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On the Lagrangian and Eulerian analyticity for the Euler equations

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Dedicated to Professor Edriss S. Titi on the occasion of his sixtieth birthday

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ABSTRACT

We address preservation of the Lagrangian analyticity radius of solutions to the Euler equations belonging to natural analytic space based on the size of Taylor (or Gevrey) coefficients. We prove that if the solution belongs to such space, then the solution also belongs to it for a positive amount of time. We also prove the local analog of this result for a sufficiently large Gevrey parameter; however, we show that the preservation holds independently of the size of the radius. Finally, we construct a solution which shows that the first result does not hold in the Eulerian setting.

1. Introduction

The motion of a fluid can be described either in reference to a fixed space–time point (x, t) (Eulerian coordinates) or by following the flow lines of individual fluid particles (Lagrangian coordinates). It is known that these two formulations are significantly different. There are examples of 3D steady flows with complicated particle paths where streamlines have the space filling property (cf. [1]). Conversely, there are cases where the Lagrangian formulation is better adapted.

In this paper we study the Lagrangian formulation of the incompressible Euler equations on $\mathbb{R}^d$, where $d \in \{2, 3\}$. It is known from [2–5] that a solution of the Euler equations remains analytic (or more generally Gevrey regular) if it is so initially, as long as the solution exists. It is an interesting question whether the solution actually belongs to the same analytic space for a positive amount of time. This is since while solutions remain analytic (or more generally Gevrey regular), the radius may actually decrease. In [6], it was shown that the Lagrangian solution of the Euler equation (or more generally Gevrey) belongs to the same analytic space as the initial data for a positive amount of time. The space in [6] requires summability of Taylor coefficients. Moreover, it was shown, that the same statement does not hold in the Eulerian setting, i.e., a solution was constructed whose sum of Taylor coefficients strictly decreases as time increases.

In this paper, we address the issue of persistence with respect to the analytic space which is however based on the natural supremum condition for the Taylor coefficient (i.e., uniform analyticity/Gevrey regularity) as studied say in [7] and many other works (cf. [8–12]). More precisely, by defining a suitable Lagrangian Gevrey-class norm, we prove that if the initial velocity gradient is of Gevrey-class s, where $s \geq 1$, then the Sobolev solution $v(\cdot, t) \in C([0, T]; H^r(\mathbb{R}^d))$ is of Gevrey-class s for all $t < T$. We would like to emphasize that the main result in [6] and Theorem 3.1 here do not imply each other; they both establish that the solution persists in a certain analytic space, which have different definitions (and are both natural in a certain sense).

In the second main result, Theorem 4.1, we establish a local version of the preservation of the Lagrangian radius. Namely, we prove that the local Gevrey radius is preserved on a positive time interval, regardless of the size of the Gevrey radius (i.e., even if it is larger than the diameter of the domain). The statement requires the usual assumption that the Sobolev norm of the velocity stays finite for a small time, i.e., $\sup_{t\in[0,T]} \|v(\cdot, t)\|_{H^r(B(0,R))} < \infty$, and that the Gevrey parameter is sufficiently large. It would be interesting to prove this for the full range of Gevrey radii, although this may be true in the analytic case only for radii sufficiently small. In a third main result, Theorem 5.1, we demonstrate that our global analyticity result does not hold in the Eulerian coordinates by constructing an explicit initial data for which the radius strictly decreases. A similar example was provided in [6]; however, the nature of our space (the supremum assumption instead of integrability), allows for a simpler construction, which can be carried out in the original variables (rather than in the Fourier space as in [6]).

The Lagrangian approach has gained a strong interest due to the possibility that the Lagrangian paths could be analytic in time, a

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fact which was first observed by Serfati [13]. Earlier, Chemin [14] proved that the Lagrangian trajectories are C^∞ smooth. More recently, in [15–17], the authors developed an elementary theory of analytic fluid particle trajectories. Their work is based on Cauchy's long-forgotten manuscript (1815) on Lagrangian formulation of 3D incompressible Euler equations in terms of Lagrangian invariants. For further results on time-analyticity of Lagrangian trajectories, we refer the reader to [18–20] and to [21] for advantages of the Lagrangian formulation in the Hölder class.

The remarkable difference in time analyticity of the two formulations suggests a further discussion on the analytic regularity of solutions. There is quite a rich history on the persistence of real-analyticity of the solutions in both two and three dimensions with many works pointing out the contrast between the Lagrangian and Eulerian analytic regularity (cf. [2,3,22–26]). In [3], the authors established a lower bound of the form $\exp(-C\exp(CT))/C$ for the radius of analyticity in 2D, with the constant C depending on the initial data. Finding explicit rates for the decay of the radius of analyticity was studied further using different methods in [27,28] for the interior and in [29] for the boundary value problem. Also, in [30,31] the authors extended the results in [28,29] to the non-analytic Gevrey classes and improved the dependence on the initial datum. For space-periodic domains, Levermore and Oliver gave in [5] a proof of persistence of analyticity for all times. Their proof is based on a characterization of Gevrey classes in terms of decay of Fourier coefficients.

Furthermore, the shear flow example (cf. [32,33]) has generated numerous constructions of explicit solutions to periodic 3D Euler equations whose radius of analyticity decays for all time. In the analytic class (Gevrey-1 class) case, cf. [30, Remark 1.3] and [6, Theorem 1.3]. Also, one can construct an example in the non-analytic Gevrey classes with $s > 1$ (cf. [34]). Moreover, in [6] the authors point out the difference of behaviors in the two formulations; the radius of analyticity is conserved locally in time for the Lagrangian formulation, whereas it deteriorates instantaneously in the Eulerian one. A similar contrast is also observed in terms of solvability in anisotropic classes. The Lagrangian formulation is locally well-posed in anisotropic classes yet the equations are ill-posed in Eulerian coordinates.

The paper is organized as follows. In Section 2, we introduce the Gevrey-class space using the supremum over Taylor coefficients (as opposed to their sum as in [6]). Furthermore, the well-posedness of Lagrangian formulation in anisotropic Gevrey classes holds for local solutions as well. In Section 3, we prove that if the Gevrey regularity parameter is sufficiently large (cf. (4.1) below), then the analyticity radius is preserved in the future, regardless of its size. It is an interesting question if this theorem can be extended for analytic class as well. Finally, in Section 5, we provide a counterexample to Theorem 3.1 in the Eulerian setting.

The paper is dedicated to Professor Edriss Titi on the occasion of his sixtieth birthday in admiration of his work and in appreciation for his support throughout the years.

2. Euler equations in Lagrangian coordinates

The incompressible homogeneous Euler equations in $\mathbb{R}^d$, for $d = 2, 3$, are given by the system of equations

$$u_t + u\cdot\nabla u + \nabla p = 0 \tag{2.1}$$
$$\nabla\cdot u = 0 \tag{2.2}$$
$$u(x,0) = u_0(x) \tag{2.3}$$

for $(x,t)\in\mathbb{R}^d\times[0,\infty)$. The above system models the flow of an incompressible, homogeneous, and inviscid fluid, where $u(x,t) = (u^1,\dots,u^d)$ denotes the fluid velocity and $p(x,t)$ the pressure.

