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A Conley index study of the evolution of the Lorenz strange set[☆]

Héctor Barge^{a,*}, José M.R. Sanjurjo^b

^a E.T.S. Ingenieros informáticos, Universidad Politécnica de Madrid, 28660 Madrid, Spain

^b Facultad de C.C. Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

HIGHLIGHTS

- The evolution of the Lorenz system from the Conley index perspective is studied.
- Morse decompositions and equations of the global attractor are calculated.
- General Poincaré–Andronov–Hopf bifurcations are analyzed.
- Bifurcations with two homoclinic loops are investigated.
- The role of traveling repellers is examined.

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ABSTRACT

In this paper we study the Lorenz equations using the perspective of the Conley index theory. More specifically, we examine the evolution of the strange set that these equations possess throughout the different values of the parameter. We also analyze some natural Morse decompositions of the global attractor of the system and the role of the strange set in these decompositions. We calculate the corresponding Morse equations and study their change along the successive bifurcations. We see how the main features of the evolution of the Lorenz system are explained by properties of the dynamics of the global attractor. In addition, we formulate and prove some theorems which are applicable in more general situations. These theorems refer to Poincaré–Andronov–Hopf bifurcations of arbitrary codimension, bifurcations with two homoclinic loops and a study of the role of the traveling repellers in the transformation of repeller–attractor pairs into attractor–repeller ones.

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1. Introduction

Edward N. Lorenz studied in the 1960s a simplified model of fluid convection dynamics in the atmosphere [1]. This model is described by the following family of differential equations, now known as the Lorenz equations,

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = rx - y - z \\ \frac{dz}{dt} = xy - bz, \end{cases}$$

where σ , r and b are three real positive parameters corresponding respectively to the Prandtl number, the Rayleigh number and an adimensional magnitude related to the region under consideration. As we vary the parameters, we change the behavior of the flow determined by the equations in \mathbb{R}^3 . The values $\sigma = 10$ and $b = 8/3$ have deserved special attention in the literature. We shall fix them from now on, and we shall consider the family of flows obtained when we vary the remaining parameter, r .

Based on numerical studies of these equations, Lorenz found sensitive dependence on initial conditions and he emphasized the importance of this property in the study of natural phenomena, observing that, even in simple models, trajectories are sensitive to small changes in the initial conditions. He was able to prove that for every value of the parameter r there is a bounded region (an ellipsoid) where every trajectory eventually enters and never thereafter leaves. As a consequence, the existence of a global attractor Ω of zero volume is established. This attractor is the intersection of the successive images of the ellipsoid by the flow for increasing times and should not be confused with the famous Lorenz attractor, which is a proper subset of Ω .

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* Corresponding author.

E-mail addresses: h.barge@upm.es (H. Barge), jose_sanjurjo@mat.ucm.es (J.M.R. Sanjurjo).

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Afraïmovič, Bykov and Sil'nikov [2], Williams [3] and Guckenheimer and Williams [4] constructed and studied a geometric model of the system based on the numerically-observed features of the solutions of the Lorenz system. From this model, the existence of a robust attractor can be derived. Tucker [5,6] proved that, in fact, the Lorenz equations define a geometric Lorenz flow and, as a consequence, they admit an attractor. In [7] it is proved that this attractor is mixing. Tucker's results were preceded by Mischaikow and Mrozek [8,9] and Mischaikow, Mrozek and Szymczak [10] who gave a computer-assisted proof of the existence of chaos in the Lorenz equations. An important contribution to the study of the equations is the book [11] by Sparrow. This book was written long before Tucker's work was available and some of the global statements made in it are only tentative. However, except for a few details, they have proved to agree with Tucker's results. The topological classification of the Lorenz attractors (for different parameter values) can be found in the paper [12] by D. Rand. A recent study of the global organization of the phase space in the transition to chaos in the Lorenz system can be found in the recent paper [13] by Doedel, Krauskopf and Osinga (see also [14,15]). See also [16,17] by R. Barrio and S. Serrano for related results. An additional reference is the book [18], where the elements of a general theory for flows on three-dimensional manifolds are presented. The main motivation for this theory was, according to the authors, the Lorenz equations.

The present paper is devoted to the study of the Lorenz equations, using the perspective of the Conley index theory. More specifically, we examine the evolution of the strange set that these equations possess throughout the different values of the parameter. We also analyze some natural Morse decompositions of the global attractor of the system and the role of the strange set in these decompositions. We calculate the corresponding Morse equations and study their change along the successive bifurcations. We see how the main features of the evolution of the Lorenz system are explained by properties of the dynamics of the global attractor. Particular importance is given to the evolution through the preturbulent stage, just before the strange set becomes an attractor. It is proved that the transition from the preturbulent stage of the system to the turbulent one is marked by a change in the internal dynamics of the strange set, namely, an attractor–repeller splitting of the strange set, which is present at the preturbulent stage, ceases to exist at the turbulent stage. On the other hand, we see that the evolution of the system from the homoclinic bifurcation to the Hopf bifurcation corresponds to a transformation of a repeller–attractor decomposition of the global attractor into an attractor–repeller one. This transformation is achieved via a “traveling repeller”. The purpose of the paper is to give a global vision from both the dynamical and the topological perspectives and, based on the features of the Lorenz equations, formulate and prove some theorems which reach well beyond the scope of these equations and are applicable in more general situations. These theorems refer to Poincaré–Andronov–Hopf bifurcations of arbitrary codimension, bifurcations with two homoclinic loops and a study of the role of the traveling repellers in the transformation of repeller–attractor pairs into attractor–repeller ones.

2. Preliminaries

Through the paper we deal with families of flows $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ depending continuously on a parameter $\lambda \in [0, 1]$. In some occasions we assume that these families are induced by families of ODE's $\dot{X} = F_\lambda(X)$ depending differentiably on the parameter. In this case, it will be implicit that, for each λ , F_λ is a C^1 map.

Trajectories, limit sets and stability

The main reference for the elementary concepts of dynamical systems will be [19] but we also recommend [20–23].

Let $\varphi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a flow. Sometimes we write xt instead of $\varphi(x, t)$ in order to simplify the notation.

We shall use the notation $\gamma(x)$ for the *trajectory* of the point x , i.e.

$$\gamma(x) = \{xt \mid t \in \mathbb{R}\}.$$

Similarly for the *positive semi-trajectory* and the *negative semi-trajectory*

$$\gamma^+(x) = \{xt \mid t \in \mathbb{R}^+\}, \quad \gamma^-(x) = \{xt \mid t \in \mathbb{R}^-\}.$$

By the *omega-limit* of a point x we understand the set

$$\omega(x) = \bigcap_{t>0} \overline{x[t, \infty)}.$$

In an analogous way, the *negative omega-limit* is the set

$$\omega^*(x) = \bigcap_{t<0} \overline{x[-\infty, t]}.$$

We recall that if $\omega(x)$ (resp. $\omega^*(x)$) is compact then it must be connected.

Attractors

In the literature there are many different definitions of attractor as it has been pointed out by Milnor [24]. Among the definitions treated by Milnor we shall use that of an asymptotically stable compactum. An invariant compactum K is *stable* if every neighborhood U of K contains a neighborhood V of K such that $V[0, \infty) \subset U$. Similarly, K is *negatively stable* if every neighborhood U of K contains a neighborhood V of K such that $V(-\infty, 0] \subset U$.

The compact invariant set K is said to be *attracting* provided that there exists a neighborhood U of K such that $\emptyset \neq \omega(x) \subset K$, for every $x \in U$, and *repelling* if there exists a neighborhood U of K such that $\emptyset \neq \omega^*(x) \subset K$ for every $x \in U$.

An *attractor* (or *asymptotically stable compactum*) is an attracting stable set and a *repeller* is a repelling negatively stable set. We stress the fact that stability (positive or negative) is required in the definition of attractor or repeller.

If K is an attractor, its region (or basin) of attraction of K is the set

$$\mathcal{A}(K) = \{x \in M \mid \emptyset \neq \omega(x) \subset K\}.$$

It is well known, that $\mathcal{A}(K)$ is an open invariant set. If in particular $\mathcal{A}(K)$ is the whole phase space we say that K is a *global attractor*.

Dissipative flows

Let M be a non-compact, locally compact metric space. A flow $\varphi : M \times \mathbb{R} \rightarrow M$ is *dissipative* provided that for each $x \in M$, $\omega(x) \neq \emptyset$ and the closure of the set

$$\Omega(\varphi) = \bigcup_{x \in M} \omega(x)$$

is compact.

The dissipativeness of φ is equivalent to the existence of a global attractor or, equivalently, to $\{\infty\}$ being a repeller for the flow extended to the Alexandroff compactification of M , leaving ∞ fixed. This was proved by Pliss [25]. Dissipative flows have been introduced by Levinson [26]. An interesting reference regarding dissipative flows is [27].

Isolated invariant sets and isolating blocks

An important class of invariant compacta is the so-called *isolated invariant sets* (see [28–30] for details). These are compact invariant sets K which possess an *isolating neighborhood*, i.e. a compact neighborhood N such that K is the maximal invariant set in N .

