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Analytic solutions for a three-level system in a time-dependent field

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ABSTRACT

This paper generalizes some known solitary solutions of a time-dependent Hamiltonian in two ways: The time-dependent field can be an elliptic function, and the time evolution is obtained for a complete set of basis vectors. The latter makes it feasible to consider arbitrary initial conditions. The former makes it possible to observe a beating caused by the non-harmonicity of the driving field.

1. Introduction

The analytic solutions of Allen and Eberly [1] for the Bloch equations are well known. Similar results for spin-one systems or three-level atoms do exist [2] and are derived in terms of the coherence vector of Hioe and Eberly [3]. We consider a three-level system with time-dependent external fields which enable transitions between two pairs of levels, between (1) and (3) and between (2) and (3), respectively. See Fig. 1. This kind of system has applications in different domains of physics. Analytic expressions for the time evolution of the density matrix are very helpful for understanding many of the phenomena observed in light scattering experiments — see for instance [4]. In the context of quantum computers the accurate manipulation of the state of a quantum system – in this case a qutrit – is important. See for instance in [5] the discussion of the technical difficulties in manipulating biphotonic qutrits.

In the present work the known solitary solutions of [2] are generalized in more than one way. The external fields are modulated with Jacobi's elliptic functions. By varying the elliptic modulus k these functions make the bridge between periodic functions ($\cos(\omega t)$ and $\sin(\omega t)$) and single pulses described by $\operatorname{sech}(\omega t)$ and $\tanh(\omega t)$. In addition, a full set of solutions is presented instead of just one solution. This makes it possible to take arbitrary initial conditions at time $t = 0$.

Section 2 presents the time-dependent Hamiltonian and the special solutions. Section 3 discusses a limiting case in order to make the connection with known results. In Section 4, a specific setting is chosen. Section 5 discusses the results. Appendix A contains the explicit expressions which are used for the generators of SU(3). Appendix B explains the method by which the special solutions were obtained from known solutions of the non-linear von Neumann equation (see the Appendix of [6])

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = [H_0, \rho_t^2]. \tag{1}$$

Next, part of the theoretical framework of [7] is used to obtain a set of linearly independent solutions. Finally, the results are transferred to a more general setting.

2. Special solutions

Consider a Hamiltonian of the form

$$H = H_0 + a\,\mathrm{cn}\,(\omega t, k)[S_4]_t + x\,\mathrm{dn}\,(\omega t, k)[S_7]_t \tag{2}$$

where $S_1, S_2, \ldots, S_8$ are the generators of SU(3) and equal half the Gell-Mann matrices – see Appendix A – and where

$$[S_j]_t \equiv \mathrm{e}^{-(it/\hbar)H_0} S_j \mathrm{e}^{(it/\hbar)H_0} \tag{3}$$

are the generators written in the interaction picture. This kind of Hamiltonian is considered in quantum optics when studying three-level systems driven by laser light, neglecting damping effects—see for instance [4,8].

The functions sn , cn , dn are Jacobi's elliptic functions. In the limit $k = 0$ the function $\mathrm{sn}\,(\omega t, k)$ converges to $\sin(\omega t)$, $\mathrm{cn}\,(\omega t, k)$

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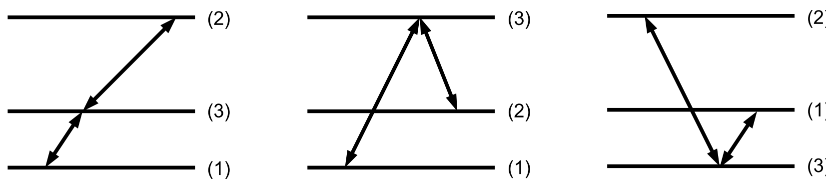

Fig. 1. Ladder (or cascade) configuration (left), Λ configuration (center), and V configuration (right).

converges to $\cos(\omega t)$, and $\mathrm{dn}\,(\omega t, k)$ converges to 1. In the limit $k = 1$ the function $\mathrm{sn}\,(\omega t, k)$ converges to $\tanh(\omega t)$ and $\mathrm{cn}\,(\omega t, k)$ and $\mathrm{dn}\,(\omega t, k)$ both converge to $\mathrm{sech}(\omega t)$. In what follows, we drop the arguments $(\omega t, k)$ of Jacobi's functions when this does not lead to ambiguities.

