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Existence and spectral stability of multi-pulses in discrete Hamiltonian lattice systems

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ABSTRACT

In the present work, we consider the existence and spectral stability of multi-pulse solutions in Hamiltonian lattice systems which are invariant under a one-parameter unitary group of symmetries. We provide a general framework for the study of such wave patterns based on a discrete analogue of Lin's method, previously used in the continuum realm. We develop explicit conditions for the existence of multi-pulse standing wave structures and subsequently develop a reduced matrix allowing us to address their spectral stability. As a prototypical example, we consider the discrete nonlinear Schrödinger equation (DNLS). Using Lin's method, we extend existence and linear stability results of multi-pulse solutions beyond the anti-continuum and continuum limits. Different families of 2- and 3-pulse solitary waves are discussed, and analytical expressions for the corresponding stability eigenvalues are obtained which are in very good agreement with numerical results.

1. Introduction and motivation

The study of multi-pulse wave structures has a time honored history in continuum systems. Attempts at a systematic formulation have taken place both at a more phenomenological, asymptotic level [1] and at a more rigorous level [2]. The development in the latter work of the so-called Lin's method for such wave patterns offered a systematic view into a reduced formulation where the characteristics of the pulses (such as their centers, or possibly also their widths) could constitute effective dynamical variables for which simpler dynamical equations, i.e. ordinary differential equations, could be derived. While Lin's method for discrete dynamical systems has been developed in [3], it has not so far been applied to the discrete multi-pulse problem. Over the following decade, methods were sought to isolate and freeze the dynamics of individual pulses within the patterns [4,5]. More recently, such freezing techniques have also been extended to other structures, including rotating waves [6].

Despite the intense interest in such multiple coherent structure patterns at the continuum limit, similar techniques have not been systematically developed at the discrete level. Parts of the relevant efforts have involved an attempt at adapting the asymptotic methodology of [1] (in the work of [7]) and also the consideration of structures systematically in the vicinity of the so-called anti-continuum limit [8]. The latter setting involves as a starting point the limit of vanishing coupling between the discrete sites, whereby suitable Lyapunov–Schmidt conditions can be brought to bear to identify persistent configurations for finite coupling strengths between the adjacent lattice sites. While works such as [9] have emerged that develop instability criteria, it would be useful to have a systematic toolbox to study the spectrum of multi-pulses in the spatially discrete setting. This would serve to both quantify the persistence conditions of the multi-structure states, and also to offer specific predictions on their spectral stability and nonlinear dynamics.

It is this void that it is the aim of the present work to fill. Following [10], we start with a Hamiltonian lattice system which is invariant under a one-parameter unitary group of symmetries $R(\theta)$ (we will comment in Section 3.5 on the far easier case of systems that do not respect continuous symmetries and also note that non-Hamiltonian systems can be tackled using an adaptation of the results in [2]). Such systems often exhibit primary pulse solutions which are standing waves that evolve only in the direction of the symmetry group. We assume that this primary pulse solution is orbitally stable in the sense of [10]. From a spatial dynamics perspective, the primary pulse is a homoclinic orbit that lies in intersection of the unstable and stable manifolds of a saddle equilibrium at the origin.

To construct multi-pulse solutions, we use Lin's method [3,11,12], a Lyapunov–Schmidt reduction that can be used to find solutions which remain near a known homoclinic orbit. Heuristically,

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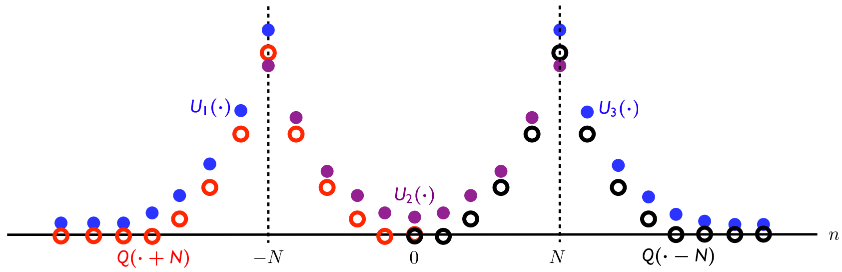

Fig. 1. Shown is an illustration of a piecewise 2-pulse $U(n)$, composed of the three pieces U_1, U_2, and U_3, and the linear superposition of two copies of the primary pulse $Q(n)$ placed at $n = -N$ and $n = N$, respectively. $U(n)$ is a genuine 2-pulse solution if and only if the two jumps at $n = \pm N$ are zero.

we can construct a 2-pulse on the integer lattice $\mathbb{Z}$ as follows. Let $Q(n)$ be the primary homoclinic orbit; then a homoclinic orbit $U(n)$ is said to be a 2-pulse if there is an $N \gg 1$ so that

$$\sup_{n\leq 0} |U(n) - Q(n+N)| + \sup_{n \geq 0} |U(n) - Q(n-N)|$$

is small, i.e. the graph of $U(n)$ resembles the sum of two copies of the primary pulse translated by N units to the left and right, respectively. For each $N \gg 1$, Lin's method produces a piecewise 2-pulse solution that is composed of three pieces

$$\begin{cases} U_1(n) & n \in (-\infty, -N] \\ U_2(n) & n \in [-N, N] \\ U_3(n) & n \in [N, \infty) \end{cases}$$

as shown in Fig. 1. This piecewise function will be a genuine 2-pulse if and only if these pieces coincide at $n = \pm N$ so that

$$U_1(-N) = U_2(-N), \qquad U_2(N) = U_3(N).$$

This approach therefore reduces the existence problem for 2-pulses to solving two jump conditions. Similarly, the existence of an m-pulse is equivalent to solving m jump conditions.

For a multi-pulse, we also use Lin's method to find the eigenvalues which result from nonlinear interactions between neighboring copies of the primary pulse. These eigenvalues are close to 0, and we call them interaction eigenvalues. To do this, we adapt the analysis in [2] to the Hamiltonian case. (For non-Hamiltonian systems, the analysis is much simpler, and in fact is the discrete analogue of [2]). Invariance under the symmetry group $R(\theta)$ induces an eigenvalue at 0 in the linearized equation. The corresponding eigenfunctions are, to leading order, piecewise linear combinations of the kernel eigenfunction.

As a concrete example for the implementation of the method, we revisit the discrete nonlinear Schrödinger (DNLS) system, for which many of the methods of the previous paragraph have been developed [13] (see also [14]). We show that known existence and linear stability results hold for regimes beyond the anti-continuum and continuum limits; in essence, we replace the requirement that the coupling between adjacent lattice sites be small by the requirement that the pulses are well separated. In particular, we give a systematic description especially of 2- and 3-pulse solutions and explain how the relevant conclusions can be generalized to arbitrary multi-pulse structures. Our presentation will be structured as follows. In Section 2, we will present the mathematical setup of the problem and of the special case example of interest (DNLS). In Section 3, we will develop Lin's method, providing the main results but deferring the proof details to later sections. In Section 4, we apply the method to the DNLS, comparing the theoretical findings to systematic computations of multi-pulse solutions. Our results are then summarized in Section 5, and some possible directions for future work are offered. Details of the proofs are presented in Sections 6–8.

While the paradigm of interest herein will be the DNLS, we note in passing that the technique considered is more broadly applicable in dispersive discrete nonlinear systems. As only one parallel worth drawing, we mention the case of multiple kinks in the context of discrete Klein–Gordon models such as the discrete sine–Gordon or the discrete ϕ^4 model [15]. While there are some nontrivial differences between this case and the one considered herein (including, e.g., the study of fronts rather than pulses, and the stability of configurations centered between adjacent sites rather than on a lattice site), the relevant setting can be adapted to such a case. Such Klein–Gordon multi-kinks are of interest both in their static [16] and in their traveling [17] form in the discrete problem. A more challenging generalization is the one to the setting of discrete breathers in Klein–Gordon lattices. The latter, however, is also worth considering in a suitable action–angle framework where the relevant persistence and stability problem can also be considered; see, e.g., [18] for a recent discussion thereof.

2. Mathematical setup

A lattice dynamical system is an infinite system of ordinary differential equations which are indexed by points (nodes) on a lattice. For the purposes of this work, we will only consider dynamical systems on the integer lattice $\mathbb{Z}$, where the differential equation for each point on the lattice is identical, and the equations are coupled by a centered, second order difference operator.

As a specific example, we will look at the discrete nonlinear Schrödinger equation (DNLS)

$$i\dot{\psi}_n + d(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + |\psi_n|^2\psi_n = 0, \tag{1}$$

which is (2.12) in [13], where we have taken $\beta = -1$ and $\sigma = 1$. The parameter d represents the coupling between nodes; $d > 0$ is the focusing case, and $d < 0$ is the defocusing case [13]. When $d = 0$, there is no coupling between nodes; this is referred to as the anti-continuum limit. Eq. (1) is Hamiltonian, with energy given by (2.17) in [13,14]. Of general interest in this type of lattice is the existence and stability of standing waves, which are bound state solutions of the form $\psi_n(t) = e^{i\omega t}\phi_n$ [19]. Making this substitution in (1) and simplifying, a standing wave solves the steady state equation

$$d(\phi_{n+1} - 2\phi_n + \phi_{n-1}) - \omega\phi_n + |\phi_n|^2\phi_n = 0. \tag{2}$$

From [20], a symmetric, real-valued, on-site soliton solution q_n exists to (2) for all $\omega \neq 0$ and $d \geq 0$. This solution q_n furthermore is differentiable in ω.

We will write DNLS as a system of two real variables $u = (v, w) \in \ell^2(\mathbb{Z}, \mathbb{R}^2)$, where $v = \text{Re}\,\psi$ and $w = \text{Im}\,\psi$. In this fashion, we can write (1) in Hamiltonian form as

$$\dot{u}_n - J[\mathcal{H}'(u)]_n = 0, \tag{3}$$

where J is the standard skew-symmetric symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the Hamiltonian $\mathcal{H} : \ell^2(\mathbb{Z}, \mathbb{R}^2) \to \mathbb{R}$ is

$$\mathcal{H}(v, w) = \sum_{n=-\infty}^{\infty} \left(\frac{d}{2} (v_n - v_{n-1})^2 + \frac{d}{2} (w_n - w_{n-1})^2 - \frac{1}{4} \left(v_n^2 + w_n^2\right)^2 \right). \tag{4}$$

The Hamiltonian $\mathcal{H}$ is invariant under the standard rotation group $R(\theta)$, given by

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{5}$$

which has infinitesimal generator $R'(0) = -J$. The corresponding conserved quantity, often called the charge [10, Section 6.C], is given by

$$\mathcal{Q}(v, w) = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \left(v_n^2 + w_n^2\right). \tag{6}$$

We note that for physical considerations, typically the opposite of this quantity appears as the relevant conserved quantity, representing, e.g., the power in optics or the number of atoms in atomic condensates [13].

Standing waves are solutions of (3) of the form $R(\omega t)u$, where u is independent of t. Substituting this into (3), we obtain the equivalent system of equations

$$-\mathcal{H}'(u) - \omega u = 0, \tag{7}$$

which for DNLS is given by

$$\begin{aligned} d(v_{n+1} - 2v_n + v_{n-1}) + v_n w_n^2 + v_n^3 - \omega v_n = 0 \\ d(w_{n+1} - 2w_n + w_{n-1}) + v_n^2 w_n + w_n^3 - \omega w_n = 0 . \end{aligned} \tag{8}$$

If u is a standing wave solution, then $R(\theta)u$ is also a solution to Eq. (7). Since $R(\theta)u$ is a rotation of u by θ, we will not distinguish between u and $R(\theta)u$, although θ will play a central role when we construct multi-pulses in Section 4. We note that the steady state system is equivalent to

$$\mathcal{H}'(u) - \omega \mathcal{Q}'(u) = 0, \tag{9}$$

which is the stationary equation [10, (2.15)]. The steady state equation (8) also has a conserved quantity E, which is the current density [21, (6)] and is given by

$$E = 2d(v_n w_{n-1} - v_{n-1} w_n) = 2d\langle u_n, J u_{n-1}\rangle. \tag{10}$$

By a conserved quantity in this setting, we mean that this quantity is independent of the lattice index n.

3. Main theorems

3.1. Setup

With DNLS as our principal motivation, we will consider the following more general setting. Consider the Hamiltonian lattice differential equation

$$\dot{u}_n = J[\mathcal{H}'(u)]_n, \tag{11}$$

where $u(t) \in \ell^2(\mathbb{Z}, \mathbb{R}^{2k})$, $\mathcal{H} : \ell^2(\mathbb{Z}, \mathbb{R}^{2k}) \to \mathbb{R}$ is smooth with $\mathcal{H}(0) = 0$ and $\mathcal{H}'(0) = 0$, and J is a $2k \times 2k$ symplectic matrix. The integer k is the lattice dimension, and $k = 1$ corresponds to the usual one-dimensional lattice. For simplicity, and again using DNLS as motivation, we will assume that $\mathcal{H}'(u)$ takes the form

$$[\mathcal{H}'(u)]_n = -d(\Delta_2 u)_n + f(u_n), \tag{12}$$

where Δ_2 is the second difference operator $(\Delta_2 u)_n = u_{n+1} - 2u_n + u_{n-1}$, d is the coupling constant, and $f : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ is smooth with $f(0) = 0$ and $Df(0) = 0$. This implies that, other than the terms from $\Delta_2 u$, the RHS of (11) only involves the lattice site u_n. We note that $Df(u(n))$ is self-adjoint since $\mathcal{H}''(u)$ is self-adjoint.

Using [10] as a guide, we make the following hypothesis concerning the invariance of the system under a group of symmetries. This hypothesis allows us to have standing wave solutions to (11), which are solutions which evolve only in the direction of the symmetry group. In addition, there is a kernel eigenvalue in the linearized problem which is a result of this symmetry invariance. We will construct the eigenfunctions associated with a multi-pulse standing wave using this kernel eigenvalue.

Hypothesis 1. There is unitary group of symmetries $\{R(\theta) : \theta \in \mathbb{R}\}$ on $\mathbb{R}^{2k}$ such that

(i) The Hamiltonian $\mathcal{H}$ is invariant under $R(\theta)$, i.e.

$$\mathcal{H}(R(\theta)u) = \mathcal{H}(u). \tag{13}$$

(ii) $R(\theta)$ satisfies the "commuting" relation

$$R(\theta)J = JR^*(-\theta). \tag{14}$$

Condition (ii) implies

$$R'(0)J = -JR'(0)^*, \tag{15}$$

where $R'(0)$ is the infinitesimal generator of the symmetry group $R(\theta)$. In addition, since $R(\theta)$ is unitary,

$$R(\theta)J = JR(\theta). \tag{16}$$

For DNLS, $R(\theta)$ is the rotation group (5). Following [10, (2.9)], since J is an invertible matrix, the corresponding conserved quantity is given by

$$\mathcal{Q}(u) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \langle J^{-1} R'(0) u_n, u_n\rangle. \tag{17}$$

It follows from (14) and $J^{-1} = -J$ that $\mathcal{Q}(u)$ is also invariant under the symmetry group $R(\theta)$, i.e. $\mathcal{Q}(R(\theta)u) = \mathcal{Q}(u)$.