A special kind of isolating neighborhoods will be useful in the sequel, the so-called *isolating blocks*, which have good topological properties. More precisely, an isolating block N is an isolating neighborhood such that there are compact sets $N^i, N^o \subset \partial N$, called the entrance and the exit sets, satisfying

- (1) $\partial N = N^i \cup N^o$;
- (2) for each $x \in N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset M \setminus N$ and for each $x \in N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset M \setminus N$;
- (3) for each $x \in \partial N \setminus N^i$ there exists $\varepsilon > 0$ such that $x[-\varepsilon, 0) \subset \dot{N}$ and for every $x \in \partial N \setminus N^o$ there exists $\delta > 0$ such that $x(0, \delta] \subset \dot{N}$.

These blocks form a neighborhood basis of K in M . If the flow is differentiable, the isolating blocks can be chosen to be manifolds which contain N^i and N^o as submanifolds of their boundaries and such that $\partial N^i = \partial N^o = N^i \cap N^o$.

Hartman-Grobman blocks and complex invariant manifolds

Let $\dot{X} = F(X)$ an ODE defined in \mathbb{R}^n and let φ be the (local) flow induced by this vector field. Suppose that $\dot{X} = F(X)$ possesses a hyperbolic fixed point p and let φ_* the flow induced by the linearization $\dot{Y} = dF(p)Y$ of $\dot{X} = F(X)$. Then, Hartman–Grobman Theorem (see [31, Chapter 2, pg. 120] or [32, Theorem II.3, pg. 53]) ensures that there exist neighborhoods U of p and V of 0 in \mathbb{R}^n and a homeomorphism $h : U \rightarrow V$ such that $h(\varphi(x, t)) = \varphi_*(h(x), t)$ if $\varphi(x, [0, t]) \subset U$. Let $\delta > 0$ be such that the closed ball $\overline{B_\delta(0)}$ of radius δ centered at 0 is contained in V . Notice that $\overline{B_\delta(0)}$ is an isolating block of $\{0\}$ for φ_* if the norm of \mathbb{R}^n is chosen properly. Then, it follows that $h^{-1}(\overline{B_\delta(0)})$ is an isolating block of $\{p\}$ for φ . We shall call this kind of blocks *Hartman–Grobman blocks* of the hyperbolic fixed point $\{p\}$.

Consider an ODE $\dot{X} = F(X)$ defined in \mathbb{R}^3 and let p be a hyperbolic fixed point having one negative eigenvalue β and two complex conjugated eigenvalues $\mu \pm vi$ with $\mu < 0$. Let $\dot{Y} = dF(p)Y$ be the linearization of $\dot{X} = F(X)$, E the invariant 2-dimensional subspace associated to the complex eigenvalues and $\delta > 0$ such that $\overline{B_\delta(0)} \subset V$. We call *local complex invariant manifold* of p to the positively invariant open 2-disk $\text{Loc } W^c = h^{-1}(E \cap \overline{B_\delta(0)})$. The *complex invariant manifold* of p is defined as the set of points whose forward trajectory eventually enters $\text{Loc } W^c$, i.e.

$$W^c = \{x \in \mathbb{R}^3 \mid xt \in \text{Loc } W^c \text{ for some } t > 0\}.$$

Algebraic topology and shape theory

We use some topological notions through this paper. We recommend the books of Hatcher and Spanier [33,34] to cover this material. We use the notation H^* for Čech cohomology. We consider cohomology taking coefficients in \mathbb{Z} . We recall that Čech and singular cohomology theories agree on polyhedra and manifolds and, more generally, on ANRs.

If a pair of spaces (X, A) satisfies that its cohomology $H^k(X, A)$ is finitely generated for each k and is non-zero only for a finite number of values of k (as it happens if (X, A) is a pair of compact manifolds), its *Poincaré polynomial* is defined as

$$P_t(X, A) = \sum_{k \geq 0} \text{rk } H^k(X, A) t^k.$$

There is a form of homotopy which has proved to be the most convenient for the study of the global topological properties of the invariant spaces involved in dynamics, namely *Borsuk’s homotopy theory* or *shape theory*, introduced and studied by Karol Borsuk. We are not going to make a deep use of shape theory but we recommend to the interested reader the books [35–37] for an exhaustive treatment of the subject, and the papers [38–45] for a short comprehensive introduction and some applications to dynamical systems. We only recall that shape theory and homotopy theory agree when dealing with manifolds, CW-complexes or, more generally, ANRs and that Čech cohomology is a shape invariant.

Conley index

Let K be an isolated invariant set. Its *Conley index* $h(K)$ is defined as the pointed homotopy type of the topological space $(N/N^o, [N^o])$, where N is an isolating block of K . A weak version of the Conley index which will be useful for us is the *cohomological index* defined as $CH^*(K) = H^*(h(K))$. It can be proved that $CH^*(K) \cong H^*(N, N^o)$. Our main references for the Conley index theory and its applications are [28,46–49]. In addition, some applications of the Conley index theory to the study of the Lorenz equations can be seen in [50–52].

Morse decompositions and equations

We recall that if K is a compact invariant set, the finite collection $\{M_1, \dots, M_n\}$ of pairwise disjoint invariant subcompacta of K is a *Morse decomposition* if it satisfies that

$$\text{for each } x \in \left(K \setminus \bigcup_{i=1}^n M_i \right), \quad \omega(x) \subset M_j \text{ and } \omega^*(x) \subset M_k$$

with $j < k$.

Each set M_i is said to be a *Morse set*.

Given a Morse decomposition $\{M_1, M_2, \dots, M_k\}$ of an isolated invariant set K , there exists a polynomial $Q(t)$ whose coefficients are non-negative integers such that

$$\sum_{i=1}^n P_t(h(M_i)) = P_t(h(K)) + (1 + t)Q(t).$$

This formula, which relates the Conley indices of the Morse sets with the Conley index of the isolated invariant set is known as the *Morse equations* of the Morse decomposition and it generalizes the classical Morse inequalities.

Hausdorff distance

Let X be a complete metric space with metric d and consider the *hyperspace* $\mathcal{H}(X)$ whose elements are the non-empty subcompacta of X . We recall that the *Hausdorff distance* in $\mathcal{H}(X)$ is defined as

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid B \subset A_\varepsilon \text{ and } A \subset B_\varepsilon\},$$

where A_ε and B_ε denote the open ε -neighborhoods of A and B , with respect to the metric d , respectively. For more information about the Hausdorff distance and its properties we recommend de book [53].

Continuations of isolated invariant sets

In this paper the concept of continuation of isolated invariant sets plays a crucial role. Let M be a locally compact metric space, and let $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ be a parametrized family of flows (parametrized by $\lambda \in [0, 1]$, the unit interval). The family $(K_\lambda)_{\lambda \in J}$, where $J \subset [0, 1]$ is a closed (non-degenerate) subinterval and, for each $\lambda \in J$, K_λ is an isolated invariant set for φ_λ is said to be a *continuation* if for each $\lambda_0 \in J$ and each N_{λ_0} isolating neighborhood for K_{λ_0} , there exists $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$. We say that the family $(K_\lambda)_{\lambda \in J}$ is a continuation of K_{λ_0} for each $\lambda_0 \in J$.

Notice that [48, Lemma 6.1] ensures that if K_{λ_0} is an isolated invariant set for φ_{λ_0} , there always exists a continuation $(K_\lambda)_{\lambda \in J_{\lambda_0}}$ of K_{λ_0} for some closed (non-degenerate) subinterval $J_{\lambda_0} \subset [0, 1]$.

There is a simpler definition of continuation based on [48, Lemma 6.2]. There, it is proved that if $\varphi_\lambda : M \times \mathbb{R} \rightarrow M$ is a parametrized family of flows and if N_1 and N_2 are isolating neighborhoods of the same isolated invariant set for φ_{λ_0} , then there exists $\delta > 0$ such that N_1 and N_2 are isolating neighborhoods for φ_λ , for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap [0, 1]$, with the property that, for every λ , the isolated invariant subsets in N_1 and N_2 which have N_1 and N_2 as isolating neighborhoods coincide.

Therefore, the family $(K_\lambda)_{\lambda \in J}$, with K_λ an isolated invariant set for φ_λ , is a continuation if for every $\lambda_0 \in J$ there are an isolating neighborhood N_{λ_0} for K_{λ_0} and a $\delta > 0$ such that N_{λ_0} is an isolating neighborhood for K_λ , for every $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$.

Notice that, since this should not lead to any confusion, sometimes we will only say that K_λ is a continuation of K_{λ_0} without specifying the subinterval $J \subset [0, 1]$ to which the parameters belong.

We shall make use of [54, Theorem 4] which states that if K_{λ_0} is an attractor for φ_{λ_0} and $(K_\lambda)_{\lambda \in J}$ is a continuation of K_{λ_0} , then there exists $\delta > 0$ such that K_λ is an attractor of the same shape of K_{λ_0} for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta) \cap J$.

3. Generalized pitchfork bifurcations

We shall use along the paper some facts about the Lorenz equations which can be found in the existing literature. We recommend, in particular, the book by Sparrow [11].