Let us assume that the parameters of the Hamiltonian satisfy

$$4k^2(\hbar\omega)^2 = a^2 + k^2x^2. \tag{4}$$

Then three orthonormal solutions ψ_0, ψ_+, ψ_- of the Schrödinger equation $i\hbar\dot{\psi} = H\psi$ are given by

$$\psi_0(t) = \frac{1}{T}\mathrm{e}^{-(it/\hbar)H_0}\begin{pmatrix} ia\,\mathrm{dn} \\ k^2x\,\mathrm{cn} \\ B\,\mathrm{sn}\end{pmatrix}, \tag{5}$$

$$\psi_\pm(t) = \frac{\mathrm{e}^{\mp i\phi(t)}}{R(t)}\mathrm{e}^{-(it/\hbar)H_0}\begin{pmatrix} k^2xT\,\mathrm{cn} \pm iaB\,\mathrm{sn}\,\mathrm{dn} \\ \pm k^2xB\,\mathrm{sn}\,\mathrm{cn} + iaT\,\mathrm{dn} \\ \mp k^4x^2\,\mathrm{cn}^2 \mp a^2\,\mathrm{dn}^2\end{pmatrix} \tag{6}$$

with $B = 2k^2\hbar\omega$ and $T = \sqrt{a^2 + k^4x^2}$. Note that by assumption one has $T^2 = B^2 + a^2(1-k^2)$. The functions $R(t)$ and $\phi(t)$ are given by

$$R(t) = \sqrt{2}\,T\sqrt{T^2 - B^2\,\mathrm{sn}^2} = \sqrt{2}\,T\sqrt{a^2(1-k^2) + B^2\,\mathrm{cn}^2} \tag{7}$$

$$\hbar\phi(t) = \frac{ax}{2}T\int_0^t \mathrm{d}s\,\frac{1-k^2}{a^2(1-k^2) + B^2\,\mathrm{cn}^2(\omega s, k)}. \tag{8}$$

One verifies the above statements by explicit calculation.

3. Limiting case

The limit $k = 0$ of solutions (5), (6) is meaningless. The Hamiltonian becomes time independent and the conditions imply that $T = B = a = 0$. On the other hand, in the limit $k = 1$ the Hamiltonian reads

$$H(t) = H_0 + [aS_4 + xS_7]_t\,\mathrm{sech}(\omega t). \tag{9}$$

The conditions imply that $a^2 + x^2 = T^2 = B^2 = 4(\hbar\omega)^2$. The solutions become

$$\psi_0(t) = \frac{1}{2\hbar\omega}\mathrm{e}^{-(it/\hbar)H_0}\left[\mathrm{sech}(\omega t)\begin{pmatrix} ia \\ x \\ 0\end{pmatrix} + \tanh(\omega t)\begin{pmatrix} 0 \\ 0 \\ 2\hbar\omega\end{pmatrix}\right], \tag{10}$$

$$\psi_\pm(t) = \frac{1}{\sqrt{2}}\frac{1}{2\hbar\omega}\mathrm{e}^{-(it/\hbar)H_0} \times\left[\begin{pmatrix} x \\ ia \\ 0\end{pmatrix} \pm \tanh(\omega t)\begin{pmatrix} ia \\ x \\ 0\end{pmatrix} \mp \mathrm{sech}(\omega t)\begin{pmatrix} 0 \\ 0 \\ 2\hbar\omega\end{pmatrix}\right]. \tag{11}$$

