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Compatibility conditions to allow some large amplitude WKB analysis for Burger's type systems

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Abstract

In this article, we discuss the problem of finding large amplitude asymptotic expansions for monophase oscillating solutions of the following multidimensional ($d > 1$) Burger's type system:

$$(\diamond) \quad \partial_t \mathbf{u} + (V \circ \mathbf{u} \cdot \nabla_x)\mathbf{u} = 0, \quad \mathbf{u} \in \mathbb{R}^d, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad V \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d).$$

More precisely, we are concerned with families $\{\mathbf{u}^\varepsilon\}_{\varepsilon \in]0,1]}$ made of solutions to $(\diamond)$ and having a development of the form $\mathbf{u}^\varepsilon(t,x) = \mathbf{H}\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right) + O(\varepsilon)$ where the function $\theta \longmapsto \mathbf{H}(t,x,\theta)$ is periodic. In general, due to the formation of shocks, such a construction is not possible on a domain Ω which does not depend on $\varepsilon \in]0,1]$. In this article, we perform a detailed analysis of the restrictions to impose on the profile $\mathbf{H}$ and on the phase Φ in order to remedy this. Among these compatibility conditions, we can isolate some new interesting system of nonlinear partial differential equations. We explain how to solve them and we describe how the underlying structure is propagated through the evolution equation.

Keywords: Burger and Euler equations; Nonlinear geometric optics

1. Introduction

Let $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and

$$|x| := \left(\sum_{j=1}^{d} x_j^2\right)^{\frac{1}{2}}, \quad \partial_j := \frac{\partial}{\partial x_j}, \quad \partial_\theta := \frac{\partial}{\partial \theta}.$$

Let $(T, \mathbf{V}, r) \in (\mathbb{R}_+^*)^3$. Work on the domain

$$\Omega^{\mathrm{T}} := \left\{(t,x) \in [0,T] \times \mathbb{R}^d; |x| + \mathbf{V}t \leq r\right\}, \quad d \in \mathbb{N} \setminus \{0,1\}.$$

Select a function $V \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ and consider the Burger's type system

$$\partial_t \mathbf{u} + (V \circ \mathbf{u} \cdot \nabla_x)\mathbf{u} = 0, \quad \mathbf{u} \in \mathbb{R}^d, \quad (t,x) \in \Omega^{\mathrm{T}}. \tag{1}$$

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Associate (1) with a family of initial data

$$\mathbf{u}^\varepsilon(0,x) = h^\varepsilon(x) = H\left(x, \frac{\varphi(x)}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \in]0,1] \tag{2}$$

defined on the ball $B(0,r] := \{x \in \mathbb{R}^d;\ |x| \le r\}$, built with

$$H(x,\theta) \in \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R}^d), \quad \varphi(x) \in \mathcal{C}^1(B(0,r]; \mathbb{R}), \quad \mathbb{T} := \mathbb{R}/\mathbb{Z}$$

and consisting of large amplitude high frequency monophase oscillating waves, which implies a nontrivial (main) profile

$$\exists (x,\theta) \in B(0,r] \times \mathbb{T}; \quad \partial_\theta W(x,\theta) \neq 0, \quad W := V \circ H \tag{3}$$

and a non stationary phase

$$\nabla_x \varphi(x) \neq 0, \quad \forall x \in B(0,r]. \tag{4}$$

To describe more precisely the expressions involved in (2), select a function

$$\begin{aligned} H : [0,1] \times B(0,r] \times \mathbb{T} &\longrightarrow \mathbb{R}^d \\ (\varepsilon, x, \theta) &\longmapsto H^\varepsilon(x,\theta) \end{aligned}$$

which is smooth with respect to the parameter $\varepsilon \in [0,1]$

$$H \in \mathcal{C}^\infty\left([0,1]; \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R}^d)\right)$$

and whose Taylor expansion near $\varepsilon = 0$ is noted as

$$H^\varepsilon(x,\theta) := H(x,\theta) + \sum_{j=1}^{m} \varepsilon^j H^j(x,\theta) + O(\varepsilon^{m+1}), \quad m \in \mathbb{N}^*. \tag{5}$$

Define:

$$h^\varepsilon(x) := H^\varepsilon\left(x, \frac{\varphi(x)}{\varepsilon}\right), \quad W^\varepsilon(x,\theta) := V \circ H^\varepsilon(x,\theta). \tag{6}$$

Associate (1) with the family of initial data $\{h^\varepsilon\}_{\varepsilon \in]0,1]}$. The evolution equation (1) is a quasilinear (diagonal) system of hyperbolic equations. The speed of propagation is finite. More precisely, it can be uniformly controlled by

$$\mathbb{R} \ni \mathbf{V} := \left\{\sup |V \circ H^\varepsilon(x,\theta)|;\ (\varepsilon,x,\theta) \in [0,1] \times B(0,r] \times \mathbb{T}\right\}.$$

Standard results [7] guarantee the existence of $T^\varepsilon > 0$ such that the Cauchy problem (1) and (2) has a local $\mathcal{C}^1$ solution $\mathbf{u}^\varepsilon(t,x)$ on the truncated cone $\Omega^{\mathrm{T}^\varepsilon}$. In the context of (1), the limitations on T^ε are due to the formation of shocks. The size of T^ε can be estimated very precisely [1,3,8] in terms of h^ε. In general, this yields

$$\limsup_{\varepsilon \longrightarrow 0} T^\varepsilon = 0. \tag{7}$$

In this article, we exhibit solutions $\mathbf{u}^\varepsilon$ on a fixed domain Ω^{T} (with $T > 0$) having the asymptotic description

$$\mathbf{u}^\varepsilon(t,x) = \mathbf{H}\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \in]0,1]. \tag{8}$$

The main novelty in comparison with usual works [5] in WKB analysis is the size of the involved oscillations. Indeed, in a quasilinear context such as (1), the standard regime is given by *weakly nonlinear geometric optics* [4], which means to consider expansions of the following form

$$\mathbf{u}^\varepsilon(t,x) = \mathbf{u}(t,x) + \varepsilon \mathbf{H}^1\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right) + O(\varepsilon), \quad \varepsilon \in]0,1]. \tag{9}$$

Of course, to deal with (8) in place of (9) requires to manage much stronger nonlinear phenomena. In particular, the interplay between the phase Φ and the profile $\mathbf{H}$ is reinforced.

In fact, the construction can be faced only if the expressions $\varphi := \Phi_{|t=0}$ and $H := \mathbf{H}_{|t=0}$ satisfy very special restrictions. The corresponding constraints in the case of the dimension $d = 2$ are brought out in the recent contribution [3]. The aim of the present paper is precisely to generalize the discussion when $d > 2$ and to study more deeply the structure to impose on φ and H.

• In the Section 2, we exhibit (Proposition 5) necessary and sufficient compatibility conditions on $\varphi(x)$ and $W(x,\theta) := V \circ H(x,\theta)$ in order to guarantee that

$$\liminf_{\varepsilon \longrightarrow 0} T^{\varepsilon} = \tilde{T} > 0. \tag{10}$$

From these compatibility conditions, we can isolate some interesting system of nonlinear partial differential equations, which we introduce below. Let $\mathbf{u} = {}^t(\mathbf{u}_1, \dots, \mathbf{u}_d) \in \mathbb{R}^d$. Note $\mathbf{u}^{\perp}$ or ${}^t\,\mathbf{u}^{\perp}$ the hyperplane of $\mathbb{R}^d$ composed with the directions orthogonal to the vector $\mathbf{u}$, that is

$$\mathbf{u}^{\perp} \equiv {}^t\,\mathbf{u}^{\perp} := \left\{ \mathbf{v} = {}^t(\mathbf{v}_1, \dots, \mathbf{v}_d) \in \mathbb{R}^d;\ {}^t\,\mathbf{v} \cdot \mathbf{u} = \sum_{j=1}^{d} \mathbf{v}_j \mathbf{u}_j = 0 \right\}.$$

Consider the orthogonal projector Π_F from $\mathbb{R}^d$ onto the vector space F, that is the operator Π_F defined by the conditions

$$\mathbf{u} = \Pi_F \mathbf{u} + (I - \Pi_F)\mathbf{u}, \quad \Pi_F \mathbf{u} \in F, \quad (I - \Pi_F)\mathbf{u} \in F^{\perp}.$$

Select $W \in \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R}^d)$. The symbol $D_x W(x,\theta)$ is for the Jacobian matrix

$$D_x W(x,\theta) = \big(\partial_j W_i(x,\theta)\big)_{1 \le i,j \le d}, \qquad W(x,\theta) = {}^t(W_1, \dots, W_d).$$

Definition 1. The couple

$$(\varphi, W) \in \mathcal{C}^2(B(0,r]; \mathbb{R}) \times \mathcal{C}^2(B(0,r] \times \mathbb{T}; \mathbb{R}^d)$$

is said to be *well prepared* if it satisfies the following system

$$\begin{cases} \partial_\theta W(x,\theta) \subset \nabla\varphi(x)^{\perp} \\ \Pi_{\partial_\theta W(x,\theta)^{\perp}} D_x W(x,\theta) \Pi_{\nabla\varphi(x)^{\perp}} = 0 \end{cases}, \qquad \forall (x,\theta) \in B(0,r] \times \mathbb{T}. \tag{11}$$

As explained before, the study of (11) is the main motivation for the present article. Indeed, the structure of (11) is new and interesting. It is not a usual quasilinear system because it is made of fully nonlinear constraints on the derivatives $\partial_j W_i$, $\partial_\theta W_i$ and $\partial_j \varphi$. It extends to the case $d \ge 3$ preliminary conditions which have been emphasized (only when $d = 2$) in the recent contribution [3].