For $r \leq 1$ the origin is a global attractor (this includes $r = 1$ although for $r = 1$ there are two negative eigenvalues and the third is equal to zero). For $r > 1$ there are two additional singularities C_1 and C_2 which are attractors until $r = 24.74$ (when a Hopf bifurcation takes place). For $r > 1$ the origin is a hyperbolic fixed point with a two-dimensional stable manifold and a one-dimensional unstable manifold. All the points not lying in $W^s(0)$ are attracted by C_1 or C_2 until the value $r = 13.926$, when a homoclinic bifurcation takes place. For all r with $1 < r < 13.926$, the unstable manifold of 0 consists of two orbits attracted by C_1 and C_2 respectively, together with 0. Hence, at $r = 1$, a pitchfork bifurcation takes place in the origin, which is an attractor for $r = 1$ and becomes a hyperbolic non stable fixed point for $r > 1$. We summarize the discussion in the following statement:

– For $r \leq 1$ the origin is a global attractor and for $r > 1$ it becomes a hyperbolic fixed point with a two-dimensional stable manifold and a one-dimensional unstable manifold.

This is a particular example of a phenomenon which can be studied in a more general form in \mathbb{R}^n with an arbitrary distribution of positive and negative eigenvalues. There are two extreme

cases: (a) when the origin becomes a hyperbolic point with dimension of $W^u(0)$ equal to 1 (which is the current situation with $n = 3$) and (b) when the origin becomes a hyperbolic point with dimension of $W^u(0)$ equal to n or, in other words, the origin becomes a repeller. The second case has been called a *generalized Poincaré–Andronov–Hopf bifurcation* [51,55]. We would like to study in detail this phenomenon for arbitrary dimension of $W^u(0)$ because, when it takes place, an interesting invariant object is created near the origin, namely an attractor with the Borsuk homotopy type (or shape) of a sphere of dimension one unit less than the dimension of $W^u(0)$.

In order to state our next result, we must introduce first a definition which is applicable in the following situation: Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a family of flows induced by a system $\dot{X} = F_\lambda(X)$ of ODE in \mathbb{R}^n which depend differentiably on a parameter $\lambda \in [0, 1]$ and suppose that 0 is an equilibrium for every λ . Suppose, additionally, that 0 is an attractor for $\lambda = 0$ and a hyperbolic fixed point with exactly k positive and $n - k$ negative eigenvalues for $\lambda > 0$ (hence, $W_\lambda^s(0)$ is an immersed $(n - k)$ -dimensional manifold). We say that the family is *rigid* at $\lambda = 0$ if there is an $\varepsilon > 0$ arbitrarily small and a $\lambda_\varepsilon > 0$ such that every trajectory of $W_\lambda^s(0)$ other than 0 leaves $\overline{B_\varepsilon(0)}$ in the past and the pair $(\overline{B_\varepsilon(0)}, \overline{B_\varepsilon(0)} \cap W_\lambda^s(0))$ is homeomorphic to the pair $(\overline{B_n}, \overline{B_{n-k}})$ (the unit closed balls of dimensions n and $(n - k)$ respectively) for every λ with $0 < \lambda < \lambda_\varepsilon$. Rigidity is a kind of uniformity condition for the local stable manifolds (which are known to be topological $(n - k)$ -balls), which prevents them from collapsing immediately after $\lambda = 0$ (it is not difficult to describe situations where that phenomenon occurs).

In the case of the Lorenz system immediately after the pitchfork bifurcation ($r > 1$), the stable manifold $W^s(0)$ of the origin can be regarded, at least near the origin, as a plane, the plane associated with the two negative eigenvalues of the flow linearized near the origin (see [11, p. 13]). Thus, for ε sufficiently small, every trajectory other than 0 leaves $\overline{B_\varepsilon(0)}$ in the past and the pair $(\overline{B_\varepsilon(0)}, \overline{B_\varepsilon(0)} \cap W^s(0))$ is homeomorphic to the pair $(\overline{B_3}, \overline{B_2})$ for all values of the parameter sufficiently close to $r = 1$. Hence the Lorenz system is rigid at the value of the pitchfork bifurcation.

Theorem 1. *Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ be a family of flows induced by a system $\dot{X} = F_\lambda(X)$ of ODE in \mathbb{R}^n depending differentiably on a parameter $\lambda \in [0, 1]$ and suppose that 0 is an equilibrium for every λ . Suppose that $\{0\}$ is an attractor for $\lambda = 0$ and a hyperbolic fixed point with exactly k positive and $n - k$ negative eigenvalues for $\lambda > 0$. We assume that $W_\lambda^s(0)$ is rigid at $\lambda = 0$. Then there exists a $\lambda_0 > 0$ such that for every λ with $0 < \lambda < \lambda_0$ there exists an attractor A_λ with the Borsuk homotopy type (shape) and the cohomology of the sphere S^{k-1} . The Conley index of A_λ is the homotopy type of $(S^{k-1} \cup \{*\}, *)$ and its cohomological Conley index is \mathbb{Z} for $i = 0, k - 1$ and $\{0\}$ otherwise if $k > 1$ and $\mathbb{Z} \oplus \mathbb{Z}$ for $i = 0$ and $\{0\}$ otherwise if $k = 1$. Moreover, the family A_λ shrinks to 0 when $\lambda \rightarrow 0$ (in particular, if $k = 2$ we have a family of attractors with the shape of S^1 shrinking to 0). Moreover, the attractor A_λ is contained in an attractor K_λ of trivial shape which contains the origin and such that $(A_\lambda, \{0\})$ is an attractor–repeller decomposition of K_λ whose Morse equations are*

$$1 + t^{k-1} + t^k = 1 + (1 + t)t^{k-1}.$$

The family (K_λ) also shrinks to 0. In the particular case of the Lorenz flow, A_λ consists of two equilibria, i.e. $A_\lambda = S^0$ and K_λ is the union of A_λ with the unstable manifold of the origin i.e. $K_\lambda \approx \overline{B_1}$ and the Morse equations are

$$2 + t = 1 + (1 + t).$$

Proof. Since $\{0\}$ is an attractor of φ_λ for $\lambda = 0$, there is a continuation (K_λ) of $\{0\}$, where K_λ is an attractor of trivial shape of φ_λ for λ sufficiently small [54, Theorem 4]. Since K_λ is the maximal invariant set of φ_λ in a neighborhood of 0 and 0 is an equilibrium of φ_λ we have that $0 \in K_\lambda$. We define $A_\lambda = K_\lambda \setminus W_\lambda^u(0)$. First we see that $W_\lambda^u(0) \subset K_\lambda$. Otherwise, there exists a point $x \in W_\lambda^u(0)$ such that $x \notin K_\lambda$ and we arrive at a contradiction as follows. Notice that x must be in the region of attraction $\mathcal{A}_\lambda(K_\lambda)$ of K_λ since $0 \in K_\lambda$ and $\mathcal{A}_\lambda(K_\lambda)$ is a neighborhood of K_λ , which implies that xt must be in $\mathcal{A}_\lambda(K_\lambda) \setminus K_\lambda$ for certain negative value of t . Since $\mathcal{A}_\lambda(K_\lambda) \setminus K_\lambda$ is invariant, the whole trajectory of x is in $\mathcal{A}_\lambda(K_\lambda) \setminus K_\lambda$ and, hence, $\emptyset \neq \omega_\lambda(x) \subset K_\lambda$. In addition, since $x \in W_\lambda^u(0)$, $\omega^*(x) = \{0\} \subset K_\lambda$. As a consequence, $K_\lambda \cup \varphi_\lambda(x, \mathbb{R})$ is a compact invariant set contained in $\mathcal{A}_\lambda(K_\lambda)$. This contradicts the stability of K_λ , since K_λ being stable, must be the maximal compact invariant set contained in its region of attraction. We shall see now that A_λ is compact. Since (K_λ) is a continuation of $\{0\}$ we have that $K_\lambda \subset B_\varepsilon(0)$ for $0 < \lambda < \lambda_0$ where $\varepsilon > 0$ is chosen using the rigidity of $W_\lambda^s(0)$. If A_λ is not compact then there exists a sequence of points $x_n \in A_\lambda = K_\lambda \setminus W_\lambda^u(0)$ such that $x_n \rightarrow x$ and $x \in W_\lambda^u(0)$. Since K_λ and $W_\lambda^u(0)$ are invariant then A_λ is also invariant and we can assume that $x_n \rightarrow 0$. This is proved as follows. Since $x \in W_\lambda^u(0)$ we have that $xt_k \rightarrow 0$ for a certain sequence $t_k \rightarrow -\infty$. For every k select a x_{n_k} such that $x_{n_k}t_k$ is $1/k$ -close to xt_k . Hence $x_{n_k}t_k \rightarrow 0$ with $x_{n_k}t_k \in A_\lambda$. Consider now a Hartman–Grobman block H_λ for φ_λ contained in $B_\varepsilon(0)$. Since the points x_n are not in $W_\lambda^u(0)$ there exists, for each n , $t_n < 0$ such that $x_n t_n \in \partial H_\lambda$ and $x_n [t_n, 0] \subset H_\lambda$. Since ∂H_λ is compact we may assume that $x_n t_n \rightarrow y \in \partial H_\lambda$. Notice that the sequence $t_n \rightarrow -\infty$ since, otherwise, we may assume that it converges to some $t_0 \leq 0$ and, hence $x_n t_n$ converges to $0t_0 = 0$ which is clearly not in ∂H_λ . Let us see that $yt \in H_\lambda$ for each $t \geq 0$ and, as a consequence, $y \in W_\lambda^s(0)$. Let $t \geq 0$, then, since $t_n \rightarrow -\infty$, there exists n_0 such that $t + t_n < 0$ for every $n \geq n_0$. Thus $x_n(t + t_n) \in H_\lambda$ for each $n \geq n_0$ and, since the sequence $x_n(t_n + t)$ converges to yt , it follows from the compactness of H_λ that $yt \in H_\lambda$. As a consequence, the rigidity condition ensures that the trajectory of y must leave $B_\varepsilon(0)$ and, thus, K_λ , which is in contradiction with the invariance of K_λ . This contradiction proves the compactness of A_λ . Moreover the pair $(A_\lambda, \{0\})$ is an attractor–repeller decomposition of K_λ . Indeed, we see that $\{0\}$ is a repeller for $\varphi_\lambda|_{K_\lambda}$. Suppose that $\{0\}$ is not a repeller for $\varphi_\lambda|_{K_\lambda}$, then [48, Lemma 3.1] ensures that any compact neighborhood U of 0 in K_λ disjoint from A_λ contains a point x , other than 0, such that $\gamma^+(x) \subset U$. Since U isolates $\{0\}$ in K_λ , it follows that $\omega_\lambda(x) = \{0\}$ and the rigidity condition ensures that the trajectory of x must leave K_λ in contradiction with the invariance of K_λ . Notice that $W_\lambda^u(0)$ is the region of repulsion of $\{0\}$ and, hence, $A_\lambda = K_\lambda \setminus W_\lambda^u(0)$ is its complementary attractor. Since K_λ is an attractor and A_λ is an attractor in K_λ then A_λ is an attractor of the flow φ_λ . We consider the attractor–repeller cohomology sequence of the decomposition $(A_\lambda, \{0\})$ of K_λ