We rewrite the Euler equations using the particle trajectory mapping $X(\cdot,t) : \alpha \mapsto X(\alpha,t)\in\mathbb{R}^d$, where $t \geq 0$. The vector $X(\alpha,t) = (X_1,\dots,X_d)$ denotes the location of a fluid particle at time t that is initially placed at the Lagrangian label α, and is given by an ODE

$$\partial_t X(\alpha,t) = u(X(\alpha,t),t) \tag{2.4}$$
$$X(\alpha,0) = \alpha. \tag{2.5}$$

Composing the velocity and the pressure with X, we obtain the Lagrangian velocity v and the pressure q by

$$v(\alpha,t) = u(X(\alpha,t),t)$$
$$q(\alpha,t) = p(X(\alpha,t),t).$$

Also, denote by Y_i^k the (k,i)th entry of the inverse of the Jacobian of X, i.e.,

$$Y(\alpha,t) = (\nabla_\alpha X(\alpha,t))^{-1}.$$

We then write the Lagrangian formulation of the Euler equations as

$$\partial_t v^i + Y_i^k\partial_k q = 0, \qquad i = 1,\dots,d \tag{2.6}$$
$$Y_i^k\partial_k v^i = 0 \tag{2.7}$$

with the summation convention on repeated indices understood. The system (2.6)–(2.7) is supplemented with the initial conditions

$$v(\alpha,0) = v_0(\alpha) = u_0(\alpha)$$
$$Y(\alpha,0) = I.$$

Differentiating (2.4) with respect to the Lagrangian labels along with using $\det(\nabla X) = 1$ and inverting the matrix in the resulting equation, we get

$$Y_t = -Y : (\nabla v) : Y \tag{2.8}$$

where the symbol : denotes the matrix multiplication. Taking the curl of the Eq. (2.1) and using $\nabla\times(u\cdot\nabla u) = u\cdot\nabla\omega - \omega\cdot\nabla u$, we obtain

$$\partial_t\omega + u\cdot\nabla\omega = \omega\cdot\nabla u.$$

Hence,

$$\frac{D\omega}{Dt} = \omega\cdot\nabla u$$

where D/Dt is the convective derivative, i.e., the derivative along the particle trajectories. In 2D flows the vorticity is conserved, i.e., $\zeta(\alpha,t) = \omega_0(a)$ for $t \geq 0$. Denoting the sign of the permutation $(1,2)\mapsto(i,j)$ by ϵ_{ij}, we may write the Euler system as

$$\epsilon_{ij}Y_i^k\partial_k v^j = Y_1^k\partial_k v^2 - Y_2^k\partial_k v^1 = \omega_0 \tag{2.9}$$
$$Y_i^k\partial_k v^i = Y_1^k\partial_k v^1 + Y_2^k\partial_k v^2 = 0. \tag{2.10}$$

If $d = 3$, we may use the vorticity-transport formula $\zeta^i(\alpha,t) = \partial_k X^i(\alpha,t)\omega_0^k(\alpha)$ and proceed as in [6] to write the Euler equations as

$$\epsilon_{ijk}Y_i^m Y_j^l\partial_l v^k = Y_i^m\zeta^i = \omega_0^m, \qquad m = 1,2,3 \tag{2.11}$$
$$Y_i^k\partial_k v^i = 0. \tag{2.12}$$

The Eq. (2.11), derived in [6], represents a way to write the Cauchy invariance formula without involving X.

3. The preservation of the Gevrey radius

We start by recalling the definition of Gevrey spaces. For any $s \geq 1$, we define the s-Gevrey norm with radius $\delta > 0$ by

$$\|f\|_{G_{s,\delta}} = \sup_{|\alpha|\geq 0}\frac{\delta^{|\alpha|}}{|\alpha|!^s}\|\partial^\alpha f\|_{H^r}$$

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where we assume $r > d/2$ so that the space $H^r = H^r(\mathbb{R}^d)$ is an algebra. When the above norm is finite for $s = 1$, we say that the function is uniformly real-analytic. The anisotropic Gevrey norm is obtained by replacing the multi-index derivative with a directional derivative, i.e., given a direction $j \in \{1, \ldots, d\}$, we define the anisotropic s-Gevrey norm with radius $\delta > 0$ by

$$\|f\|_{G^{(j)}_{s,\delta}} = \sup_{m\geq 0} \frac{\delta^m}{m!^s} \|\partial_j^m f\|_{H^r}$$

and $G^{(j)}_{s,\delta}$ denotes the set of functions f for which $\|f\|_{G^{(j)}_{s,\delta}}$ is finite.

Our first main theorem addresses preservation of the Lagrangian analyticity radius on a positive time interval.

Theorem 3.1. *Assume that $v_0 \in H^{r+1}(\mathbb{R}^d)$ for a fixed direction $j \in \{1, \ldots, d\}$ and that*

$$\nabla v_0 \in G^{(j)}_{s,\delta}$$

for some radius $\delta > 0$ and Gevrey index $s \geq 1$. Then there exists $T > 0$ and a unique solution $(v, Y) \in C([0, T], H^{r+1}(\mathbb{R}^d)) \times C([0, T], H^r(\mathbb{R}^d))$ of the system (2.6)–(2.7), which satisfies

$$\nabla v, Y \in L^\infty([0, T], G^{(j)}_{s,\delta}).$$

We emphasize that the solution belongs to the same Gevrey space $G^{(j)}_{s,\delta}$ for all $t \in [0, T]$, i.e., there is no reduction in the Gevrey radius as in [3,30].

Proof of Theorem 3.1. We provide the proof for $d = 2$ and outline the necessary changes for $d = 3$ in Remark 3.2 below. Without loss of generality, we may assume $j = 1$. For $m \in \mathbb{N}_0$, denote

$$\Omega_m = \|\partial_1^m \nabla v_0\|_{H^r}.$$

Assume that $\delta > 0$ satisfies

$$\Omega_0,\ \Omega_1\delta,\ \Omega_2\delta^2 \leq M_0 \tag{3.1}$$

as well as

$$\Omega_m \frac{\delta^m}{(m-3)!^s} \leq M_0, \qquad m \in \{3, 4, \ldots\} \tag{3.2}$$

for some $M_0 > 0$. We follow the setting in [6, Section 3]. We fix a time $T > 0$ and define

$$V_m(T) = \sup_{t\in[0,T]} \|\partial_1^m \nabla v(t)\|_{H^r},$$
$$Z_m(T) = \sup_{t\in[0,T]} t^{-1/2} \|\partial_1^m (Y(t) - I)\|_{H^r}.$$

As in [6], we bound V_m and Z_m using the div–curl system (2.9)–(2.10) and the Lagrangian evolution (2.8) in the integrated form, respectively. Denote by

$$I = \{(j, k) \in \mathbb{N}_0^2 : 0 < |(j, k)| \leq m\} \setminus \{(m, 0), (0, m)\} \tag{3.3}$$

the index set. Using the same argument as in [6], we get for all $m \in \mathbb{N}$

$$\begin{aligned} V_m &\leq C\Omega_m + CT^{1/2} Z_0 V_m + CT^{1/2} Z_m V_0 \\ &\quad + CT^{1/2} \sum_{j=1}^{m-1} \binom{m}{j} Z_j V_{m-j} \end{aligned} \tag{3.4}$$

$$\begin{aligned} Z_m &\leq CT^{1/2}(TZ_0^2 V_m + TZ_m Z_0 V_0 + T^{1/2} Z_0 V_m + T^{1/2} Z_m V_0 + V_m) \\ &\quad + CT^{3/2} \sum_{(j,k)\in I} \binom{m}{j,k} Z_j Z_k V_{m-j-k} + CT \sum_{j=1}^{m-1} \binom{m}{j} Z_j V_{m-j} \end{aligned} \tag{3.5}$$

where C denotes a generic positive constant for which we always assume $C \geq 1$ for convenience. For $m = 0$ and any $t \in (0, T]$ we have

$$V_0(t) \leq C_0\Omega_0 + C_0 t^{1/2} Z_0(t) V_0(t)$$
$$Z_0(t) \leq C_0 t^{1/2} \sup_{\tau\in[0,t)} (V_0(\tau)(1 + \tau^{1/2} Z_0(\tau))^2)$$

where $C_0 \geq 1$ is fixed. We then define

$$S_m = V_m + Z_m = \sup_{t\in[0,T]} (V_m(t) + Z_m(t))$$

for all $m \geq 0$. By adding (3.4) and (3.5), we obtain an estimate on S_m which reads

$$\begin{aligned} S_m &\leq C\Omega_m + CT^{1/2}(1 + S_0 + T^{1/2} S_0 + TS_0^2) S_m \\ &\quad + CT^{1/2}(1 + T^{1/2}) \sum_{0<j<m} \binom{m}{j} S_j S_{m-j} \\ &\quad + CT^{3/2} \sum_{(j,k)\in I} \binom{m}{j,k} S_j S_k S_{m-j-k} \end{aligned} \tag{3.6}$$