Equilibrium solutions to (11) satisfy

$$\mathcal{H}'(u) = 0. \tag{18}$$

Differentiating the symmetry invariance (13) as in [10], we obtain the symmetry relations

$$\begin{aligned} &\mathcal{H}'(R(\theta)u) = R(\theta)\mathcal{H}'(u) \\ &\mathcal{H}''(R(\theta)u) = R(\theta)\mathcal{H}''(u)R(\theta)^* , \end{aligned} \tag{19}$$

from which it follows that u is a solution to (18) if and only if $R(\theta)u$ is a solution. We also note that $f(R(\theta)u) = f(u)R(\theta)$.

We are interested in bound states (also referred to as standing waves), which are solutions to (11) of the form $u(t) = R(\omega t)u$, where $u \in \ell^2(\mathbb{Z}, \mathbb{R}^{2k})$ is independent of t. Following [10], bound states satisfy the equilibrium equation

$$\mathcal{H}'(u) - \omega \mathcal{Q}'(u) = 0. \tag{20}$$

Since $\mathcal{Q}'(u) = J^{-1}R'(0)u$, this is equivalent to

$$\mathcal{H}'(u) - \omega J^{-1} R'(0) u = 0. \tag{21}$$

We note that if q is a bound state, $R(\theta)q$ is also a solution to (20), i.e.

$$\mathcal{H}'(R(\theta)u) - \omega \mathcal{Q}'(R(\theta)u) = 0. \tag{22}$$

Let q be a bound state solution to (20). The linearization of (11) about q is the linear operator

$$L(q) = J\mathcal{H}''(q) - \omega J \mathcal{Q}''(q). \tag{23}$$

Since $\mathcal{Q}''(u) = J^{-1}R'(0)$, this is equivalent to

$$L(q) = J\mathcal{H}''(q) - \omega R'(0). \tag{24}$$

Taking $u = q$ in (22) and differentiating with respect to θ at $\theta = 0$,

$$L(q)R'(0)q = 0, \tag{25}$$

thus $R'(0)q$ is an eigenfunction of $L(q)$ with eigenvalue 0.

As in [10, Assumption 2], we take the following hypothesis about the existence of bound state solutions. We also assume that a stability criterion is satisfied.

Hypothesis 2. For $\omega \in (\omega_1, \omega_2)$, there exists an injective C^1 map $\omega \mapsto q$ such that $q \in \ell^2(\mathbb{Z}, \mathbb{R}^{2k})$ is a bound state solution to (20). Furthermore, for all $\omega \in (\omega_1, \omega_2)$, $M > 0$, where

$$M = -\frac{d}{d\omega}\mathcal{Q}(q) = -\sum_{n=-\infty}^{\infty} \langle J^{-1}R'(0)u_n, \partial_\omega u_n \rangle. \tag{26}$$

We note that for the map $\omega \mapsto q$ to be well defined, we do not distinguish between solutions $R(\theta)q$ for $\theta \in \mathbb{R}$. The case of $M < 0$ is connected to an instability of the primary pulse and hence is not further considered herein.

Remark 1. The condition $M > 0$ is the Vakhitov Kolokolov stability criterion [22], which is generalized to abstract Hamiltonian systems in [10]. $M > 0$ implies stability of the primary pulse [10]. In general, the condition $M > 0$ can only be verified numerically.

Remark 2. We use the notation M for the stability criterion (26) since M will appear in the proof of Theorem 2 as a Melnikov sum which measures a jump in a specific direction.

By Hypothesis 2, $\partial_\omega q$ exists for $\omega \in (\omega_1, \omega_2)$. Differentiating (20) with respect to ω and multiplying by J, $\partial_\omega q$ satisfies

$$L(q)\partial_\omega q = R'(0)q. \tag{27}$$

3.2. Spatial dynamics formulation

Using Eq. (12), we write the bound state equation (21) as the first order difference equation

$$U(n+1) = F(U(n)), \tag{28}$$

where $U(n) = (u(n), \tilde{u}(n)) = (u_n, u_{n-1}) \in \mathbb{R}^{4k}$ and $F : \mathbb{R}^{4k} \to \mathbb{R}^{4k}$ is smooth and defined by

$$F\begin{pmatrix} u \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 2u + \frac{1}{d}\left(f(u) - \omega J^{-1}R'(0)u\right) - \tilde{u} \\ u \end{pmatrix}. \tag{29}$$

We note that $F(0) = 0$. Reformulation of the problem as a first order difference equation is required by standard implementations of Lin's method (see Eq. (1.1) in [3] for the discrete case). For systems where the coupling between nodes is nonlinear, a different approach would be needed.

It is straightforward to verify the symmetry relation

$$F(T(\theta)U) = F(U)T(\theta), \tag{30}$$

where

$$T(\theta) = \begin{pmatrix} R(\theta) & 0 \\ 0 & R(\theta) \end{pmatrix}. \tag{31}$$

We can similarly write the eigenvalue problem $(L(q) - \lambda I)v = 0$, where L is defined by Eq. (23), as the first order difference equation

$$V(n+1) = DF(Q(n))V(n) + \lambda BV(n), \tag{32}$$

where

$$DF(Q(n)) = \begin{pmatrix} 2I + \frac{1}{d}Df(q_n) - \frac{1}{d}\omega J^{-1}R'(0) & -I \\ I & 0 \end{pmatrix} \tag{33}$$

and B is the constant-coefficient block matrix

$$B = \frac{1}{d}\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}. \tag{34}$$

Since $R(\theta)J = JR(\theta)$, $T(\theta)$ commutes with B. It follows from (25) and (27) that

$$T'(0)Q(n+1) = DF(Q(n))T'(0)Q(n) \tag{35}$$

$$\partial_\omega Q(n+1) = DF(Q(n))\partial_\omega Q(n) + BT'(0)Q(n). \tag{36}$$

Since $F(0) = 0$, the origin is an equilibrium point for the dynamical system (28). Fix $\omega \in (\omega_1, \omega_2)$, and let q be the bound state from Hypothesis 2 corresponding to ω. Let $Q(n) = (q_n, q_{n-1})$. Since $q \in \ell^2(\mathbb{Z}, \mathbb{R}^{2k})$, $q_n \to 0$ as $n \to \pm\infty$, and $Q(n)$ is a homoclinic orbit solution to (28) connecting the equilibrium at 0 to itself. We will refer to this as the primary pulse solution. Since $f(0) = Df(0) = 0$, for the equilibrium at 0 we have

$$DF(0) = \begin{pmatrix} u \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 2I - \frac{1}{d}\omega J^{-1}R'(0) & -I \\ I & 0 \end{pmatrix}. \tag{37}$$

In the next hypothesis, we assume the equilibrium at 0 is hyperbolic. This facilitates the analysis by allowing us to decompose the linearized problem in an exponential dichotomy. This also allows us to characterize the eigenvalues of $DF(0)$.

Hypothesis 3. The equilibrium at 0 is hyperbolic, i.e. there are no eigenvalues ν of $DF(0)$ on the unit circle.

We can use standard determinant identities to compute the characteristic polynomial of $DF(0)$.

$$\det(DF(0) - \nu I) = \det(-\nu I)\det\left(2I - \frac{1}{d}\omega J^{-1}R'(0) - \left(\nu + \frac{1}{\nu}\right)I\right). \tag{38}$$

It follows from (38) that the eigenvalues of $DF(0)$ come in pairs $\{\nu, 1/\nu\}$. By (15), the matrix $2I - \frac{1}{d}\omega J^{-1}R'(0)$ is self-adjoint, thus $DF(0)$ has $2k$ pairs of real eigenvalues $\{(\nu_j, 1/\nu_j) : j = 1, \ldots, 2k\}$, where $\nu_j > 1$. Let

$$r = \min_{j=1,\ldots,2k} \nu_j. \tag{39}$$

The homoclinic orbit $Q(n)$ lies in the intersection of the stable and unstable manifolds. In the following hypothesis, we assume that this intersection is non-degenerate.

Hypothesis 4. The tangent spaces of the stable manifold $W^s(0)$ and the unstable manifold $W^u(0)$ have a one-dimensional intersection at $Q(n)$.

By (25), this intersection is spanned by $T'(0)Q(n)$. By the stable manifold theorem, we have the decay rate

$$|Q(n)| \leq Cr^{-|n|}. \tag{40}$$

By Hypothesis 4, $T'(0)Q(n)$ is the unique bounded solution to the variational equation

$$V(n+1) = D_U F(Q(n))V(n),$$

where

$$T'(0)Q(n) = \begin{pmatrix} R'(0)q(n) \\ R'(0)q(n-1) \end{pmatrix}. \tag{41}$$

It follows that there exists a unique bounded solution $Z_1(n)$ to the adjoint variational equation

$$Z(n) = D_U F(Q(n))^* Z(n+1).$$

We can verify directly that

$$Z_1(n) = \begin{pmatrix} -R'(0)q(n-1) \\ R'(0)q(n) \end{pmatrix}. \tag{42}$$

In both of these cases, uniqueness is up to scalar multiples.

3.3. Existence of multi-pulses

We are interested in multi-pulses, which are bound states that resemble multiple, well separated copies of the primary pulse $Q(n)$. In this section, we give criteria for the existence of multi-pulses. We will characterize a multi-pulse solution in the following way. Let $m > 1$ be the number of copies of $Q(n)$; N_i ($i = 1, \dots, m-1$) be the distances (in lattice points) between consecutive copies; and $\theta_i \in \mathbb{R}$ ($i = 1, \dots, m$) be symmetry parameters associated with each copy of $Q(n)$. We seek a solution which can be written piecewise in the form

$$\begin{aligned} U_i^-(n) &= T(\theta_i)Q(n) + \tilde{Q}_i^-(n) \quad n \in [-N_{i-1}^-, 0] \\ U_i^+(n) &= T(\theta_i)Q(n) + \tilde{Q}_i^+(n) \quad n \in [0, N_i^+], \end{aligned} \tag{43}$$

where $N_i^+ = \lfloor \frac{N_i}{2} \rfloor$, $N_i^- = N_i - N_i^+$, $N_0^- = N_m^+ = \infty$, and

$$N = \frac{1}{2}\min\{N_i\}. \tag{44}$$

The individual pieces are joined together end-to-end as in [2]. The functions $\tilde{Q}_i^{\pm}(n)$ are remainder terms, which we expect to be small; see the estimates in Theorem 3. The process of constructing a double pulse in terms of overlapping, well-separated single pulses resembles the construction of multi-site breathers in [23] (see [23, Lemma 1] for a two-site breather in the discrete Klein–Gordon equation).

In addition to satisfying (28), the pieces $U_i^{\pm}(n)$ must match at the endpoints of consecutive intervals. Thus, in order to have a multi-pulse solution, $U_i^{\pm}(n)$ must satisfy the system of equations

$$\begin{aligned} U_i^{\pm}(n+1) &= F(U_i^{\pm}(n)) \\ U_i^+(N_i^+) - U_{i+1}^-(-N_i^-) &= 0 \\ U_i^+(0) - U_i^-(0) &= 0 \end{aligned} \tag{45}$$

for $i = 1, \dots, m$. The first equation in (45) states that the individual pulses are solutions to the difference equation (28) on the appropriate domains; the second equation glues together the individual pulses at their tails; and the third equation is a matching condition at the centers of the pulses.

We will solve (45) using Lin's method. Lin's method yields a solution which has m jumps in the direction of $Z_1(0)$. An $m-$pulse solution exists if and only if all m jumps are 0. These jump conditions are given in the next theorem.

Theorem 1. *Assume Hypotheses 1, 2, 3, and 4, and let $Q(n)$ be the primary pulse solution to (28). Then there exists a positive integer N_0 with the following property. For all $m > 1$, pulse distances $N_i \geq N_0$ and symmetry parameters θ_i, there exists a unique $m-$pulse solution $Q_m(n)$ to (28) if and only if the m jump conditions*

$$\begin{aligned} \xi_1 &= \langle T(\theta_1)Z_1(N_1^+), T(\theta_2)Q(-N_1^-)\rangle + R_1 = 0 \\ \xi_i &= \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})Q(-N_i^-)\rangle \\ &\quad - \langle T(\theta_i)Z_1(-N_{i-1}^-), T(\theta_{i-1})Q(N_{i-1}^+)\rangle + R_i = 0 \\ &\quad \text{for } i = 2, \dots, m-1 \\ \xi_m &= -\langle T(\theta_m)Z_1(-N_{m-1}^-), T(\theta_{m-1})Q(N_{m-1}^+)\rangle + R_m = 0 \end{aligned} \tag{46}$$

are satisfied, where the remainder terms have uniform bound

$$|R_i| \leq Cr^{-3N},$$

and r is defined in Eq. (39). $Q_m(n)$ can be written piecewise in the form (43), and the following estimates hold:

$$\begin{aligned} \|\tilde{Q}_i^{\pm}\| &\leq Cr^{-N} \\ \tilde{Q}_i^+(N_i^+) &= T(\theta_{i+1})Q(-N_i^-) + \mathcal{O}(r^{-2N}) \\ \tilde{Q}_{i+1}^-(-N_i^-) &= T(\theta_i)Q(N_i^+) + \mathcal{O}(r^{-2N}). \end{aligned} \tag{47}$$

Remark 3. If Eq. (28) has a conserved quantity, i.e. a function $E : \mathbb{R}^{4k} \to \mathbb{R}$ such that $E(F(U)) = E(U)$, we can remove one of the jump conditions in (46) as is done in [24]. For DNLS, this conserved quantity is the current density (10).

3.4. Eigenvalue problem

We will now turn to the spectral stability of multi-pulses. In particular, we will locate the interaction eigenvalues. Let $Q_m(n) = (q_m(n), q_m(n-1))$ be an $m-$pulse solution to (28) constructed according to Theorem 1. By Theorem 1, $Q_m(n)$ can be written piecewise in the form (43). The eigenvalue problem is

$$V(n+1) = DF(q_m(n))V(n) + \lambda BV(n), \tag{48}$$

where $DF(q_m(n))$ and B are given by (33) and (34). Since $q_m(n)$ decays exponentially to 0 and F is smooth, $DF(q_m(n))$ is exponentially asymptotic to the constant coefficient matrix $DF(0)$, which is hyperbolic.

We can now state the following theorem, in which we locate the eigenvalues of Eq. (32) resulting from interactions between neighboring pulses.