From these solutions one can derive the unitary operator $U(t)$ which describes the time evolution. One obtains

$$U(t) = \frac{1}{4\hbar^2\omega^2}\begin{pmatrix} x^2 & -iax & 0 \\ iax & a^2 & 0 \\ 0 & 0 & 0\end{pmatrix} + \frac{1}{4\hbar^2\omega^2}\mathrm{sech}(\omega t)\begin{pmatrix} a^2 & iax & 0 \\ -iax & x^2 & 0 \\ 0 & 0 & 4\hbar^2\omega^2\end{pmatrix} + \frac{1}{4\hbar^2\omega^2}\tanh(\omega t)\begin{pmatrix} 0 & 0 & -2i\hbar\omega a \\ 0 & 0 & -2\hbar\omega x \\ -2i\hbar\omega a & 2\hbar\omega x & 0\end{pmatrix}. \tag{12}$$

Finally, the time dependence of a density matrix ρ now follows from $\rho(t) = U(t)\rho U(t)^\dagger$. The result can then be compared with the expressions (3.22) found at the end of [2].

4. Exciting the ground state

Let us consider a wave function $\psi(t)$ which at $t = 0$ satisfies $\psi(0) = (1, 0, 0)^{\mathrm{T}}$. In the Ladder or the Λ configuration this means that the system is in the ground state. It can be decomposed into the basis of special solutions as

$$\psi(0) = -i\frac{a}{T}\psi_0(0) + \frac{k^2x}{\sqrt{2}\,T}(\psi_+(0) + \psi_-(0)). \tag{13}$$

The time-dependent solution is then

$$\begin{aligned}\psi(t) &= -i\frac{a}{T}\psi_0(t) + \frac{k^2x}{\sqrt{2}\,T}(\psi_+(t) + \psi_-(t)) \\ &= \mathrm{e}^{-(it/\hbar)H_0}\left[\frac{a}{T^2}\begin{pmatrix} a\,\mathrm{dn} \\ -ik^2x\,\mathrm{cn} \\ -iB\,\mathrm{sn}\end{pmatrix} + \sqrt{2}\,\frac{k^2x}{R(t)}\cos(\phi(t))\begin{pmatrix} k^2x\,\mathrm{cn} \\ ia\,\mathrm{dn} \\ 0\end{pmatrix}\right. \\ &\quad\left. + i\sqrt{2}\,\frac{k^2x}{TR(t)}\sin(\phi(t))\begin{pmatrix} -iaB\,\mathrm{sn}\,\mathrm{dn} \\ -k^2xB\,\mathrm{sn}\,\mathrm{cn} \\ k^4x^2\,\mathrm{cn}^2 + a^2\,\mathrm{dn}^2\end{pmatrix}\right].\end{aligned} \tag{14}$$

Clearly, all the three independent solutions are needed to obtain the time evolution for the given initial condition. It is also clear that the phase factor $\mathrm{e}^{\mp i\phi(t)}$ which appears in (6) when $k \neq 1$, although not so relevant for the special solutions, becomes highly relevant in the above quantum superposition.

As expected, the time-dependent interaction populates the two other states. See Figs. 2 and 3. Note that level (1) does not go below half occupation. This can be understood as follows. In each of the three solutions $\psi_0(t)$, $\psi_\pm(t)$ the occupation of level (1) cannot go much below 1. In other words, for the chosen parameters the system is in a regime of a weak and inefficient pumping. However, due to interference effects in the superposition (14), pronounced dips appear in the population of level (1).

5. Discussion

We obtained solitary solutions for a three-level system with periodic time-dependent external fields. Two aspects are novel.

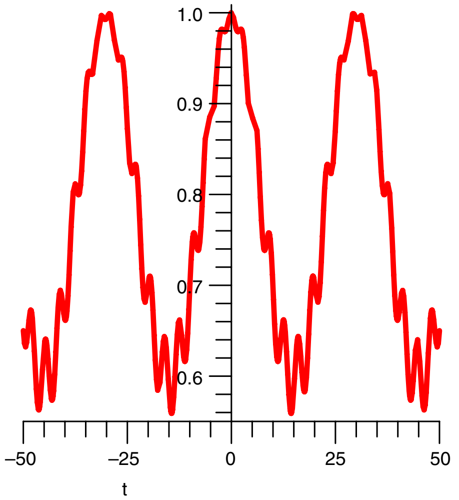

Fig. 2. Occupational probability of level (1) as a function of time for $k = 0.25$, $\hbar = \omega = 1$, $a = 0.3$ and $x = 1.6$.