for all $m \geq 1$. In view of the initial conditions

$$V_0(0) = \Omega_0 \leq C_0 M_0$$
$$Z_0(0) = 0$$

and the continuity in time of $V_0(t)$ and $Z_0(t)$, we have

$$S_0 = V_0 + Z_0 \leq 3C_0 M_0 \tag{3.7}$$

if

$$0 \leq t \leq T \leq \min\left\{\frac{1}{4C_0^4}, \frac{1}{16C_0^2}\right\}.$$

We may further reduce T so that

$$CT^{1/2}(1 + S_0 + T^{1/2} S_0 + TS_0^2) \leq \frac{1}{2} \tag{3.8}$$

where C is as in (3.6). Using (3.7) and (3.8) in (3.6) we obtain the estimate

$$\begin{aligned} S_m &\leq C\Omega_m + CT^{1/2}(1 + T^{1/2}) \sum_{0<j<m} \binom{m}{j} S_j S_{m-j} \\ &\quad + CT^{3/2} \sum_{(j,k)\in I} \binom{m}{j,k} S_j S_k S_{m-j-k} \end{aligned} \tag{3.9}$$

for $m \in \mathbb{N}$. One last arrangement may be performed regarding the index set I in (3.3). Namely, every term with $(j, k) \in I$ for which either $j = 0$ or $k = 0$ may be absorbed into the summation in the second term, that is using the bound on S_0 in (3.7) for any $k \in \{1, \ldots, m-1\}$; for instance,

$$\binom{m}{0,k} S_0 S_k S_{m-k} \leq 3C_0 M_0 \binom{m}{k} S_k S_{m-k}.$$

Thus we may rewrite (3.9) as

$$\begin{aligned} \frac{S_m}{(m-3)!^s} &\leq \frac{C\Omega_m}{(m-3)!^s} + \frac{C}{(m-3)!^s} T^{1/2}(1 + T^{1/2} + M_0 T) \\ &\quad \times \sum_{j=1}^{m-1} \binom{m}{j} S_j S_{m-j} \\ &\quad + \frac{C}{(m-3)!^s} T^{3/2} \sum_{(j,k)\in\widetilde{I}} \binom{m}{j,k} S_j S_k S_{m-j-k} \end{aligned} \tag{3.10}$$

for all $m \geq 3$, where

$$\widetilde{I} = \{(j, k) \in \mathbb{N}^2 : 0 < |(j, k)| \leq m\}.$$

For $m = 1, 2$, the estimate (3.9) reads as

$$S_1 \leq C\Omega_1$$

and

$$S_2 \le C\Omega_2 + CT^{1/2}(1+T^{1/2})S_1^2 + CT^{3/2}S_0S_1^2.$$

Using the hypothesis on Ω_1, Ω_2 we have

$$S_1\delta \le CM_0$$

and

$$S_2\delta^2 \le CM_0 + CT^{1/2}M_0^2(1+T^{1/2}+S_0T).$$

Now, assuming that

$$\frac{S_j}{(j-3)!^s}\delta^j \le M, \qquad j=3,\dots,m-1$$

for some $M>0$, to be determined below, we estimate (3.10) from above. By (3.2),

$$\begin{aligned}\frac{S_m}{(m-3)!^s} &\le C\delta^{-m}M_0 + CT^{1/2}(1+M_0T)\sum_{j=1}^{m-1}M^2\delta^{-m}\frac{m!}{(m-3)!^s}\\ &\quad\times\frac{(j-3)!^s}{j!}\frac{(m-j-3)!^s}{(m-j)!}\\ &\quad + CT^{3/2}\sum_{(j,k)\in I}M^3\delta^{-m}\frac{m!}{(m-3)!^s}\\ &\quad\times\frac{(j-3)!^s}{j!}\frac{(k-3)!^s}{k!}\frac{(m-j-k-3)!^s}{(m-j-k)!}.\end{aligned} \tag{3.11}$$

We use the agreement $s!=1$ if $s\le 0$. Keeping this in mind, we simply estimate

$$\frac{(s-3)!}{s!} \le \frac{9}{s^3}, \qquad s\in\mathbb{N}.$$

As $s\ge 1$, we may write

$$\begin{aligned}&\frac{m!}{(m-3)!^s}\frac{(j-3)!^s}{j!}\frac{(m-j-3)!^s}{(m-j)!}\\ &\quad\le Cm^3\left(\frac{(j-3)!}{j!}\right)\left(\frac{(m-j-3)!}{(m-j)!}\right)\\ &\quad\times\left(\frac{(m-3)!}{(j-3)!(m-j-3)!}\right)^{1-s}\\ &\quad\le C\frac{m^3}{j^3(m-j)^3}\left(\frac{(m-6)!}{(j-3)!(m-j-3)!}\right)^{1-s}(m-3)^{3(1-s)}\\ &\quad\le C\frac{m^3}{j^3(m-j)^3}(m-3)^{3(1-s)}\end{aligned}$$

and

$$\begin{aligned}&\frac{m!}{(m-3)!^s}\frac{(j-3)!^s}{j!}\frac{(k-3)!^s}{k!}\frac{(m-j-k-3)!^s}{(m-j-k)!}\\ &\quad\le Cm^3\frac{1}{j^3}\frac{1}{k^3}\frac{1}{(m-j-k)^3}\\ &\quad\times\left(\frac{(m-9)!}{(j-3)!(k-3)!(m-3-j-k)!}\right)^{1-s}(m-3)^{6(1-s)}\\ &\quad\le Cm^3\frac{1}{j^3}\frac{1}{k^3}\frac{1}{(m-j-k)^3}(m-3)^{6(1-s)}.\end{aligned}$$

Now, we use

$$\sum_{j=1}^{m-1}\frac{(j-3)!}{j!}\frac{(m-j-3)!}{(m-j)!} \le 9^2\sum_{j=1}^{m-1}\frac{1}{j^3}\frac{1}{(m-j)^3} \le \frac{C}{m^3} \tag{3.12}$$

and

$$\begin{aligned}&\sum_{(j,k)\in I}\frac{(j-3)!}{j!}\frac{(k-3)!}{k!}\frac{(m-j-k-3)!}{(m-j-k)!}\\ &\quad\le 9^3\sum_{|(j,k)|=2}^{m-1}\frac{1}{j^3}\frac{1}{k^3}\frac{1}{(m-j-k)^3}\le\frac{C}{m^3}.\end{aligned} \tag{3.13}$$

Thus, from (3.11) we obtain

$$\begin{aligned}\frac{S_m}{(m-3)!^s}\delta^m &\le CM_0 + CT^{1/2}(1+M_0T)\frac{1}{(m-3)^{3(s-1)}}M^2\\ &\quad + CT^{3/2}C\frac{1}{(m-3)^{6(s-1)}}M^3\\ &\le C(M_0 + T^{1/2}(1+M_0T)M^2 + T^{3/2}M^3)\end{aligned}$$

for all $m\ge 3$. Setting $M\ge CM_0$ for a sufficiently large $C\ge 1$ and choosing a sufficiently small time

$$0<T<\frac{1}{C}\min\left\{\frac{M_0^2}{M^4},\frac{1}{M^{4/3}},\frac{M_0^{2/3}}{M^2}\right\},$$

we obtain the desired result. □

Remark 3.2 (*Justification of Theorem 3.1 for the 3D Case*)**.** In 3D, the only change arises when estimating V_m. As the vorticity is not conserved along the particle trajectories, the 2D elliptic div–curl system (2.9)–(2.10) that we used above is replaced by (2.11)–(2.12). We then write

$$\begin{aligned}(\operatorname{curl} v)^m &= \epsilon_{mlk}\partial_l v^k = \omega_0^m + \epsilon_{ilk}(\delta_{im}-Y_i^m)\partial_l v^k + \epsilon_{mjk}(\delta_{jl}-Y_j^l)\partial_l v^k\\ &\quad - \epsilon_{ijk}(\delta_{im}-Y_i^m)(\delta_{jl}-Y_j^l)\partial_l v^k,\\ &\qquad m=1,2,3\\ \operatorname{div} v &= (\delta_{ik}-Y_i^k)\partial_k v^i.\end{aligned}$$