Theorem 2. *Assume Hypotheses 1, 2, 3, and 4. Let $Q_m(n)$ be an $m-$pulse solution to (28) constructed according to Theorem 1 with pulse distances $\{N_1, \dots, N_{m-1}\}$ and symmetry parameters $\{\theta_1, \dots, \theta_m\}$. Then there exists $\delta > 0$ small with the following property. There exists a bounded, nonzero solution $V(n)$ of the eigenvalue problem (48) for $|\lambda| < \delta$ if and only if $E(\lambda) = 0$, where*

$$E(\lambda) = \det\left(A - \frac{1}{d}M\lambda^2 I + R(\lambda)\right). \tag{49}$$

M is defined in (26), and A is the tridiagonal $m \times m$ matrix

$$A = \begin{pmatrix} -a_1 & a_1 & & & \\ -\tilde{a}_1 & \tilde{a}_1 - a_2 & a_2 & & \\ & -\tilde{a}_2 & \tilde{a}_2 - a_3 & a_3 & \\ & \ddots & & \ddots & \\ & & & -\tilde{a}_{m-1} & \tilde{a}_{m-1} \end{pmatrix}, \tag{50}$$

where

$$\begin{aligned} a_i &= \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})T'(0)Q(-N_i^-)\rangle \\ \tilde{a}_i &= \langle T(\theta_{i+1})Z_1(-N_i^-), T(\theta_i)T'(0)Q(N_i^+)\rangle. \end{aligned}$$

The remainder term has uniform bound

$$|R(\lambda)| \leq C\left((r^{-N} + |\lambda|)^3\right), \tag{51}$$

where $N = \frac{1}{2}\min\{N_1, \dots, N_{m-1}\}$ and r is defined in Eq. (39).

3.5. Transverse intersection

We present one more result, which concerns the existence of multi-pulse solutions in the case where the stable manifold $W^s(0)$ and unstable manifold $W^u(0)$ intersect transversely, as opposed to the one-dimensional intersection in Hypothesis 4. This is particularly useful for DNLS, as this occurs when we consider its real-valued solutions. In the transverse intersection case, we have a much more general result. Consider the difference equation

$$U(n+1) = F(U(n)), \tag{52}$$

where $F : \mathbb{R}^k \to \mathbb{R}^k$ is smooth. We make the following assumptions about F.

Hypothesis 5. The following hold concerning the function F.

(i) There exists a finite group G (which may be the trivial group) for which the group action is a unitary group of symmetries $T(\theta)$ on $\mathbb{R}^k$ such that

$$F(T(\theta)U) = F(U)T(\theta) \tag{53}$$

for all $\theta \in G$ and all $U \in \mathbb{R}^k$.

(ii) The origin is a hyperbolic equilibrium for F, thus there exists a radius $r > 1$ such that for all eigenvalues ν of $DF(0)$, $|\nu| \leq 1/r$ or $|\nu| \geq r$. Furthermore, $\dim E^s \geq 1$ and $\dim E^u \geq 1$, where E^s and E^u are the stable and unstable eigenspaces of $DF(0)$.

(iii) There exists a primary pulse homoclinic orbit solution $Q(n)$ to (52) which connects the equilibrium at 0 to itself.

(iv) The stable and unstable manifolds $W^s(0)$ and $W^u(0)$ intersect transversely.

We note that for DNLS, the group G is $(\{\pm 1\}, \cdot)$. In this case, Lin's method yields a unique $m-$pulse solution to (52).

Theorem 3. *Assume Hypothesis 5, and let $Q(n)$ be the primary pulse solution to (28). Then there exists a positive integer N_0 with the following property. For all $m > 1$, pulse distances $N_i \geq N_0$ and symmetry parameters θ_i, there exists a unique $m-$pulse solution $Q_m(n)$ to (28) which can be written in the form (43). The remainder terms $\tilde{Q}_i^{\pm}(n)$ have the same estimates as in Theorem 1.*

Note that if the discrete system comes from a discrete Hamiltonian system, the Hessian of the Hamiltonian $\mathcal{H}$ about the primary pulse will not have an eigenvalue at the origin. If the Hessian of the Hamiltonian $\mathcal{H}$ for the primary pulse is positive definite, the primary pulse will be orbitally stable. Lin's method then shows that the multi-pulses we constructed in Theorem 3 will not have any eigenvalues near the origin either: their Hessians will therefore also be positive definite, and the multi-pulses are orbitally stable.

4. Discrete NLS equation

4.1. Background

We will now apply the results of the previous section to the DNLS to illustrate the impact of the discrete Lin's method. Before we do that, we will give a brief overview what is already known. Many more details can be found in [13,14].

At the anti-continuum limit, Eq. (2) reduces to a system of decoupled algebraic equations. Any u_n with $u_n \in \{0, \pm\sqrt{\omega}\}$ is a solution. For $d > 0$, the DNLS possesses two real-valued, symmetric, single pulse solutions (up to rotation): on-site solutions, which are centered on a single lattice point; and off-site solutions, which are centered between two adjacent lattice points [13]. The on-site solution has a single eigenvalue at 0 from rotational symmetry. The off-site solution has an additional pair of real eigenvalues; since the off-site solution is spectrally unstable, we will only consider the on-site solution from here on as the foundation for the single pulse state.

For sufficiently small d, m-pulse solutions exist to Eq. (2) for any pulse distances as long as the phase differences satisfy $\Delta\theta_i \in \{0, \pi\}$ [8, Proposition 2.1]. For sufficiently small d, this $m-$pulse is spectrally unstable unless all of the phase differences $\Delta\theta_i$ are π; in that case there are $m-1$ pairs of purely imaginary eigenvalues with negative Krein signature [8, Theorem 3.6]. This means that these eigendirections, although neutrally stable, are prone to instabilities upon collision with other eigenvalues, which can occur when parameters (such as d) are varied. However, they may also lead to instabilities at a purely nonlinear level (despite potential spectral stability) as a result of the mechanism explored, e.g., in [25,26]. For any d for which the $m-$pulse exists, if one or more phase differences $\Delta\theta_i$ is 0, it was shown in [9] that there is at least one positive, real eigenvalue.

4.2. Main results

Let $q(n)$ be the on-site, real-valued soliton solution to (2). The existence of $q(n)$ is discussed in Section 2. The symmetry group $R(\theta)$, which is the rotation group (5), satisfies the commuting relation (14). Since $J^{-1}R'(0) = -I$, the conserved charge is given by

$$\mathcal{Q}(u) = -\frac{1}{2}\sum_{n=-\infty}^{\infty}\langle u_n, u_n\rangle, \tag{54}$$

which is the same as (6). The stability criterion M from (26) is given by

$$M = \partial_\omega\left(\frac{1}{2}\sum_{n=-\infty}^{\infty} q_n^2\right) = \sum_{n=-\infty}^{\infty} q_n\partial_\omega q_n. \tag{55}$$

We will assume that $M > 0$, as confirmed by numerical computation [13]. Following Section 3.2, Eq. (8) can be written as $U(n+1) = F(U(n))$, where

$$F\begin{pmatrix} u \\ \tilde{u}\end{pmatrix} = \begin{pmatrix}\left(2+\frac{\omega}{d}\right)u + \frac{1}{d}f(u) - \tilde{u} \\ u\end{pmatrix}. \tag{56}$$

The origin is a hyperbolic equilibrium of F, and the eigenvalues of $DF(0)$ are $\nu = \{r, 1/r\}$, each with multiplicity 2, where

$$r = 1 + \frac{\omega}{2d}\left(1+\sqrt{1+\frac{4d}{\omega}}\right). \tag{57}$$

For $\omega, d > 0$, we have $r > 1$.

We will characterize an $m-$pulse solution to (2) in terms of the $m-1$ pulse distances $\{N_1, \ldots, N_{m-1}\}$ and phase differences $\{\Delta\theta_1, \ldots, \Delta\theta_{m-1}\}$ between consecutive copies of $q(n)$. Using Theorem 3, the m jump conditions necessary for an $m-$pulse to exist can be written in terms of the phase differences $\Delta\theta_i$ as

$$\begin{aligned}\xi_i = {}& \langle T(-\Delta\theta_i)Z_1(N_i^+), Q(-N_i^-)\rangle - \langle T(\Delta\theta_{i-1})Z_1(-N_{i-1}^-), Q(N_{i-1}^+)\rangle \\ & + R_i \qquad \text{for } i = 1, \ldots, m,\end{aligned} \tag{58}$$

where we take $\Delta\theta_0 = \Delta\theta_m = 0$. We have the following theorem regarding the existence of $m-$pulse solutions.

Theorem 4. *There exists a positive integer N_0 (which depends on ω and d), with the following property. For any $m \geq 2$, pulse distances $N_i \geq N_0$, and phase differences $\Delta\theta_i \in \{0, \pi\}$, there exists a unique $m-$pulse solution $q_m(n)$ to (2) which resembles m consecutive copies of the on-site pulse $q(n)$. No other phase differences are possible.*

Remark 4. For $\omega, d > 0$, r is an increasing function of both ω and d, thus the threshold N_0 decreases as ω and d increase.

In addition, we note that an alternative approach to the existence of pulse solutions in DNLS using the underlying reversibility of the system can be found in [27].

By (25) and (27), the linearization (48) about $q_m(n)$ has a kernel with algebraic multiplicity at least 2 and geometric multiplicity at least 1 which is a result of rotational invariance. Assuming Hypothesis 5, the geometric multiplicity is exactly 1. The following theorem locates the small eigenvalues of the linearization about $q_m(n)$ resulting from interaction between consecutive copies of $q(n)$.

Theorem 5. *Let $q_m(n)$ be an $m-$pulse solution to (2) with pulse distances N_i and phase differences $\Delta\theta_i$. Assume that $M > 0$, where M is given by (55). Let $N = \frac{1}{2}\min\{N_1, \dots, N_{m-1}\}$. Then for N sufficiently large, there exist $m-1$ pairs of interaction eigenvalues $\{\pm\lambda_1, \dots, \pm\lambda_{m-1}\}$, which can be grouped as follows. There are k_π pairs of purely imaginary eigenvalues and k_0 pairs of real eigenvalues, where k_π is the number of phase differences $\Delta\theta_i$ which are π, and k_0 is the number of phase differences $\Delta\theta_i$ which are 0. The λ_j are close to 0 and are given by the following formula*

$$\lambda_j = \sqrt{\frac{d\mu_j}{M}} + \mathcal{O}(r^{-2N}) \qquad \text{for } j = 1, \dots, m-1, \tag{59}$$

where r is defined in Eq. (57), d is the coupling constant and $\{\mu_1, \dots, \mu_{m-1}\}$ are the distinct, real, nonzero eigenvalues of the symmetric, tridiagonal matrix

$$A = \begin{pmatrix} -a_1 & a_1 & & & \\ a_1 & -a_1 - a_2 & a_2 & & \\ & a_2 & -a_2 - a_3 & a_3 & \\ & \ddots & & \ddots & \\ & & & a_{m-1} & -a_{m-1} \end{pmatrix}, \tag{60}$$

where $a_i = \cos(\Delta\theta_i)b_i$ and

$$b_i = \begin{cases} q\left(\frac{N_i}{2}\right)\left[q\left(\frac{N_i}{2}+1\right) - q\left(\frac{N_i}{2}-1\right)\right] & N_i \text{ even} \\ q\left(\frac{N_i-1}{2}\right)\left(\frac{N_i+3}{2}\right) - \left(\frac{N_i+1}{2}\right)q\left(\frac{N_i-3}{2}\right) & N_i \text{ odd.} \end{cases} \tag{61}$$

Remark 5. If all the nonzero eigenvalues μ_j of A are larger than $\mathcal{O}(r^{-4N})$, then the formula (59) is the sum of a leading order term and a small remainder term. A good approximation for the eigenvalues λ_j can be obtained by computing the eigenvalues of A. A sufficient condition for this is $N_{\max} < 2N$, where $N_{\max} = \frac{1}{2}\max\{N_1, \dots, N_{m-1}\}$.

In addition, we remark that if $b_i = b$ for all i, $A = -b\mathcal{M}_1$, where the matrix $\mathcal{M}_1$ is defined in [13, (2.84)] and represents interactions between neighboring sites. The scaling of the interaction eigenvalues with respect to d and N is similar to that of small eigenvalues of multi-site breathers in the discrete Klein–Gordon equation [23, Lemma 2], where we note that N in our analysis corresponds to $2N$ in [23].

We can compute the nonzero eigenvalues of Eq. (60) in several special cases. In the first corollary, we consider the case where the pulse distances N_i are equal.

Corollary 1. *Let $q_m(n)$ be an $m-$pulse solution to (2) with pulse distances $N_i = 2N$ and phase differences $\Delta\theta_i$. Then the interaction eigenvalues λ_j are as follows.*

(i) For $m = 2$, we have

$$\lambda_1 = \begin{cases} \sqrt{2}\nu + \mathcal{O}(r^{-2N}) & \Delta\theta_1 = 0 \\ \sqrt{2}\nu i + \mathcal{O}(r^{-2N}) & \Delta\theta_1 = \pi. \end{cases} \tag{62}$$

(ii) For $m = 3$, we have

$$\lambda_{1,2} = \begin{cases} \nu, \sqrt{3}\nu + \mathcal{O}(r^{-2N}) & (\Delta\theta_1, \Delta\theta_2) = (0, 0) \\ 3^{1/4}\nu, 3^{1/4}\nu i + \mathcal{O}(r^{-2N}) & (\Delta\theta_1, \Delta\theta_2) = (0, \pi) \\ \nu i, \sqrt{3}\nu i + \mathcal{O}(r^{-2N}) & (\Delta\theta_1, \Delta\theta_2) = (\pi, \pi). \end{cases} \tag{63}$$

(iii) For $m > 3$, if $\Delta\theta_i = \Delta\theta$ for all i,

$$\lambda_j = \begin{cases} \sqrt{2\left(\cos\frac{\pi j}{m} - 1\right)\nu} + \mathcal{O}(r^{-2N}) & \Delta\theta = 0 \\ \sqrt{2\left(\cos\frac{\pi j}{m} - 1\right)\nu i} + \mathcal{O}(r^{-2N}) & \Delta\theta = \pi \end{cases}$$

for $j = 1, \dots, m-1$.

where $\nu = \sqrt{\dfrac{|b|d}{M}} = \mathcal{O}(r^{-N})$, and b is given by Eq. (61).

The case of equal pulse distances in DNLS resembles the multi-site breathers considered in [23, Lemma 2]. In the second corollary, we give a general formula for the eigenvalues for a 3-pulse.

Corollary 2. *Let $q_3(n)$ be an $3-$pulse solution to (2) with pulse distances N_1, N_2 and phase differences $\Delta\theta_1, \Delta\theta_2$. Then the interaction eigenvalues λ_1, λ_2 are given by*

$$\lambda_{1,2} = \sqrt{\frac{d}{M}}\Big(-b_1\cos\Delta\theta_1 - b_2\cos\Delta\theta_2 \pm\sqrt{b_1^2 + b_2^2 - b_1 b_2 \cos\Delta\theta_1\cos\Delta\theta_2}\Big)^{1/2} + \mathcal{O}(r^{-2N}). \tag{64}$$

If $N_1 < N_2 < 2N_1$, then, to leading order, these have magnitude

$$\begin{aligned} |\lambda_1| &= \sqrt{\frac{2|b_1|d}{M}} = \mathcal{O}(r^{-N_1/2}) \\ |\lambda_2| &= \sqrt{\frac{3|b_2|d}{2M}} = \mathcal{O}(r^{-N_2/2}), \end{aligned} \tag{65}$$

where b_1 and b_2 are given by Eq. (61).

4.3. Numerical results

In this section, we provide numerical verification for the results in the previous section. We first construct multi-pulse solutions to the steady state DNLS problem by using Matlab for parameter continuation in the coupling constant d from the anti-continuum limit. We then find the eigenvalues of the linearization about this solution using Matlab's `eig` function.