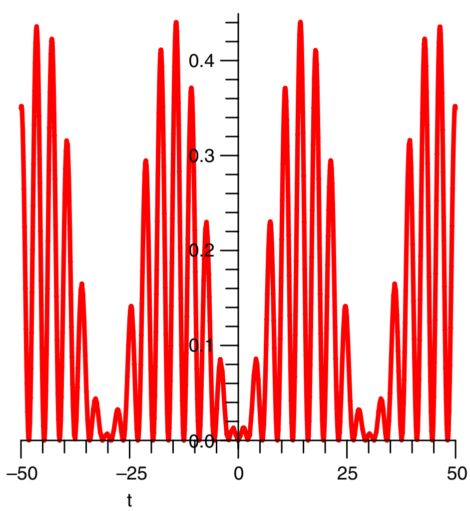

Fig. 3. Occupational probability of level (3) as a function of time for the same parameter values as in the previous figure.

The external fields are anharmonic in the sense that Jacobi's elliptic functions are used as deformations of the usual harmonic functions. In addition, a full set of special solutions is obtained so that arbitrary initial conditions can be considered.

Note that the three-level system under consideration is non-degenerate. The only matrices commuting with both S_4 and S_7, the two generators appearing in the Hamiltonian (2), are the multiples of the identity matrix. Only in the $k = 1$ limit (see Section 3) there exists a conserved quantity. In this limit the system reduces to a two-level problem, and a dark state appears.

The additional phase factor $\exp(\mp i\phi(t))$ appearing in solutions (5), (6) is known as a dynamical phase factor. It was first considered in [7]. It is not very relevant for the special solutions themselves, but has an effect on their superpositions. The function $\sin(\phi(t))$ is plotted in Fig. 4. Its frequency is slightly lower than the frequency $\omega/2\pi$ of the driving field. As a consequence, a low frequency beat appears when the special solutions are superimposed. This is dominantly visible in Figs. 2 and 3. For the chosen set of parameters the beat period is about 5 times the frequency of the external field. Note also the frequency doubling by which the exchange of population occurs between levels (2) and (3).

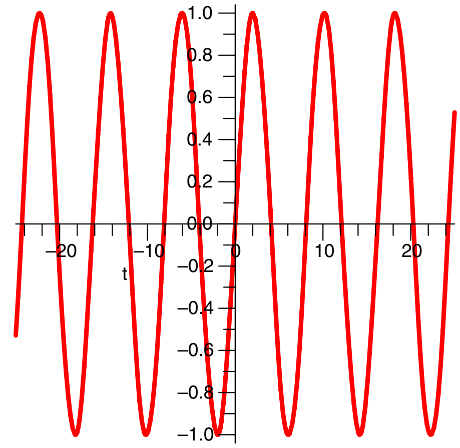

Fig. 4. The function $\sin(\phi(t))$ for the same parameter values as in the previous figures.

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We acknowledge discussions with Dr. Maciej Kuna about solving the non-linear von Neumann equation in the SU(3) case.

Appendix A. Generators

The following expressions are used for the generators of SU(3).

$$S_1 = \frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad S_2 = \frac{1}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$S_3 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad S_4 = \frac{1}{2}\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},$$

$$S_5 = \frac{1}{2}\begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}, \qquad S_6 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

$$S_7 = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix}, \qquad S_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}. \tag{A.1}$$

Appendix B. Method

Solutions (5), (6) were obtained starting from a known solution of the non-linear von Neumann equation

$$i\hbar\dot{\rho}_t = \frac{3}{2}[\{H_0, \rho_t\}, \rho_t], \tag{B.1}$$

where H_0 is given by

$$H_0 = \frac{2}{3}\begin{pmatrix}-\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \lambda\end{pmatrix}. \tag{B.2}$$

The three different configurations, V, Ladder, and Lambda, are obtained by taking $\lambda < -\mu < 0$, $|\lambda| < \mu$, $\lambda > \mu > 0$, respectively.

