]. A variational approach was used in [12] and in [13] where the correct power law behavior was obtained but not the magnitude of the shift. The method of geometric singular perturbation has been applied to calculate rigorously the exact leading order behavior of shift in the speed due to a cut-off for the Nagumo equation $f(u) = u(1-u)(u-a)$ in the bistable regime $0 < a < 1/2$, and for general bistable fronts with non-vanishing derivative at the origin [14]. The reaction term $f(u) = u^m(1-u)$ was studied, using the same methods, in [15].

The purpose of this work is to give a general expression for the shift in the speed for bistable and pushed fronts for arbitrary reaction terms. We use an integral variational principle for the speed and find that two cases have to be distinguished, depending on the vanishing or non-vanishing derivative of the uncut reaction term at the origin. We give a general formula for each case, both of which depend on the solution to the original front. The shift in the speed due to the cut-off is found to depend on the cut-off parameter and on the speed and rate of approach to the leading edge of the front in the absence of the cut-off.

We show that the shift in the speed is given by

$$\Delta c = -Kf'(0)\frac{\epsilon^{2+c_0/k_2}}{(2+c_0/k_2)^{1+c_0/k_2}} \quad \text{when } f'(0) \neq 0 \tag{3}$$

where c_0 is the speed of the front in the absence of a cut-off, and k_2 is the rate of approach of the front to the leading edge in the absence of a cut-off, that is $u \approx e^{k_2 z}$ as $z \to \infty$ with $k_2 = -c_0/2 - \sqrt{c_0^2 - 4f'(0)}/2$. K is a constant which depends on an integral of the exact solution of the front without cut-off.

If the derivative of the reaction term vanishes at the origin we show that the shift in the speed is obtained integrating the leading order of

$$\frac{d\Delta c}{d\epsilon} = -K\left(\frac{f(\epsilon)}{\epsilon}\right). \tag{4}$$

These two expressions for Δc, Eqs. (3) and (4), can be written as the single formula

$$\frac{d\Delta c}{d\epsilon} = -K\, Q\, \epsilon^{c_0/k_2} f(\epsilon) \tag{5}$$

where

$$Q = \left[\frac{c_0 + \sqrt{c_0^2 - 4f'(0)}}{2\sqrt{c_0^2 - 4f'(0)}}\right]^{c_0/k_2} = \left(\frac{1}{2 + c_0/k_2}\right)^{c_0/k_2}.$$

In the case $f'(0) \neq 0$ we use $f(\epsilon) = \epsilon f'(0)$ and integrate (5) to obtain (3). When $f'(0) = 0, k_2 = -c_0, Q = 1$ and (5) reduces to (4).

In Section 2 we describe the problem and state known results which are needed to obtain the shift in the speed. A brief derivation of them is given in the Appendix. In this section the main results are derived. In Section 3 we apply the general formula to cases where the exact solution in the absence of a cut-off is known, which allows the complete determination of the shift in the speed. We conclude with a comparison with previous results and indicate possible extensions of this work.

2. Speed of the fronts

In this section we begin recalling properties of the traveling fronts of the reaction–diffusion equation. It was shown in [2,4] that for a wide class of reaction terms $f(u)$ the solution of (1) starting from a sufficiently localized initial condition evolves into a monotonic traveling front $u(x,t) = U(z = x - ct)$ which obeys the ordinary differential equation

$$U_{zz} + cU_z + f(U) = 0, \qquad \lim U_{z\to-\infty} = 1, \quad \text{and} \quad \lim U_{z\to\infty} = 0. \tag{6}$$

In this work we consider reaction terms of bistable type, that is $f(u) < 0$ for u in $(0, a)$, $f > 0$ on $(a, 1)$ with $\int_0^1 f(u)\,du > 0$. We also consider pushed fronts, that is, fronts for which $f > 0$ in $(0, 1)$, with speed greater than the KPP value $2\sqrt{f'(0)}$. In both cases the front approaches $U = 0$ exponentially with the decay rate [4]

$$U \approx e^{-[c+\sqrt{c^2-4f'(0)}]z/2} \quad \text{as } z \to \infty.$$

Throughout the rest of this work we will denote derivatives with respect to U by a prime, and derivatives with respect to the space variable z by a subscript.