First, we look at multi-pulses where the pulse distances are equal. The left and center panels of Fig. 2 show the pulse profile and eigenvalue pattern for the two double pulses (of relative phase 0 and π). Eq. (62) from Corollary 1 states that for fixed ω and d, the interaction eigenvalues decay as r^{-N}. In the right panel of Fig. 2, we plot $\log\lambda$ vs. N for the two possible double pulses and construct a least-squares linear regression line. In both cases, the relative error of the slope of this line (which is predicted to be $-\log r$) is order 10^{-4}. This result provides theoretical and numerical support to the earlier observations of [9].

We do the same for triple pulses with equal pulse distances in Fig. 3. Since the pulse distances are equal, both sets of interaction eigenvalues decay as r^{-N} by Eq. (63) from Corollary 1. In the right panel of Fig. 3, we plot $\log\lambda$ vs. N for the three triple pulses and construct a least-squares linear regression line. In all three cases, namely the in-phase (or $+++$) pulse, the out-of-phase (or $+-+$) and finally the intermediate/mixed phase case (or $++-$), the relative error of the slope of the least squares linear regression line is of order 10^{-4}.

We can also look at triple pulses with unequal pulse distances N_1 and N_2. If $N_1 < N_2 < 2N_1$, then by Corollary 2, there are two pairs of eigenvalues of order $r^{-N_1/2}$ and $r^{-N_2/2}$. We can similarly verify these decay rates numerically.

Finally, we can compute the leading order term in Eq. (59) and compare that to the numerical result. A value for ω is chosen, and the single pulse solution $q(n;\omega)$ is constructed numerically

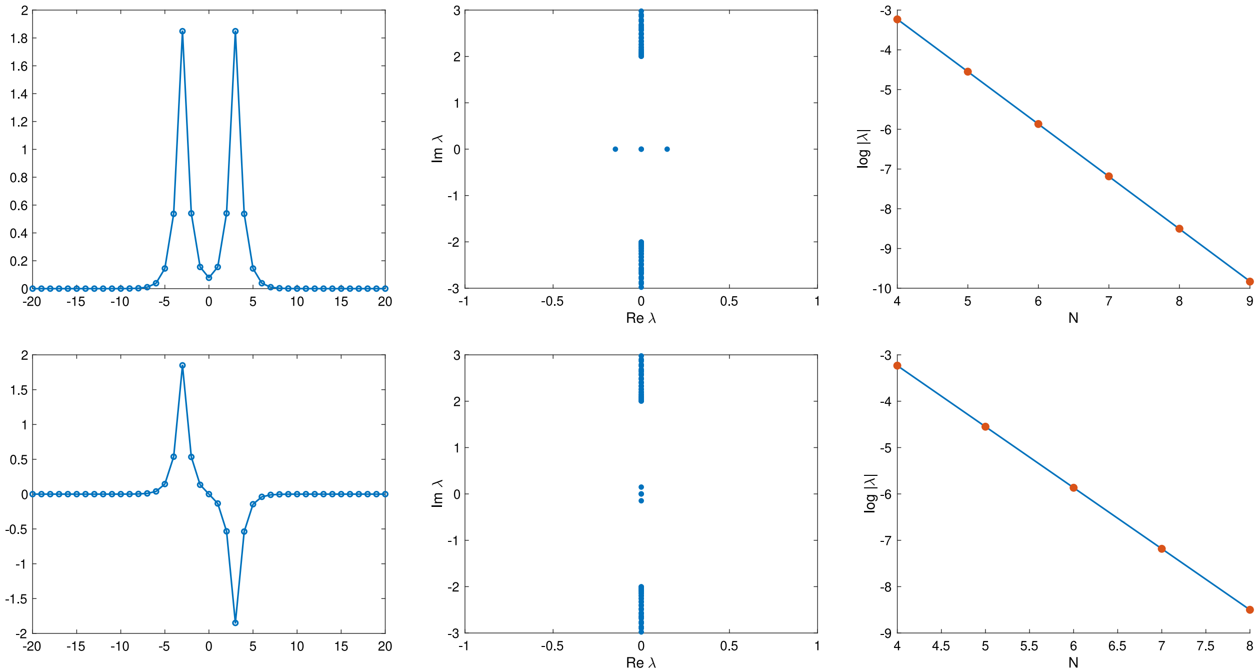

Fig. 2. Solution profile (left panel), spectral plane eigenvalue pattern (center panel), and plot of log(λ) vs. N with least squares linear regression line (right panel) for $++$ (top) and $+-$ (bottom) pulses. The symbolic notation here and below follows that of [19], referring with a symbolic sign representation to the positive or negative value of the peak of the pulse. Parameters $\omega = 2$ and $d = 1.0$.

using parameter continuation from the anti-continuum limit until the desired coupling parameter d is reached. The terms b_i from the matrix A are computed by using Eq. (61) with the numerically constructed solution $q(n;\omega)$. For the derivative $\partial_\omega q(n;\omega)$, solutions $q(n;\omega+\epsilon)$ and $q(n;\omega-\epsilon)$ are constructed numerically for small ϵ by parameter continuation from the anti-continuum limit to the same value of d. The derivative $\partial_\omega q(n;\omega)$ is computed from these via a centered finite difference method; this is used together with $q(n;\omega)$ to calculate M.

First, we consider the case of equal pulse distances. We use the expressions from Corollary 1 to compute the leading order term for the interaction eigenvalues, and we compare this to the results from Matlab's `eig` function. In Fig. 4 we fix the inter-pulse distances and plot the log of the relative error of the eigenvalues vs. the coupling parameter d. For intermediate values of d, the relative error is less than 10^{-3}. Since the results of Theorem Theorem 2 are not uniform in d, i.e. they hold for sufficiently large N once d and ω are chosen, we do not expect to have a nice relationship between the error and d. This is furthermore complicated by the fact that additional sources of error arise from numerically approximating b_i and M. In principle, though, the method (and the asymptotic prediction) yields satisfactory results except for the vicinity of the anti-continuum limit and the near-continuum limit (where the role of discreteness is too weak). It is interesting to point out that at a "middle ground" between these two limits, namely around $d = 0.5$, we observe the optimal performance of the theoretical prediction.

We can also do this for triple pulses with unequal pulse distances. In this case, we use Corollary 2 to compute the eigenvalues to leading order. Fig. 5 shows the log of the relative error of the eigenvalues vs. the coupling parameter d. For intermediate values of d, the relative error is again less than 10^{-3}. Once again this validates the relevance of the method especially so for the case of intermediate ranges of the coupling parameter d.

5. Conclusions and future challenges

In this paper we used Lin's method to construct multi-pulses in discrete systems and to find the small eigenvalues resulting from interaction between neighboring pulses in these structures. In doing so, we are able to extend known results about DNLS to parameter regimes which are further from the anti-continuum limit. In essence, we replace the requirement that the coupling parameter d be small by the condition that the pulses are well separated. This method also allows us to estimate these interaction eigenvalues to a good degree of accuracy for intermediate values of d.

The theoretical results we obtained will apply to many other Hamiltonian systems, as long as the coupling between nodes is via the discrete second order centered difference operator Δ_2. Since these restrictions were motivated partly by mathematical convenience, future work could extend these results to a broader class of Hamiltonian systems. Indeed, there exist numerous examples worth considering, ranging from simpler ones such as discrete multiple-kink states in the discrete sine–Gordon equation [28], to settings of first order PDE discretizations related, e.g., to the Burgers model [29] or even discretizations of third order models such as the Korteweg–de Vries equation [30]. This work could also be applied to the spectral stability of multi-site breathers, a wider class of equations, such as that considered in [31].

Another direction for future work is characterizing the family of multi-pulse solutions which arises as the coupling parameter d is varied. Recent work [32] has investigated stationary, spatially localized patterns in lattice dynamical systems which change as a parameter is varied; the coupling parameter in this case is fixed. In some cases, these patterns exist along a closed bifurcation curve known as an isola. Numerical continuation with AUTO in the coupling parameter d suggests that multi-pulse solutions in DNLS exist on an isola. The parameter d varies over a bounded interval which includes the origin, thus the isola contains solutions to both the focusing and defocusing equation.

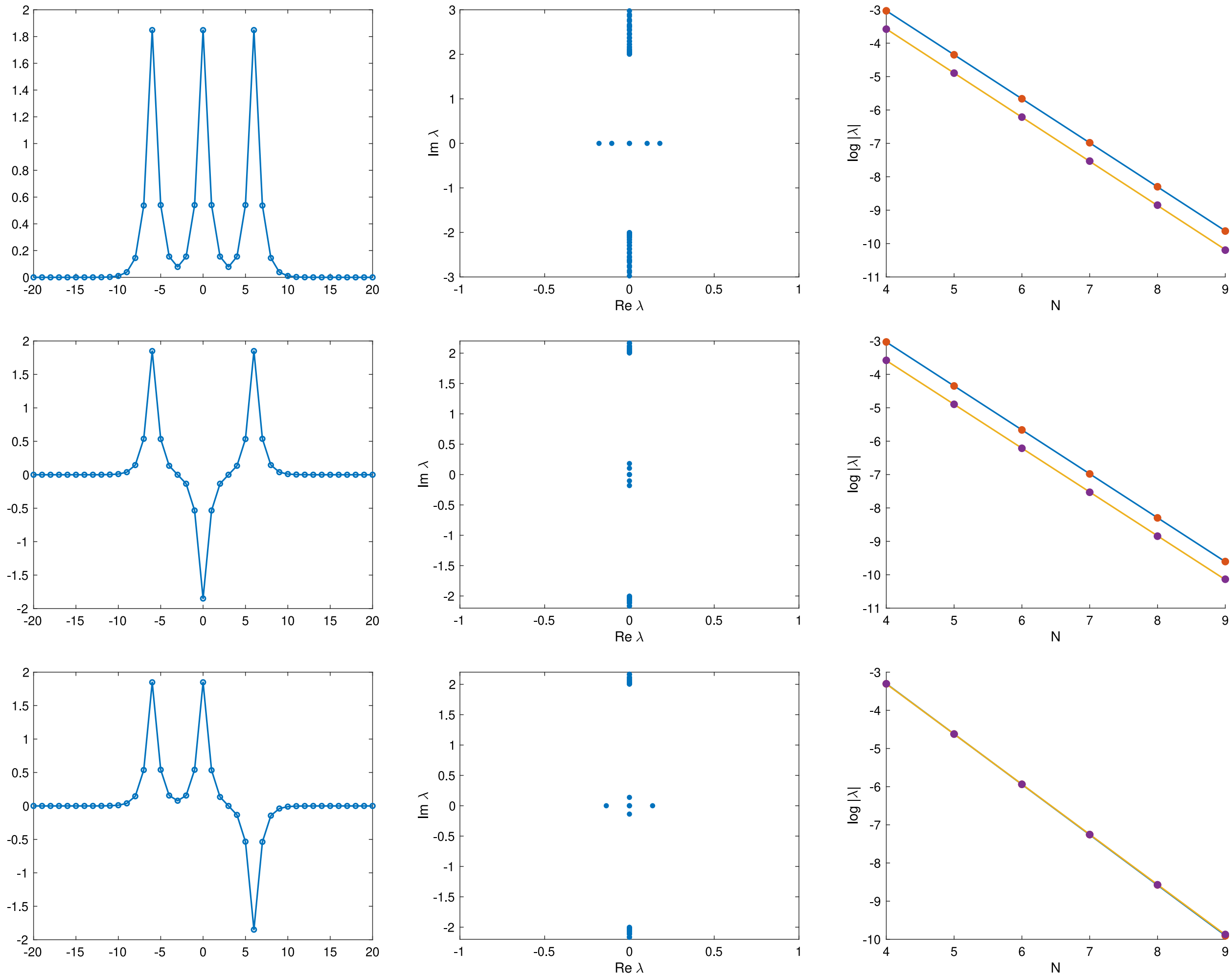

Fig. 3. Solution profile (left panel), spectral plane eigenvalue pattern (center panel), and plot of $\log(\lambda)$ vs. N with least squares linear regression line (right panel) for the three triple pulse cases: $+++$ (top), $+-+$ (middle), and $++-$ (bottom) pulses. Parameters $\omega = 2$ and $d = 1.0$.

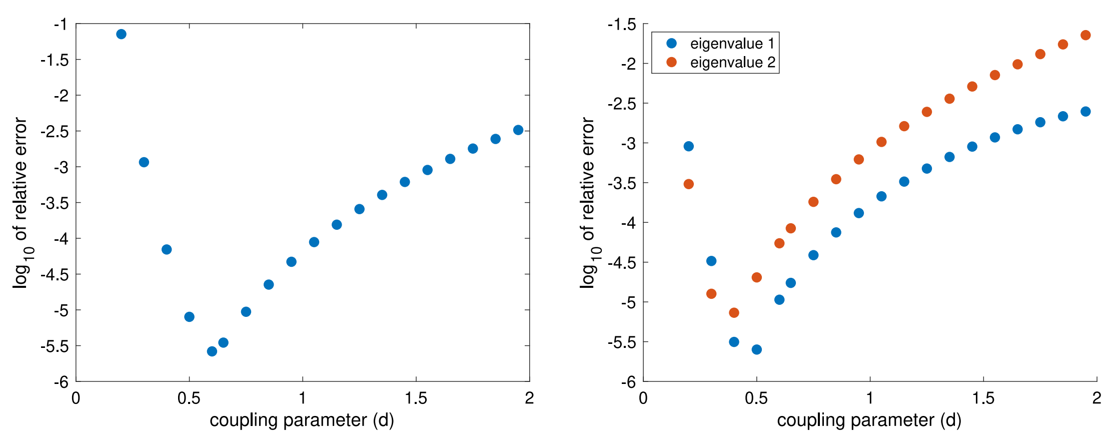

Fig. 4. Log of relative error of eigenvalues vs. coupling parameter d for double (in phase) pulse $++$ ($N_1 = 10$) and triple (out-of-phase) pulse $+-+$ ($N_1 = N_2 = 8$). $\omega = 2$ in both cases.

A final direction for future works would concern the consideration of higher dimensional settings. Here, the interaction between pulses would involve the geometric nature of the configuration they form and the "line of sight" between them. The latter is expected (from the limited observations that exist [33]) to determine the nature of the interaction eigenvalues. Here, however, the scenarios can also be fundamentally richer as coherent states involving topological charge/vorticity may come into play [13]. In the latter case, it is less straightforward to identify what the conclusions may be and considering such more complex configurations (given also their experimental observation [34,35]) may be of particular interest.

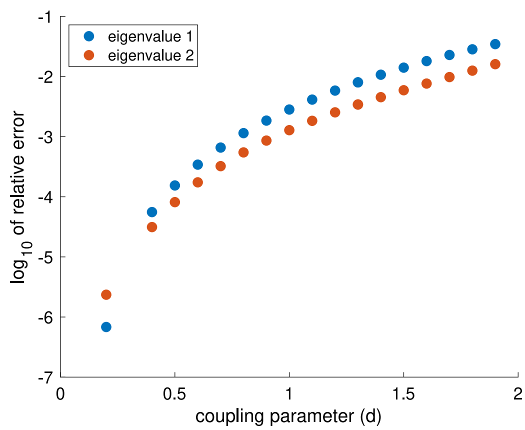

Fig. 5. Log of relative error of eigenvalues vs. coupling parameter d for triple in-phase pulse $+++$ with unequal pulse distances ($N_1 = 8, N_2 = 6$), $\omega = 2$.