• In the Section 3, we work under natural assumptions on φ and W. In this framework, we succeed in classifying all the solutions of (11). The fact that such a complete discussion is available is very surprising. At all events, this confirms the coherence of (11).

The first observation is that any phase φ involved in (11) inherits some affine structure. Its level surfaces must be spanned by pieces of vector spaces (see Lemmas 9 and 10). This geometrical particularity seems to always play an important part when dealing with phase involved in a supercritical WKB calculus, as here. Once φ is determined, it becomes possible to identify all the profiles $W(x,\theta)$ which are subjected to (11). This is done in Proposition 11. Quite a lot freedom is available in the construction of $W(x,\theta)$.

The function $W(x,\theta)$ can be put in the form

$$W(x,\theta) = W_{\parallel}(\varphi(x), \psi(x,\theta)) + W_{\perp}(\varphi(x))$$

where $W_{\parallel} \in \mathcal{C}^1(\mathbb{R}^2; \mathbb{R}^d)$ and $W_{\perp} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^d)$ are conveniently well-polarized vector fields whereas $\psi \in \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R})$ is any scalar function. Define

$$\langle W \rangle(x) \equiv \bar{W}(x) := \int_{\mathbb{T}} W(x,\theta)\,\mathrm{d}\theta, \qquad W^*(x,\theta) := W(x,\theta) - \bar{W}(x).$$

The construction of large amplitude oscillating solutions to system (1) – or to variants of system (1) – is a delicate problem which has recently attracted some attention. The article [6] and the related contributions are mainly concerned with *time* oscillations. In the continuity of the works [1–3], we are faced here with the case of *spatial* oscillations.

According to Section 2, any family $\{h^{\varepsilon}\}_{\varepsilon} \in \mathcal{C}^1\left(B(0,r]; \mathbb{R}^d\right)^{]0,1]}$ issued from a well prepared couple (φ, W) leads to a family $\{\mathbf{u}^{\varepsilon}\}_{\varepsilon}$ which is composed of $\mathcal{C}^1$ solutions $\mathbf{u}^{\varepsilon}$ of (1) on $\Omega^{\tilde{T}}$. Now, the question is to determine the asymptotic behaviour of $\{\mathbf{u}^{\varepsilon}\}_{\varepsilon}$ when ε goes to 0. In particular, we want to understand how the constraint (11) is propagated through the evolution equation (1).

• Explanations are given in the Section 4. They can be obtained just by looking at the simple wave solutions of (1).

Theorem 2. *Suppose that the couple*

$$(\varphi, W) \in \mathcal{C}^2(B(0,r]; \mathbb{R}) \times \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R}^d), \qquad W := V \circ H$$

is well prepared. Then, the Cauchy problem consisting in the (apparently overdetermined) system

$$\begin{cases} \partial_t \mathbf{H} + V \circ \mathbf{H} \cdot \nabla_x \mathbf{H} = 0, \\ \partial_t \Phi + \langle V \circ \mathbf{H} \rangle \cdot \nabla_x \Phi = 0, \\ (V \circ \mathbf{H})^* \cdot \nabla_x \Phi = 0, \end{cases} \tag{12}$$

associated with the initial data

$$\mathbf{H}(0, x, \theta) = H(x, \theta), \quad \Phi(0, x) = \varphi(x) \tag{13}$$

has a unique solution on $\Omega^{\mathrm{T}} \times \mathbb{T}$ for some $T > 0$. For all $\varepsilon \in]0, 1]$, the simple wave $\mathbf{u}^\varepsilon(t, x) := \mathbf{H}\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right)$ is a solution of (1) on Ω^{T}. Moreover, for all $t \in [0, T]$, the trace $(\Phi(t, \cdot), \mathbf{H}(t, \cdot))$ is still subjected to (11).

At the time $t = 0$, it is also possible to take into account (small) perturbations of $H\left(x, \frac{\varphi(x)}{\varepsilon}\right)$. For instance, we can select

$$h^\varepsilon(x) = H^\varepsilon\left(x, \frac{\varphi(x)}{\varepsilon}\right), \quad \varepsilon \in]0, 1]$$

where $H^\varepsilon(x, \theta)$ is like in (5). Again, the discussion of the Section 2 indicates that corresponding $\mathcal{C}^1$ solutions $\mathbf{u}^\varepsilon(t, x)$ of (1) are still available on Ω^{T}. When ε goes to 0, the expression $\mathbf{u}^\varepsilon(t, x)$ remains close (in a convenient sense) to the simple wave $\mathbf{H}\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right)$. This result can be proved by adapting and extending the method presented in [3]. The related analysis will not be developed here.

2. Analysis of the compatibility conditions

Introduce the curves $t \longmapsto (X(t; x, \lambda), \Lambda(t; x, \lambda))$ associated with the integration of (1) along the relevant characteristics. They are defined by the ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X = V(\Lambda), \quad X(0; x, \lambda) = x, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} \Lambda = 0, \quad \Lambda(0; x, \lambda) = \lambda. \end{cases} \tag{14}$$

The corresponding solutions are

$$X(t; x, \lambda) = x + t V(\lambda), \quad \Lambda(t; x, \lambda) = \lambda. \tag{15}$$

Define

$$\mathbb{X}^\varepsilon(t, x) := X\left(t; x, h^\varepsilon(x)\right) = x + t W^\varepsilon\left(x, \frac{\varphi(x)}{\varepsilon}\right), \quad W^\varepsilon := V \circ H^\varepsilon. \tag{16}$$

Any smooth $\mathcal{C}^1$ solution of (1)–(2) must be subjected to the relation

$$\mathbf{u}^\varepsilon\left(t, \mathbb{X}^\varepsilon(t, x)\right) = \mathbf{u}^\varepsilon\left(t, x + t V \circ h^\varepsilon(x)\right) = h^\varepsilon(x). \tag{17}$$

Fix $\varepsilon \in]0, 1]$. For t small enough, say for $t \in [0, \tilde{T}^\varepsilon]$ with $\tilde{T}^\varepsilon > 0$, the implicit theorem guarantees that the application

$$\begin{aligned} &\mathbb{X}_t^\varepsilon : B(0, r] \longrightarrow \mathbb{X}^\varepsilon\,(t, B(0, r]) \\ &x \longmapsto \mathbb{X}^\varepsilon(t, x) \end{aligned}$$

is a $\mathcal{C}^1$ diffeomorphism. Then, due to the definition of the maximal speed of propagation $\mathbf{V}$, any point (t, x) contained in $\Omega^{\tilde{T}^\varepsilon}$ is sure to be realized as $(t, x) = (t, \mathbb{X}^\varepsilon(t, y))$ for some unique $y \in B(0, r]$. We can define

$$\mathbf{u}^\varepsilon(t, x) := h^\varepsilon \circ (\mathbb{X}_t^\varepsilon)^{-1}(x), \quad (t, x) \in \Omega^{\tilde{T}^\varepsilon} \tag{18}$$

which yields a $\mathcal{C}^1$ solution on $\Omega^{\tilde{T}^\varepsilon}$ of the Cauchy problem (1) and (2). The relation (18) implies that

$$D_x \mathbf{u}^\varepsilon(t, x) := D_x h^\varepsilon \circ (\mathbb{X}_t^\varepsilon)^{-1}(x) \mathrm{Co}\left[D_x \mathbb{X}^\varepsilon(t, x)\right] / \det D_x \mathbb{X}^\varepsilon(t, x) \tag{19}$$

where $\mathrm{Co}[M]$ stands for the co-matrix of M. We have

$$D_x \mathbb{X}^\varepsilon(t, x) = \varepsilon^{-1} t \partial_\theta W^\varepsilon\left(x, \frac{\varphi(x)}{\varepsilon}\right) \otimes {}^t\nabla \varphi(x) + \mathrm{I} + t D_x W^\varepsilon\left(x, \frac{\varphi(x)}{\varepsilon}\right), \quad W^\varepsilon := V \circ H^\varepsilon \tag{20}$$

where we adopt the following convention

$$u \otimes v = (u_i v_j)_{1\le i,j\le d}, \quad u = {}^t(u_1,\ldots,u_d), \quad v = {}^t(v_1,\ldots,v_d).$$

Classical results – see for instance [7] – assert that a $\mathcal{C}^1$ solution $\mathbf{u}^\varepsilon(t,x)$ on Ω^{T} can be extended in time as long as the matrix $D_x\mathbf{u}^\varepsilon(t,x)$ is bounded. In view of the formula (19), to recover a $\mathcal{C}^1$ solution $\mathbf{u}^\varepsilon(t,x)$ on Ω^{T}, it is necessary and sufficient to have

$$\det D_x\mathbb{X}^\varepsilon(t,x) > 0, \quad \forall (t,x) \in \Omega^{\mathrm{T}}.$$

Therefore, the life span of a $\mathcal{C}^1$ solution on a domain of propagation is bounded below by

$$T^\varepsilon := \sup\left\{T > 0; \det D_x\mathbb{X}^\varepsilon(t,x) > 0, \forall (t,x) \in [0,T]\times B(0,r[\right\}.$$

In general, due to the presence in (20) of the (singular) term with ε^{-1} in factor, only (7) can be asserted. Now, the opposite situation is still possible providing that the family $\{h^\varepsilon\}_\varepsilon$ is conveniently adjusted. This situation is distinguished below.