In the previous work [16] we showed that the speed of the front satisfies the integral variational principle (see the Appendix)

$$c^2 = \sup\left(2\,\frac{\int_0^1 f(U)\,g(U)\,du}{\int_0^1 (-g^2(U)/g'(U))dU}\right), \tag{7}$$

where the supremum is taken over all positive decreasing functions g in $(0, 1)$ for which the integrals exist. Moreover there is always a maximizing g for bistable reaction functions and for pushed fronts. The optimal g, say $\hat{g}$, is the solution of

$$\frac{\hat{g}'}{\hat{g}} = -\frac{c}{p} \quad \text{where } p(U) = -\frac{dU}{dz}. \tag{8}$$

This variational principle is the starting point for our derivation. Since we are interested in bistable and pushed fronts, we know that a maximizing g exists [16] and the Feynman–Hellmann theorem holds. The Feynman–Hellmann theorem states that if the reaction term f depends on a parameter α then (see the Appendix)

$$\frac{\partial c^2}{\partial \alpha} = 2\frac{\int_0^1 \frac{\partial f}{\partial \alpha}(U,\alpha)\,\hat{g}(U,\alpha)\,dU}{\int_0^1 (-\hat{g}^2/\hat{g}')dU}, \tag{9}$$

where $\hat{g}(U,\alpha)$ is the function (unique up to a multiplicative constant) that yields the maximum in (7) at the given parameter α. Notice that the Feynman–Hellmann theorem holds only if the maximum is attained, which is not the case for KPP fronts.

We apply now the Feynman–Hellmann theorem to pushed or bistable reaction terms $f(u)$ to which we apply a cut-off, that is the reaction term becomes $f(u)\Theta(u-\epsilon)$ (here, $\Theta(x)$ denotes the Heaviside step function). The Feynman–Hellmann theorem tells us that

$$\frac{\partial c^2}{\partial\epsilon} = 2\frac{\int_0^1 \frac{\partial f(U)\Theta(U-\epsilon)}{\partial\epsilon}\,\hat{g}(U,\epsilon)\,dU}{\int_0^1(-\hat{g}^2/\hat{g}')dU} = -2\frac{f(\epsilon)\hat{g}(U=\epsilon,\epsilon)}{\int_0^1(-\hat{g}^2/\hat{g}')dU}. \quad (10)$$

In the expression above $\hat{g}=\hat{g}(U,\epsilon)$ is the optimizing function for the speed of the front with the reaction term $f(U)\Theta(u-\epsilon)$. Let us call c_0 and $U_0(z)$ respectively the speed and the front profile in the absence of the cut-off, which from here on we call the unperturbed problem. Since the cut-off is small we expect the shift in the speed to be small as well. We write then

$$c = c_0 + \Delta c$$

so that, to leading order, (10) implies

$$\frac{d\Delta c}{d\epsilon} = -Kf(\epsilon)\hat{g}(\epsilon,\epsilon) \quad \text{with } K = \frac{1}{c_0\int_0^1(-\hat{g}^2/\hat{g}')dU}. \quad (11)$$

In order to calculate $\hat{g}(\epsilon,\epsilon)$ we notice that Eq. (8) can be solved explicitly in terms of the space variable z. Replacing p by its definition we observe that $\hat{g}$ is given by

$$\hat{g} = \mathrm{e}^{cz} \quad (12)$$

up to a multiplicative constant. This constant can be set equal to 1 due to the translation invariance of Eq. (6) and invariance of the variational principle (7) under scaling in g. It is convenient to calculate $\hat{g}(u=\epsilon,\epsilon)$ by solving the problem in the space variables, for which the leading order profile $U(z)$ including the cut-off must be obtained.