6. Proof of existence theorems

In this section, we will prove Theorems 1 and 3. Since the proofs are very similar, we will prove Theorem 1 then state what modifications are necessary for the proof of Theorem 3. Throughout this section, we will assume Hypotheses 1, 2, 3, and 4. We begin with setting up the exponential dichotomy necessary for the proof. The technique of the proof is very similar to that in [36].

6.1. Discrete exponential dichotomy

First, we define the discrete evolution operator for linear difference equations.

Lemma 1 (*Discrete Evolution Operator*)**.** *Consider the difference equation together with its adjoint*

$$V(n+1) = A(n)V(n) \tag{66}$$

$$Z(n+1) = [A(n)^{-1}]^* Z(n), \tag{67}$$

where $n \in \mathbb{Z}$, $V(n) \in R^d$, and the matrix $A(n)$ is invertible for all n. Define the discrete evolution operator by

$$\Phi(m,n) = \begin{cases} I & m = n \\ A(m-1)\cdots A(n+1)A(n) & m > n \\ A^{-1}(m)\cdots A^{-1}(n-2)A^{-1}(n-1) & m < n. \end{cases} \tag{68}$$

(i) The evolution operators Φ of Eq. (66) and Ψ of Eq. (67) are related by

$$\Psi(m,n) = \Phi(n,m)^*. \tag{69}$$

(ii) If $V(n)$ is a solution to Eq. (66) and $Z(n)$ is a solution to Eq. (67), then the inner product $\langle V(n), Z(n)\rangle$ is constant in n.

Proof. For (i), the result holds trivially for $m = n$. For, $m < n$ we have

$$\begin{aligned} \Psi(m,n) &= A(m)^* \cdots A(n-2)^* A(n-1)^* \\ &= [A(n-1)A(n-2)\cdots A(m)]^* \\ &= \Phi(n,m)^*. \end{aligned}$$

The case for $m > n$ is similar.

For (ii), we have

$$\begin{aligned} \langle V(n+1), Z(n+1)\rangle &= \langle A(n)V(n), [A(n)^{-1}]^* Z(n)\rangle \\ &= \langle A(n)^{-1}A(n)V(n), Z(n)\rangle \\ &= \langle V(n), Z(n)\rangle. \quad \square \end{aligned}$$

Next, we give a criterion for an exponential dichotomy.

Lemma 2 (*Exponential Dichotomy*)**.** *Consider the difference equation*

$$V(n+1) = A(n)V(n). \tag{70}$$

Suppose there exist a constant $r > 1$ and a constant coefficient matrix A such that

$$|A(n) - A| \leq C r^{-|n|} \tag{71}$$

and $|\lambda| \geq r$ or $|\lambda| \leq 1/r$ for all eigenvalues λ of A. Then (70) has exponential dichotomies on $\mathbb{Z}^\pm$. Specifically, there exist projections $P_\pm^s$ and $P_\pm^u$ defined on $\mathbb{Z}^\pm$ such that the following are true.

(i) Let $\Phi(m,n)$ be the evolution operator for (70). Then

$$P_\pm^{s/u}(m)\Phi(m,n) = \Phi(m,n)P_\pm^{s/u}(n). \tag{72}$$

(ii) Let $\Phi_\pm^{s/u}(m,n) = \Phi(m,n)P_\pm^{s/u}(n)$ for $m, n \geq 0$ and $m, n \leq 0$ (respectively). Then we have the estimates

$$\begin{aligned} |\Phi_+^s(m,n)| &\leq C r^{-(m-n)} & 0 \leq n \leq m \\ |\Phi_+^u(m,n)| &\leq C r^{-(n-m)} & 0 \leq m \leq n \\ |\Phi_-^s(m,n)| &\leq C r^{-(m-n)} & n \leq m \leq 0 \\ |\Phi_-^u(m,n)| &\leq C r^{-(n-m)} & m \leq n \leq 0\,, \end{aligned}$$

where the evolution operator $\Phi(m,n)$ is defined in Lemma 1.

(iii) Let $E^{s/u}$ be the stable and unstable eigenspaces of A, and let $Q^{s/u}$ be the corresponding eigenprojections. Then we have

$$\begin{aligned} \dim \operatorname{ran} P_\pm^s(n) &= \dim E^s \\ \dim \operatorname{ran} P_\pm^u(n) &= \dim E^u \end{aligned}$$

and the exponential decay rates

$$|P_\pm^{s/u}(n) - Q^{s/u}| \leq C r^{-|n|}. \tag{73}$$

Proof. We will consider the problem on $\mathbb{Z}^+$. Since A is constant coefficient and hyperbolic, the difference equation $W(n+1) = AW(n)$ has an exponential dichotomy on $\mathbb{R}^+$. All the results except for Eq. (73) follow directly from [37, Proposition 2.5]. Eq. (73) follows from using the estimate (71) in the proof of [37, Proposition 2.5]. $\square$

The last thing we will need is a version of the variation of constants formula for the discrete setting.

Lemma 3 (*Discrete Variation of Constants*)**.** *The solution $V(n)$ to the initial value problem*

$$\begin{aligned} V(n+1) &= A(n)V(n) + G(V(n),n) \\ V(n_0) &= V_{n_0} \end{aligned}$$

can be written in summation form as

$$V(n) = \begin{cases} V_{n_0} & n = n_0 \\ \Phi(n,n_0)V_{n_0} + \sum_{j=n_0}^{n-1} \Phi(n,j+1)G(V(j),j) & n > n_0 \\ \Phi(n,n_0)V_{n_0} - \sum_{j=n}^{n_0-1} \Phi(n,j+1)G(V(j),j) & n < n_0. \end{cases} \tag{74}$$

Proof. For $n = n_0 + 1$,

$$\begin{aligned} V(n_0+1) &= A(n_0)V(n_0) + G(V(n_0),n_0) \\ &= \Phi(n_0+1,n_0)V(n_0) + \Phi(n_0,n_0)G(V(n_0),n_0). \end{aligned}$$

Iterate this to get the result for $n > n_0$. The case for $n < n_0$ is similar. $\square$

6.2. Fixed point formulation

To find a solution to the system of equations (45), we will rewrite the system as a fixed point problem. First, we expand F in a Taylor series about $T(\theta_i)Q(n)$ to get

$$\begin{aligned} F(U_i^\pm(n)) &= F(T(\theta_i)Q(n) + \tilde{Q}_i^\pm(n)) \\ &= F(T(\theta_i)Q(n)) + DF(T(\theta_i)Q(n))\tilde{Q}_i^\pm(n) + G(\tilde{Q}_i^\pm(n)) \\ &= T(\theta_i)DF(Q(n))T(\theta_i)^{-1}\tilde{Q}_i^\pm(n) + G(\tilde{Q}_i^\pm(n)), \end{aligned}$$

where $G(\tilde{Q}_i^\pm(n)) = \mathcal{O}(|\tilde{Q}_i^\pm|^2)$ with $G(0) = 0$ and $DG(0) = 0$, and we used the symmetry relation (19) in the last line. Finally, let

$$d_i = T(\theta_{i+1})Q(-N_i^-) - T(\theta_i)Q(N_i^+). \tag{75}$$

Substituting these into (45), we obtain the following system of equations for the remainder functions $\tilde{Q}_i^\pm$.

$$\tilde{Q}_i^\pm(n+1) = T(\theta_i)DF(Q(n))T(\theta_i)^{-1}\tilde{Q}_i^\pm(n) + G(\tilde{Q}_i^\pm(n)) \tag{76}$$

$$\tilde{Q}_i^+(N_i^+) - \tilde{Q}_{i+1}^-(-N_i^-) = d_i \tag{77}$$

$$\tilde{Q}_i^+(0) - \tilde{Q}_i^-(0) = 0. \tag{78}$$

Next, we look at the variational and adjoint variational equations associated with (28), which are

$$V(n+1) = DF(Q(n))V(n) \tag{79}$$

$$Z(n+1) = [DF(Q(n))^*]^{-1}Z(n). \tag{80}$$

The variational equation (79) has a bounded solution $T'(0)Q(n)$, thus we can decompose the tangent spaces to $W^s(0)$ and $W^u(0)$ at $Q(0)$ as

$$\begin{aligned} T_{Q(0)}W^u(0) &= Y^- \oplus \mathbb{R}T'(0)Q(0) \\ T_{Q(0)}W^s(0) &= Y^+ \oplus \mathbb{R}T'(0)Q(0). \end{aligned}$$

The adjoint variational equation also has a unique bounded solution $Z_1(n)$ given by Eq. (42). By Lemma 1, $Z_1(0) \perp T'(0)Q(0) \oplus Y^- \oplus Y^+$, thus we can decompose $\mathbb{R}^{4k}$ as

$$\mathbb{R}^{4k} = \mathbb{R}T'(0)Q(0) \oplus Y^+ \oplus Y^- \oplus \mathbb{R}Z_1(0). \tag{81}$$

Since $T(\theta)$ is unitary, we also have the decomposition

$$\mathbb{R}^{4k} = \mathbb{R}T(\theta_i)T'(0)Q(0) \oplus T(\theta_i)Y^+ \oplus T(\theta_i)Y^- \oplus \mathbb{R}T(\theta_i)Z_1(0). \tag{82}$$

Finally, since perturbations in the direction of $T(\theta_i)T'(0)Q(0)$ are handled by the symmetry parameter θ_i, we may without loss of generality choose $\tilde{Q}_i^\pm$ so that

$$\tilde{Q}_i^\pm(0) \in T(\theta_i)Y^+ \oplus T(\theta_i)Y^- \oplus \mathbb{R}T(\theta_i)Z_1(0). \tag{83}$$

Let $\Phi(m,n;\theta)$ be the evolution operator for

$$V(n+1;\theta) = T(\theta)DF(Q(n))T(\theta)^{-1}V(n;\theta). \tag{84}$$

We note that $T(\theta)T'(0)Q(n)$ is a solution to Eq. (84). Using Eq. (19), the evolution operators are related to those for $\theta = 0$ by

$$\Phi(m,n;\theta) = T(\theta)\Phi(m,n;0)T(\theta)^{-1}. \tag{85}$$

Since $T(\theta)DF(Q(n))T(\theta)^{-1}$ decays exponentially to $T(\theta)DF(0)T(\theta)^{-1}$ and $DF(0)$ is hyperbolic, Eq. (84) has exponential dichotomies on $\mathbb{Z}^+$ and $\mathbb{Z}^-$ by Lemma 2, and we note that the estimates from Lemma 2 do not depend on θ. Let $P_{s/u}^\pm(m;\theta)$ and $\Phi_{s/u}^\pm(m,n;\theta)$ be the projections and evolutions for this exponential dichotomy on $\mathbb{Z}^\pm$. The projections $P_{s/u}^\pm(m;\theta)$ are related to those for $\theta = 0$ by

$$P_{s/u}^\pm(m;\theta) = T(\theta)P_{s/u}^\pm(m;0)T(\theta)^{-1}.$$

Finally, let $E^s(\theta)$ and $E^u(\theta)$ be the stable and unstable eigenspaces of $T(\theta)DF(0)T(\theta)^{-1}$, and let $P_0^s(\theta)$ and $P_0^u(\theta)$ be the corresponding eigenprojections.

Next, as in [36] and [3], we write Eq. (76) in fixed-point form using the discrete variation of constants formula (74) together with projections on the stable and unstable subspaces of the exponential dichotomy.

$$\begin{aligned} \tilde{Q}_i^-(n) &= \Phi_s^-(n,-N_{i-1}^-;\theta_i)a_{i-1}^- + \Phi_u^-(n,0;\theta_i)b_i^- \\ &\quad + \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j)) \\ &\quad - \sum_{j=n}^{-1} \Phi_u^-(n,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j)) \\ \tilde{Q}_i^+(n) &= \Phi_u^+(n,N_i^+;\theta_i)a_i^+ + \Phi_s^+(n,0;\theta_i)b_i^+ \\ &\quad + \sum_{j=0}^{n-1} \Phi_s^+(n,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j)) \\ &\quad - \sum_{j=n}^{N_i^+-1} \Phi_u^+(n,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j)), \end{aligned} \tag{86}$$

where $\tilde{Q}_i^-(n) \in \ell^\infty([-N_{i-1}^-, 0])$, $\tilde{Q}_i^+(n) \in \ell^\infty([0, N_i^+])$, and the sums are defined to be 0 if the upper index is smaller than the lower index. For the initial conditions,

$$\begin{aligned} &a_i^- \in E^s(\theta_i),\ a_i^+ \in E^u(\theta_i), \text{ and } a_0^- = a_m^+ = 0 \\ &b_i^+ \in T(\theta_i)Y^+ \text{ and } b_i^- \in T(\theta_i)Y^-. \end{aligned}$$

We note that we do not need to include a component in $T'(0)Q(0)$ in $b_i^\pm$, since that direction is handled by the symmetry parameter θ_i.

Since we wish to construct a homoclinic orbit to the rest state at 0, we take the initial conditions $a_0^- = 0$ and $a_m^+ = 0$. For these cases, the fixed point equations are given by

$$\begin{aligned} \tilde{Q}_1^-(n) &= \Phi_u^-(n,0;\theta_i)b_i^- + \sum_{j=-\infty}^{n-1} \Phi_s^-(n,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j)) \\ &\quad - \sum_{j=n}^{-1} \Phi_u^-(n,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j)) \\ \tilde{Q}_m^+(n) &= \Phi_s^+(n,0;\theta_i)b_i^+ + \sum_{j=0}^{n-1} \Phi_s^+(n,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j)) \\ &\quad - \sum_{j=n}^{\infty} \Phi_u^+(n,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j)), \end{aligned}$$

where the infinite sums converge due to the exponential dichotomy.

6.3. Inversion

As in [36], we will solve Eqs. (76), (77), and (78) in stages. In the first lemma of this section, we solve Eq. (76) for $\tilde{Q}_i^\pm$.