Definition 3 (*See* (6) *and* (16) *for the Definitions of* h^ε *and* $\mathbb{X}^\varepsilon$). The family $\{h^\varepsilon\}_\varepsilon$ is said to be *compatible* if there exists $T > 0$ and $c > 0$ such that

$$\det D_x\mathbb{X}^\varepsilon(t,x) \ge c > 0, \quad \forall (t,x,\varepsilon) \in [0,T]\times B(0,r]\times]0,1]. \tag{21}$$

The preceding discussion can be summarized by the following statement.

Proposition 4 (*See* (6) *for the Definition of* h^ε). *Suppose that the family* $\{h^\varepsilon\}_\varepsilon$ *is compatible. Then, for all* $\varepsilon \in]0,1]$, *the expression* $\mathbf{u}^\varepsilon(t,x)$ *defined through* (18) *is a* $\mathcal{C}^1$ *solution on* Ω^{T} *of the Cauchy problem* (1) *and* (2).

Our aim now is to transcribe (21) in terms of constraints to impose on $\varphi(x)$ and $W(x,\theta)$. To this end, introduce

$$\mathcal{V} := \{(x,\theta) \in B(0,r]\times\mathbb{T};\ \partial_\theta W(x,\theta) \neq 0\}, \quad W := V \circ H. \tag{22}$$

We assume (3), that is $\mathcal{V} \neq \emptyset$.

Proposition 5 (*See* (6) *for the Definitions of* h^ε *and* W^ε). *The family* $\{h^\varepsilon\}_\varepsilon$ *can be compatible only if:*

$$^t\nabla\varphi(x)\cdot\partial_\theta W(x,\theta) = 0, \quad \forall (x,\theta) \in B(0,r]\times\mathbb{T} \tag{23}$$

where we recall that

$$W(x,\theta) = W^0(x,\theta) = V \circ H(x,\theta).$$

Proof. The reasoning is based on the identity (20) which can be formulated as

$$\varepsilon D_x\mathbb{X}^\varepsilon(t,x) = M^0\left(t,x,\frac{\varphi(x)}{\varepsilon}\right) + \varepsilon M^1\left(t,x,\frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^2 t R^\varepsilon\left(t,x,\frac{\varphi(x)}{\varepsilon}\right)$$

where

$$M^0(t,x,\theta) := t\partial_\theta W(x,\theta)\otimes {}^t\nabla\varphi(x),$$
$$M^1(t,x,\theta) := \mathrm{I} + tD_x W(x,\theta) + t\partial_\theta\left[D_{\mathbf{u}}V\left(H^0(x,\theta)\right)H^1(x,\theta)\right]\otimes {}^t\nabla\varphi(x),$$

whereas the matrix $R^\varepsilon(t,x,\theta)$ is a continuous function of all the variables $(\varepsilon,t,x,\theta) \in [0,1]\times\mathbb{R}\times B(0,r]\times\mathbb{T}$. Assume that the restriction (21) is satisfied for some $T > 0$ and some $c > 0$. We start by showing

$$^t\nabla\varphi(x)\cdot\partial_\theta W(x,\theta) \ge 0, \quad \forall (x,\theta) \in \mathcal{V}. \tag{24}$$

To this end, we argue by contradiction. We suppose that we can find a point $(\bar{x},\bar{\theta}) \in \mathcal{V}$ such that

$$^t\nabla\varphi(\bar{x})\cdot\partial_\theta W(\bar{x},\bar{\theta}) < 0. \tag{25}$$

This information allows us to express the matrices $M^0(t,\bar{x},\bar{\theta})$ and $M^1(t,\bar{x},\bar{\theta})$ in a basis of $\mathbb{R}^d$ having the form $(e_1,e_2,\ldots,e_d)$ where $e_1 := \partial_\theta W(\bar{x},\bar{\theta})$ and where $(e_2,\ldots,e_d)$ is a basis of $\nabla\varphi(\bar{x})^\perp$.

In this special basis, the matrices M^0 and M^1 look like

$$M^0 = \begin{pmatrix} t\,{}^t\nabla\varphi\cdot\partial_\theta W & 0 & \ldots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad M^1 = \begin{pmatrix} m^1_{11} & \ldots & m^1_{1d} \\ m^1_{21} & \ldots & m^1_{2d} \\ \vdots & & \vdots \\ m^1_{d1} & \ldots & m^1_{dd} \end{pmatrix}.$$

It follows that

$$\begin{aligned}\det D_x \mathbb{X}^\varepsilon(t,\bar{x}) &= \varepsilon^{-d} \det \left[M^0 \left(t, \bar{x}, \frac{\varphi(\bar{x})}{\varepsilon} \right) + \varepsilon M^1 \left(t, \bar{x}, \frac{\varphi(\bar{x})}{\varepsilon} \right) + O(\varepsilon^2) \right] \\ &= \varepsilon^{-1} t\, {}^t\nabla\varphi(\bar{x}) \cdot \partial_\theta W \left(\bar{x}, \frac{\varphi(\bar{x})}{\varepsilon} \right) \det M^\flat \left(t, \bar{x}, \frac{\varphi(\bar{x})}{\varepsilon} \right) + O(1)\end{aligned}$$

with

$$M^\flat = M^\flat(t,\bar{x},\bar{\theta}) = \begin{pmatrix} m^1_{22} & \dots & m^1_{2d} \\ \vdots & & \vdots \\ m^1_{d2} & \dots & m^1_{dd} \end{pmatrix}.$$

When $t = 0$, we have $M^1(0,\bar{x},\bar{\theta}) = \mathrm{I}$ so that $M^\flat = \mathrm{I}_{\mathbb{R}^{d-1}}$ and $\det M^\flat = 1$. By continuity, for t small enough (say $t \in [0,\tilde{T}]$ with $\tilde{T} > 0$), it remains

$$\det M^\flat \left(t, \bar{x}, \frac{\varphi(\bar{x})}{\varepsilon} \right) \geq \frac{1}{2}, \quad \forall (t,\varepsilon) \in [0,\tilde{T}] \times]0,1].$$

Choose $t \in]0,\tilde{T}]$ and a sequence $\{\varepsilon_n\}_n \in]0,1]^{\mathbb{N}}$ tending to 0 and such that

$$\forall n \in \mathbb{N}, \quad \exists k_n \in \mathbb{Z}; \quad \varphi(\bar{x}) = \varepsilon_n(\bar{\theta} + 2k_n\pi).$$

Then, by construction, we have

$$\exists C \in \mathbb{R}; \quad \det D_x \mathbb{X}^{\varepsilon_n}(t,\bar{x}) \leq \frac{t}{2\varepsilon_n}\, {}^t\nabla\varphi(\bar{x}) \cdot \partial_\theta W(\bar{x},\bar{\theta}) + C, \quad \forall n \in \mathbb{N}.$$

For n large enough, the right hand side becomes negative. This is not compatible with (21). This means that the case (25) must be excluded. Now, because the function $\theta \longmapsto W(x,\theta)$ is periodic, we have

$$\int_0^1 {}^t\nabla\varphi(x) \cdot \partial_\theta W(x,\theta) \mathrm{d}\theta = {}^t\nabla\varphi(x) \cdot W(x,1) - {}^t\nabla\varphi(x) \cdot W(x,0) = 0.$$

Combining this with (24), we see that the restriction (23) is necessary. $\square$

Below, up to the end of the proof of Proposition 6, we select $(x,\theta) \in \mathcal{V}$ such that ${}^t\nabla\varphi(x) \cdot \partial_\theta W(x,\theta) = 0$. Introduce the notations

$$\tilde{e}_1 := \partial_\theta W(x,\theta), \qquad \tilde{e}_d := {}^t\nabla\varphi(x), \qquad {}^t\tilde{e}_1 \cdot \tilde{e}_d = 0.$$