$$\dots \rightarrow CH^{i-1}(K_\lambda) \rightarrow CH^{i-1}(A_\lambda) \rightarrow CH^i(\{0\}) \rightarrow CH^i(K_\lambda) \rightarrow \dots$$

Since K_λ is a continuation of the attractor $\{0\}$ of φ_0 we know its cohomology index and the cohomology index of $\{0\}$ for φ_λ (because 0 is now a hyperbolic point). We deduce from this the cohomology index of A_λ which is \mathbb{Z} in dimension $k - 1$ when $k > 1$ and $\mathbb{Z} \oplus \mathbb{Z}$ in dimension 0 when $k = 1$. On the other hand by the rigidity condition $\text{Loc } W_\lambda^s(0)$ is uniformly locally flat and this implies that if we take δ sufficiently small we have $\overline{B_\delta(0)}$ is contained in the region of attraction of K_λ and $\overline{B_\delta(0)} \setminus W_\lambda^s(0)$ is homeomorphic to $\overline{B_n} \setminus \overline{B_{n-k}}$ which is homotopy equivalent to S^{k-1} . Using the flow we can define a sequence of maps $r_k : \overline{B_\delta(0)} \setminus W_\lambda^s(0) \rightarrow \mathbb{R}^n$ by

$$r_k(x) = \varphi_\lambda(x, k).$$

Since A_λ is an attractor and $\overline{B_\delta(0)} \setminus W_\lambda^s(0)$ is contained in its region of attraction, it follows that given any neighborhood U of A_λ there exists $k_0 \in \mathbb{N}$ such that the image of r_k is contained in U for every $k \geq k_0$. In addition, the flow defines, in a natural way, a homotopy between r_k and r_{k+1} , for each $k \geq k_0$, taking place in U . As a consequence, this family of maps defines an *approximative sequence*

$$\mathbf{r} = \{r_k, \overline{B_\delta(0)} \setminus W_\lambda^s(0) \rightarrow A_\lambda\},$$

in the sense of Borsuk [35] and, hence, a shape morphism. Since $r_k|_{A_\lambda}$ is homotopic to the identity for each k , it follows that the shape morphism induced by the inclusion $i : A_\lambda \hookrightarrow \overline{B_\delta(0)} \setminus W_\lambda^s(0)$ is a left inverse for \mathbf{r} and, therefore

$$\text{Sh}(S^{k-1}) = \text{Sh}(\overline{B_\delta(0)} \setminus W_\lambda^s(0)) \geq \text{Sh}(A_\lambda).$$

On the other hand, since the cohomology Conley index of A_λ is \mathbb{Z} in dimension $k - 1$ if $k > 1$ and $\mathbb{Z} \oplus \mathbb{Z}$ in dimension 0 if $k = 1$, it follows that $H^*(A_\lambda) \neq H^*(\{*\})$. Now since $\text{Sh}(S^{k-1}) \geq \text{Sh}(A_\lambda)$ and $H^*(A_\lambda) \neq H^*(\{*\})$ Borsuk–Holsztyński Theorem [56], which ensures that if a compactum K satisfies that $\text{Sh}(K) \leq \text{Sh}(S^n)$ and K does not have the shape of a point then $\text{Sh}(K) = \text{Sh}(S^n)$, applies and we have that, in fact, $\text{Sh}(S^{k-1}) = \text{Sh}(A_\lambda)$. A direct consequence of this fact is that the Conley index of A_λ is the homotopy type of $(S^{k-1} \cup \{*\}, *)$.

From the previous discussion it readily follows that $CH^i(A_\lambda)$ is \mathbb{Z} for $i = 0, k - 1$ and zero otherwise when $k > 1$ and $\mathbb{Z} \oplus \mathbb{Z}$ for $i = 0$ and zero otherwise when $k = 1$, $CH^i(K_\lambda)$ is \mathbb{Z} for $i = 0$ and zero otherwise and $CH^i(\{0\})$ is \mathbb{Z} for $i = k - 1$ and zero otherwise. Combining all of this we get the desired Morse equations for the attractor–repeller decomposition $(A_\lambda, 0)$ of K_λ . \square

Remark 2. Our previous result can be looked at as describing either a generalized pitchfork bifurcation or a generalized Poincaré–Andronov–Hopf bifurcation of arbitrary codimension.

4. Transient chaos

For a parameter value approximately equal to 13.926..., the behavior of the flow experiments an important change. At this critical value the stable manifold of the origin includes the unstable manifold of the origin; i.e. trajectories started in the unstable manifold of the origin tend, in both positive and negative time, to the origin. As a consequence, a couple of homoclinic orbits is produced, one for every branch of the unstable manifold and we say that a *homoclinic bifurcation* has taken place at the parameter value $r_H = 13.926...$ This parameter value signals the appearance of a phenomenon known as *preturbulence* or *transient chaos*, whose study was carried out by Kaplan and Yorke and by Yorke and Yorke in [57,58]. This phenomenon is characterized by the fact that certain trajectories behave chaotically for a while, before escaping to an external attractor. Turbulent trajectories also exist but represent a set of measure zero. By using arguments similar, to a certain extent, to Smale’s horseshoe [59] they proved that for $r > r_H$ a countable infinity of periodic orbits is created together with an uncountable infinity of bounded trajectories that are asymptotically periodic (in either forwards or backwards time) and an uncountable infinity of bounded aperiodic trajectories. These aperiodic trajectories were termed as *turbulent* by Ruelle and Takens [60] because their limit sets are neither points, nor periodic orbits, nor manifolds. Sparrow remarked that also an uncountable infinity of *bounded trajectories which terminate in the origin* is produced. The union of all these trajectories together with the origin forms an invariant “strange set” K_r which exhibits sensitive dependence on initial conditions. By studying a return map of the flow with respect to a suitable Poincaré section, Sparrow proved, relying on Kaplan and Yorke’s results, that the

intersections of the trajectories of K_r with the Poincaré section can be coded by bisequences of two symbols S and T such that repeating sequences correspond to periodic orbits, sequences which terminate on the right correspond to trajectories which terminate in the origin and aperiodic sequences correspond to trajectories which oscillate aperiodically.

In the sequel we analyze the nature and the evolution of the strange sets K_r from the point of view of Conley's index theory. To get our conclusions, we use some facts that have been established by Kaplan–Yorke [57] and Sparrow [11].

1. *The strange sets are isolated invariant sets and they define a continuation (in the sense of Conley's theory) of the double homoclinic loop.*

Every K_r is a compact isolated invariant set. As a matter of fact, if we take a neighborhood \mathcal{N}_r of the double homoclinic loop, consisting of a small box B around the origin, together with two tubes, S and T around the two branches of the loop (see [11, Appendix D, pg. 199]), we have that K_r is the maximal invariant set inside this neighborhood for values of r close to that of the homoclinic bifurcation. The passage of the trajectories of K_r through the tubes is in correspondence with the codification with two symbols previously stated, and this is the explanation for the use of the same notation. We clearly have that the family (K_r) , for $r > r_H$, is a continuation (in the sense of Conley's theory), of the double homoclinic loop which originates the homoclinic bifurcation at $r = r_H$.