Lemma 4. *For $i = 1, \ldots, m$ there exist unique bounded functions $\tilde{Q}_i^{\pm}(n)$ such that Eq. (76) is satisfied. These solutions depend smoothly on the initial conditions $a_i^{\pm}$ and $b_i^{\pm}$, and we have the estimates*

$$\begin{aligned} \|\tilde{Q}_i^-\| &\le C(|a_{i-1}^-| + |b_i^-|) \\ \|\tilde{Q}_i^+\| &\le C(|a_i^+| + |b_i^+|)\,. \end{aligned} \tag{87}$$

For the interior pieces, we have the piecewise estimates

$$\begin{aligned} |\tilde{Q}_i^-(n)| &\le C(r^{-(N_{i-1}^- + n)}|a_{i-1}^-| + r^n|b_i^-|) \quad n \in [-N_{i-1}^-, 0] \\ |\tilde{Q}_i^+(n)| &\le C(r^{-(N_i^+ - n)}|a_i^+| + r^{-n}|b_i^+|) \quad n \in [0, N_i^+]\,. \end{aligned} \tag{88}$$

Proof. First, we show that the RHS of the fixed point equations (86) defines a smooth map from ℓ^∞ (on the appropriate interval) to itself. For the terms in (86), we have the estimates

$$|\Phi_s^-(n, -N_{i-1}^-; \theta_i)a_{i-1}^-| + |\Phi_u^-(n, 0; \theta_i)b_i^-| \le C(|a_{i-1}^-| + |b_i^-|) \tag{89}$$

and

$$\left| \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n, j+1; \theta_i) G_i^-(\tilde{Q}_i^-(j)) \right| + \left| \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_i) G_i^-(\tilde{Q}_i^-(j)) \right| \le C\|\tilde{Q}_i^-\|^2_{\ell^\infty([-N_{i-1}, 0])}\,,$$

both of which are independent of n. Define the map $K_i^- : \ell^\infty([-N_{i-1}, 0]) \times E^s \times Y^- \to \ell^\infty([-N_{i-1}, 0])$ by

$$\begin{aligned} K_i^-(\tilde{Q}_i^-(n), a_{i-1}^-, b_i^-) &= \tilde{Q}_i^-(n) - \Phi_s^-(n, -N_{i-1}^-; \theta_i)a_{i-1}^- - \Phi_u^-(n, 0; \theta_i)b_i^- \\ &\quad - \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n, j+1; \theta_i) G_i^-(\tilde{Q}_i^-(j)) \\ &\quad + \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_i) G_i^-(\tilde{Q}_i^-(j)). \end{aligned} \tag{90}$$

Since 0 is an equilibrium, $K(0, 0, 0) = 0$. It is straightforward to show that the Fréchet derivative of K_i^- with respect to $\tilde{Q}_i^-$ at $(\tilde{Q}_i^-(n), a_{i-1}^-, b_i^-) = (0, 0, 0)$ is a Banach space isomorphism on $l^\infty([-N_{i-1}, 0])$. Thus we can solve for $\tilde{Q}_i^-(x)$ in terms of (a_{i-1}^-, b_i^-) using the implicit function theorem. This dependence is smooth, since the map K_i^- is smooth. The estimate (87) on $\tilde{Q}_i^-$ comes from (89), since the terms in (90) involving sums are quadratic in $\tilde{Q}_i^{\pm}$. The case for $\tilde{Q}_i^+$ is similar. It is straightforward to obtain the piecewise estimates (88) for the interior pieces. □

Next, we use the center matching conditions at $N_i^{\pm}$ to solve Eq. (77). This will give us the initial conditions $a_i^{\pm}$.

Lemma 5. *For $i = 1, \ldots m-1$ there is a unique pair of initial conditions $(a_i^+, a_i^-) \in E^u(\theta_i) \times E^s(\theta_i)$ such that the matching conditions (77) are satisfied. (a_i^+, a_i^-) depends smoothly on (b_i^+, b_{i+1}^-, d_i), and we have the following expressions for a_i^- and a_i^+.*

$$\begin{aligned} a_i^- &= -P_0^s(\theta_i)d_i + \tilde{a}_i^- \\ a_i^+ &= P_0^u(\theta_i)d_i + \tilde{a}_i^+, \end{aligned} \tag{91}$$

where

$$\tilde{a}_i^{\pm} = \mathcal{O}\left(r^{-N}(|b_i^+| + |b_{i+1}^-|) + |b_i^+|^2 + |b_{i+1}^-|^2 \right). \tag{92}$$

In terms of $Q(\pm N_i^{\pm})$, we can write $P_0^s(\theta_i)d_i$ and $P_0^u(\theta_i)d_i$ as

$$\begin{aligned} P_0^s(\theta_i)d_i &= -T(\theta_i)Q(N_i^+) + \mathcal{O}(r^{-2N}) \\ P_0^u(\theta_i)d_i &= T(\theta_{i+1})Q(-N_i^-) + \mathcal{O}(r^{-2N}). \end{aligned} \tag{93}$$

Proof. Evaluating the fixed point equations (86) at $\pm N_i^{\pm}$ and subtracting, solving Eq. (77) is equivalent to solving $K_i(a_i^+, a_i^-, b_i^+, b_{i+1}^-, d_i) = 0$, where $K_i : E^s \times E^u \times Y^+ \times Y^- \times \mathbb{R}^{4k} \to \mathbb{R}^{4k}$ is defined by

$$\begin{aligned} K_i(a_i^+, a_i^-, b_i^+, b_{i+1}^-, d_i) &= a_i^+ - a_i^- - d_i + (P_u^+(N_i^+; \theta_i) - P_0^u)a_i^+ \\ &\quad - (P_s^-(-N_i^-; \theta_{i+1}) - P_0^s)a_i^- \\ &\quad + \Phi_s^+(N_i^+, 0; \theta_i)b_i^+ - \Phi_u^-(-N_i^-, 0; \theta_{i+1})b_{i+1}^- \\ &\quad + \sum_{j=0}^{N_i^+-1} \Phi_s^+(N_i^+, j+1; \theta_i)G_i^+(\tilde{Q}_i^+(j; a_i^+, b_i^+)) \\ &\quad + \sum_{j=-N_i^-}^{-1} \Phi_u^-(-N_i^-, j+1; \theta_{i+1})G_i^-(\tilde{Q}_{i+1}^-(j; a_i^-, b_{i+1}^-)), \end{aligned}$$

and we substituted $\tilde{Q}_{i+1}^-(n; a_i^-, b_{i+1}^-)$ and $\tilde{Q}_i^+(n; a_i^+, b_i^+)$ from Lemma 4. Next, we note that $K_i(0, 0, 0, 0, 0) = 0$ and that

$$\begin{aligned} \frac{\partial}{\partial a_i^-} K_i(0, 0, 0, 0, 0) &= -1 + \mathcal{O}(r^{-N_i^-}) \\ \frac{\partial}{\partial a_i^+} K_i(0, 0, 0, 0, 0) &= 1 + \mathcal{O}(r^{-N_i^+}), \end{aligned}$$

since the derivatives of the terms in K_i involving sums will be 0 since $G_i^{\pm}$ is quadratic in $\tilde{Q}_i^{\pm}$, thus quadratic order in $a_i^{\pm}$ by Lemma 4. For sufficiently large N, $D_{a_i^{\pm}}K(0, 0, 0, 0, 0)$ is invertible in a neighborhood of $(0, 0, 0, 0, 0)$. Thus, since $(a_i^+, a_i^-) \in E^s(\theta_i) \oplus E^u(\theta_i) = \mathbb{R}^{4k}$, we can use the implicit function theorem to solve for $a_i^{\pm}$ in terms of (b_i^+, b_{i+1}^-, d_i) for (b_i^+, b_{i+1}^-, d_i) sufficiently small.

To get the estimates on and expressions for $a_i^{\pm}$, we project $K_i(a_i^+, a_i^-, b_i^+, b_{i+1}^-) = 0$ onto $E^s(\theta_i)$ and $E^u(\theta_i)$ in turn to get

$$\begin{aligned} a_i^+ &= P_0^u(\theta_i)d_i + \mathcal{O}(r^{-N}(|b_i^+| + |b_{i+1}^-|) + |b_i^+|^2 + |b_{i+1}^-|^2) \\ a_i^- &= -P_0^s(\theta_i)d_i + \mathcal{O}(r^{-N}(|b_i^+| + |b_{i+1}^-|) + |b_i^+|^2 + |b_{i+1}^-|^2), \end{aligned}$$

which we can write in the form (91) with estimates (92).

To get the first equation in (93), we apply the projection $P_0^s(\theta_i)$ to each term in (75). For the second term in (75),

$$\begin{aligned} P_0^s(\theta_i)T(\theta_i)Q(N_i^+) &= (P_0^s(\theta_i) - P_s^+(N_i^+; \theta_i))T(\theta_i)Q(N_i^+) \\ &\quad + P_s^+(N_i^+; \theta_i)T(\theta_i)Q(N_i^+) \\ &= T(\theta_i)(P_0^s(0) - P_s^+(N_i^+; 0))Q(N_i^+) \\ &\quad + P_s^+(N_i^+; \theta_i)T(\theta_i)Q(N_i^+) \\ &= T(\theta_i)Q(N_i^+) + \mathcal{O}(r^{-2N}), \end{aligned}$$

where in the second line we used Eq. (73). Similarly, for the first term in (75), we can show that

$$P_0^s(\theta_i)T(\theta_{i+1})Q(-N_i^-) = \mathcal{O}(r^{-2N}).$$

The second equation in (93) can be similarly obtained by applying the projection $P_0^u(\theta_i)$ to (75). □

It only remains to satisfy (78), which is the jump condition at 0. We will not in general be able to solve Eq. (78). In the next lemma, we will solve for the initial conditions $b_i^{\pm}$. This will give us a unique solution which will generically have m jumps in the direction of $T(\theta)Z_1(0)$. We will obtain a set of m jump conditions which will depend on the symmetry parameters θ_i. Satisfying the jump conditions, which solves (78), can be accomplished by adjusting the symmetry parameters.

Recall that for all $\theta \in \mathbb{R}$ we have the decomposition

$$\mathbb{R}^{4k} = \mathbb{R}T(\theta)T'(0)Q(0) \oplus T(\theta)Y^+ \oplus T(\theta)Y^- \oplus \mathbb{R}T(\theta)Z_1(0).$$

Projecting in these directions, we can write Eq. (78) as the system of equations

$$P_{T(\theta_i)T'(0)Q(0)}\left(\tilde{Q}_i^+(0)-\tilde{Q}_i^-(0)\right)=0 \tag{94}$$

$$P_{T(\theta_i)Y^+\oplus T(\theta_i)Y^-}\left(\tilde{Q}_i^+(0)-\tilde{Q}_i^-(0)\right)=0 \tag{95}$$

$$P_{\mathbb{R}T(\theta_i)Z_1(0)}\left(\tilde{Q}_i^+(0)-\tilde{Q}_i^-(0)\right)=0. \tag{96}$$

Since $\tilde{Q}_i^\pm(0) \in T(\theta_i)Y^+ \oplus T(\theta_i)Y^- \oplus \mathbb{R}T(\theta_i)Z_1(0)$, Eq. (94) is automatically satisfied. Since $b_i^+ \in T(\theta)Y^+$ and $b_i^- \in T(\theta)Y^-$, we will be able to satisfy Eq. (95) by solving for the $b_i^\pm$, which we do in the following lemma.

Lemma 6. *For $i=1,\dots m$ there is a unique pair of initial conditions $(b_i^-, b_i^+) \in T(\theta_i)Y^- \times T(\theta_i)Y^+$ such that Eq. (95) is satisfied. We have the uniform bound*

$$b=\mathcal{O}(r^{-2N}). \tag{97}$$

Proof. For convenience, let $X_i = T(\theta_i)Y^+ \oplus T(\theta_i)Y^-$. Evaluating the fixed point equations (86) at 0, subtracting, and applying the projection P_{X_i} to both sides, we have

$$\begin{aligned}P_{X_i}(\tilde{Q}_i^+(0)-\tilde{Q}_i^-(0)) &= b_i^+ - b_i^- + P_{X_i}(\Phi_u^+(0,N_i^+;\theta_i)a_i^+)\\ &\quad - P_{X_i}(\Phi_s^-(0,-N_{i-1}^-;\theta_i)a_{i-1}^-)\\ &\quad - P_{X_i}\left(\sum_{j=0}^{N_i^+-1}\Phi_u^+(0,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j))\right.\\ &\quad \left.-\sum_{j=-N_{i-1}^-}^{-1}\Phi_s^-(0,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j))\right).\end{aligned}$$

Next, substitute $\tilde{Q}_i^\pm$ from Lemma 4 and $a_i^\pm$ from Lemma 5. Define the spaces

$$Y=\bigoplus_{i=1}^{m}(T(\theta)Y^+\oplus T(\theta)Y^-)=\bigoplus_{i=1}^{m}\mathbb{R}^{4k-2} \tag{98}$$

$$Z=\bigoplus_{i=1}^{m-1}\mathbb{R}^{4k}. \tag{99}$$

Let $b=(b_1^+,b_1^-,\dots,b_m^+,b_m^-)\in Y$ and $d=(d_1,\dots,d_{m-1})\in Z$. Define the function $K: Y\times Z\to Y$ component-wise by

$$\begin{aligned}K_i(b,d) &= b_i^+ - b_i^- + P_{X_i}\left(\Phi_u^+(0,N_i^+;\theta_i)P_0^u d_i\right.\\ &\quad \left.+\Phi_s^-(0,-N_{i-1}^-;\theta_i)P_0^s d_{i-1}\right)\\ &\quad + P_{X_i}\left(\Phi_u^+(0,N_i^+;\theta_i)\tilde{a}_i^+(b_i^+,b_{i+1}^-)\right.\\ &\quad \left.-\Phi_s^-(0,-N_{i-1}^-;\theta_i)\tilde{a}_{i-1}^-(b_{i-1}^+,b_i^-)\right)\\ &\quad - P_{X_i}\sum_{j=0}^{N_i^+-1}\Phi_u^+(0,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j;b_i^+,b_{i+1}^-))\\ &\quad - P_{X_i}\sum_{j=-N_{i-1}^-}^{-1}\Phi_s^-(0,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j;b_{i-1}^+,b_i^-)),\end{aligned}$$

where $d_0=d_m=0$, and we have indicated the dependencies on the $b_i^\pm$. Using the estimates from Lemmas 4 and 5, $K(0,0)=0$. For the partial derivatives with respect to $b_i^\pm$, we have

$$\frac{\partial}{\partial b_i^+}K_i(0)=1+\mathcal{O}(r^{-N})$$

$$\frac{\partial}{\partial b_i^-}K_i(0)=-1+\mathcal{O}(r^{-N})$$

$$\frac{\partial}{\partial b_{i-1}^+}K_i(0),\ \frac{\partial}{\partial b_{i+1}^-}K_i(0)=\mathcal{O}(r^{-N}).$$

For all other indices,

$$\frac{\partial}{\partial b_j^\pm}K_i(0)=0.$$

Thus, for sufficiently large N, the matrix $D_bK(0,0)$ is invertible. Using the implicit function theorem, there exists a unique smooth function $b: Z\to Y$ with $b(0)=0$ such that $K(b(d),d)=0$ for d sufficiently small, which is the case for N sufficiently large, since $d=\mathcal{O}(r^{-N})$. The bound for b comes from projecting $K_i(b(d),d)=0$ onto $T(\theta)Y^+$ and $T(\theta)Y^-$ together with the estimate $d=\mathcal{O}(r^{-N})$. □

Finally, we will use (96) to derive the jump conditions in the direction of $T(\theta_i)Z_1$.