We can complete $\tilde{e}_1$ and $\tilde{e}_d$ into some orthonormal basis $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{d-1}, \tilde{e}_d)$ of $\mathbb{R}^d$. In this special basis, the matrix $\mathrm{I}+t D_x W(x,\theta)$ looks like:

$$\mathrm{I} + t D_x W(x,\theta) = \begin{pmatrix} \tilde{m}^1_{11} & \dots & \tilde{m}^1_{1(d-1)} & \tilde{m}^1_{1d} \\ \tilde{m}^1_{21} & \dots & \tilde{m}^1_{2(d-1)} & \tilde{m}^1_{2d} \\ \vdots & & \vdots & \\ \tilde{m}^1_{d1} & \dots & \tilde{m}^1_{d(d-1)} & \tilde{m}^1_{dd} \end{pmatrix}.$$

We can extract the $(d-1) \times (d-1)$ matrix:

$$\mathcal{L}(t,x,\theta) = \begin{pmatrix} \tilde{m}^1_{21} & \dots & \tilde{m}^1_{2(d-1)} \\ \vdots & & \vdots \\ \tilde{m}^1_{d1} & \dots & \tilde{m}^1_{d(d-1)} \end{pmatrix}.$$

Observe that $\mathcal{L}$ is the realisation (in some specific basis) of the linear application:

$$\begin{aligned} &\mathcal{L} : \nabla\varphi(x)^\perp \longrightarrow \partial_\theta W(x,\theta)^\perp \\ &\mathbf{u} \longmapsto \Pi_{\partial_\theta W(x,\theta)^\perp} (\mathrm{I} + t\, D_x W(x,\theta))\, \mathbf{u}. \end{aligned}$$

Proposition 6. *The family $\{h^\varepsilon\}_\varepsilon$ can be compatible only if there is $T > 0$ such that for all $t \in [0,T]$, we have:*

$$(-1)^d \det \mathcal{L}(t,x,\theta) \geq 0. \tag{26}$$

Proof. Assume again that the restriction (21) is satisfied for some $T > 0$ and some $c > 0$. We already know that (23) is verified. In the basis $(\tilde{e}_1, \dots, \tilde{e}_d)$ of $\mathbb{R}^d$, the matrices M^0 and M^1 take the form

$$M^0 = \begin{pmatrix} 0 & \cdots & 0 & t|\nabla\varphi|^2 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad M^1 = \begin{pmatrix} m^1_{11} & \cdots & m^1_{1d} \\ m^1_{21} & \cdots & m^1_{2d} \\ \vdots & & \vdots \\ m^1_{d1} & \cdots & m^1_{dd} \end{pmatrix}.$$

It follows that

$$\det D_x \mathbb{X}^\varepsilon(t,x) = \varepsilon^{-1}(-1)^d t|\nabla\varphi(x)|^2 \det M^\sharp\left(t, x, \frac{\varphi(x)}{\varepsilon}\right) + 1 + t g^\varepsilon\left(t, x, \frac{\varphi(x)}{\varepsilon}\right)$$

with

$$M^\sharp(t,x,\theta) = \begin{pmatrix} m^1_{21} & \cdots & m^1_{2(d-1)} \\ \vdots & & \vdots \\ m^1_{d1} & \cdots & m^1_{d(d-1)} \end{pmatrix} \equiv \Pi_{\partial_\theta W(x,\theta)^\perp} M^1 \Pi_{\nabla\varphi(x)^\perp}$$

whereas the scalar application $g^\varepsilon(t,x,\theta)$ is a continuous function of all the variables $(\varepsilon, t, x, \theta) \in [0,1] \times \mathbb{R} \times B(0,r] \times \mathbb{T}$. Observe that

$$\left[\mathbf{u} \otimes {}^t\nabla\varphi(x)\right] \mathbf{v} = 0, \quad \forall (\mathbf{u}, \mathbf{v}) \in \mathbb{R}^d \times \nabla\varphi(x)^\perp.$$

Therefore, the expression of $M^\sharp$ can be simplified according to

$$M^\sharp(t,x,\theta) = \mathcal{L}(t,x,\theta) \equiv \Pi_{\partial_\theta W(x,\theta)^\perp} \left(\mathrm{I} + t D_x W(x,\theta)\right) \Pi_{\nabla\varphi(x)^\perp}.$$

Follow the argument of the preceding proof, using a well adjusted sequence $\{\varepsilon_n\}_n$, in order to extract the necessary condition

$$(-1)^d \det M^\sharp(t,x,\theta) \geq 0, \quad \forall (t,x,\theta) \in [0,T] \times B(0,r] \times \mathbb{T}$$

which is exactly (26). □

Remark 2.1. In the basis $(\tilde{e}_1, \dots, \tilde{e}_d)$, we can get the decomposition

$$\mathcal{L}(t,x,\theta) = \mathcal{L}_0 + t\tilde{\mathcal{L}}(x,\theta), \quad \mathcal{L}_0 := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $\tilde{\mathcal{L}}(x,\theta) \equiv \Pi_{\partial_\theta W(x,\theta)^\perp} D_x W(x,\theta) \Pi_{\nabla\varphi(x)^\perp}$. This special structure implies the existence of coefficients $\alpha_j(x,\theta)$ such that

$$(-1)^d \det \mathcal{L}(t,x,\theta) = \sum_{j=1}^{d-1} \alpha_j(x,\theta) t^j.$$

Noting

$$J(x,\theta) := \begin{cases} \min \mathcal{J} & \text{if } \mathcal{J} := \left\{j;\, \alpha_j(x,\theta) \neq 0\right\} \neq \emptyset, \\ d-1 & \text{if } \mathcal{J} = \emptyset, \end{cases}$$

the condition (26) is equivalent to the restriction

$$\alpha_{J(x,\theta)}(x,\theta) \geq 0, \quad \forall (x,\theta) \in B(0,r] \times \mathbb{T}. \tag{27}$$

On the one hand, especially when $d \gg 1$, the conditions (27) can be technically difficult to deal with. On the other hand, nothing guarantees that they are intrinsic. Instead of looking at (27), we will consider

$$\Pi_{\partial_\theta W(x,\theta)^\perp} D_x W(x,\theta) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (x,\theta) \in \mathcal{V}. \tag{28}$$

This relation is clearly intrinsic and, if it is satisfied, we are sure that

$$\det \mathcal{L}(t,x,\theta) = \det \mathcal{L}_0 = 0. \quad \triangle$$

We can summarize the preceding discussion by:

Proposition 7. *Suppose that the relations (23) and (28) are verified. Then, the family $\{h^\varepsilon\}_\varepsilon$ is compatible.*

Proof. Under conditions (23) and (28), it remains the case that

$$\det D_x \mathbb{X}^\varepsilon(t,x) = 1 + t g^\varepsilon \left(t, x, \frac{\varphi(x)}{\varepsilon} \right), \quad g^\varepsilon \in C^0\left([0,1] \times \mathbb{R} \times B(0,r] \times \mathbb{T}; \mathbb{R}\right).$$

In particular, we get:

$$\det D_x \mathbb{X}^\varepsilon(t,x) \geq 1 - C(T)t, \quad \forall (t,x,\varepsilon) \in [0,T] \times B(0,r] \times]0,1]$$

where the function $T \longmapsto C(T)$ is increasing. Now, it suffices to choose $T > 0$ small enough to recover (21). $\square$

Remark 2.2. Suppose that $V : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a $\mathcal{C}^1$ diffeomorphism. Then, it is equivalent to solve (1) or

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla_x)\mathbf{w} = 0, \quad \mathbf{w} := V \circ \mathbf{u} \tag{29}$$

completed with the initial data

$$\mathbf{w}(0,x) = W\left(x, \frac{\varphi(x)}{\varepsilon} \right), \quad \varepsilon \in]0,1]. \tag{30}$$

The system (11) can also be interpreted as a compatibility condition in order to solve the Cauchy problem (29) and (30) in the class of $\mathcal{C}^1$ solutions, locally in time, on some domain Ω^{T} with $T > 0$ independent of $\varepsilon \in]0,1]$. This interpretation explains why the relevant constraint is concerned with $V \circ H$ instead of dealing separately with V and H. $\triangle$

From now on, we consider functions φ and W satisfying (23) and (28). In other words, we will concentrate on well prepared couples (φ, W).

3. Existence of compatible families

The goal of this subsection is to show through a constructive proof that the system (11) actually admits nontrivial solutions. We want also to understand the structure of its generic solutions.