In the region $0 < U \le \epsilon$ where the reaction term is zero, which we call the inner region (the leading edge of the front) the front profile satisfies

$$U_{1zz} + cU_{1z} = 0, \quad \text{with } \lim U_1 = 0 \text{ as } z\to\infty,$$

the solution of which is $U_1(z) = A\mathrm{e}^{-cz}$. If we let $c = c_0 + \Delta c$, to leading order we may write

$$U_1(z) = A\mathrm{e}^{-c_0 z}. \quad (13)$$

Let us call $z = z^*$ the spatial coordinate where $U_1 = \epsilon$. It follows from (12) that, in leading order,

$$\hat{g}(U=\epsilon,\epsilon) = \mathrm{e}^{c_0 z^*} = \frac{A}{\epsilon}, \quad (14)$$

where the constant A has to be determined.

Far from the cut-off, which we call the outer region, $\epsilon < U \le 1$ the solution is in leading order the unperturbed solution $U_0(z)$. It was proved in [4] that the front approaches the leading edge as

$$U_0(z) = \mathrm{e}^{k_2 z} \quad \text{as } z\to\infty, \text{ with } k_2 = -\frac{c_0}{2} - \frac{1}{2}\sqrt{c_0^2 - 4f'(0)}. \quad (15)$$

Moreover, since the profile for the front is a solution of the second-order differential equation $U_{zz} + cU_z + f(U) = 0$, we know that the profile and its derivative are continuous. Therefore the solution and its derivative in each region have to be matched. We find that two cases have to be distinguished according to the value of $f'(0)$.

2.1. The case $f'(0) = 0$

In this case the rate of exponential approach to the leading edge of the unperturbed profile is $k_2 = -c_0$, so that $U_0(z) \approx \mathrm{e}^{-c_0 z}$ as $z \to \infty$ which matches smoothly to the leading order inner solution U_1 choosing $A = 1$. In this case then,

$$\hat{g}(U=\epsilon,\epsilon) = \frac{1}{\epsilon}, \quad \text{when } f'(0) = 0.$$

It is worth mentioning that due to translation invariance, we could have set $U_0(z) \approx \text{constant} \times \mathrm{e}^{-c_0 z}$ as $z\to\infty$ which would lead to a different value for A. Since the problem is invariant under translations in z and the variational principle (7) and Eq. (11) are invariant with respect to a scaling in g such a constant cancels out in the final result. Therefore, with no loss of generality, we may set it equal to 1.

Using this result in (11) we obtain

$$\frac{d\Delta c}{d\epsilon} = -K\frac{f(\epsilon)}{\epsilon}. \quad (16)$$

Finally, since we are interested in the leading order correction of Δc, and since $f(\epsilon)/\epsilon$ is in this case $f''(0)\epsilon/2$ or smaller, we may approximate K by its leading order value,

$$K = \frac{1}{c_0\int_0^1(-\hat{g}_0^2/\hat{g}_0')dU}, \quad (17)$$

where $\hat{g}_0$ is the optimizing function for the unperturbed problem. The shift to the speed in this case is obtained integrating (16).

2.2. The case $f'(0) \ne 0$

In this case the inner solution (13) cannot be matched directly to the outer solution as the value of k_2 does not allow smooth matching of the profiles. This indicates the existence of a transition layer in the region $U \gtrsim \epsilon$.