Lemma 7. *The jumps in the direction of $T(\theta_i)Z_1$ are given by*

$$\begin{aligned}\xi_1 &= \langle T(\theta_1)Z_1(N_1^+), T(\theta_2)Q(-N_1^-)\rangle + R_1 = 0\\ \xi_i &= \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})Q(-N_i^-)\rangle\\ &\quad - \langle T(\theta_i)Z_1(-N_{i-1}^-), T(\theta_{i-1})Q(N_{i-1}^+)\rangle + R_i = 0\\ &\quad \textit{for } i=2,\dots,m-1\\ \xi_m &= -\langle T(\theta_m)Z_1(-N_{m-1}^-), T(\theta_{m-1})Q(N_{m-1}^+)\rangle + R_m = 0,\end{aligned}$$

where the remainder term has bound

$$|R_i|\le Cr^{-3N}. \tag{100}$$

Proof. Evaluating the fixed point equations (86) at 0 and substituting Eq. (91) from Lemma 4, we get

$$\begin{aligned}\tilde{Q}_i^+(0)-\tilde{Q}_i^-(0) &= \Phi_u^+(0,N_i^+;\theta_i)P_0^u(\theta_i)d_i\\ &\quad + \Phi_s^-(0,-N_{i-1}^-;\theta_i)P_0^s(\theta_{i-1})d_{i-1}\\ &\quad + b_i^+ - b_i^- + \Phi_u^+(0,N_i^+;\theta_i)\tilde{a}_i^+ - \Phi_s^-(0,-N_{i-1}^-;\theta_i)\tilde{a}_{i-1}^-\\ &\quad - \sum_{j=0}^{N_i^+-1}\Phi_u^+(0,j+1;\theta_i)G_i^+(\tilde{Q}_i^+(j))\\ &\quad - \sum_{j=-N_{i-1}^-}^{-1}\Phi_s^-(0,j+1;\theta_i)G_i^-(\tilde{Q}_i^-(j)).\end{aligned}$$

Next, we project on $\mathbb{R}T(\theta_i)Z_1(0)$ by taking the inner product with $T(\theta_i)Z_1(0)$. Since $b_i^\pm \in T(\theta_i)Y_i^\pm$, these terms are eliminated by the projection. For the leading order terms in (96), using Eq. (93) from Lemma 5, we have

$$\begin{aligned}&\langle T(\theta_i)Z_1(0), \Phi_u^+(0,N_i^+;\theta_i)P_0^u(\theta_i)d_i\rangle\\ &\quad = \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})Q(-N_i^-)\rangle + \mathcal{O}(r^{-3N})\\ &\langle T(\theta_i)Z_1(0), \Phi_s^-(0,-N_{i-1}^-;\theta_i)P_0^s(\theta_{i-1})d_{i-1}\rangle\\ &\quad = -\langle T(\theta_i)Z_1(-N_{i-1}^-), T(\theta_{i-1})Q(N_{i-1}^+)\rangle + \mathcal{O}(r^{-3N}).\end{aligned}$$

For the higher order terms in Eq. (96), we substitute $\tilde{Q}_i^\pm$ from Lemma 4, $\tilde{a}_i^\pm$ from Lemma 5, and $b_i^\pm$ from Lemma 6. This gives us the remainder bound (100). Since $N_0^- = N_m^+ = \infty$, one of the two inner product terms vanishes in the jumps ξ_1 and ξ_m. □

6.4. Proof of Theorem 1

The existence statement follows from the jump conditions in Lemma 7. The uniform bound $\|\tilde{Q}_i^\pm\| \le Cr^{-N}$ in (47) follows from Lemma 4 together with the estimates on $a_i^\pm$ and $b_i^\pm$. For the second estimate in (47), recall that in Lemma 5 we solved the matching condition at the pulse tails

$$\tilde{Q}_i^+(N_i^+)-\tilde{Q}_{i+1}^-(-N_i^-)=T(\theta_{i+1})Q(-N_i^-)-T(\theta_i)Q(N_i^+). \tag{101}$$

Apply the projection $P_-^u(-N_i^-;\theta_{i+1})$, noting that it acts as the identity on $T(\theta_{i+1})Q(-N_i^-)$. We look at the three remaining terms in Eq. (101) one at a time. For $T(\theta_i)Q(N_i^+)$, we follow the proof of Lemma 5 and use the estimate (73) to get

$$P_-^u(-N_i^-;\theta_{i+1})T(\theta_i)Q(N_i^+) = \mathcal{O}(r^{-2N}).$$

For $\tilde{Q}_i^+(N_i^+)$, we use the fixed point equations (86) and the uniform bound on $\tilde{Q}_i^\pm$ from Lemma 4 to get

$$(I - P_-^u(-N_i^-;\theta_{i+1}))\tilde{Q}_i^+(N_i^+) = P_-^s(-N_i^-;\theta_{i+1})\tilde{Q}_i^+(N_i^+) = \mathcal{O}(r^{-2N}),$$

from which it follows that

$$P_-^u(-N_i^-;\theta_{i+1})\tilde{Q}_i^+(N_i^+) = \tilde{Q}_i^+(N_i^+) + \mathcal{O}(r^{-2N}).$$

For $\tilde{Q}_{i+1}^-(-N_i^-)$, we follow a similar procedure to conclude that

$$P_-^u(-N_i^-;\theta_{i+1})\tilde{Q}_{i+1}^-(-N_i^-) = \mathcal{O}(r^{-2N}).$$

Combining all of these gives us the second estimate in (47). For the third estimate in (47), we apply the projection $P_+^s(N_i^+;\theta_i)$ to Eq. (101) and follow the same procedure.

6.5. Proof of Theorem 3

In the transverse intersection case, we can decompose $\mathbb{R}^k$ as $\mathbb{R}^k = Y^+ \oplus Y^-$, where $Y^+ = T_{Q(0)}W^s(0)$ and $Y^- = T_{Q(0)}W^u(0)$. Lemmas 4 and 5 are identical. To obtain a multi-pulse, all that remains to do is solve

$$\tilde{Q}_i^+(0) - \tilde{Q}_i^-(0) = P_{T(\theta_i)Y^+ \oplus T(\theta_i)Y^-}(\tilde{Q}_i^+(0) - \tilde{Q}_i^-(0)) = 0,$$

which is done in Lemma 6. There are no remaining jump conditions to satisfy.

7. Proof of Theorem 2

In this section, we will prove Theorem 2, which provides a means of locating the interaction eigenvalues associated with a multi-pulse. Throughout this section, we will assume Hypotheses 1, 2, 3, and 4. The technique of the proof is similar to the proof of [2, Theorem 2].

7.1. Setup

Let $Q_m(n)$ be an $m-$pulse solution to (28), constructed using Theorem 1 with pulse distances N_i and symmetry parameters θ_i. Write $Q_m(n)$ piecewise as

$$\begin{aligned} Q_i^-(n) &= T(\theta_i)Q(n) + \tilde{Q}_i^-(n) \quad n \in [-N_{i-1}^-, 0] \\ Q_i^+(n) &= T(\theta_i)Q(n) + \tilde{Q}_i^+(n) \quad n \in [0, N_i^+]. \end{aligned} \tag{102}$$

From Theorem 1 and (40), we have the following bounds:

$$\begin{aligned} Q(n) &= \mathcal{O}(r^{-|n|}) \\ \|\tilde{Q}\| &\leq Cr^{-N} \\ |\tilde{Q}_{i+1}^-(-N_i^-) - T(\theta_i)Q(N_i^+)| &\leq Cr^{-2N} \\ |\tilde{Q}_i^+(N_i^+) - T(\theta_{i+1})Q(-N_i^-)| &\leq Cr^{-2N}. \end{aligned} \tag{103}$$

Recall that the eigenvalue problem is given by

$$V(n+1) = DF(Q_m(n))V(n) + \lambda BV(n), \tag{104}$$

where B is defined in (34). Following (35) and (36), we have

$$\begin{aligned} T'(0)Q_m(n+1) &= DF(Q_m(n))T'(0)Q_m(n) \\ \partial_\omega Q_m(n+1) &= DF(Q_m(n))\partial_\omega Q_m(n) + BT'(0)Q_m(n). \end{aligned} \tag{105}$$

As in [2], we will take an ansatz for the eigenfunction $V(n)$ which is a piecewise perturbation of the kernel eigenfunction. If we follow [2] and use an ansatz of the form

$$V_i^\pm(n) = d_i T'(0)Q_m(n) + W_i^\pm(n),$$

we will obtain a Melnikov sum of the form

$$M_1 = \sum_{j=-\infty}^{\infty} \langle Z_1(j+1), BT'(0)Q(j) \rangle,$$

which is zero. Instead, we will take a piecewise ansatz of the form

$$V_i^\pm(n) = d_i[T'(0)Q_m(n) + \lambda\partial_\omega Q_m(n)] + W_i^\pm(n), \tag{106}$$

where $d_i \in \mathbb{C}$. Substituting this into (104), and simplifying by using (105), the eigenvalue problem becomes

$$\begin{aligned} W_i^\pm(n+1) &= DF(T(\theta_i)Q(n))W_i^\pm(n) + G_i^\pm(n)W_i^\pm(n) + \lambda BW_i^\pm(n) \\ &\quad + \lambda^2 d_i B\partial_\omega Q_i^\pm(n), \end{aligned} \tag{107}$$

where

$$G_i^\pm(n) = DF(Q_m(n)) - DF(T(\theta_i)Q(n)). \tag{108}$$

In addition to solving (107), the eigenfunction must satisfy matching conditions at $n = \pm N_i$ and $n = 0$. Thus the system of equations we need to solve is

$$\begin{aligned} W_i^\pm(n) &= DF(T(\theta_i)Q(n))W_i^\pm(n) + (G_i^\pm(n) + \lambda B)W_i^\pm(n) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^\pm(n) \\ W_i^+(N_i^+) - W_{i+1}^-(-N_i^-) &= D_i d \\ W_i^\pm(0) &\in \mathbb{C}T(\theta_i)Y^+ \oplus T(\theta_i)Y^- \oplus T(\theta_i)Z_1(0) \\ W_i^+(0) - W_i^-(0) &= 0, \end{aligned} \tag{109}$$

where

$$\begin{aligned} D_i d &= [T(\theta_{i+1})T'(0)Q(-N_i^-) + T'(0)\tilde{Q}_{i+1}^-(-N_i^-)]d_{i+1} \\ &\quad - [T(\theta_i)T'(0)Q(N_i^+) + T'(0)\tilde{Q}_i^+(N_i^+)]d_i \\ &\quad + \lambda[\partial_\omega Q_{i-1}^-(-N_i^-)d_{i+1} - \partial_\omega Q_i^+(N_i^+)d_i] \end{aligned} \tag{110}$$

and

$$\begin{aligned} \tilde{H}_i^\pm(n) &= \partial_\omega Q_i^\pm(n) \\ H_i(n) &= T(\theta_i)\partial_\omega Q(n). \end{aligned} \tag{111}$$

We can require the third condition in the system (109) since perturbations in the direction of $T(\theta_i)T'(0)Q(0)$ are handled by the $d_iT'(0)Q_m(0) = d_iT(\theta_i)T'(0)Q(0) + d_iT'(0)\tilde{Q}_i^\pm(0)$ term in Eq. (106).

As in [2] and the previous section, we will generally not be able to solve the system (109). Instead, we will relax the fourth condition in (109) to get the system

$$\begin{aligned} W_i^\pm(n) &= DF(T(\theta_i)Q(n))W_i^\pm(n) + (G_i^\pm(n) + \lambda B)W_i^\pm(n) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^\pm(n) \end{aligned} \tag{112}$$

$$W_i^+(N_i^+) - W_{i+1}^-(-N_i^-) = D_i d \tag{113}$$

$$W_i^\pm(0) \in T(\theta_i)Y^+ \oplus T(\theta_i)Y^- \oplus \mathbb{C}T(\theta_i)Z_1(0) \tag{114}$$

$$W_i^+(0) - W_i^-(0) \in \mathbb{C}T(\theta_i)Z_1(0). \tag{115}$$

Using Lin's method, we will be able to find a unique solution to this system. This solution, however, will generically have m jumps at $n = 0$. Thus a solution to this system is eigenfunction if and only if the m jump conditions

$$\xi_i = \langle T(\theta_i)Z_1(0), W_i^+(0) - W_i^-(0) \rangle = 0$$

are satisfied. Using the bounds from (103), we have the estimates

$$\begin{aligned} \|G_i^\pm\| &\leq Cr^{-N} \\ \|\tilde{H}_i^\pm - H_i\| &\leq Cr^{-N}. \end{aligned} \tag{116}$$

7.2. Fixed point formulation

As in [2], we write Eq. (112) as a fixed point problem using the discrete variation of constants formula from Lemma 3 together with projections on the stable and unstable subspaces of the exponential dichotomy from Lemma 2. Let $\delta > 0$ be small, and choose N sufficiently large so that $r^{-N} < \delta$. Let $\Phi(m, n; \theta_i)$ be the family of evolution operators for Eqs. (84). Define the spaces

$$\begin{aligned}
V_W &= \ell^\infty([-N_{i-1}, 0]) \oplus \ell^\infty([0, N_i]) \\
V_a &= \bigoplus_{i=1}^{m-1} E^u \oplus E^s \\
V_b &= \bigoplus_{i=1}^{m} \operatorname{ran} P_-^u(0; \theta_i) \oplus \operatorname{ran} P_+^s(0; \theta_i) \\
V_\lambda &= B_\delta(0) \subset \mathbb{C} \\
V_d &= \mathbb{C}^m.
\end{aligned}$$

Then for

$$\begin{aligned}
W &= (W_i^-, W_i^+) \in V_W \\
a &= (a_i^-, a_i^+) \in V_a \\
b &= (b_i^-, b_i^+) \in V_b \\
\lambda &\in V_\lambda,
\end{aligned}$$

the fixed point equations for the eigenvalue problem are

$$\begin{aligned}
W_i^-(n) &= \Phi_s^-(n, -N_{i-1}^-; \theta_i) a_{i-1}^- + \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n, j+1; \theta_i)[(G_i^-(j) \\
&\quad + \lambda B) W_i^-(j) + \lambda^2 d_i B \tilde{H}_i^-(j)] \\
&\quad + \Phi_u^-(n, 0; \theta_i) b_i^- - \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_i)[(G_i^-(j) \\
&\quad + \lambda B) W_i^-(j) + \lambda^2 d_i B \tilde{H}_i^-(j)] \\
W_i^+(n) &= \Phi_s^+(n, 0; \theta_i) b_i^+ + \sum_{j=0}^{n-1} \Phi_s^+(n, j+1; \theta_i)[(G_i^+(j) \\
&\quad + \lambda B) W_i^+(j) + \lambda^2 d_i B \tilde{H}_i^+(j)] \\
&\quad + \Phi_u^+(n, N_i^+; \theta_i) a_i^+ - \sum_{j=n}^{N_i^+-1} \Phi_u^+(n, j+1; \theta_i)[(G_i^+(j) \\
&\quad + \lambda B) W_i^+(j) + \lambda^2 d_i B \tilde{H}_i^+(j)],
\end{aligned} \tag{117}$$

where $a_0^- = a_m^+ = 0$, and the sums are defined to be 0 if the upper index is smaller than the lower index. Since we are taking $a_0^- = a_m^+ = 0$, the corresponding equations are

$$\begin{aligned}
W_1^-(n) &= \sum_{j=-\infty}^{n-1} \Phi_s^-(n, j+1; \theta_1)[(G_1^-(j) + \lambda B) W_1^-(j) + \lambda^2 d_1 B \tilde{H}_1^-(j)] \\
&\quad + \Phi_u^-(n, 0; \theta_1) b_1^- - \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_1)[(G_1^-(j) \\
&\quad + \lambda B) W_1^-(j) + \lambda^2 d_1 B \tilde{H}_1^-(j)] \\
W_m^+(n) &= \Phi_s^+(n, 0; \theta_m) b_m^+ \\
&\quad + \sum_{j=0}^{n-1} \Phi_s^+(n, j+1; \theta_m)[(G_m^+(j) + \lambda B) W_m^+(j) + \lambda^2 d_m B \tilde{H}_m^+(j)] \\
&\quad - \sum_{j=n}^{\infty} \Phi_u^+(n, j+1; \theta_m)[(G_m^+(j) + \lambda B) W_m^+(j) \\
&\quad + \lambda^2 d_m B \tilde{H}_m^+(j)].
\end{aligned}$$

7.3. Inversion

We will now solve the eigenvalue problem in a series of lemmas. This is very similar to the procedure in [2]. First, we use the fixed point equations (117) to solve for $W_i^\pm$.