Estimating the gradient with the divergence and the curl, we have

$$\begin{aligned}\|\partial^m\nabla v\|_{H^r} &\le C\|\partial^m\omega_0^n\|_{H^r} + C\|\partial^m(\epsilon_{ijk})(\delta_{in}-Y_i^n)(\delta_{jl}-Y_j^l)\partial_l v^k\|_{H^r}\\ &\quad + C\|\partial^m(\epsilon_{njk}(\delta_{jl}-Y_j^l)\partial_l v^k)\|_{H^r}\\ &\quad + C\|\partial^m(\epsilon_{ijk}(\delta_{in}-Y_i^n)\partial_j v^k)\|_{H^r}\\ &\quad + C\|\partial^m((\delta_{ik}-Y_i^k)\partial_k v^i)\|_{H^r}.\end{aligned}$$

Once again applying the Leibniz rule and taking the supremum over $t\in[0,T]$ we get the estimate

$$\begin{aligned}V_m &\le C\Omega_m + CT^{1/2}Z_0V_m + CT^{1/2}Z_mV_0 + CTZ_0^2V_m + CTZ_mZ_0V_0\\ &\quad + CT\sum_{(j,k)\in I}\binom{m}{j,k}Z_jZ_kV_{m-j-k} + CT^{1/2}\sum_{j=1}^{m-1}Z_jV_{m-j}.\end{aligned}$$

Note that the estimate (3.5) for Z_m remains the same. Therefore, the bounds for S_m change only slightly. Namely,

$$\begin{aligned}S_m &\le C\Omega_m + CT^{1/2}(1+T^{1/2})\sum_{0<j<m}\binom{m}{j}S_jS_{m-j}\\ &\quad + CT(1+T^{1/2})\sum_{(j,k)\in I}\binom{m}{j,k}S_jS_kS_{m-j-k}\end{aligned}$$

for $m\in\mathbb{N}$. As a result, the conclusion of Theorem 3.1 remains valid in the 3D case as well.

4. The preservation of the local Gevrey radius

The purpose of this section is to provide a local regularity result for (2.1)–(2.3), showing preservation of the local Gevrey radius. As in the previous section, denote by $d\in\{2,3\}$ the space dimension.

Theorem 4.1. *Let $0 < \rho < R$. Assume that $v_0 \in H^{r+1}(B_R)$, and let $j \in \{1, \dots, d\}$. Also suppose*

$$\nabla v_0 \in G^{(j)}_{s,\delta}(B_R)$$

with the index s satisfying

$$s \geq \max\left\{r+1, \frac{r}{[r](1-d/2r)}\right\} \tag{4.1}$$

and with a Gevrey radius $\delta > 0$. Also, assume that v is a smooth solution of the Euler equations which satisfies

$$\sup_{t\in[0,T_0]} \|v(t)\|_{H^r(B_R)} < \infty \tag{4.2}$$

for some time $T_0 > 0$. Then we have

$$\nabla v, Y \in L^\infty([0,T], G^j_{s,\delta}(B_\rho))$$

for some $T \in (0, T_0]$.

Observe that the Gevrey radius remains unchanged for the positive time, regardless of the size of the radius, even though the solution is assumed to be only locally Gevrey.

Proof of Theorem 4.1. We prove the theorem in the case $d = 2$; the adjustments for $d = 3$ are as in Remark 3.2 above. Using dilation and coverings, it is sufficient to prove the theorem for $\rho = 1$ and $R = 3$.

Fix two radii $1 \leq \rho < \rho + \overline{\rho} \leq 2$, and let η be a smooth nonnegative cut-off function such that

$$\eta(x) = 1, \qquad x \in B_{\rho+\overline{\rho}/3}$$
$$\eta(x) = 0, \qquad x \in \mathbb{R}^d \setminus B_{\rho+2\overline{\rho}/3}$$

with

$$\|\eta\|_{H^r} \leq \frac{C}{\overline{\rho}^r}$$

and

$$\|\nabla\eta\|_{H^r} \leq \frac{C}{\overline{\rho}^{r+1}}$$

for some universal constant $C \geq 1$. Additionally, by (4.2), suppose that

$$\sup_{t\in[0,T_0]} \|v(t)\|_{H^r(B_3)} \leq M_0.$$

Let $m \in \mathbb{N}$. Using the Helmholtz decomposition, we have

$$\begin{aligned}
\|\nabla((\partial_1^m v)\eta)\|_{H^r} &\leq C\|\mathrm{curl}((\partial_1^m v)\eta)\|_{H^r} + C\|\mathrm{div}((\partial_1^m v)\eta)\|_{H^r} \\
&\leq C\|(\mathrm{curl}\,\partial_1^m v)\eta\|_{H^r} + C\|\eta \mathrm{div}\,\partial_1^m v\|_{H^r} \\
&\quad + C\|\partial_1^m v\|_{H^r}\|\nabla\eta\|_{H^r}.
\end{aligned} \tag{4.3}$$

After commuting ∂_1^m with the operators curl and div, we use (2.9)–(2.10) on the right side of (4.3) to get

$$\begin{aligned}
\|\nabla(\eta\partial_1^m v)\|_{H^r} \leq{}& C\|\eta\partial_1^m(\omega_0 + \epsilon_{ij}(\delta_{ik} - Y_i^k)\partial_k v^j)\|_{H^r} \\
&+ C\|\eta\partial_1^m((\delta_{ik} - Y_i^k)\partial_k v^i)\|_{H^r} \\
&+ C\|\partial_1^m v\|_{H^r}\|\nabla\eta\|_{H^r}.
\end{aligned} \tag{4.4}$$

Distributing the gradient on the left side of (4.4) leads to

$$\|\eta\partial_1^m\nabla v\|_{H^r} \leq \|\nabla(\eta\partial_1^m v)\|_{H^r} + \|\partial_1^m v\nabla\eta\|_{H^r}.$$

In addition, we use that H^r is an algebra and apply the Leibniz rule on the right side of (4.4) to obtain

$$\begin{aligned}
\|\eta\partial_1^m\nabla v\|_{H^r} \leq{}& C\|\eta\partial_1^m\omega_0\|_{H^r} + C\|Y - I\|_{H^r}\|\eta\partial_1^m\nabla v\|_{H^r} \\
&+ C\|\eta\partial_1^m(Y-I)\|_{H^r}\|\nabla v\|_{H^r} \\
&+ C\sum_{j=1}^{m-1}\binom{m}{j}\|\partial_1^j(Y-I)\|_{H^r}\|\partial_1^{m-j}\nabla v\|_{H^r}\|\eta\|_{H^r} \\
&+ C\|\partial_1^{m-1}\nabla v\|_{H^r}\|\nabla\eta\|_{H^r}
\end{aligned} \tag{4.5}$$

with all the norms on the right side taken on $B_{\rho+\overline{\rho}}$.

Now, fix $T > 0$ and consider the decreasing sequence of radii

$$\rho_m = 2 - \frac{1}{\overline{C}}\sum_{j=1}^m \frac{1}{j^\kappa}, \qquad m \in \mathbb{N}_0$$

where

$$1 < \kappa < 2$$

to insure convergence and $\overline{C} > 0$ is such that $\rho_m \to 1$ as $m \to \infty$, i.e., $\overline{C} = \sum_{j=1}^\infty 1/j^\kappa$. For $m \in \mathbb{N}_0$, let

$$V_m = \sup_{t\in[0,T]} \|\partial_1^m\nabla v(t)\|_{H^r(B_{\rho_m})} \tag{4.6}$$

$$Z_m = \sup_{t\in[0,T]} \frac{1}{t^{1/2}}\|\partial_1^m(Y(t) - I)\|_{H^r(B_{\rho_m})}. \tag{4.7}$$

Applying (4.5) to (4.6)–(4.7) with $\rho = \rho_m$ and $\overline{\rho} = \rho_{m-1} - \rho_m$ we get

$$\begin{aligned}
\|\eta\partial_1^m\nabla v\|_{H^r} \leq{}& C\|\eta\partial_1^m\omega_0\|_{H^r} + \frac{C}{t^{1/2}}\|\eta\partial_1^m(Y-I)\|_{H^r}t^{1/2}V_0 \\
&+ CT^{1/2}Z_0\|\eta\partial_1^m\nabla v\|_{H^r} \\
&+ CT^{1/2}\sum_{j=1}^{m-1}\binom{m}{j}Z_jV_{m-j}m^{\kappa r} + CV_{m-1}m^{\kappa(r+1)}.
\end{aligned} \tag{4.8}$$