In this region, for sufficiently low values of ϵ the reaction term can be approximated by the linear form $f(U) \approx Uf'(0)$. In leading order the front satisfies the equation

$$U_{2zz} + c_0 U_{2z} + Uf'(0) = 0,$$

the solution of which is

$$U_2(z) = B_1\mathrm{e}^{k_1 z} + B_2\mathrm{e}^{k_2 z}$$

where

$$k_1 = -\frac{c_0}{2} + \frac{1}{2}\sqrt{c_0^2 - 4f'(0)}, \qquad k_2 = -\frac{c_0}{2} - \frac{1}{2}\sqrt{c_0^2 - 4f'(0)}.$$

We must match U_1 to U_2 at $U = \epsilon$ to obtain the coefficients B_1 and B_2 in terms of A. Let us call z^* the value of z when $U = \epsilon$ as before. The matching conditions are then

$$U_2(z^*) = U_1(z^*) = \epsilon \quad \text{and} \quad U_{1z}(z^*) = U_{2z}(z^*)$$
$$\text{at } z^* = \ln\left(\frac{A}{\epsilon}\right)^{1/c_0},$$

where the value of z^* follows from (14) or can be read from (13) evaluating at $U_1 = \epsilon$. The solution of this system yields

$$B_1 = \epsilon\frac{k_2 + c_0}{k_2 - k_1}\mathrm{e}^{-k_1 z^*}, \qquad B_2 = -\epsilon\frac{k_1 + c_0}{k_2 - k_1}\mathrm{e}^{-k_2 z^*}.$$

Replacing the expressions for k_1, k_2 and z^* this is

$$B_1 = \frac{\epsilon[\sqrt{c_0^2 - 4f'(0)} - c_0]}{2\sqrt{c_0^2 - 4f'(0)}}\left(\frac{\epsilon}{A}\right)^{k_1/c_0}$$

$$B_2 = \frac{\epsilon[c_0 + \sqrt{c_0^2 - 4f'(0)}]}{2\sqrt{c_0^2 - 4f'(0)}}\left(\frac{\epsilon}{A}\right)^{k_2/c_0}.$$

Having matched this intermediate solution to the inner solution U_1 we now match U_2 to the outer solution valid farther from the cut-off. This matching condition follows from (15),

$$U_2(z) \to \mathrm{e}^{k_2 z} \quad \text{as } z\to-\infty. \quad (18)$$

This implies $B_2 = 1$ and we obtain

$$A = \epsilon^{1+c_0/k_2} \left[\frac{c_0 + \sqrt{c_0^2 - 4f'(0)}}{2\sqrt{c_0^2 - 4f'(0)}} \right]^{c_0/k_2}.$$

Replacing this value of A in (14) we obtain

$$\hat{g}(U = \epsilon, \epsilon) = \epsilon^{c_0/k_2} \left[\frac{c_0 + \sqrt{c_0^2 - 4f'(0)}}{2\sqrt{c_0^2 - 4f'(0)}} \right]^{c_0/k_2},$$

when $f'(0) \neq 0$.

To calculate the shift in the speed, we go back to (11), with $f(\epsilon) = \epsilon f'(0)$, using the leading order value (17) for K. Integrating with respect to ϵ we obtain the shift in the speed

$$\Delta c = -Kf'(0) \left[\frac{c_0 + \sqrt{c_0^2 - 4f'(0)}}{2\sqrt{c_0^2 - 4f'(0)}} \right]^{c_0/k_2} \frac{\epsilon^{2+c_0/k_2}}{2 + c_0/k_2}. \tag{19}$$

Again the leading order correction is obtained with K evaluated at the unperturbed value $\hat{g}_0$ given in (17). Replacing the value of k_2 in the expression above, the shift in the speed can be written in the compact form (3).

Eqs. (16) and (19) (or their equivalent (5)) together with (17) constitute our main result.

3. Examples

We will apply the results obtained in the previous section to the Nagumo equation,

$$f(u) = u(1-u)(u-a) \tag{20}$$

for which an exact solution of the unperturbed case is known and K can be calculated explicitly. The shift in the speed due to the cut-off has been calculated rigorously [14] with a geometric approach.