Of course, to face (11), preliminary assumptions are needed. We select some phase $\varphi \in \mathcal{C}^2\left(B(0,r]; \mathbb{R}\right)$ with no critical point in $B(0,r]$. Without loss of generality (relabelling the coordinates and diminishing r if necessary) we can adjust φ so that

$$\partial_d \varphi(x) \neq 0, \quad \forall x \in B(0,r], \quad r > 0. \tag{31}$$

We take $W = V \circ H \in \mathcal{C}^2\left(B(0,r] \times \mathbb{T}, \mathbb{R}^d\right)$. Introduce the linear subspace of $\mathbb{R}^d$ spanned by the vectors $\partial_\theta W(x,\theta)$ with $\theta \in \mathbb{T}$, that is

$$\mathbf{E}(x) := \left\{ \sum_{j=1}^{N} \mu_j \partial_\theta W(x,\theta_j); (\mu_1,\ldots,\mu_N) \in \mathbb{R}^N, (\theta_1,\ldots,\theta_N) \in \mathbb{T}^N, N \in \mathbb{N} \right\}. \tag{32}$$

Choose $N = 1$, $\mu_1 = 1$ and $\theta_1 = \theta$ in this definition to see that

$$\partial_\theta W(x,\theta) \in \mathbf{E}(x) \subset \mathbb{R}^d, \quad \forall (x,\theta) \in B(0,r] \times \mathbb{T}.$$

Because $\mathbf{E}(x)$ is of finite dimension, we can find J^x numbers $\theta_1^x, \ldots, \theta_{J^x}^x$ such that

$$\mathbf{E}(x) = \left\{ \sum_{j=1}^{J_x} \mu_j \partial_\theta W(x,\theta_j^x); (\mu_1,\ldots,\mu_{J^x}) \in \mathbb{R}^{J^x} \right\}, \quad J^x := \dim \mathbf{E}(x).$$

Then, in view of the first line of (11), we must have

$$\mathbf{E}(x) \subset \nabla\varphi(x)^\perp, \quad \forall (x,\theta) \in B(0,r] \times \mathbb{T}.$$

On the one hand, the case $J^x = \dim \mathbf{E}(x) = 0$ is not interesting because this situation corresponds to the absence of oscillations. On the other hand, we have, necessarily,

$$J^x \leq \dim \nabla\varphi(x)^\perp = d - 1, \quad \forall x \in B(0,r].$$

Due to the continuity of $\partial_\theta W$, the application $x \longmapsto \dim \mathbf{E}(x)$ is lower semi-continuous. In particular, the set

$$\left\{ x \in B(0,r[; J^x > d - \frac{3}{2} \right\} = \left\{ x \in B(0,r[; J^x = d - 1 \right\}$$

is open. Now, suppose that $J^0 = d - 1$. By restricting $r > 0$ if necessary, we can always suppose that $J^x = d - 1$ for all $x \in B(0, r[$. More generally, in what follows, we will retain the generic case where the application $x \longmapsto J^x = \dim \mathbf{E}(x)$ is constant (not necessarily equal to $d - 1$) on $B(0, r]$:

$$\exists J \in \{1, \dots, d-1\}; \quad \dim \mathbf{E}(x) = J, \quad \forall x \in B(0, r]. \tag{33}$$

Denote by the symbol $\mathcal{G}_d^J$ the *Grassmanian* manifold of linear subspaces of $\mathbb{R}^d$ with dimension J.

Lemma 8. *Assume* $W \in \mathcal{C}^2\left(B(0, r[\times\mathbb{T}, \mathbb{R}^d\right)$ *and* (33). *Then* $\mathbf{E} \in \mathcal{C}^1\left(B(0, r[, \mathcal{G}_d^J\right)$.

Proof. Let $x_0 \in B(0, r]$. By hypothesis, we can find $\theta_1^{x_0}, \dots, \theta_J^{x_0}$ in $\mathbb{T}$ such that $\left(\partial_\theta W(x_0, \theta_1^{x_0}), \dots, \partial_\theta W(x_0, \theta_J^{x_0})\right)$ is a basis of $\mathbf{E}(x_0)$. Hence, we can extract a $J \times J$ determinant

$$\delta(x_0) := \det\left(\partial_\theta W_{i_j}(x_0, \theta_k^{x_0})\right)_{1 \le j,k \le J}, \quad i_j \in [\![1, d]\!]$$

such that $\delta(x_0) \neq 0$. Since $\partial_\theta W$ is continuous, the function $x \longmapsto \delta(x)$ is continuous. Therefore, we can isolate some small open neighborhood Ω of x_0 such that

$$\delta(x) \neq 0, \quad \forall x \in \Omega, \quad x_0 \in \Omega.$$

For $x \in \Omega$, the family $\left(\partial_\theta W(x, \theta_1^{x_0}), \dots, \partial_\theta W(x, \theta_J^{x_0})\right)$ is still linearly independent, and it is built with J vectors of $\mathbf{E}(x)$. Since by hypothesis $\mathbf{E}(x)$ is of dimension J, this is in fact a basis of $\mathbf{E}(x)$. Obviously, the application

$$x \longmapsto \left(\partial_\theta W(x, \theta_1^{x_0}), \dots, \partial_\theta W(x, \theta_J^{x_0})\right)$$

is of class $\mathcal{C}^1$ in Ω. This remark gives the expected local regularity of $\mathbf{E}$. Finally, since $x_0 \in B(0, r]$ can be chosen arbitrarily, the Lemma 8 is proved. □

Recall that

$$\partial_\theta W(x, \theta) \in \mathbf{E}(x) \subset \nabla\varphi(x)^\perp, \quad \forall (x, \theta) \in B(0, r] \times \mathbb{T}. \tag{34}$$

The second line of (11) implies that

$$\Pi_{\mathbf{E}(x)^\perp} D_x W(x, \theta) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (x, \theta) \in B(0, r] \times \mathbb{T}.$$

Observe that, in this formulation, the two projectors (on the left and on the right) do not depend any more on the variable $\theta \in \mathbb{T}$. This allows us to extract the mean value to get

$$\Pi_{\mathbf{E}(x)^\perp} D_x W^*(x, \theta) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (x, \theta) \in B(0, r] \times \mathbb{T}. \tag{35}$$

Lemma 9. *Let* $\varphi \in \mathcal{C}^2(B(0, r], \mathbb{R})$ *and* $W \in \mathcal{C}^1(B(0, r] \times \mathbb{T}, \mathbb{R}^d)$, *satisfying respectively the conditions* (31) *and* (33). *Suppose that the relations* (34) *and* (35) *are satisfied. Then, the application* $x \longmapsto \mathbf{E}(x)$ *is constant on the level surfaces of* φ. *More precisely*

$$\exists \mathbb{E} \in \mathcal{C}^1(\mathbb{R}, \mathcal{G}_d^J); \quad \mathbf{E}(x) = \mathbb{E} \circ \varphi(x), \quad \forall x \in B(0, r]. \tag{36}$$

Proof. Let us denote by δ_{ij} the usual Dirichlet symbol, and by $\delta^{(k)}$ the vector of $\mathbb{R}^d$ whose components are $(\delta_{ik})_{1 \le i \le d}$. The $d - 1$ vectors

$$v_k(x) = -\delta^{(k)} + \partial_k\varphi(x)/\partial_d\varphi(x)\delta^{(d)}, \quad 1 \le k \le d-1$$

form a $\mathcal{C}^1$ basis of $\nabla\varphi(x)^\perp$. By permuting the components of $\mathbb{R}^d$ and by diminishing r if necessary, we can always arrange the datas so that

$$\mathbf{E}(x) \oplus \langle v_1(x), \dots, v_{d-J-1}(x)\rangle = \nabla\varphi(x)^\perp, \quad \forall x \in B(0, r].$$

Therefore, for all $j \in [\![1, J]\!]$, the vector $v_{d-j}(x) \in \nabla\varphi(x)^\perp$ can be decomposed according to

$$v_{d-j}(x) = e_j(x) - \sum_{k=1}^{d-J-1} \alpha_j^k(x) v_k(x), \quad e_j(x) \in \mathbf{E}(x)$$

where, due to the assumptions related to the regularity of φ and $\mathbf{E}$, we have

$$e_j = (e_j^1, \dots, e_j^d) \in \mathcal{C}^1(B(0, r]; \mathbb{R}^d), \qquad \alpha_j^k \in \mathcal{C}^1(B(0, r]; \mathbb{R}).$$

The vectors e_j with $j \in [\![1, J]\!]$ are necessarily independent. They form a basis of $\mathbf{E}(x)$. Besides, we have the general formula

$$W(x,\theta) = \bar{W}(x) + \int_0^\theta \partial_\theta W(x,\tilde{\theta})\mathrm{d}\tilde{\theta} - \int_{\mathbb{T}} \left(\int_0^\theta \partial_\theta W(x,\tilde{\theta})\mathrm{d}\tilde{\theta} \right) \mathrm{d}\theta$$

that, in view of (34), implies

$$W(x,\theta) = \bar{W}(x) + \sum_{j=1}^{J} w_j^*(x,\theta) e_j(x), \quad w_j^* \in \mathcal{C}^1(B(0,r] \times \mathbb{T}; \mathbb{R}).$$