In order to estimate Z_m, we apply ∂_1^m to the Lagrangian evolution (2.8) in the integrated form and then multiply both sides by η. We obtain

$$\begin{aligned}
\eta\partial_1^m(I - Y(t)) &= \eta\partial_1^m\int_0^t Y : \nabla v : Y d\tau \\
&= \eta\int_0^t \sum_{|(j,k)|\leq m}\binom{m}{j,k}\partial_1^j(Y-I) : \partial_1^k\nabla v : \partial_1^{m-j-k}(Y-I)d\tau \\
&\quad + \eta\int_0^t\sum_{j=0}^k\binom{m}{j}\partial_1^j(Y-I) : \partial_1^{m-j}\nabla v d\tau \\
&\quad + \eta\int_0^t\sum_{j=0}^m\binom{m}{j}\partial_1^j\nabla v : \partial_1^{m-j}(Y-I)d\tau + \eta\int_0^t\partial_1^m\nabla v d\tau
\end{aligned} \tag{4.9}$$

for all $t \in [0, T]$. Using the notation (4.6)–(4.7), we arrive at

$$\begin{aligned}
&\frac{1}{t^{1/2}}\|\eta\partial_1^m(Y(t) - I)\|_{H^r} \\
&\quad\leq CT^{3/2}Z_0^2\sup_{t\in[0,T]}\|\eta\partial_1^m\nabla v\|_{H^r} \\
&\qquad + CTZ_0\sup_{t\in[0,T]}\left(\frac{1}{t^{1/2}}\|\eta\partial_1^m(Y-I)\|_{H^r}\right)V_0 \\
&\qquad + CTZ_0\sup_{t\in[0,T]}\|\eta\partial_1^m\nabla v\|_{H^r} \\
&\qquad + CTV_0\sup_{t\in[0,T]}\left(\frac{1}{t^{1/2}}\|\eta\partial_1^m(Y-I)\|_{H^r}\right) \\
&\qquad + CT^{1/2}\sup_{t\in[0,T]}\|\eta\partial_1^m\nabla v\|_{H^r} \\
&\qquad + Cm^{\kappa r}T^{3/2}\sum_{(j,k)\in I}\binom{m}{j,k}Z_jZ_kV_{m-j-k}
\end{aligned}$$

$$+ Cm^{\kappa r} T \sum_{j=1}^{m-1} \binom{m}{j} Z_j V_{m-j} \tag{4.10}$$

where the index set I is as in (3.3). Let

$$S_m = V_m + Z_m.$$

Adding (4.8)–(4.10) we obtain an estimate for S_m, which reads

$$\begin{aligned}
&\|\eta\partial_1^m \nabla v\|_{H^r} + \frac{1}{t^{1/2}}\|\eta\partial_1^m(Y(t)-I)\|_{H^r} \\
&\quad \le C\|\eta\partial_1^m\omega_0\|_{H^r} + C\left(\sup_{t\in[0,T]}\|\eta\partial_1^m\nabla v\|_{H^r}\right. \\
&\qquad \left. + \sup_{t\in[0,T]}\left(\frac{1}{t^{1/2}}\|\eta\partial_1^m(Y-I)\|_{H^r}\right)\right) \\
&\qquad \times \left(S_0T^{1/2} + S_0T + S_0^2T + T^{1/2}\right) \\
&\qquad + Cm^{\kappa r}T^{1/2}(1+T^{1/2})\sum_{j=1}^{m-1}\binom{m}{j}S_jS_{m-j} \\
&\qquad + Cm^{\kappa r}T^{3/2}\sum_{(j,k)\in I}\binom{m}{j,k}S_jS_kS_{m-j-k} + Cm^{\kappa(r+1)}S_{m-1}.
\end{aligned} \tag{4.11}$$

In order to initiate the induction, we recall the initial conditions. We have

$$V_0(0) = \|\nabla v_0\|_{H^r(B_r)} \le M_0$$
$$Z_0(0) = 0.$$

Writing (4.3) for $m = 0$ with a cut-off function η which is identically 1 in a neighborhood of $\overline{B}_2$ and vanishing in the neighborhood of $\mathbb{R}^d \setminus B_3$, we observe that for all $t \in [0, T]$

$$\|\nabla(v\eta)\|_{H^r(B_2)} \le C\Omega_0 + CT^{1/2}Z_0V_0 + C\sup_{t\in[0,T]}\|v(t)\|_{H^r(B_3)}$$

where

$$\Omega_m = \|\partial_1^m\nabla v_0\|_{H^r(B_3)}, \qquad m \in \mathbb{N}_0$$

and assume that (3.1) and (3.2) hold for some radius $\widetilde{\delta} > 0$ and $M_0 > 0$.

Note that (4.9) is still valid for $m = 0$. By continuity we may thus assume

$$Z_0(t) \le CM_0$$
$$V_0(t) \le CM_0$$

whence

$$S_0 \le CM_0.$$

Having S_0 bounded from above, we assume that $0 < T(C, M_0) \le T_0$ is sufficiently small so that

$$S_0T^{1/2} + S_0T + S_0^2T + T^{1/2} \le \frac{1}{C}. \tag{4.12}$$

This implies that all the mth order terms in (4.11) may be absorbed on the left side. Furthermore, as in the proof of Theorem 3.1, we may switch to the index set $\widetilde{I} = \{(j,k) \in \mathbb{N}^2 : 0 < |(j,k)| < m\}$. Together with (4.12), this turns (4.11) to

$$\begin{aligned}
S_m &\le C\|\eta\partial_1^m\omega_0\|_{H^r} + Cm^{\kappa r}T^{1/2}(1+T^{1/2}+M_0T)\sum_{j=1}^{m-1}\binom{m}{j}S_jS_{m-j} \\
&\quad + Cm^{\kappa r}T^{3/2}\sum_{(j,k)\in\widetilde{I}}\binom{m}{j,k}S_jS_kS_{m-j-k} + Cm^{\kappa(r+1)}S_{m-1}.
\end{aligned} \tag{4.13}$$

Note that (4.13) is very similar to (3.9), the only difference being the factors $m^{\kappa r}$ and $m^{\kappa(r+1)}$.

In order to obtain the recursion relation for S_m, we need to estimate the first term on the right side of (4.13). First,

$$\begin{aligned}
\|\eta\partial_1^m\omega_0\|_{H^r} &\le C\|\eta\|_{L^\infty}\|\partial_1^m\omega_0\|_{H^r} \\
&\quad + C\|\eta\|_{H^r}\|\partial_1^m\omega_0\|_{H^r}^{d/2r}\|\partial_1^m\omega_0\|_{L^2}^{1-d/2r} \\
&\le C\|\partial_1^m\omega_0\|_{H^r} + Cm^{\kappa r}\|\partial_1^m\omega_0\|_{H^r}^{d/2r}\|\partial_1^m\omega_0\|_{L^2}^{1-d/2r}.
\end{aligned} \tag{4.14}$$