For $0 < a < 1/2$ this is a bistable reaction term. For $-1/2 < a < 0$ it gives rise to a pushed front. The case $a = 0$ is an example with vanishing derivative at the origin. The speed without the cut-off is given by

$$c_0 = \frac{1}{\sqrt{2}} - a\sqrt{2} \tag{21}$$

which is obtained from the variational principle (7) with the trial function [16]

$$\hat{g}_0(U) = \left(\frac{1-U}{U} \right)^{1-2a}. \tag{22}$$

For this reaction term $f'(0) = -a$. The value of K is

$$K = \left[c_0 \int_0^1 (-g_0^2/g_0')dU \right]^{-1} = \frac{\sqrt{2}\,\Gamma(4)}{\Gamma(1+2a)\Gamma(3-2a)}. \tag{23}$$

In the bistable and in the pushed regime replacing the value of c_0 and $f'(0) = -a$, in (19) we obtain

$$\Delta c = \frac{\sqrt{2}\,\Gamma(4)a}{\Gamma(1+2a)\Gamma(3-2a)} \frac{\epsilon^{1+2a}}{(1+2a)^{2a}}$$

in agreement with the result of [14].

When $a = 0$, $f'(0) = 0$ and the shift is obtained using (16). In this case $f(\epsilon) = \epsilon^2$ to leading order, $c_0 = 1/\sqrt{2}$ and $g_0(u) = (1-u)/u$ so that

$$K = 3\sqrt{2}.$$

The shift in the speed is the solution of

$$\frac{d\Delta c}{d\epsilon} = -3\sqrt{2}\epsilon$$

that is,

$$\Delta c = -\frac{3}{\sqrt{2}}\epsilon^2$$

in agreement with the result obtained in [15] by geometric analysis.

As a third example take the family of reaction terms also considered in [15]

$$f_m(u) = u^m(1-u).$$

The effect of the cut-off is to shift the speed according to (16) so that

$$\frac{d\Delta c}{d\epsilon} = -K\frac{\epsilon^m}{\epsilon}$$

from where we recover Theorem 1.1 of [15] obtained via geometric analysis,

$$\Delta c = -K\frac{\epsilon^m}{m}$$

with the identification $\gamma_m = -K/m$. We cannot compute explicitly the value of K since the unperturbed solution is not known except in the case $m = 2$ already described above.

Finally we apply the results to the class of exactly solvable reaction terms, which have not been considered elsewhere,

$$f(u) = f'(0) \left(u + \frac{n+1-\lambda}{\lambda-1}u^n - \frac{n}{\lambda-1}u^{2n-1} \right),$$

with $\lambda \neq 1\, \lambda < 2$, $n > 1$, and $\lambda + n > 1$.

The traveling front solution to (6) is given by [17]

$$U_0(z) = \frac{e^{-a_1 z}}{(1 + e^{-(n-1)a_1 z})^{1/(n-1)}}, \quad \text{where } a_1 = \sqrt{\frac{f'(0)}{\lambda-1}} \tag{24}$$

which travels with speed

$$c_0 = \lambda\sqrt{\frac{f'(0)}{\lambda-1}}.$$

The phase space solution for this front is

$$p_0(U) = -\frac{dU}{dz}(U) = a_1 U(1-U^{n-1}).$$

For the values of λ specified, this solution corresponds to pushed or bistable fronts [17]. Using Eqs. (12) and (24) we obtain the optimal trial function for the exact front $\hat{g}_0(U)$,

$$g_0(U) = e^{c_0 z} = \left(\frac{1-U^{n-1}}{U^{n-1}} \right)^{\lambda/(n-1)}.$$

In order to calculate the speed shift we need to calculate

$$\frac{1}{K} = c_0 \int_0^1 -\frac{\hat{g}_0^2}{\hat{g}_0'}dU = \int_0^1 g_0(U)p_0(U)dU$$
$$= a_1 \frac{\Gamma(2+\frac{\lambda}{n-1})\Gamma(\frac{2-\lambda}{n-1})}{(n+1)\Gamma(\frac{n+1}{n-1})}, \tag{25}$$

where we used (8) as an intermediate step. Replacing this result in Eq. (3) or its equivalent (19), and using $k_2 = -a_1$ we obtain the final expression for the shift in the speed to be

$$\Delta c = -K\frac{\epsilon^{2-\lambda}}{(2-\lambda)^{1-\lambda}}$$

with K given in (25). The breakdown of this expression at $\lambda = 2$ is expected and it is due to the transition from a pushed to a KPP front that occurs at $\lambda = 2$ [17]. The Nagumo equation is a special case with $n = 2$, $\lambda = 1 - 2a$.