2. *The continuation is continuous in the Hausdorff metric for $r = r_H$.*

As a matter of fact, each of tubes S and T contains exactly one periodic orbit which does not wind around the z -axis. The notation S and T is also used to designate these two simplest orbits. Then, if we fix $\varepsilon > 0$ we have that the neighborhood \mathcal{N}_r can be chosen to be contained in the ε -neighborhood of the double homoclinic loop for values of r sufficiently close to r_H and, hence, so is K_r . On the other hand, the ε -neighborhood of the orbits S and T (and hence the ε -neighborhood of K_r) contains the double homoclinic loop for r sufficiently close to r_H . This proves that K_r converges to the double homoclinic loop when $r \rightarrow r_H$.

3. *The strange sets have the cohomological Conley index of the circle.*

The cohomological Conley index of K_r is isomorphic to $H^*(S^1, *)$, where S^1 is the circle. This is a consequence of the fact that the origin $\{0\}$ is a continuation of the double homoclinic loop for $r < r_H$. Since the cohomological Conley index is preserved by continuation and the index of the origin is isomorphic to $H^*(S^1, *)$ then the index of the double homoclinic loop and also that of its continuation K_r for $r > r_H$ must be the same.

4. *The strange sets are repellers in an attractor–repeller decomposition of the global attractor Ω_r of the flow.*

The strange set K_r is contained in the global attractor Ω_r . Since all the trajectories in Ω_r not contained in K_r terminate in C_1 or C_2 we must have that the ω^* -limit of these trajectories (else than C_1 or C_2) must be contained in K_r . As a consequence $(\{C_1, C_2\}, K_r)$ is an attractor–repeller decomposition of the global attractor Ω_r .

5. *The strange sets K_r are not chaotic but they admit an attractor–repeller decomposition $(\{0\}, L_r)$ where L_r is chaotic. the set L_r is the suspension of a smale horseshoe but the strange set K_r is not.*

Contrarily to some statements in the literature, the strange set K_r is not chaotic, since there is not a single trajectory in K_r whose closure contains the trajectories terminating in the origin. This was remarked by Sparrow in [11]. However, if we consider all the trajectories in K_r except those terminating in the origin we obtain a chaotic invariant set L_r . As a matter of fact, this was the set discovered and studied by Kaplan and Yorke in [57] where they proved that L_r has sensitive dependence on initial conditions, the set of periodic orbits is dense in L_r and it contains an uncountable infinity of aperiodic dense trajectories. This set is the suspension of a return map of the flow with respect to a suitable Poincaré section studied by Sparrow, whose dynamics is that of the Smale horseshoe. On the other hand the existence of a fixed point in K_r prevents the strange set from being a suspension. As we prove in our next result, L_r is an isolated invariant set with trivial cohomological index and the pair $(\{0\}, L_r)$ defines an attractor–repeller decomposition of K_r . We deduce from this that $(\{C_1, C_2\}, \{0\}, L_r)$ is a Morse decomposition of the global attractor Ω_r . The Morse equations of this decomposition are obtained also in our next result, where we analyze a situation which is more general than the one described here.

6. *The strange sets K_r have the cohomology of the figure eight.*

In spite of its dynamical and topological complexity, the strange set K_r has the cohomology of the figure eight. This is a consequence of a more general result proved in our next theorem.

Our study of the evolution of the strange set concerns mainly asymptotic properties of its internal structure and of the structure of the global attractor of the flow. Recently, E.J. Doedel, B. Krauskopf and H.M. Osinga [13] have performed a study of the global organization of the phase space in the transition to chaos where they show how global invariant manifolds of equilibria and periodic orbits change with the parameters.

The following is a result of a general nature which has been suggested by the previous discussion on the evolution of the Lorenz strange set. Some of the remarks previously made are consequences of this theorem.

Theorem 3. *Let $\varphi_\lambda : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ be a dissipative family of flows induced by a system $\dot{X} = F_\lambda(X)$ of ODE in \mathbb{R}^3 depending differentiably on a parameter $\lambda \in [0, 1]$. Suppose that 0 is a hyperbolic equilibrium for every λ with exactly one positive and two negative eigenvalues and that there are two other hyperbolic equilibria C_1 and C_2 , both of them having one real negative eigenvalue β_λ and two conjugate complex eigenvalues $\mu_\lambda \pm \nu_\lambda i$ with $\mu_\lambda < 0$ for every λ . Suppose that for $\lambda = 0$ the fixed point 0 has two homoclinic trajectories corresponding with the two branches of its unstable manifold and that the points C_1 and C_2 attract all bounded orbits of \mathbb{R}^3 not lying in $W_0^s(0)$ and suppose, additionally, that for $\lambda > 0$ the two branches of $W_\lambda^u(0)$ connect the point 0 with C_1 and C_2 respectively and that $W_\lambda^s(0) \setminus \{0\}$ contains at least one bounded orbit. Then:*

- (a) *For $\lambda = 0$, the ω^* -limit of every bounded orbit different from the stationary orbits C_1 and C_2 is contained in the double homoclinic loop $W_0^u(0)$.*
- (b) *For $\lambda > 0$ the set of bounded trajectories of φ_λ other than those finishing in C_1 or C_2 is a non-empty isolated invariant set K_λ whose cohomology Conley index is isomorphic to $H^*(S^1, *)$. Moreover $(\{C_1, C_2\}, K_\lambda)$ is an attractor–repeller decomposition of the global attractor Ω_λ and K_λ itself has*

a finer attractor–repeller decomposition $(\{0\}, L_\lambda)$ where L_λ consists of all bounded trajectories not ending neither in the origin nor in C_1 or C_2 . The set L_λ has trivial cohomology index and the triple $\{\{C_1, C_2\}, \{0\}, L_\lambda\}$ is a Morse decomposition of the global attractor whose Morse equations are

$$2 + t = 1 + (1 + t).$$

- (c) If the complex invariant manifolds of the points C_1 and C_2 consist of all the bounded orbits finishing in C_1 and C_2 respectively (as is the case in the Lorenz equations) then the cohomology of K_λ agrees with that of the figure eight.

Proof. To prove part (a) consider the global attractor Ω_0 of the flow φ_0 . Since $\{C_1, C_2\}$ is an attractor contained in Ω_0 there exists a dual repeller for the flow φ_0 restricted to Ω_0 . Obviously the double loop $W_0^u(0)$ is contained in this repeller. Moreover, for every point $x \in \Omega_0$ with $x \neq C_i$, $i = 1, 2$, we have that $\emptyset \neq \omega^*(x) \subset W_0^u(0)$ since, otherwise, there would be a bounded orbit in $\omega^*(x)$ not lying in $W_0^u(0)$ and not attracted by C_1 or C_2 , contrarily to our hypothesis. As a consequence, $W_0^u(0)$ is, in fact, the dual repeller of $\{C_1, C_2\}$ for the flow φ_0 restricted to Ω_0 .

To prove part (b) we use the fact that the attractor–repeller decomposition $(\{C_1, C_2\}, W_0^u(0))$ of Ω_0 has a continuation to an attractor–repeller decomposition of the global attractor Ω_λ of the flow φ_λ . The continuation of $\{C_1, C_2\}$ is the attractor $\{C_1, C_2\}$ itself. And the continuation of the repeller $W_0^u(0)$ is the set K_λ formed by the union of all bounded orbits not ending in C_1 or C_2 , which is the dual repeller of $\{C_1, C_2\}$ for the restriction of φ_λ to Ω_λ . Since the Conley index continues, the cohomology index of K_λ for the flow φ_λ must agree with that of $W_0^u(0)$ for the flow φ_0 . We see that the cohomology index of $W_0^u(0)$ is isomorphic to $H^*(S^1, *)$. Let N be a compact manifold with boundary which is a positively invariant neighborhood of the global attractor Ω_0 . It is possible to get such a neighborhood by using a Lyapunov function. Then, (N, \emptyset) is an index pair for Ω_0 . Notice that by [38, Theorem 3.6] N is acyclic. Since $(\{C_1, C_2\}, W^u(0))$ is an attractor–repeller decomposition for Ω_0 [48, Corollary 4.4] ensures the existence of $N_0 \subset N$ such that (N_0, \emptyset) is an index pair for $\{C_1, C_2\}$ and (N, N_0) is an index pair for $W^u(0)$. Taking into account that N_0 is a positively invariant neighborhood of $\{C_1, C_2\}$, [38, Theorem 3.6] it follows that $H^k(N_0)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ for $k = 0$ and zero otherwise. Consider the long exact sequence of reduced cohomology of the pair (N, N_0)

$$\dots \rightarrow \tilde{H}^k(N, N_0) \rightarrow \tilde{H}^k(N) \rightarrow \tilde{H}^k(N_0) \rightarrow \dots$$

This exact sequence, together with the previous discussion, ensures that $H^*(N, N_0)$ is isomorphic to 0 if $i \neq 1$. On the other hand, for $i = 1$ we have

$$0 \cong \tilde{H}^0(N) \rightarrow \mathbb{Z} \cong \tilde{H}^0(N_0) \xrightarrow{\partial} \tilde{H}^1(N, N_0) \rightarrow \tilde{H}^1(N) \cong \{0\}$$

hence, ∂ is an isomorphism and, as a consequence, $H^1(N, N_0) \cong \mathbb{Z}$.