When m is sufficiently large compared to r, we use the interpolation on the L^2 norm on the right side we get

$$\begin{aligned}
\|\partial_1^m\omega_0\|_{L^2} &\le C\|\partial_1^{[r]+1}(\partial_1^{m-[r]-1}\omega_0)\|_{L^2} \\
&\le C\|\partial_1^{m-[r]-1}\omega_0\|_{H^r}^{1-\alpha}\|\partial_1^{m-[r]}\omega_0\|_{H^r}^{\alpha} \le C\Omega_{m-[r]-1}^{(1-\alpha)}\Omega_{m-[r]}^{\alpha}
\end{aligned}$$

where $\alpha = 1-(r-[r])$ under the assumption $m \ge [r]+1$. If, on the other hand, $m \le [r]$, then we simply use

$$\|\partial_1^m\omega_0\|_{L^2} \le C\Omega_{[r]}. \tag{4.15}$$

Then, combining (4.14)–(4.15), we obtain

$$\|\eta\partial_1^m\omega_0\|_{H^r} \le C\left(\Omega_m + m^{\kappa r}\Omega_m^{d/2r}(\Omega_{m-[r]-1}^{(1-\alpha)}\Omega_{m-[r]}^{\alpha})^{1-d/2r}\right) \tag{4.16}$$

if $m \ge [r]+1$ and

$$\|\eta\partial_1^m\omega_0\|_{H^r} \le C\left(\Omega_m + m^{\kappa r}\Omega_m^{d/2r}\Omega_r^{1-d/2r}\right)$$

otherwise. For the initial datum, we assume

$$\Omega_m\frac{\delta^m}{(m-3)!^s} \le M_0, \qquad m \in \mathbb{N}_0$$

and we now intend to show inductively that

$$\frac{S_j}{(j-3)!^s}\delta^m \le M \tag{4.17}$$

holds for all $m \in \mathbb{N}_0$ with some sufficiently large M. It is sufficient to verify the induction step when $m \ge [r]+1$. First, by (4.13) and (4.16) and assuming (4.17) for $j = 0, 1, \dots, m-1$, we have

$$\begin{aligned}
\frac{S_m}{(m-3)!^s}\delta^m &\le CM_0 + Cm^{\kappa r}M_0^{d/2r}M_0^{1-d/2r}\left(\frac{\delta}{m^{\kappa s}}\right)^{[r](1-d/2r)} \\
&\quad + Cm^{\kappa r}T^{1/2}M^2\sum_{j=1}^{m-1}\binom{m}{j}\frac{(j-3)!^s(m-j-3)!^s}{(m-3)!^s} \\
&\quad + Cm^{\kappa r}T^{3/2}M^3\sum_{\substack{|(j,k)|=2,\\ j>0,k>0}}^{m}\binom{m}{j,k} \\
&\qquad \times \frac{(j-3)!^s(k-3)!^s(m-j-k-3)^s}{(m-3)!^s} \\
&\quad + C\delta Mm^{\kappa(r+1)}\frac{(m-4)!^s}{(m-3)!^s}
\end{aligned} \tag{4.18}$$

if $m \ge [r]+1$, where we also assumed that T is so small that $T^{1/2}+M_0T \le 1/C$. Denote the above sums by I_1 and I_2, respectively. For the first sum, we have

$$\begin{aligned}
I_1 &= \sum_{j=1}^{m-1}\frac{m!}{j!(m-j)!}\left(\frac{(j-3)!(m-j-3)!}{(m-3)!}\right)^s \\
&= \sum_{j=1}^{m-1}\binom{m-6}{j-3}^{1-s}\frac{(j-3)!}{j!}\frac{(m-j-3)!}{(m-j)!}\frac{m!}{(m-6)!^{1-s}(m-3)!^s} \\
&\le Cm^{6-3s}\sum_{j=1}^{m-1}\frac{1}{j^3(m-j)^3} \\
&\le Cm^{3-3s}
\end{aligned}$$

while for the second sum

$$I_2 = \sum_{\substack{(j,k)=2,\\ j>0,k>0}}^{m} \binom{m}{j,k} \left(\frac{(j-3)!(k-3)!(m-j-k-3)!}{(m-3)!} \right)^s$$
$$\le C \sum_{\substack{(j,k)=2,\\ j>0,k>0}}^{m} \binom{m-9}{j-3,k-3}^{1-s} \frac{m^{9-6s}}{j^3k^3(m-j-k)^3}$$
$$\le Cm^{6-6s}$$

where we used the second inequalities in (3.12) and (3.13) respectively. Finally, we may rewrite (4.18) as

$$\frac{S_m}{(m-3)!^s}\delta^m \le CM_0 m^{\kappa r}\left(\frac{\delta}{m^{\kappa s}}\right)^{[r](1-d/2r)} + CT^{1/2}M^2\frac{K_1 m^{\kappa r}}{m^{3s-3}} + CT^{3/2}M^3\frac{K_2 m^{\kappa r}}{m^{6s-6}} + C\delta M m^{\kappa(r+1-s)}. \quad (4.19)$$

In order to bound the four terms on the right side of (4.19), we need to make sure that the powers of m are not positive. Assuming

$$s \ge \frac{r}{[r](1-d/2r)}$$

and

$$s \ge r+1$$

all the exponents of m in (4.19) are indeed negative, and the induction step is in tact if M is sufficiently large. □

5. A counterexample

In this section, we construct a solution $u(x,t)$ of the system (2.1)–(2.3) whose radius of analyticity in Eulerian coordinates is not conserved in time. Compared to [6], the summability assumption is removed and replaced by the uniform bound on Taylor coefficients.

Theorem 5.1. *There exists a divergence-free periodic function u_0 such that*

$$\|u_0\|_{G_{1,1}} < \infty \quad (5.1)$$

while the corresponding solution u satisfies

$$\|u(t)\|_{G_{1,1}} = \infty \quad (5.2)$$

for any $t > 0$.

In order to build such a solution, we appeal to the example introduced in [32,33]. Recall that given two periodic functions f and g on $\mathbb{T} = [-\pi, \pi]$, the function

$$u(x_1,x_2,x_3,t) = (f(x_2), 0, g(x_1 - tf(x_2))) \quad (5.3)$$

is an explicit solution of the Euler equations on $\mathbb{T}^3$, with the initial data

$$u_0(x_1,x_2,x_3,t) = (f(x_2), 0, g(x_1)).$$

The proof of Theorem 5.1 is divided into two steps. We start with a real analytic function in a strip whose holomorphic extension has simple poles at $\pm i$. Under (5.3) these poles move toward the real axis. Since we would like to control the supremum of the Taylor coefficients, it suffices to integrate such a real valued function twice for the holomorphic extension to have a sufficient regularity up to the boundary (rather than four times as in [6]). Then we multiply the resulting function with a Gaussian kernel in order to obtain an integrable version, which is a necessary condition for periodization. After the periodization, we show that the initial datum constructed indeed yields the solution described in Theorem 5.1. The simplicity of the space allows the construction to take place in the phase space, rather than in the Fourier space as in [6].

Proof of Theorem 5.1. First, set

$$f(x) = \sin x \quad (5.4)$$

in (5.3). As f is entire, we need to construct a 2π-periodic function g as described above with $\|g\|_{G_{1,1}} < \infty$ so that we get (5.1). With

$$H(x) = \int_0^x \int_0^w \frac{1}{1+y^2}\,dy\,dw = x\arctan x - \frac{1}{2}\log(1+x^2) \quad (5.5)$$

consider

$$\varphi(x) = H(x)e^{-s^2x^2} \quad (5.6)$$

where $s \in (0,1)$ is a fixed constant to be determined. Note that φ is a smooth function that belongs to $L^1(\mathbb{R})$. Using the Poisson summation on φ, we build a 1-periodic function

$$\Phi(x) = \sum_{m\in\mathbb{Z}} \varphi(x-m).$$

We claim that $\Phi \in G_{1,1}$. First, we have

$$\|\Phi\|_{G_{1,1}} = \sup_{n\ge 0}\frac{1^n}{n!}\left\|\frac{d^n\Phi}{dx^n}\right\|_{H^2([0,1])} \le C\sup_{n\ge 0}\frac{1}{n!}\left\|\frac{d^{n+2}\Phi}{dx^{n+2}}\right\|_{L^2([0,1])}$$
$$= \sup_{n\ge 0}\left\|\sum_{m\in\mathbb{Z}}\frac{1}{n!}\frac{d^{n+2}\varphi}{dx^{n+2}}(x-m)\right\|_{L^2([0,1])}.$$