4. Summary

We studied the change in speed of a reaction–diffusion front due to a cut-off in the case when the original front without a cut-off is bistable or pushed. We used a variational principle in order to calculate the shift in the speed. We distinguished two cases, that in which the derivative of the reaction term vanishes at the origin and that in which it does not. This last case includes bistable and positive non-KPP fronts. While we considered only the simplest classical reaction–diffusion equation (1) the method we have used can be extended to generalized reaction–diffusion equations for which a variational principle for the speed has been formulated [18,19]. It is of interest to compare the expression for the shift in the speed for arbitrary bistable reaction terms derived by Dumortier, Popovic and Kaper in [14] and that obtained here via a variational approach. In [14] it is shown that for a general bistable term the shift in the speed due to the cut-off is given by $\Delta c = K_{DPK}\epsilon^s$ where $s = 2\sqrt{c_0^2 - 4f'(0)}/(c_0 + 2\sqrt{c_0^2 - 4f'(0)})$ and K_{DPK} a constant that can be calculated explicitly when the solution of the unperturbed problem and the solution of a partial differential equation for $\partial p/\partial c$ are known. It is straightforward to verify that $s = 2 + c_0/k_2$ so that both formulations give the correct power of ϵ dependence. In both cases the exact value of the accompanying constant requires knowledge of the exact solution of the unperturbed problem $p(u; c_0)$. In the present case knowledge of the exact unperturbed solution allows us to calculate $\hat{g}_0$ and as a consequence K. It is beyond the scope of this work to prove that the formulas coincide; we rely on the agreement obtained in the exactly solvable cases.

The variational formulation presented in this work allows a unified approach to pushed and bistable fronts, with vanishing or non-vanishing derivative at the leading edge. We have only considered one type of cut-off, a complete cut-off at the edge of the front. While a Heaviside cut-off has been the most widely studied case, a gradual cut-off is also possible. The effect of a linear cut-off has been studied for the Nagumo equation in the case of vanishing speed $a = 1/2$ [20]. It is an open problem to study whether a general cut-off for pushed and bistable fronts can be studied via the variational approach presented in this work.

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Appendix

In this appendix we recall the derivation of the variational principle Eq. (7) and derive the Feynman–Hellmann theorem as it applies to this specific problem.

A.1. Variational principle

We follow closely the derivation of the variational principle given in [16]. The traveling front of the reaction–diffusion equation (1) obeys Eq. (6). Since it is known that sufficiently localized initial conditions evolve into a decaying monotonic front, it is convenient and customary to define $p(U) = -dU/dz$, where the minus sign is introduced so that $p > 0$. Monotonic fronts are then a solution of

$$p(U)\frac{dp}{dU} - cp(U) + f(U) = 0, \quad \text{with } p(0) = p(1) = 0. \tag{A.1}$$

Let $g(U)$ be an arbitrary positive decaying function in $(0, 1)$ so that $h = -g' > 0$. Multiplying (A.1) by g and integrating in U between 0 and 1 we obtain after integrating by parts,

$$\int_0^1 f(U)g(U)dU = c\int_0^1 p(U)g(U)dU - \frac{1}{2}\int_0^1 h(U)p^2(U)dU.$$