Now consider the subset L_λ of K_λ consisting of all bounded trajectories of φ_λ not ending neither in the origin nor in C_1 or C_2 . We shall prove that L_λ is a repeller for the flow φ_λ restricted to K_λ . We remark that $W_\lambda^u(0) \cap K_\lambda = \{0\}$ for $\lambda > 0$ since the two branches of $W_\lambda^u(0)$ connect the point 0 with C_1 and C_2 respectively and the stationary points C_1 and C_2 do not belong to K_λ . Since 0 is a hyperbolic equilibrium for every λ with exactly one positive and two negative eigenvalues, it possesses a Hartman–Grobman block H_λ of 0 (which can be arbitrarily small). We claim that there exists an $\varepsilon > 0$ such that for every $x \in H_\lambda \cap K_\lambda$ with $x \in B_\varepsilon(0)$ its positive semitrajectory $\gamma^+(x)$ is contained in H_λ and, hence, ends in 0. Otherwise there is a sequence of points $x_n \in K_\lambda$, $x_n \rightarrow 0$, such that $\gamma^+(x_n)$ leaves H_λ . This produces an orbit in K_λ which leaves H_λ in the future and whose ω^* -limit is $\{0\}$, which is in contradiction with the fact that $W_\lambda^u(0) \cap K_\lambda = \{0\}$. Then there

is an $\varepsilon > 0$ such that all points of $H_\lambda \cap K_\lambda$ contained in the ball $B_\varepsilon(0)$ go to 0. Hence, we have a neighborhood $H_\lambda \cap K_\lambda \cap B_\varepsilon(0)$ of 0 in K_λ attracted by $\{0\}$ and such that the orbits of its points do not leave H_λ in the future. This proves that $\{0\}$ is an attractor in K_λ whose dual repeller is obviously L_λ . The attractor–repeller cohomology exact sequence of the decomposition $(\{0\}, L_\lambda)$ of K_λ takes the form

$$\dots \xrightarrow{\partial} CH^*(L_\lambda) \xrightarrow{j^*} CH^*(K_\lambda) \xrightarrow{i^*} CH^*(\{0\}) \xrightarrow{\partial} \dots$$

and, taking into account that the cohomology indices of $\{0\}$ and K_λ are both $H^*(S^1, *)$, we readily get that $CH^i(L_\lambda)$ is trivial for $i \neq 1, 2$. To see that $CH^i(L_\lambda)$ is trivial for $i = 1, 2$ we analyze the following segment of the long exact sequence

$$0 \rightarrow CH^1(L_\lambda) \xrightarrow{j^*} CH^1(K_\lambda) \xrightarrow{i^*} CH^1(\{0\}) \xrightarrow{\partial} CH^2(L_\lambda) \rightarrow 0$$

Let us see that i^* is an isomorphism. Let \bar{N} be a compact manifold with boundary which is a positively invariant neighborhood of the global attractor Ω_λ . Then, (\bar{N}, \emptyset) is an index pair for Ω_λ . Since $(\{C_1, C_2\}, K_\lambda)$ is an attractor–repeller decomposition for Ω_λ and $(\{0\}, L_\lambda)$ is an attractor–repeller decomposition of K_λ , it easily follows that $(\{C_1, C_2\}, \{0\}, L_\lambda)$ is a Morse decomposition of Ω_λ . Hence, [48, Corollary 4.4] ensures the existence of a filtration $\bar{N}_0 \subset \bar{N}_1 \subset \bar{N}$ such that (\bar{N}_0, \emptyset) is an index pair for $\{C_1, C_2\}$, (\bar{N}, \bar{N}_0) is an index pair for K_λ , (\bar{N}, \bar{N}_1) is an index pair for L_λ and (\bar{N}_1, \bar{N}_0) is an index pair for $\{0\}$. Notice that the homomorphism i^* is induced by the inclusion $i : (\bar{N}_1, \bar{N}_0) \hookrightarrow (\bar{N}, \bar{N}_0)$. Taking into account that \bar{N}_0 is a positively invariant neighborhood of $\{C_1, C_2\}$, [38, Theorem 3.6] ensures that this inclusion induces the following commutative diagram of short exact sequences in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \cong \tilde{H}^0(\bar{N}_0) & \xrightarrow{\partial} & \mathbb{Z} \cong H^1(\bar{N}, \bar{N}_0) & \longrightarrow & H^1(\bar{N}) \longrightarrow 0 \\ & & \downarrow & & i^* \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} \cong \tilde{H}^0(\bar{N}_0) & \xrightarrow{\bar{\partial}} & \mathbb{Z} \cong H^1(\bar{N}_1, \bar{N}_0) & \longrightarrow & H^1(\bar{N}_1) \longrightarrow 0 \end{array}$$

Since $H^1(\bar{N}) = 0$ by [38, Theorem 3.6], it follows that ∂ is an isomorphism. Let us see that $\bar{\partial}$ is also an isomorphism. From the fact that the lower right arrow is an epimorphism, it follows that $H^1(\bar{N}_1)$ is either 0, \mathbb{Z} or a finite cyclic group. The exactness of the second row ensures that $\bar{\partial}$ is a monomorphism and, hence, the lower right arrow cannot be an isomorphism. As a consequence $H^1(\bar{N}_1)$ cannot be \mathbb{Z} . In addition, the Universal Coefficient Theorem ensures that $H^1(\bar{N}_1)$ must be torsion free and, as a consequence, it cannot be finite cyclic either. Hence, $H^1(\bar{N}_1) = 0$ and $\bar{\partial}$ is also an isomorphism. By combining this with the fact that the leftmost vertical arrow is the identity homomorphism, it follows that i^* is an isomorphism and, hence, it readily follows, from the exactness of the attractor–repeller sequence, that $CH^i(L_\lambda) = 0$ for $i = 1, 2$.

We see that the Morse equations of the decomposition $(\{C_1, C_2\}, \{0\}, L_\lambda)$ of the global attractor Ω_λ are

$$2 + t = 1 + (1 + t).$$

Since $\{C_1, C_2\}$ is an attractor consisting of two fixed points and N_1 is a positively invariant neighborhood, it easily follows that $CH^*(\{C_1, C_2\}) \cong H^*(S^0)$ which contributes with the term 2 of the lefthand side of the equation. The term t of the lefthand side comes from the fact that $CH^*(\{0\}) \cong H^*(S^1, *)$ is a hyperbolic fixed point with one real positive eigenvalue and two complex conjugate eigenvalues with negative real part. L_λ does not contribute to the equations since its cohomology index is trivial. Finally, the term 1 from the righthand side of the equation comes from the fact that $CH^*(\Omega_\lambda) \cong H^*(\bar{N})$ which is acyclic by [38, Theorem 3.6].

To prove part (c) we remark that our hypothesis ensures the existence of arbitrarily small positively invariant neighborhoods

\tilde{N}_1 and \tilde{N}_2 of C_1 and C_2 in Ω_λ that are topological closed disks. If we call $\tilde{N} = \tilde{N}_1 \cup \tilde{N}_2$ and consider the exact cohomology sequence of the pair $(\Omega_\lambda, \tilde{N})$ we readily see that $H^k(\Omega_\lambda, \tilde{N}) = \{0\}$ for every $k \neq 1$ and $H^1(\Omega_\lambda, \tilde{N}) = \mathbb{Z}$. Now consider smaller positively invariant closed disks \hat{N}_1 and \hat{N}_2 contained in the interiors of \tilde{N}_1 and \tilde{N}_2 respectively. By excision $H^k(\Omega_\lambda, \tilde{N}) \cong H^k(\Omega_\lambda \setminus \hat{N}, \tilde{N} \setminus \hat{N})$, where $\hat{N} = \hat{N}_1 \cup \hat{N}_2$. By the choice of the disks \hat{N}_1 and \hat{N}_2 we have that $\Omega_\lambda \setminus \hat{N}$ is negatively invariant and, since K_λ is the complementary repeller of $\{C_1, C_2\}$ in Ω_λ , the cohomology of K_λ agrees with that of $\Omega_\lambda \setminus \hat{N}$ (see [38, Theorem 3.6]). By combining this with the fact that $\Omega_\lambda \setminus \hat{N}$ is connected, since otherwise \hat{N}_i would disconnect \tilde{N}_i for $i = 1, 2$, it follows that K_λ is connected. If we consider now the exact cohomology sequence of the pair $(\Omega_\lambda \setminus \hat{N}, \tilde{N} \setminus \hat{N})$

$$\begin{aligned} \dots \rightarrow H^{k-1}(\tilde{N} \setminus \hat{N}) \rightarrow H^k(\Omega_\lambda \setminus \hat{N}, \tilde{N} \setminus \hat{N}) \rightarrow H^k(\Omega_\lambda \setminus \hat{N}) \\ \rightarrow H^k(\tilde{N} \setminus \hat{N}) \rightarrow \dots \end{aligned}$$

and take into account that $\tilde{N} \setminus \hat{N}$ is homotopy equivalent to the union of two disjoint circles we readily get that the homology of K_λ is that of the figure eight. \square

There is some recent literature dedicated to the study of *transient chaos*. According to Capeáns, Sabuco, Sanjuán and Yorke [61] “this is a characteristic behavior in nonlinear dynamics where trajectories in a certain region of phase space behave chaotically for a while, before escaping to an external attractor. In some situations the escapes are highly undesirable, so that it would be necessary to avoid such a situation”. These authors have developed control methods which prevent the escapes of the trajectories to external attractors, in such a way that they stay in the chaotic region forever. See [62–68] for some contributions on this subject.