Now, the relation (35) reads

$$\sum_{j=1}^{J} w_j^*(x,\theta) \Pi_{\mathbf{E}(x)^\perp} D_x e_j(x) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (x,\theta) \in B(0,r] \times \mathbb{T}.$$

Recall that the dimension of $\mathbf{E}(x)$ is J. This implies that

$$\exists (\theta_1^x, \ldots, \theta_J^x) \in \mathbb{T}^J; \quad \det \left[w_i^*(x, \theta_j^x) \right]_{1 \le i,j \le J} \neq 0.$$

Combining the preceding informations, we see that (35) is equivalent to

$$\Pi_{\mathbf{E}(x)^\perp} D_x e_j(x) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (j,x) \in [\![1, J]\!] \times B(0,r]. \tag{37}$$

The vector space $\mathbf{E}(x)^\perp$ is of dimension $d - J$. It is generated by the vector $e_d(x) := \nabla\varphi(x)$ and the $d - J - 1$ vectors

$$e_j(x) = -\delta^{(j-J)} + \sum_{k=1}^{J} \alpha_k^{j-J}(x) \delta^{(d-k)}, \quad j \in [\![J+1, d-1]\!].$$

Therefore (37) is exactly the same as

$$^t e_l(x) D_x e_j(x) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (l,j,x) \in [\![J+1, d]\!] \times [\![1, J]\!] \times B(0,r]. \tag{38}$$

For $j \in [\![1, J]\!]$, compute

$$D_x e_j(x) = \sum_{k=1}^{d-J-1} \nabla_x \alpha_j^k(x) v_k(x) + \left[\sum_{k=1}^{d-J-1} \alpha_j^k(x) \nabla_x \left(\partial_k\varphi(x)/\partial_d\varphi(x)\right) + \nabla_x \left(\partial_{d-j}\varphi(x)/\partial_d\varphi(x)\right) \right] \delta^{(d)}.$$

Applying on the left $^t e_l(x)$ with $l \in [\![J+1, d-1]\!]$, yields

$$^t e_l(x) D_x e_j(x) = \nabla_x \alpha_j^{l-J}(x), \quad 1 \le j \le J < l \le d-1.$$

We can extract from (38) that

$$\nabla_x \alpha_j^{l-J}(x) \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall (l,j,x) \in [\![J+1, d-1]\!] \times [\![1, J]\!] \times B(0,r]. \tag{39}$$

Independent statement. *Let $\varphi \in \mathcal{C}^1(B(0,r], \mathbb{R})$ satisfying (31). Let $\alpha \in \mathcal{C}^1(B(0,r], \mathbb{R})$ be a function which is subjected to the relation (39). Then, restricting $r > 0$ if necessary, we can always find some function $Z \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that*

$$\alpha(x) = Z \circ \varphi(x), \quad \forall x \in B(0,r]. \tag{40}$$

Proof of the independent statement. The geometric reason for (40) is the following. The relation (39) means that either the vectors $\nabla_x\alpha(x)$ and $\nabla\varphi(x)$ are parallel or that the tangent spaces at x to the level surfaces associated with the scalar functions α and φ coincide. Since the level surfaces associated with α and φ are spanned by these tangent spaces, we can deduce that α and φ have common level surfaces. Moreover, the relation (31) allows us to characterize (locally near 0) these level surfaces through the different values of φ. This is why we have (40).

Now, we can also proceed as follows. Due to (31), the functions $x_1, x_2, \ldots, x_{d-1}$ and $\varphi(x)$ form locally (near 0) a system of coordinates. Therefore, we can find $Z \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ such that

$$\alpha(x) = Z\left(\hat{x}, \varphi(x)\right), \quad \hat{x} := (x_1, x_2, \ldots, x_{d-1}), \quad \forall x \in B(0,r].$$

Decompose $\nabla\varphi(x)$ according to

$$\nabla\varphi(x) = \left(\nabla_{\hat{x}}\varphi(x), \partial_d\varphi(x)\right), \quad \nabla_{\hat{x}}\varphi(x) = \left(\partial_1\varphi(x), \ldots, \partial_{d-1}\varphi(x)\right) \in \mathbb{R}^{d-1}.$$

Given $\hat{h} \in \mathbb{R}^{d-1}$, define

$$h_d(x, \hat{h}) := -\partial_d \varphi(x)^{-1} \nabla_{\hat{x}} \varphi(x) \cdot \hat{h}.$$

Observe that

$$\left(\hat{h}, h_d(x, \hat{h})\right) \in \nabla\varphi(x)^{\perp}, \quad \forall \hat{h} \in \mathbb{R}^{d-1}.$$

Testing (39) with such choices gives rise to

$$\nabla_{\hat{x}} Z\left(x_1, x_2, \ldots, x_{d-1}, \varphi(x)\right) \cdot \hat{h} = 0, \quad \forall \hat{h} \in \mathbb{R}^{d-1}.$$

This information clearly implies that the function Z does not depend on its $d-1$ first variables. We have (40). $\square$

Applying the independent statement to the functions α_j^{l-J}, we see that we can exhibit functions

$$Z_j^k \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}), \quad (k, j) \in [\![1, d-J-1]\!] \times [\![1, J]\!]$$

such that, for all $(k, j) \in [\![1, d-J-1]\!] \times [\![1, J]\!]$, we have

$$e_j^k(x) = -\alpha_j^k(x) = Z_j^k \circ \varphi(x), \quad \forall x \in B(0, r]. \tag{41}$$

This construction of the Z_j^k is not classical and it is one of the main difficulties in the proof of Lemma 9. Finally, the remaining conditions to consider are obtained by taking $j \in [\![1, J]\!]$ and $l = d$. Specifically

$$\nabla\varphi(x) D_x e_j(x) \Pi_{\nabla\varphi(x)^{\perp}} = 0, \quad \forall (j, x) \in [\![1, J]\!] \times B(0, r].$$

Use (31) and (41) to simplify this into

$$\nabla_x e_j^d(x) \Pi_{\nabla\varphi(x)^{\perp}} = 0, \quad \forall (j, x) \in [\![1, J]\!] \times B(0, r]$$

where we recall that

$$e_j^d(x) = -\sum_{k=1}^{d-J-1} Z_j^k \circ \varphi(x) \partial_k \varphi(x) / \partial_d \varphi(x) + \partial_{d-j} \varphi(x) / \partial_d \varphi(x).$$

Again, this means the existence of $Z_j^d \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$e_j^d(x) = Z_j^d \circ \varphi(x), \quad \forall (j, x) \in [\![1, J]\!] \times B(0, r].$$

Briefly, we have obtained, for all $j \in [\![1, J]\!]$, that

$$e_j(x) = Z_j \circ \varphi(x), \quad Z_j = {}^t(Z_j^1, \ldots, Z_j^{d-J-1}, 0, \ldots, 0, -1, 0, \ldots, 0, Z_j^d).$$

The vector space $\mathbf{E}$ is spanned by the e_j with $j \in [\![1, J]\!]$. Therefore, it depends only on φ, in a $\mathcal{C}^1$ way. This gives rise to (36). In particular, $\mathbf{E}$ is constant on the level surfaces of φ. $\square$

Combining (34) and (36), we can produce the necessary condition

$$\nabla\varphi(x) \in \mathbb{E} \circ \varphi(x)^{\perp} = \mathbf{E}(x)^{\perp}, \quad \forall x \in B(0, r]. \tag{42}$$

The condition (42) is a geometrical constraint on φ underlying the resolution of (11). We explain below how to solve it.

Lemma 10. *Select:*

- *a curve $\mathbb{E} \in \mathcal{C}^2(\mathbb{R}, \mathcal{G}_d^J)$ of J-dimensional vector spaces of $\mathbb{R}^d$,*
- *a $\mathcal{C}^2$ submanifold $\mathcal{S} \subset \mathbb{R}^d$ of dimension $d-J$, containing $0 \in \mathbb{R}^d$,*
- *a $\mathcal{C}^2$ scalar function $\chi : \mathcal{S} \longrightarrow \mathbb{R}$.*

Note $T_0\mathcal{S}$ the tangent space of $\mathcal{S}$ at the point $0 \in \mathbb{R}^d$. We suppose that

$$T_0\mathcal{S} + \mathbb{E}\left(\chi(0)\right) = \mathbb{R}^d. \tag{43}$$

Then, we can find $r > 0$ such that the nonlinear equation (42) completed with $\varphi_{|\mathcal{S} \cap B(0,r]} \equiv \chi$ has a unique $\mathcal{C}^2$ solution on $B(0, r]$. We will say that the phase φ is generated *by* $(\mathbb{E}, \mathcal{S}, \chi)$.