Let $H(t)$ be defined by $H(t) = \frac{3}{2}\{H_0, \rho_t\}$. Then $\rho(t)$ is a solution of the linear von Neumann equation with time-dependent Hamiltonian $H(t)$.

A known solution of the non-linear equation (B.1) is of the form [6,9]

$$\rho(t) = \frac{1}{3}\mathbb{I} + A\,\mathrm{cn}\,(\omega t, k)[S_4]_t + B\,\mathrm{sn}\,(\omega t, k)[S_1]_t + C\,\mathrm{dn}\,(\omega t, k)[S_7]_t$$

$$= \mathrm{e}^{-(it/\hbar)H_0} \begin{pmatrix} \frac{1}{3} & \frac{1}{2}B\,\mathrm{sn} & \frac{1}{2}A\,\mathrm{cn} \\ \frac{1}{2}B\,\mathrm{sn} & \frac{1}{3} & -\frac{i}{2}C\,\mathrm{dn} \\ \frac{1}{2}A\,\mathrm{cn} & \frac{i}{2}C\,\mathrm{dn} & \frac{1}{3} \end{pmatrix} \mathrm{e}^{(it/\hbar)H_0}, \qquad \text{(B.3)}$$

The coefficients A, B, and C, are real. They must satisfy the set of conditions

$$\begin{aligned} \hbar\omega B &= \mu AC \\ 2\hbar\omega k^2 C &= (\lambda+\mu)AB \\ -2\hbar\omega A &= (\lambda-\mu)BC. \end{aligned} \qquad \text{(B.4)}$$

This set of equations can be solved in a straightforward manner when $0 < |\lambda| < \mu$.

Next, a unitary matrix $V(t)$, satisfying

$$\rho_t = V(t)\rho_0 V(t)^\dagger \qquad \text{(B.5)}$$

is calculated. The fastest way to find $V(t)$ is by first diagonalizing ρ_t. Note that the eigenvalues of ρ_t do not depend on time. They are given by

$$\frac{1}{3} \quad \text{and} \quad \frac{1}{3} \mp \frac{1}{2}T \qquad \text{(B.6)}$$

with $T = \sqrt{A^2 + C^2}$. The result is

$$V(t) = \mathrm{e}^{-(it/\hbar)H_0} G(t) G(0)^\dagger \qquad \text{(B.7)}$$

with

$$G(t) = \frac{1}{\sqrt{2}TR(t)} \times \begin{pmatrix} i\sqrt{2}CR(t)\,\mathrm{dn} & -AT\,\mathrm{cn} - iBC\,\mathrm{sn}\,\mathrm{dn} & AT\,\mathrm{cn} - iBC\,\mathrm{sn}\,\mathrm{dn} \\ -\sqrt{2}AR(t)\,\mathrm{cn} & AB\,\mathrm{sn}\,\mathrm{cn} + iCT\,\mathrm{dn} & AB\,\mathrm{sn}\,\mathrm{cn} - iCT\,\mathrm{dn} \\ \sqrt{2}BR(t)\,\mathrm{sn} & R(t)^2 & R(t)^2 \end{pmatrix} \qquad \text{(B.8)}$$

and

$$R(t) = \sqrt{A^2\,\mathrm{cn}^2 + C^2\,\mathrm{dn}^2} = \sqrt{T^2 - B^2\,\mathrm{sn}^2}. \qquad \text{(B.9)}$$

However, $V(t)$ does not necessarily describe the unitary time evolution $U(t)$. But the latter can be related to $V(t)$ by the method of [7]. The details of the calculation of $U(t)$ are omitted here. The knowledge of $U(t)$ implies the time evolution of wave functions $\psi(t)$ for arbitrary initial conditions $\psi(t) = U(t)\psi(0)$. It turns out that the special solutions (5), (6) are the columns of the matrix $G(t)$, taken in the interaction picture. Two of the three solutions are multiplied with the time-dependent phase factor $\exp(\mp i\phi(t))$. Finally, note that conditions (B.4) are needed during the above derivation but are not required for (5), (6) to hold. They rather are replaced by the single condition (4).

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