5. Traveling repellers: The creation and evolution of the Lorenz attractor

The attractor–repeller decomposition $(\{0\}, L_r)$ of the strange set ceases to exist at $r = 24.06$, when the two branches of the unstable manifold of the origin are absorbed by K_r . As a matter of fact, they asymptotically converge (*only at this value of r*) to the original periodic orbits S and T , responsible in the future for the Hopf bifurcation. Immediately afterwards, the strange set K_r expels the simple periodic orbits S and T and it becomes an attractor (the Lorenz attractor), while the unstable manifold of the origin remains in K_r . We remark, however, that at the parameter value $r = 24.06$ the strange set K_r is still a repeller relative to the flow restricted to the global attractor Ω_r . Hence, the creation of the Lorenz attractor is the result of a repeller–attractor bifurcation in Ω_r at $r = 24.06$.

The Conley index theory tells us that if we restrict ourselves to the consideration of the flow $\varphi_r|_{\Omega_r}$, then the repeller $K_{24.06}$ continues to a family of repellers \hat{K}_r for parameter values $r > 24.06$. The Lorenz attractor K_r is a proper subset of \hat{K}_r , and it must have a dual repeller R_r . This repeller is the union of the two original periodic orbits S and T . We then have an attractor–repeller decomposition (K_r, R_r) of \hat{K}_r for $r > 24.06$. This discussion can be summarized as follows.

1. If we consider the flow restricted to the global attractor Ω_r then the Lorenz attractor is created at a repeller–attractor bifurcation of the strange set K_r at the parameter value $r = 24.06$: the strange set K_r is a repeller for $r = 24.06$ and is an attractor for $r > 24.06$. The continuation \hat{K}_r of $K_{24.06}$ for $r > 24.06$ is a repeller for $\varphi_r|_{\Omega_r}$ which contains the Lorenz attractor K_r and has an attractor–repeller decomposition (K_r, R_r) , where R_r is the union of the two original periodic orbits S and T .

We remark that the creation of the repeller R_r is a necessary consequence of the bifurcation at $r = 24.06$. We can state a much more general result, which shows that the complexity of this repeller is in some dimensions higher than the complexity of the strange set $K_{24.06}$ (from the point of view of Conley’s theory), although its topological structure is much simpler:

Theorem 4. *Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $\lambda \in \mathbb{R}$, be a continuous family of flows and let Ω_λ , with $\lambda_0 \leq \lambda \leq \lambda_1$, be a continuation of isolated invariant sets. Suppose that K_{λ_0} is a repeller for the restricted flow $\varphi_{\lambda_0}|_{\Omega_{\lambda_0}}$ and that there exists a family of compacta K_λ , with $\lambda_0 < \lambda \leq \lambda_1$, such that K_λ is an attractor for the restricted flow $\varphi_\lambda|_{\Omega_\lambda}$ and K_λ converges to K_{λ_0} in the Hausdorff metric (or, more generally, K_λ converges upper-semicontinuously to K_{λ_0}). Then a family of repellers R_λ of $\varphi_\lambda|_{\Omega_\lambda}$, with $R_\lambda \cap K_\lambda = \emptyset$, is created for $\lambda > \lambda_0$ which upper-semicontinuously converge to K_{λ_0} . Moreover, if K_λ has trivial cohomological Conley index in one dimension (as it is the case for the Lorenz attractor for dimensions other than 0 or 1), then the cohomological index of K_{λ_0} in that dimension is a direct summand of that of R_λ . Finally, the cohomological indices of K_{λ_0} and R_λ agree in dimension k if K_λ has trivial indices in dimensions $k - 1$ and k .*

Proof. Since the family of isolated invariant compacta Ω_λ is a continuation of Ω_{λ_0} we have that the repeller K_{λ_0} of $\varphi_{\lambda_0}|_{\Omega_{\lambda_0}}$ continues to a family of repellers \hat{K}_λ of $\varphi_\lambda|_{\Omega_\lambda}$. Then, for every sufficiently small neighborhood U of K_{λ_0} in \mathbb{R}^n , the compactum \hat{K}_λ is the maximal invariant set contained in U for the flow $\varphi_\lambda|_{\Omega_\lambda}$ with λ sufficiently close to λ_0 . Since the family of attractors K_λ converges upper-semicontinuously to K_{λ_0} , they must be contained in U , also for λ sufficiently small. But, since \hat{K}_λ is maximal invariant, then K_λ is, in fact, contained in \hat{K}_λ . Now, the fact that K_λ is an attractor for $\varphi_\lambda|_{\Omega_\lambda}$, and hence for $\varphi_\lambda|_{\hat{K}_\lambda}$, implies the existence of a dual repeller $R_\lambda \subset \hat{K}_\lambda$. Since \hat{K}_λ is itself a repeller then R_λ is also a repeller for the flow $\varphi_\lambda|_{\Omega_\lambda}$ (not only for $\varphi_\lambda|_{\hat{K}_\lambda}$). Moreover, the family of repellers R_λ clearly converges upper-semicontinuously to K_{λ_0} (since the family \hat{K}_λ do) and, obviously, $R_\lambda \cap K_\lambda = \emptyset$.

We have now for $\lambda > \lambda_0$ an attractor–repeller decomposition (K_λ, R_λ) of the isolated invariant compactum \hat{K}_λ . If we write the cohomological exact sequence of this decomposition

$$\begin{aligned} \dots \rightarrow CH^{k-1}(K_\lambda) \xrightarrow{\delta} CH^k(R_\lambda) \rightarrow CH^k(\hat{K}_\lambda) \rightarrow CH^k(K_\lambda) \\ \xrightarrow{\delta} CH^{k+1}(R_\lambda) \rightarrow \dots \end{aligned}$$

and take into consideration the fact that (\hat{K}_λ) is a continuation of K_{λ_0} (and, thus, their Conley indices agree) we see that, if $CH^k(K_\lambda)$ vanishes then $CH^k(R_\lambda) \rightarrow CH^k(\hat{K}_\lambda)$ is an epimorphism and, hence, $CH^k(K_{\lambda_0}) \cong CH^k(\hat{K}_\lambda)$ is a direct summand of $CH^k(R_\lambda)$. Moreover, if $CH^{k-1}(K_\lambda)$ also vanishes then $CH^k(R_\lambda) \cong CH^k(\hat{K}_\lambda)$. \square

We remark again that, in the case of the Lorenz equations, for $r = 24.06$ the strange invariant set $K_{24.06}$ is not yet an attractor. In fact it is a repeller for the restricted flow $\varphi_{24.06}|_{\Omega_{24.06}}$, which contains the two branches of the unstable manifold of the origin and the original periodic orbits S and T . Immediately after, the strange invariant set expels these periodic orbits (while retaining the unstable manifold) and becomes an attractor (the Lorenz attractor). The periodic orbits S and T “travel” through the global attractor and, finally, are absorbed by the fixed points C_1 and C_2 at the parameter value $r = 24.74$, when a Hopf bifurcation takes place.

From the point of view of the global attractor Ω_r , we have that the pair $(K_{24.06}, \{C_1, C_2\})$ defines a repeller–attractor decomposition of Ω while the pair $(K_{24.74}, \{C_1, C_2\})$ defines an attractor–repeller decomposition. The mechanism which makes possible

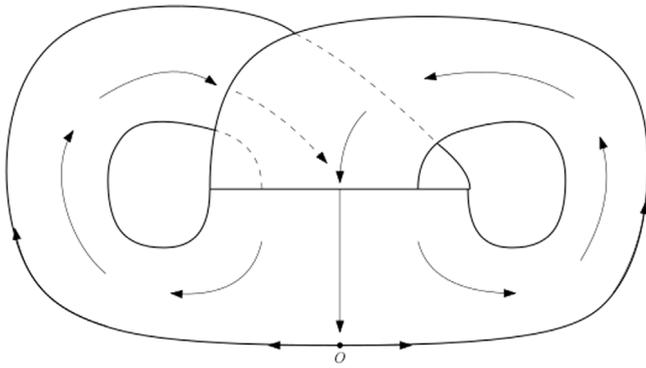


Fig. 1. Branched manifold.

this sharp transformation is the expulsion by $K_{24.06}$ of the original periodic orbits S and T and its posterior absorption by C_1 and C_2 at the parameter value $r = 24.74$. In other words, the “traveling repeller” $R_r = S \cup T$ is responsible for the transition. We summarize the process in the following statement.

2. (From repeller–attractor to attractor–repeller decompositions of Ω_r). The strange set $K_{24.06}$ is a repeller relative to the restricted flow $\varphi_{24.06}|_{\Omega_{24.06}}$, which contains the two branches of the unstable manifold of the origin and the original periodic orbits S and T . The pair $(K_{24.06}, \{C_1, C_2\})$ defines a repeller–attractor decomposition of the global attractor $\Omega_{24.06}$. Immediately after (i.e. for $r > 24.06$), the strange invariant set expels these periodic orbits (while retaining the unstable manifold) and becomes an attractor (the Lorenz attractor). The set $R_r = S \cup T$ is a repeller relative to the flow $\varphi_r|_{\Omega_r}$ and “travels” through Ω_r until finally is absorbed by $\{C_1, C_2\}$ at the parameter value $r = 24.74$ of the Hopf bifurcation. At this value, the pair $(K_{24.74}, \{C_1, C_2\})$ defines an attractor–repeller decomposition of $\Omega_{24.74}$.