Proof. Select $\delta > 0$ and J functions

$$Z_j \in \mathcal{C}^2\left(]\chi(0)-\delta, \chi(0)+\delta[;\mathbb{R}\right), \quad j \in [\![1, J]\!]$$

adjusted such that, for all $t \in]\chi(0)-\delta, \chi(0)+\delta[$, $(Z_1(t), \dots, Z_J(t))$ is a basis of $\mathbb{E}(t)$. Note that

$$\Omega_{\mathcal{S}}^{\delta} := \chi^{-1}\left(]\chi(0)-\delta, \chi(0)+\delta[\right) \subset \mathcal{S}, \quad z = {}^t(z^1, \dots, z^J) \in \mathbb{R}^J.$$

Consider the $\mathcal{C}^2$ application

$$\begin{aligned} \Xi : \Omega_{\mathcal{S}}^{\delta} \times \mathbb{R}^J &\longrightarrow \mathbb{R}^d \\ (y, z) &\longmapsto \Xi(y, z) := y + \sum_{j=1}^{J} z^j Z_j \circ \chi(y). \end{aligned}$$

Because of (43), the linear operator

$$\begin{aligned} D_x \Xi(0,0) : T_0\mathcal{S} \times \mathbb{R}^J &\longrightarrow \mathbb{R}^d \\ (h, k) &\longmapsto h + \sum_{j=1}^{J} k^j Z_j \circ \chi(y) \end{aligned}$$

is invertible. The inverse mapping Theorem can be applied at the point $(0,0) \in \mathcal{S} \times \mathbb{R}^J$. It guarantees the existence of an $r > 0$ such that Ξ is a $\mathcal{C}^2$ diffeomorphism from a neighbourhood of $(0,0) \in \mathcal{S} \times \mathbb{R}^J$ onto $B(0, r]$. Introduce the projection

$$\begin{aligned} \Gamma : \mathcal{S} \times \mathbb{R}^J &\longrightarrow \mathcal{S} \\ (y, z) &\longmapsto \Gamma(y, z) := y. \end{aligned}$$

Now, we can define

$$\varphi := \chi \circ \Gamma \circ \Xi^{-1} \in \mathcal{C}^2\left(B(0, r]; \mathbb{R}\right).$$

Since $(\Gamma \circ \Xi^{-1})_{|\mathcal{S} \cap B(0,r]} = Id$, we have $\varphi_{|\mathcal{S} \cap B(0,r]} \equiv \chi_{|\mathcal{S} \cap B(0,r]}$. Moreover, the function φ is constant on the set

$$\mathcal{F}_y := (y + \langle Z_1 \circ \chi(y), \dots, Z_J \circ \chi(y) \rangle) \cap B(0, r], \quad y \in \mathcal{S} \cap B(0, r].$$

More precisely, $\mathcal{F}_y$ is a piece of an affine manifold with direction $\mathbb{E} \circ \chi(y)$, on which φ takes the value $\chi(y)$. In particular

$$\nabla\varphi(x) \in (T_x\mathcal{F}_y)^{\perp} = \mathbb{E} \circ \chi(y)^{\perp} = \mathbb{E} \circ \varphi(x)^{\perp}, \quad \forall x \in \mathcal{F}_y.$$

Since the $\mathcal{F}_y$ with $y \in \mathcal{S} \cap B(0, r]$ form a foliation of $B(0, r]$, we have obtained the expected relation (42). □

Proposition 11. *Let φ be generated by $(\mathbb{E}, \mathcal{S}, \chi)$. The couple (φ, W) is well prepared if and only if there exist two functions $W_{\parallel} \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^d)$ and $W_{\perp} \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^d)$ satisfying*

$$W_{\parallel}(t, s) \in \mathbb{E}(t), \quad W_{\perp}(t) \in \mathbb{E}(t)^{\perp}, \quad \forall (t, s) \in \mathbb{R}^2 \tag{44}$$

and a scalar function $\psi \in \mathcal{C}^1\left(B(0, r] \times \mathbb{T}; \mathbb{R}\right)$ such that

$$W(x, \theta) = W_{\parallel}\left(\varphi(x), \psi(x, \theta)\right) + W_{\perp}\left(\varphi(x)\right), \quad \forall (x, \theta) \in B(0, r] \times \mathbb{T}. \tag{45}$$

Proof. Let $(Z_1(t), \dots, Z_J(t))$ be some orthonormal basis of $\mathbb{E}(t)$ with a $\mathcal{C}^1$ regularity with respect to $t \in \mathbb{R}$. Complete it with some $\mathcal{C}^1$ orthonormal basis $(e_{J+1}(t), \dots, e_d(t))$ of $\mathbb{E}(t)^{\perp}$, again of class $\mathcal{C}^1$. In view of (34), the definition of $\mathbf{E}(x)$ and Lemma 9, the profile $W(x, \theta)$ can be decomposed according to

$$W(x, \theta) = \sum_{k=1}^{J} w_j(x, \theta) Z_j \circ \varphi(x) + \sum_{k=J+1}^{d} w_j(x) e_j \circ \varphi(x)$$

with

$$\begin{aligned} &w_j \in \mathcal{C}^1\left(B(0, r] \times \mathbb{T}; \mathbb{R}\right), \quad \forall j \in [\![1, J]\!], \\ &w_j \in \mathcal{C}^1\left(B(0, r]; \mathbb{R}\right), \quad \forall j \in [\![J+1, d]\!]. \end{aligned}$$

Compute the derivative of $W(x, \theta)$ with respect to the variable x and compose on the right with $\Pi_{\nabla\varphi(x)^{\perp}}$. It remains

$$D_x W(x, \theta) \Pi_{\nabla\varphi(x)^{\perp}} = \sum_{k=1}^{J} \nabla_x w_j(x, \theta) \cdot \Pi_{\nabla\varphi(x)^{\perp}} \times Z_j \circ \varphi(x) + \sum_{k=J+1}^{d} \nabla_x w_j(x) \cdot \Pi_{\nabla\varphi(x)^{\perp}} \times e_j \circ \varphi(x).$$

Select a point $(x, \theta) \in \mathcal{V}$ which means that $\partial_\theta W(x, \theta) \neq 0$. Without loss of generality, we can suppose that $\partial_\theta W_J(x, \theta) \neq 0$. Otherwise, just permute the components of $\mathbb{R}^d$ to obtain this condition. By construction, the hyperplane $\partial_\theta W(x, \theta)^\perp$ is generated by the $d - J$ vectors $e_j \circ \varphi(x)$ with $j \in [\![J + 1, d]\!]$ and the $J - 1$ vectors

$$\partial_\theta w_J(x, \theta) Z_j \circ \varphi(x) - \partial_\theta w_j(x, \theta) Z_J \circ \varphi(x), \quad j \in [\![1, J - 1]\!].$$

The requirement (28) is equivalent to the conditions

$$\nabla_x w_j(x) \cdot \Pi_{\nabla\varphi(x)^\perp} = 0, \quad \forall j \in [\![J + 1, d]\!], \tag{46}$$

$$\left(\partial_\theta w_J \nabla_x w_j - \partial_\theta w_j \nabla_x w_J\right)(x, \theta) = 0, \quad \forall j \in [\![1, J - 1]\!]. \tag{47}$$

On the one hand, from (46), we deduce that

$$\exists \tilde{w}_j \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}); \quad w_j(x) = \tilde{w}_j \circ \varphi(x), \quad \forall j \in [\![J + 1, d]\!].$$

On the other hand, it follows from the relations (47) that the mappings Υ_t parameterized by $t \in \mathbb{R}$ and defined on the level sets

$$\mathcal{G}_t := \{x \in B(0, r]; \varphi(x) = t\}$$

by the formulas

$$\begin{aligned} \Upsilon_t : \mathcal{G}_t \times \mathbb{T} &\longrightarrow \mathbb{R}^J \\ (x, \theta) &\longmapsto {}^t(w_1, \ldots, w_J) \end{aligned}$$

have rank one. Thus, to each Υ_t corresponds a foliation of $\mathcal{G}_t \times \mathbb{T}$ by submanifolds of dimension $d-1$. Each such foliation depends on the parameter t. It can be described by using a function $\psi \in \mathcal{C}^1\left(B(0, r] \times \mathbb{T}, \mathbb{R}\right)$, so that

$$w_j(x, \theta) = \tilde{w}_j\left(\varphi(x), \psi(x, \theta)\right), \quad \forall j \in [\![1, J]\!].$$

Define

$$W_\perp(t) := \sum_{j=J+1}^{d} \tilde{w}_j(t) e_j(t), \quad W_\parallel(t, s) := \sum_{j=1}^{J} \tilde{w}_j(t, s) Z_j(t).$$

By construction, we have both (44) and (45).