We now estimate the higher order derivatives of the function $H(x)$ and the Gaussian $e^{-s^2x^2}$. For a fixed $m \in \mathbb{Z}$ and $k \ge 2$, we have

$$\frac{d^k}{dx^k}H(x-m) = \frac{1}{2}\frac{d^{k-2}}{dx^{k-2}}\left(\frac{1}{1-i(x-m)} + \frac{1}{1+i(x-m)}\right)$$
$$= \frac{1}{2}\left(\frac{(i)^{k-2}(k-2)!}{(1-i(x-m))^{k-1}} + \frac{(-i)^{k-2}(k-2)!}{(1+i(x-m))^{k-1}}\right) \quad (5.7)$$

and

$$\frac{d^k}{dx^k}\left(e^{-s^2(x-m)^2}\right) = e^{-s^2(x-m)^2}s^k(-1)^kH_k(sx-sm),$$

where $H_k(x)$ denotes the kth Hermite polynomial. Using the recursion relation on Hermite polynomials and induction we can derive a useful pointwise bound

$$|H_k(x)| \le 4^k(1+|x|)^k k!^{1/2}, \qquad x \in \mathbb{R},$$

which yields

$$\left|\frac{d^k}{dx^k}\left(e^{-s^2(x-m)^2}\right)\right| \le e^{-s^2(x-m)^2}s^k4^k(1+s|x-m|)^k k!^{1/2}. \quad (5.8)$$

Then by the Leibniz rule

$$\frac{d^{n+2}}{dx^{n+2}}\left(H(x-m)e^{-s^2(x-m)^2}\right)$$
$$= \sum_{j=0}^{n+2}\binom{n+2}{j}\frac{d^{n+2-j}}{dx^{n+2-j}}\left(H(x-m)\right)\frac{d^j}{dx^j}\left(e^{-s^2(x-m)^2}\right)$$
$$= J_m + \sum_{j=0}^{n}\binom{n+2}{j}\frac{1}{2}\frac{d^{n-j}}{dx^{n-j}}$$
$$\times\left(\frac{1}{1-i(x-m)} + \frac{1}{1+i(x+m)}\right)\frac{d^j}{dx^j}\left(e^{-s^2(x-m)^2}\right),$$

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where

$$J_m = (n+2)H'(x-m)\frac{d^{n+1}}{dx^{n+1}}\left(e^{-s^2(x-m)^2}\right) + H(x-m) \times \frac{d^{n+2}}{dx^{n+2}}\left(e^{-s^2(x-m)^2}\right).$$

Using (5.7) and (5.8) and canceling in the binomial terms we get

$$\begin{aligned}
&\frac{1}{n!}\left|\frac{d^{n+2}}{dx^{n+2}}\left(H(x-m)e^{-s^2(x-m)^2}\right)\right| \\
&\le \frac{1}{n!}|J_m| + \frac{C}{n!}\frac{e^{-s^2(x-m)^2}(n+2)!}{1+|x-m|^{n+1}}\sum_{j=0}^{n}\frac{1}{(n+2-j)(n+1-j)} \\
&\quad\times \frac{1}{j!^{1/2}}(4s)^j(1+|x-m|)^j(1+s|x-m|)^j \\
&\le \frac{1}{n!}|J_m| + C\frac{e^{-s^2(x-m)^2}n^2}{1+|x-m|^{n+1}}\sum_{j=0}^{n}\frac{1}{(n+2-j)(n+1-j)} \\
&\quad\times \frac{1}{j!^{1/2}}2^j(4s)^j(1+|x-m|^j+s^j|x-m|^{2j})
\end{aligned}$$

where in the second inequality we used $(a+b)^j \le 2^j(a^j+b^j)$ for $a, b \ge 0$ and $j \in \mathbb{N}$. Then we have

$$\begin{aligned}
\|\Phi\|_{G_{1,1}} \le \sup_n \sum_{m\in\mathbb{Z}}\Bigg(&\frac{1}{n!}\|J_m\|_{L^2([0,1])} \\
&+ \Big\|\frac{e^{-s^2(x-m)^2}n^2}{1+(x-m)^{n+1}}\sum_{j=0}^{n}\frac{1}{(n+1-j)^2} \\
&\times \frac{1}{j!^{1/2}}(16s)^j(1+|x-m|^j+s^j|x-m|^{2j})\Big\|_{L^2([0,1])}\Bigg).
\end{aligned} \tag{5.9}$$

Before estimating (5.9), we observe

$$\sum_{m\in\mathbb{Z}} e^{-s^2(x-m)^2}|x-m|^k \le \frac{C}{s^k}\left(\frac{k+1}{2}\right)^{\frac{k+1}{2}}, \qquad x\in[0,1] \tag{5.10}$$

which is obtained by splitting the sum into two parts $|m| \le \sqrt{k/2s^2}$ and $|m| > \sqrt{k/2s^2}$, where it is increasing and decreasing, respectively. Indeed, we have

$$\begin{aligned}
&\sum_{m\in\mathbb{Z},|m|\le\sqrt{k/2s^2}} e^{-s^2(x-m)^2}|x-m|^k \\
&\le \sum_{m\in\mathbb{Z},|m|\le\sqrt{k/2s^2}} |x-m|^k \le 2\int_0^{\sqrt{k/2s^2}}(y+1)^k dy \\
&\le C\left(\frac{1}{s^2}\right)^{k/2}\left(\frac{k+1}{2}\right)^{(k+1)/2}
\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}
&\sum_{m\in\mathbb{Z},|m|>\sqrt{k/2s^2}} e^{-s^2(x-m)^2}|x-m|^k \le 2\int_0^{\infty} y^k e^{-s^2y^2}dy \\
&\le C\left(\frac{1}{\sqrt{2}s}\right)^k k!!
\end{aligned} \tag{5.12}$$

where $(2n)!! = 2^n n!$ and $(2n+1)!! = (2n+1)!/2^n n!$. Using Stirling's formula, we have $n!! \le Cn^{n+1/2}(2/e)^n$ and the inequality (5.10) then follows from adding (5.11) and (5.12).

Confirming (5.10), we go back to bounding (5.9). Splitting the sum into three parts, we start with the term J_m. Using $|H(x-m)| \le C|x-m|$ and $|H'(x-m)| \le C$, we write

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}|J_m| &\le C\sum_{m\in\mathbb{Z}}\Bigg((n+2)\left|\frac{d^{n+1}}{dx^{n+1}}\left(e^{-s^2(x-m)^2}\right)\right| \\
&\quad + (1+|x-m|)\left|\frac{d^{n+2}}{dx^{n+2}}\left(e^{-s^2(x-m)^2}\right)\right|\Bigg) \\
&\le C(n+2)(n+1)!^{1/2}(4s)^n\sum_{m\in\mathbb{Z}}e^{-s^2(x-m)^2}(1+|x-m|) \\
&\quad\times(1+s|x-m|)^{n+2} \\
&\le C(4s)^n n^{3/2}n!^{1/2}\sum_{m\in\mathbb{Z}}e^{-s^2(x-m)^2}2^n(1+|x-m| \\
&\quad + s^{n+2}|x-m|^{n+3}).
\end{aligned}$$

Then, using (5.10) we see that for every $x\in[0,1]$

$$\begin{aligned}
\sum_{m\in\mathbb{Z}}\frac{J_m}{n!} &\le C\frac{n^{3/2}(4^2s)^n}{n!^{1/2}} + C(4^2s)^n n^{3/2}\frac{s^n}{s^{n+3}}\left(\frac{n+4}{2}\right)^{(n+4)/2} \\
&\quad\times\frac{1}{n^{(n/2+1/4)}e^{-n/2}} \\
&\le C\frac{n^{3/2}(4^2s)^n}{n!^{1/2}} + C(4^2s)^n\left(\frac{e}{2}\right)^{n/2}n^{3+1/4}\left(\frac{n+4}{n}\right)^{n/2},
\end{aligned}$$

which converges to 0 as $n \to \infty$ provided $s < (1/16)\sqrt{2/e}$. Next, we estimate the terms in (5.9) the terms in the finite sum for smaller j, i.e.,

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\frac{e^{-s^2(x-m)^2}}{1+|x-m|^{n+1}}\sum_{j=0}^{[n/2]}\frac{n^2}{(n+1-j)^2} \\
&\quad\times\frac{1}{j!^{1/2}}(4^2s)^j(1+|x-m|^j+s^j|x-m|^{2j}) \\
&\le C\sum_{m\in\mathbb{Z}}\frac{e^{-s^2(x-m)^2}}{1+|x-m|^{n+1}}(1+|x-m|^n)\sum_{j=0}^{\infty}\frac{(4^2s)^j}{j!^{1/2}},
\end{aligned}$$

and we note that the right side is uniformly convergent in x and n provided $s < 1/16$. Lastly, we bound the terms with $[n/2] < j \le n$:

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}}\frac{e^{-s^2(x-m)^2}}{1+|x-m|^{n+1}}\sum_{j=[n/2]+1}^{n}\frac{n^2}{(n+1-j)^2} \\
&\quad\times\frac{1}{j!^{1/2}}(4^2s)^j(1+|x-m|^j+s^j|x-m|^{2j}) \\
&\le \sum_{m\in\mathbb{Z}}\frac{e^{-s^2(x-m)^2}}{1+|x-m|^{n+1}}\sum_{j=[n/2]}^{n}\frac{n^2}{j!^{1/2}}(4^2s)^j \\
&\quad\times(1+|x-m|^j+s^j|x-m|^{2j}) \\
&\le n^2\sum_{j=[n/2]+1}^{n}\frac{(4^2s)^j}{j!^{1/2}}\left(\sum_{m\in\mathbb{Z}}e^{-s^2|x-m|^2}\right) \\
&\quad + n^2\sum_{j=[n/2]+1}^{n}\frac{(4^2s^2)^j}{j!^{1/2}}\left(\sum_{m\in\mathbb{Z}}e^{-s^2|x-m|^2}|x-m|^{(2j-n-1)}\right).
\end{aligned} \tag{5.13}$$

Selecting $s < 1/16$, we guarantee that the first sum converges to 0 for all x as $n \to \infty$. In order to bound the second term on the right side of (5.13), we use Stirling's formula on $j!^{1/2}$, and apply (5.10) to the sum in m. We then get

$$Cn^2\sum_{j=[n/2]+1}^{n}\frac{(4^2s^2)^j}{j^{j/2+1/4}e^{-j/2}}\frac{1}{s^{2j-n-1}}\left(j-\frac{n}{2}\right)^{j-n/2}$$

$$
\begin{aligned}
&\leq Cn^2 s^n \sum_{j=1}^{[n/2]} \left(16\sqrt{e}\right)^{j+[n/2]} \frac{j^j}{(j+[n/2])^{j/2+(n+1)/4}} \\
&\leq Cn^2 s^n (16\sqrt{e})^n \sum_{j=1}^{[n/2]} \left(\frac{1}{2}\right)^j \left(\frac{1}{j+[n/2]}\right)^{(n+1)/4-j/2} \\
&\leq Cn^2 s^n (16\sqrt{e})^n \sum_{j=1}^{[n/2]} \left(\frac{1}{2}\right)^j. \qquad (5.14)
\end{aligned}
$$

We again note that if $s < 1/16\sqrt{e}$, then (5.14) converges to 0 uniformly in n. Thus, combining the estimates (5.9)–(5.14) together, which are uniform in $x \in [0, 1]$, we arrive at

$$\|\Phi\|_{G_{1,1}} < \infty \qquad (5.15)$$

provided s is a sufficiently small constant. This concludes the construction needed in the first step. Considering the conditions on s, we fix the value $s = 1/16$.

Now, going back to Eq. (5.3), we define

$$g(x) = \Phi(2\pi x).$$

Together with (5.15) and (5.4), we get

$$\|u_0\|_{G_{1,1}(H^2(\mathbb{T}^3))} < \infty.$$

For (5.2), we proceed by the way of contradiction , and assume that $\|u(t)\|_{G_{1,1}} < \infty$ for some $t \in (0, 1/10]$. Then, define

$$\psi(x_1, x_2) = \partial_{x_1} u_3(x, t) = 2\pi\Phi'(2\pi x_1 - 2\pi t f(x_2))$$

for fixed $t \in (0, 1/10]$. Due to the hypothesis $\|u(t)\|_{G_{1,1}} < \infty$ and the fact that $f(x) = \sin x$ is an entire function, we obtain

$$\sup_{\alpha \geq 0} \frac{\|\partial^\alpha \psi\|_{H^2(\mathbb{T}^2)}}{(|\alpha| + 1)!} < \infty.$$

Note that supremum above is taken over the multi-indices $\alpha \in \mathbb{N}^2$. It follows that ψ is analytic in the closed square $\{|x_1| \leq R, |x_2| \leq R\}$ for any $0 < R < 1$. Then the complex extension defines an analytic function

$$\psi(z_1, z_2) = \psi(x_1 + iy_1, x_2 + iy_2)$$

whose power series converges absolutely in $\{|z_1| < R, |z_2| < R\}$ for $0 < R < 1$. We restrict our attention in $\{|z_1| < 1/2, |z_2| < 1/2\}$ and show that

$$\lim_{y_1 \to -R_t} |\psi(iy_1, i\log 2)| = \infty \qquad (5.16)$$

for $R_t = 1/2\pi - 3t/4 \in (0, 1/2)$. Note that $|z_2| = \log 2 < 1/2$ and $\sin(i\log 2) = 3i/4$, so (5.16) is equivalent to

$$\lim_{y \to -\frac{1}{2\pi}^+} |\Phi'(i2\pi y)| = \lim_{y \to -1^+} |\Phi'(iy)| = \infty \qquad (5.17)$$

which establishes the contradiction to the assertion $\|u(t)\|_{G_{1,1}} < \infty$, and completes the proof. In relation to (5.17), we observe that

$$
\begin{aligned}
\Phi'(z) &= \frac{d}{dz} \sum_{m \in \mathbb{Z}} \varphi(z - m) \\
&= \frac{d}{dz}\left(e^{-z^2/16} H(z)\right) + \sum_{m \in \mathbb{Z}\setminus\{0\}} \varphi'(z - m), \qquad (5.18)
\end{aligned}
$$

where H and φ were defined in (5.5) and (5.6), respectively. Then, noting that $H'(z) = \arctan z$ and $\varphi'(z) = e^{-z^2/16}(H'(z) - (z/8)H(z))$, we set

$$
\begin{aligned}
\mathscr{H}(z) &= H'(z) - \frac{z}{8} H(z) \\
&= \arctan z - \frac{z^2}{8}\arctan z + \frac{z}{16}\log(1 + z^2). \qquad (5.19)
\end{aligned}
$$

Evaluating (5.19) at $z = iy$ and using that $\arctan(iy) = i\,\mathrm{arctanh}\, y$ we get

$$\mathscr{H}(iy) = i\,\mathrm{arctanh}\, y + i\frac{y^2}{8}\mathrm{arctanh}\, y + \frac{iy}{16}\log(1 - y^2)$$

and taking the limit as $y \to -1^+$ above, we obtain

$$\lim_{y \to -1^+} |\mathscr{H}(iy)| = \infty,$$

so the first term in (5.18) is not finite. It remains to show that

$$\lim_{y \to -1^+} \sum_{m \in \mathbb{Z}\setminus\{0\}} |\varphi'(iy - m)| < \infty. \qquad (5.20)$$

The limit above holds, as

$$|\mathscr{H}(iy - m)| \leq P(m) \qquad (5.21)$$

uniformly for $1/2 < |y| < 1$, where P is a polynomial. In order to prove this claim, we use the identity

$$\arctan z = \frac{1}{2i}\left(\log\left|\frac{i - z}{i + z}\right| + i\arg\left(\frac{i - z}{i + z}\right)\right)$$

together with

$$\frac{i - (iy - m)}{i + (iy - m)} = \frac{1 - |y|^2 - m^2 - 2im}{m^2 + (y + 1)^2},$$

and obtain that

$$|\arctan(iy - m)| + \frac{|(iy - m)^2|}{8}|\arctan(iy - m)| \leq C(m^2 + 1), \quad y \in [1/2, 1].$$

Similarly,

$$
\begin{aligned}
\left|\log(1 + (iy - m)^2)\right| &= \big|\log(|1 + m^2 - y^2 - i2ym|) \\
&\quad + i\arg(1 + (iy - m)^2)\big| \leq C(m^2 + 1)
\end{aligned}
$$

uniformly in $y \in [1/2, 1]$. By (5.19), the estimates above confirm the inequality (5.21), and thus in turn (5.20) is established. □

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