Now a few comments about the topological properties of the Lorenz attractor are in order. Some global properties of the Lorenz attractor have been studied in [51]. In particular, the Borsuk homotopy type (or shape) of the attractor is calculated there and from this calculation all the homological and cohomological invariants follow. Another possibility for studying the global properties of the attractor is to use the branched manifold (see Fig. 1). We give only a brief, informal, indication on how this can be done.

The branched manifold is a two-dimensional manifold with singularities (the branch points) on which the forward flow (i.e. a semi-flow) is defined. In spite of its name, it is not a manifold but an Absolute Neighborhood Retract (ANR), an important notion of the Theory of Retracts also studied by Borsuk. The class of ANRs has homotopical properties similar to those of the manifolds. The semi-flow in the branched manifold comes from the Lorenz flow after collapsing to a point certain segments, all whose points share a common future (see [11, Appendix G, pg. 229] for a discussion). The semi-flow has a global attractor whose Borsuk homotopy type is the same as that of the Lorenz attractor, since the above mentioned identification preserves the global properties of the attractor. By [38, Theorem 3.6] (see also [39,40,69,70]) the inclusion of the attractor in the manifold is a (Borsuk) homotopy equivalence. It follows from this that the Borsuk homotopy type (or shape) of the Lorenz attractor is that of the branched manifold, which turns out to be that of the figure eight. This agrees with the results found in [51]. A consequence of this is that the cohomology of the Lorenz attractor and its cohomological Conley indices are isomorphic to \mathbb{Z} in dimension zero, to $\mathbb{Z} \oplus \mathbb{Z}$ in dimension one and zero otherwise.

Our previous Theorem 4 implies that the 2-dimensional cohomological (Conley) complexity of the traveling repeller R_r is higher than that of $K_{24.06}$. As a matter of fact, $K_{24.06}$ has the cohomological Conley indices of the (pointed) circle and hence $CH^2(K_{24.06}) = \{0\}$, as it has been remarked in Section 4. On the other hand, it follows immediately from our next result that $CH^2(R_r) = \mathbb{Z} \oplus \mathbb{Z}$.

The following theorem addresses a more general situation. We simplify the hypotheses slightly to make the exposition simpler.

Theorem 5. Let $\varphi_\lambda : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$, $\lambda \in \mathbb{R}$, be a continuous family of flows and let Ω be a global attractor for all the flows φ_λ . Suppose that K and C are isolated invariant sets for every λ and that (K, C) is a repeller–attractor decomposition of Ω for φ_{λ_0} and (K, C) is an attractor–repeller decomposition of Ω for φ_{λ_1} , where $\lambda_0 < \lambda_1$. Suppose, additionally, that the isolated invariant set R_λ is a repeller of φ_λ for $\lambda_0 < \lambda < \lambda_1$ and that $(K \cup C, R_\lambda)$ is an attractor–repeller decomposition of Ω . Denote by r_k the rank of $H^k(K)$ and by r'_k the rank of $H^k(C)$. Then we have the following Morse equations

$$r'_0 + (r'_1 + r'_0 - 1)t + \sum_{k \geq 2} (r'_k + r'_{k-1})t^k = 1 + (1+t)Q_1(t),$$

for the repeller–attractor decomposition (K, C) of $\varphi_{\lambda_0}|_\Omega$,

$$r_0 + r'_0 + (r_1 + r_0 + r'_1 + r'_0 - 1)t + \sum_{k \geq 2} (r_k + r_{k-1} + r'_k + r'_{k-1})t^k = 1 + (1+t)Q_2(t),$$

for the attractor–repeller decomposition $(K \cup C, R_\lambda)$ of $\varphi_\lambda|_\Omega$ with $\lambda_0 < \lambda < \lambda_1$ and

$$r_0 + (r_1 + r_0 - 1)t + \sum_{k \geq 2} (r_k + r_{k-1})t^k = 1 + (1+t)Q_3(t),$$

for the attractor–repeller decomposition (K, C) of $\varphi_{\lambda_1}|_\Omega$.

To prove Theorem 5 we shall make use of the following lemma.

Lemma 6. In the conditions of Theorem 5 we have that

- (a) $CH^k(C) \cong H^k(C) \cong CH^{k+1}(K)$ if $k > 0$ and $CH^0(C) \cong H^0(C) \cong \mathbb{Z} \oplus CH^1(K)$ for the flow φ_{λ_0} .
- (b) $CH^k(K) \cong H^k(K) \cong CH^{k+1}(C)$ if $k > 0$ and $CH^0(K) \cong H^0(K) \cong \mathbb{Z} \oplus CH^1(C)$ for the flow φ_{λ_1} .
- (c) $CH^{k+1}(R_\lambda) \cong CH^k(K \cup C) \cong H^k(K \cup C)$ if $k > 0$ and $CH^0(K \cup C) \cong H^0(K \cup C) \cong \mathbb{Z} \oplus CH^1(R_\lambda)$ for the flow φ_λ with $\lambda_0 < \lambda < \lambda_1$.

Proof. We shall prove a more general result which encompasses (a), (b) and (c). Let $\varphi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ a dissipative flow with global attractor Ω . Suppose that (A, R) is an attractor repeller decomposition of Ω and consider the cohomology long exact sequence associated to the decomposition (A, R) ,

$$\dots \xrightarrow{\partial} CH^*(R) \xrightarrow{j^*} CH^*(\Omega) \xrightarrow{i^*} CH^*(A) \xrightarrow{\partial} \dots$$

since Ω is a global attractor, it follows that $CH^k(\Omega) = \mathbb{Z}$ if $k = 0$ and zero if $k > 0$. Taking this into account in the exact sequence it readily follows that $CH^k(A) \cong CH^{k+1}(R)$ if $k > 0$. On the other hand, since none of the components R is an attractor, it follows that $CH^0(R) = 0$ and, hence, the initial part of the sequence looks like

$$0 \rightarrow CH^0(\Omega) \rightarrow CH^0(A) \rightarrow CH^1(R) \rightarrow 0$$

The Universal Coefficient ensures that $CH^1(R)$ must be torsion free and, as a consequence, the short exact sequence splits. Then

$$CH^0(A) \cong CH^0(\Omega) \oplus CH^1(R) \cong \mathbb{Z} \oplus CH^1(R)$$

Notice that, since A is an attractor for the flow φ restricted to the global attractor Ω , then A is an attractor for φ . Therefore $CH^*(A) \cong H^*(A)$. The result follows by replacing A and R by the corresponding sets. \square

Proof of Theorem 5. The proof follows from Lemma 6 combined with the fact that $CH^k(\Omega)$ is \mathbb{Z} if $k = 0$ and zero if $k > 0$, Ω being a global attractor for each λ . \square

Concerning the previous lemma, it is interesting to note that, when Ω is a global attractor, then the topological properties of K and C determine the cohomological Conley indices and the Morse equations of all the involved isolated invariant sets, including R_λ . It is also interesting to see how the transition from repeller–attractor to attractor–repeller is reflected in the Morse equations.

Another situation, not applicable to the Lorenz equations but provided of theoretical interest, is when we have a flow in a compact manifold M and a similar transition for a pair (K, L) . Then McCord duality for attractor–repeller pairs [71,72] is applicable and the equations are determined by the topology of K and M alone (the Conley index properties of C being dual to those of M).

We finally point out that the evolution of the Lorenz attractor that we have just studied has a nice counterpart from the analytical point of view. The following statement summarizes the situation.

3. The transition from the repeller–attractor decomposition $(K_{24.06}, \{C_1, C_2\})$ (creation of the Lorenz attractor) to the attractor–repeller decomposition $(K_{24.74}, \{C_1, C_2\})$ (Hopf bifurcation) through the decomposition $(K_r \cup \{C_1, C_2\}, R_r = S \cup L)$ (involving the traveling repellers R_r) of the global attractor Ω is reflected in the Morse equations shown in Theorem 5.

Applying Theorem 5 to this situation we get that, for $r = 24.06$ the Morse equations associated to the repeller–attractor decomposition $(K_{24.06}, \{C_1, C_2\})$ of $\varphi_{24.06}|_{\Omega_{24.06}}$ are

$$2 + t = 1 + (1 + t),$$

for r with $24.06 < r < 24.74$ the Morse equations associated to the attractor–repeller decomposition $(K_r \cup \{C_1, C_2\}, R_r = S \cup L)$ of $\varphi_r|_{\Omega_r}$ are

$$3 + 4t + 2t^2 = 1 + (1 + t)(2 + 2t),$$

and, for $r = 24.74$ the Morse equations associated to the attractor–repeller decomposition $(K_{24.74}, \{C_1, C_2\})$ of $\varphi_{24.74}|_{\Omega_{24.74}}$ are

$$1 + 2t + 2t^2 = 1 + (1 + t)2t.$$

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