Conversely, suppose that $W(x, \theta)$ has the form (45) with $W_\parallel(x, \theta)$ and $W_\perp(x, \theta)$ as in (44). Then

$$\partial_\theta W(x, \theta) = \partial_\theta \psi(x, \theta) \times \partial_s W_\parallel\left(\varphi(x), \psi(x, \theta)\right) \in \mathbb{E}\left(\varphi(x)\right) \equiv \mathbf{E}(x)$$

which is (34) and gives rise to the first part of (11). Moreover

$$D_x W(x, \theta) \Pi_{\nabla\varphi(x)^\perp} = \nabla_x \psi(x, \theta) \cdot \Pi_{\nabla\varphi(x)^\perp} \times \partial_s W_\parallel\left(\varphi(x), \psi(x, \theta)\right).$$

Since $\partial_\theta W$ and $\partial_s W_\parallel$ are collinear, we get the second equation of (11). $\square$

4. Simple wave solutions

The aim of this last part is to explain how the initial oscillating data $h^\varepsilon(x)$ is transformed through the evolution equation (1). Below, we consider this question in a simplified context, by looking only at simple wave solutions.

Definition 12. Let $\varepsilon \in]0, 1]$. We say that $\mathbf{u}^\varepsilon \in \mathcal{C}^1(\Omega^{\mathrm{T}}; \mathbb{R})$ is a *simple wave* if it can be put in the following form

$$\mathbf{u}^\varepsilon(t, x) = \mathbf{H}\left(t, x, \frac{\Phi(t, x)}{\varepsilon}\right), \quad \mathbf{H} \in \mathcal{C}^1(\Omega^{\mathrm{T}} \times \mathbb{T}; \mathbb{R}^d), \quad \Phi \in \mathcal{C}^1(\Omega^{\mathrm{T}}; \mathbb{R}).$$

The Theorem 2 explains how to associate with a well prepared couple (φ, W) a simple wave $\mathbf{u}^\varepsilon(t, x)$ which is a solution on Ω^{T} of the Burger's type system (1). It remains to show this statement Theorem 2.

Proof of Theorem 2. Compose the first equation of (12) with $D_{\mathbf{u}} V \circ \mathbf{H}$ in order to extract

$$\begin{cases} \partial_t \mathbf{W} + (\mathbf{W} \cdot \nabla_x)\mathbf{W} = 0, \\ \partial_t \Phi + (\bar{\mathbf{W}} \cdot \nabla_x)\Phi = 0, \qquad \mathbf{W} := V \circ \mathbf{H}. \\ \mathbf{W}^* \cdot \nabla_x \Phi = 0, \end{cases} \tag{48}$$

This must be associated with the initial data

$$\mathbf{W}(0, x, \theta) = W(x, \theta), \quad \Phi(0, x) = \varphi(x). \tag{49}$$

First, we discuss (48) and (49). From Proposition 11, we can write

$$W(x,\theta) = W_{\parallel}\left(\varphi(x), \psi(x,\theta)\right) + W_{\perp}\left(\varphi(x)\right).$$

Solve locally in time, say on Ω^{T} for some $T > 0$, the scalar conservation law

$$\partial_t \Phi + W_{\perp}(\Phi) \cdot \nabla_x \Phi = 0, \qquad \Phi(0,x) = \varphi(x). \tag{50}$$

Recall that $\mathbf{E}(x) = \mathbb{E} \circ \varphi(x)$ is spanned by the J vectors $e_j(x) = Z_j \circ \varphi(x)$ where the Z_j are defined at the end of the proof of Lemma 9. Now, fix any $j \in [\![1, J]\!]$ and compute

$$[\partial_t + W_{\perp}(\Phi) \cdot \nabla_x]\,(Z_j \circ \Phi \cdot \nabla_x \Phi) = -(\nabla_x \Phi \cdot W'_{\perp} \circ \Phi) \times (Z_j \circ \Phi \cdot \nabla_x \Phi).$$

Combining (13) and (42), we can extract

$$(Z_j \circ \Phi \cdot \nabla_x \Phi)(0,x) = 0, \quad \forall (j,x) \in [\![1, J]\!] \times B(0,r].$$

In view of the preceding equation, this polarization identity is propagated in time, which means that

$$Z_j \circ \Phi(t,x) \cdot \nabla_x \Phi(t,x) = 0, \quad \forall (t,x) \in [0,T] \times B(0,r]$$

or equivalently that

$$\nabla_x \Phi(t,x) \subset \mathbb{E} \circ \Phi(t,x)^{\perp}, \quad \forall (t,x) \in [0,T] \times B(0,r]. \tag{51}$$

Now, introduce the function

$$\tilde{W}(t,s) := W_{\parallel}(t,s) + W_{\perp}(t), \quad (t,s) \in \mathbb{R}^2$$

and the scalar conservation law

$$\partial_t \Psi + \tilde{W}\left(\Phi(t,x), \Psi\right) \cdot \nabla_x \Psi = 0. \tag{52}$$

Complete (52) with the initial data

$$\Psi(0,x,\theta) = \psi(x,\theta), \quad \psi \in \mathcal{C}^1\left(B(0,r] \times \mathbb{T}; \mathbb{R}\right). \tag{53}$$

In (52), the variable $\theta \in \mathbb{T}$ plays the part of a parameter. For $T > 0$ small enough, the Cauchy problem (52) and (53) has a local solution on Ω^{T}. Finally, define the profile $\mathbf{W}$ through

$$\mathbf{W}(t,x,\theta) := \tilde{W}\left(\Phi(t,x), \Psi(t,x,\theta)\right), \qquad \mathbf{W}(0,x,\theta) = W(x,\theta).$$

By construction, we have

$$\mathbf{W}^*(t,x,\theta) = W_{\parallel}\left(\Phi(t,x), \Psi(t,x,\theta)\right)^*.$$

The information from (44) and (51) implies that

$$\mathbf{W}^*(t,x,\theta) \cdot \nabla_x \Phi(t,x) = 0, \quad \forall (t,x) \in \Omega^{\mathrm{T}}.$$

Taking into account (44) and (50), we have also

$$\partial_t \Phi + \mathbf{W} \cdot \nabla_x \Phi = \partial_t \Phi + W_{\perp} \circ \Phi \cdot \nabla_x \Phi = 0.$$

Then, with (52), we can deduce that

$$\partial_t \mathbf{W} + \mathbf{W} \cdot \nabla_x \mathbf{W} = \partial_s \tilde{W}(\partial_t \Psi + \mathbf{W} \cdot \nabla_x \Psi) = 0, \qquad \mathbf{W}(0,x,\theta) = W(x,\theta). \tag{54}$$

To sum up, we have constructed functions Φ and $\mathbf{W}$ satisfying (48).

Now, we concentrate on (12). First, solve separately (on some domain Ω^{T} with $T > 0$) the Cauchy problem

$$\partial_t \mathbf{H} + V \circ \mathbf{H} \cdot \nabla_x \mathbf{H} = 0, \quad \mathbf{H}(0,x,\theta) = H(x,\theta). \tag{55}$$

Observe that the expression $\tilde{\mathbf{W}} := V \circ \mathbf{H}$ is, by Construction, subjected to

$$\partial_t \tilde{\mathbf{W}} + \tilde{\mathbf{W}} \cdot \nabla_x \tilde{\mathbf{W}} = 0, \quad \tilde{\mathbf{W}}(0,x,\theta) = W(x,\theta). \tag{56}$$

The Cauchy problems (54) and (56) are made of the same quasilinear constraints and the same initial data. Since the corresponding $\mathcal{C}^1$ solutions must coincide, we have necessarily that $\tilde{\mathbf{W}} = V \circ \mathbf{H} \equiv \mathbf{W}$ on Ω^{T}.

Briefly, the first equation of (12) is verified because this is precisely (55), whereas the two other conditions of (12) are satisfied because they correspond exactly to the two last conditions in (48). This explains why the apparently overdetermined system (12) and (13) has a unique solution on $\Omega^{\mathrm{T}} \times \mathbb{T}$ for some $T > 0$.

Finally, define the simple wave $\mathbf{u}^\varepsilon(t,x) := \mathbf{H}\left(t, x, \frac{\Phi(t,x)}{\varepsilon}\right)$. Compute

$$\partial_t \mathbf{u}^\varepsilon + V(\mathbf{u}^\varepsilon) \cdot \nabla_x \mathbf{u}^\varepsilon = (\partial_t \mathbf{H} + V \circ \mathbf{H} \cdot \nabla_x \mathbf{H}) \left(t, x, \frac{\varphi(x)}{\varepsilon}\right) + \frac{1}{\varepsilon} [(\partial_t \Phi + V \circ \mathbf{H} \cdot \nabla_x \Phi) \partial_\theta \mathbf{H}] \left(t, x, \frac{\varphi(x)}{\varepsilon}\right).$$

The fact that $\mathbf{u}^\varepsilon(t,x)$ is a solution of (1) becomes a direct consequence of the equations inside (12). Moreover, the definition of $\mathbf{W}$ indicates clearly that the structure (45) is conserved for $t \in [0, T]$. Therefore (see the end of the proof of Proposition 11), for all $t \in [0, T]$, the trace $(\Phi(t, \cdot), \mathbf{W}(t, \cdot))$ is still well prepared. This last remark concludes the proof of Theorem 2. $\square$

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