The right side of the previous formula has a maximum at

$$p_{\max} = c\frac{g}{h}$$

and it is easy to verify that $\Phi(p) = cp(U)g(U) - (1/2)h(U)p^2(U) \le \Phi(p_{\max}) = c^2g^2/(2h)$. It follows then

$$c^2 \ge 2\,\frac{\int_0^1 f(U)\,g(U)\,du}{\int_0^1(-g^2(U)/g'(U))dU}.$$

Equality is achieved when $p_{\max} = p(U)$, the solution of (A.1), that is, equality is achieved for a function $\hat{g}$ which satisfies

$$p(U) = -c\frac{\hat{g}}{\hat{g}'}. \tag{A.2}$$

It can be shown that for pushed and bistable fronts $\hat{g}$ exists; the details of the proof are given in [16]. We know then that

$$c^2 = 2\,\frac{\int_0^1 f(U)\,\hat{g}(U)\,dU}{\int_0^1(-\hat{g}^2(U)/\hat{g}'(U))dU} \quad \text{for pushed and bistable fronts.}$$

A.2. Feynman–Hellmann theorem

Although this is a general property of integral variational principles, it is commonly referred to as the Feynman–Hellmann theorem due to its standard use in quantum mechanics. Rather than giving the derivation for a general Lagrangian we derive it for our specific problem.

Consider a reaction term which depends on a parameter α, that is $f(U, \alpha)$. The solution to (A.1) will depend on α and therefore $\hat{g}$ and the speed itself will depend on α. Writing this explicitly we have

$$c^2(\alpha) = 2\,\frac{\int_0^1 f(U,\alpha)\,\hat{g}(U,\alpha)\,dU}{\int_0^1(-\hat{g}^2(U,\alpha)/\hat{g}'(U,\alpha))dU}.$$

Taking the derivative with respect to α we obtain

$$2\int_0^1\left(f\frac{\partial\hat{g}}{\partial\alpha} + \frac{\partial f}{\partial\alpha}\hat{g}\right) + \frac{dc^2}{d\alpha}\int_0^1\frac{\hat{g}^2}{\hat{g}'}dU + c^2\int_0^1\left(\frac{2\hat{g}}{\hat{g}'}\frac{\partial\hat{g}}{\partial\alpha} - \frac{\hat{g}^2}{\hat{g}'^2}\frac{\partial\hat{g}'}{\partial\alpha}\right)dU = 0. \tag{A.3}$$

Next we may integrate the last term by parts to obtain

$$\int_0^1\frac{\hat{g}^2}{\hat{g}'^2}\frac{\partial\hat{g}'}{\partial\alpha}dU = \left.\frac{\hat{g}^2}{\hat{g}'^2}\frac{\partial\hat{g}}{\partial\alpha}\right|_0^1 - \int_0^1\frac{d}{dU}\left(\frac{\hat{g}^2}{\hat{g}'^2}\right)\frac{\partial\hat{g}}{\partial\alpha} = -\int_0^1\frac{d}{dU}\left(\frac{\hat{g}^2}{\hat{g}'^2}\right)\frac{\partial\hat{g}}{\partial\alpha}$$

where we use (A.2) and the boundary conditions $p(0) = p(1) = 0$ to verify that the boundary terms vanish. Replacing this result in (A.3) and grouping terms we obtain

$$2\int_0^1\left(\frac{\partial f}{\partial\alpha}\hat{g}\right) + \frac{dc^2}{d\alpha}\int_0^1\frac{\hat{g}^2}{\hat{g}'}dU + \int_0^1\frac{\partial\hat{g}}{\partial\alpha}\left[2f + 2c^2\frac{\hat{g}}{\hat{g}'} + c^2\frac{d}{dU}\left(\frac{\hat{g}^2}{\hat{g}'^2}\right)\right]dU = 0. \tag{A.4}$$

Finally, using (A.2) and (A.1) one can verify that

$$2f + 2c^2\frac{\hat{g}}{\hat{g}'} + c^2\frac{d}{dU}\left(\frac{\hat{g}^2}{\hat{g}'^2}\right) = 2f - 2cp + \frac{dp^2}{dU} = 0;$$

therefore, (9) holds.

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