Lemma 8. *There exists an operator $W_1 : V_\lambda \times V_a \times V_b \times V_d \to V_W$ such that*

$$W = W_1(\lambda)(a, b, d)$$

is a solution to (112) for (a, b, d) and λ. The operator W_1 is analytic in λ, linear in (a, b, d), and has bound

$$\|W_1(\lambda)(a, b, d)\| \leq C \left(|a| + |b| + |\lambda|^2 |d| \right). \tag{118}$$

Proof. Rewrite the fixed point equations (117) as

$$(I - L_1(\lambda)) W = L_2(\lambda)(a, b, d),$$

where $L_1(\lambda) : V_W \to V_W$ is the linear operator composed of terms in the fixed point equations involving W

$$\begin{aligned}
(L_1(\lambda) W)_i^-(n) &= \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n, j+1; \theta_i)(G_i^-(j) + \lambda B) W_i^-(j) \\
&\quad - \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_i)(G_i^-(j) + \lambda B) W_i^-(j) \\
(L_1(\lambda) W)_i^+(n) &= \sum_{j=0}^{n-1} \Phi_s^+(n, j+1; \theta_i)(G_i^+(j) + \lambda B) W_i^+(j) \\
&\quad - \sum_{j=n}^{N_i^+-1} \Phi_u^+(n, j+1; \theta_i)(G_i^+(j) + \lambda B) W_i^+(j)
\end{aligned}$$

and $L_2(\lambda) : V_\lambda \times V_a \times V_b \times V_d \to V_W$ is the linear operator composed of terms in the fixed point equations not involving W.

$$\begin{aligned}
&(L_2(\lambda)(a, b, d))_i^-(n) \\
&\quad = \Phi_s^-(n, -N_{i-1}^-; \theta_i) a_{i-1}^- + \sum_{j=-N_{i-1}^-}^{n-1} \Phi_s^-(n, j+1; \theta_i) \lambda d_i B \tilde{H}_i^-(j) \\
&\quad\quad + \Phi_u^-(n, 0; \theta_i) b_i^- - \sum_{j=n}^{-1} \Phi_u^-(n, j+1; \theta_i) \lambda d_i B \tilde{H}_i^-(j) \\
&(L_2(\lambda)(a, b, d))_i^+(n) \\
&\quad = \Phi_s^+(n, 0; \theta_i) b_i^+ + \sum_{j=0}^{n-1} \Phi_s^+(n, j+1; \theta_i) \lambda^2 d_i B \tilde{H}_i^+(j) \\
&\quad\quad + \Phi_u^+(n, N_i^+; \theta_i) a_i^+ - \sum_{j=n}^{N_i^+-1} \Phi_u^+(n, j+1; \theta_i) \lambda^2 d_i B \tilde{H}_i^+(j).
\end{aligned}$$

Using the exponential dichotomy bounds from Lemma 2, we obtain the following uniform bounds for L_1 and L_2.

$$\begin{aligned}
\|L_1(\lambda) W\| &\leq C (\|G\| + |\lambda|) \|W\| \leq C\delta \|W\| \\
\|L_2(\lambda)(a, b, d)\| &\leq C \left(|a| + |b| + |\lambda|^2 |d| \right).
\end{aligned}$$

For sufficiently small δ, $\|(L_1(\lambda) W)\| < 1$, thus $I - L_1(\lambda)$ is invertible on V_W. The inverse $(I - L_1(\lambda))^{-1}$ is analytic in λ, and we obtain the solution

$$W = W_1(\lambda)(a, b, d) = (I - L_1(\lambda))^{-1} L_2(\lambda)(a, b, d),$$

which is analytic in λ, linear in (a, b, d), and for which we have the estimate

$$\|W_1(\lambda)(a, b, d)\| \leq C \left(|a| + |b| + |\lambda|^2 |d| \right). \quad \square$$

In the next lemma, we solve Eq. (113), which is the matching condition at the tails of the pulses.

Lemma 9. *There exist operators*

$$A_1 : V_\lambda \times V_b \times V_d \to V_a$$
$$W_2 : V_\lambda \times V_b \times V_d \to V_W$$

such that $(a, w) = (A_1(\lambda)(b, d), W_2(\lambda)(b, d))$ *solves* (112) *and* (113) *for any* (b, d) *and* λ. *These operators are analytic in* λ, *linear in* (b, d), *and have bounds*

$$|A_1(\lambda)(b, d)| \leq C \left((r^{-N} + \|G\| + |\lambda|)|b| + (|\lambda|^2 + |D|)|d| \right) \tag{119}$$
$$\|W_2(\lambda)(b, d)\| \leq C \left(|b| + (|\lambda|^2 + |D|)|d| \right). \tag{120}$$

Furthermore, we can write

$$a_i^+ = P_0^u(\theta_i) D_i d + A_2(\lambda)_i^+(b, d)$$
$$a_i^- = -P_0^s(\theta_i) D_i d + A_2(\lambda)_i^-(b, d),$$

where A_2 *is a bounded linear operator with uniform bound*

$$\begin{aligned} &|A_2(\lambda)(b, d)| \\ &\leq C \left((r^{-N} + \|G\| + |\lambda|)|b| + (r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2 |d| \right). \end{aligned} \tag{121}$$

Proof. Substituting the fixed point equations (117) into Eq. (113) and recalling that $\Phi_s^-(-N_i^-, -N_i^-; \theta_{i+1}) = P_-^s(-N_i^-; \theta_{i+1})$, $\Phi_u^+(N_i^+, N_i^+; \theta_i) = P_+^u(N_i^+; \theta_i)$, $a_i^- \in E^s(\theta_i)$, and $a_i^+ \in E^u(\theta_i)$, we have

$$\begin{aligned} D_i d = a_i^+ - a_i^- &+ (P_u^+(N_i^+; \theta_i) - P_0^u) a_i^+ - (P_s^-(-N_i^-; \theta_{i+1}) - P_0^s) a_i^- \\ &+ \Phi_s^+(N_i^+, 0; \theta_i) b_i^+ - \Phi_u^-(-N_i^-, 0; \theta_{i+1}) b_i^- \\ &+ \sum_{j=0}^{N_i^+ - 1} \Phi_s^+(N_i^+, j+1; \theta_i)[(G_i^+(j) + \lambda B) W_i^+(j) \\ &+ \lambda^2 d_i B \tilde{H}_i^+(j)] \\ &- \sum_{j=-N_i^-}^{-1} \Phi_u^-(-N_i^-, j+1; \theta_{i+1})[(G_i^-(j) + \lambda B) W_i^-(j) \\ &+ \lambda^2 d_i B \tilde{H}_i^-(j)]. \end{aligned}$$

Substituting $W = W_1(\lambda)(a, b, d)$ from Lemma 8, we obtain an equation of the form

$$D_i d = (a_i^+ - a_i^-) + L_3(\lambda)_i(a, b, d). \tag{122}$$

Using Lemma 2, the bound for W_1 from Lemma 8, and the estimate (73) from Lemma 2, the linear operator L_3 has uniform bound

$$\begin{aligned} L_3(\lambda)(a, b, d)| &\leq C \left((r^{-N} + \|G\| + |\lambda|)(|a| + |b|) + |\lambda|^2 |d| \right) \\ &\leq C\delta |a| + C \left((r^{-N} + \|G\| + |\lambda|)|b| + |\lambda|^2 |d| \right). \end{aligned} \tag{123}$$

Define the map

$$J_1 : V_a \to \bigoplus_{j=1}^{m-1} \mathbb{C}^{4k}$$

by $(J_1)_i(a_i^+, a_i^-) = a_i^+ - a_i^-$. Since $E^s \oplus E^u = \mathbb{C}^{4k}$, the map J_1 is a linear isomorphism. Let

$$K_1(a) = J_1(a) + L_3(\lambda)(a, 0, 0) = J_1 \left(I + J_1^{-1} L_3(\lambda)(a, 0) \right).$$

For sufficiently small δ, $\|J_1^{-1} L_3(\lambda)(a, 0, 0)\| < 1$, thus the operator $K_1(a)$ is invertible. We can then solve for a to get

$$a = A_1(\lambda)(b, d) = K_1^{-1} \left(-Dd - L_3(\lambda)(b, d) \right),$$

which has uniform bound

$$|A_1(\lambda)(b, d)| \leq C \left((r^{-N} + \|G\| + |\lambda|)|b| + (|\lambda|^2 + |D|)|d| \right).$$

We plug this estimate into W_1 to get $W_2(\lambda)(b, d)$, which satisfies the bound

$$\|W_2(\lambda)(b, d)\| \leq C \left(|b| + (|\lambda|^2 + |D|)|d| \right).$$

Finally, we project Eq. (122) onto $E^s(\theta_i)$ and $E^u(\theta_i)$ to get

$$a_i^+ = P_0^u(\theta_i) D_i d - P_0^u(\theta_i) L_3(\lambda)_i(a, b, d)$$
$$a_i^- = -P_0^s(\theta_i) D_i d + P_0^s(\theta_i) L_3(\lambda)_i(a, b, d).$$

Substituting $A_1(\lambda)(b, d)$ for a we obtain the equations

$$a_i^+ = P_0^u(\theta_i) D_i d + A_2(\lambda)_i^+(b, d)$$
$$a_i^- = -P_0^s(\theta_i) D_i d + A_2(\lambda)_i^-(b, d).$$

Substituting the bound for A_1 into the bound for L_3, we obtain the uniform bound

$$\begin{aligned} |A_2(\lambda)(b, d)| &\leq C \left((r^{-N} + \|G\| + |\lambda|)|b| \right. \\ &\quad \left. + (r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2 |d| \right). \quad \square \end{aligned}$$

The last step in the inversion is to satisfy Eqs. (114) and (115). Since we have the decomposition

$$\mathbb{C}^{4k} = \mathbb{C} T(\theta_i) Z_1(0) \oplus \mathbb{C} T(\theta_i) T'(0) Q(0) \oplus T(\theta_i) Y^+ \oplus T(\theta_i) Y^-, \tag{124}$$

these two equations are equivalent to the three projections

$$\begin{aligned} P(T(\theta_i) T'(0) Q(0)) W_i^- &= 0 \\ P(T(\theta_i) T'(0) Q(0)) W_i^+ &= 0 \\ P(T(\theta_i) Y^+ \oplus T(\theta_i) Y^-)(W_i^+ - W_i^-) &= 0, \end{aligned} \tag{125}$$

where the kernel of each projection consists of the remaining elements of the direct sum decomposition (124). Since we have eliminated any component in $T(\theta_i) T'(0) Q(0)$ in the first two projections, we do not need it in the third projection.

We decompose $b_i^\pm$ uniquely as $b_i^\pm = x_i^\pm + y_i^\pm$, where $x_i^\pm \in \mathbb{C} T(\theta_i) T'(0) Q(0)$ and $y_i^\pm \in T(\theta_i) Y^\pm$. In the next lemma, we solve the system of equations (125).

Lemma 10. *There exist operators*

$$B_1 : V_\lambda \times V_d \to V_b$$
$$A_3 : V_\lambda \times V_d \to V_a$$
$$W_3 : V_\lambda \times V_d \to V_W$$

such that $(a, b, W) = (A_3(\lambda)(d), B_1(\lambda)(d), W_2(\lambda)(d))$ *solves* (112), (113), (114), *and* (115) *for any* d *and* λ. *These operators are analytic in* λ, *linear in* d, *and have bounds*

$$|B_1(\lambda)(d)| \leq C \left((r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2 |d| \right) \tag{126}$$
$$|A_3(\lambda)(d)| \leq C \left(|\lambda|^2 + |D| \right) |d| \tag{127}$$
$$\|W_3(\lambda)(d)\| \leq C \left(|\lambda|^2 + |D| \right) |d|. \tag{128}$$

Furthermore, we can write

$$a_i^+ = P_0^u D_i d + A_4(\lambda)_i^+(d)$$
$$a_i^- = -P_0^s D_i d + A_4(\lambda)_i^-(d),$$

where A_4 is a bounded linear operator with estimate

$$|A_4(\lambda)(d)| \leq C\left((r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2|d|\right). \tag{129}$$

Proof. At $n=0$, the fixed point equations (117) become

$$\begin{aligned} W_i^-(0) &= x_i^- + y_i^- + \Phi_s^-(0, -N_{i-1}^-; \theta_i)a_{i-1}^- \\ &\quad + \sum_{j=-N_{i-1}^-}^{-1} \Phi_s^-(0, j+1; \theta_i)[(G_i^-(j) + \lambda B)W_i^-(j) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^-(j)] \\ W_i^+(0) &= x_i^+ + y_i^+ + \Phi_u^+(0, N_i^+; \theta_i)a_i^+ \\ &\quad - \sum_{j=0}^{N_i^+-1} \Phi_u^+(0, j+1; \theta_i)[(G_i^+(j) + \lambda B)W_i^+(j) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^+(j)]. \end{aligned}$$

Eqs. (125) can thus be written as

$$\begin{pmatrix} x_i^- \\ x_i^+ \\ y_i^+ - y_i^- \end{pmatrix} = (L_4(\lambda)(b,d))_i\,. \tag{130}$$

Using the exponential dichotomy estimates from Lemma 2 and $(a, W) = (A_1(\lambda)(b,d), W_2(\lambda)(b,d))$ from Lemma 9, we get the uniform bound on L_4

$$\begin{aligned} &|L_4(\lambda)(b,d)| \\ &\leq C\left((r^{-2N} + \|G\| + |\lambda|)|b| + (r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2|d|\right) \\ &\leq C\delta(|x| + |y|) + C\left((r^{N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2|d|\right). \end{aligned}$$

Define the map

$$\begin{aligned} J_2 : \left(\bigoplus_{j=1}^m \mathbb{C}T'(0)Q(0) \oplus \mathbb{C}T'(0)Q(0)\right) \oplus \left(\bigoplus_{j=1}^m Y^- \oplus Y^+\right) \\ \to \bigoplus_{j=1}^m \mathbb{C}T'(0)Q(0) \oplus \mathbb{C}T'(0)Q(0) \oplus (Y^- \oplus Y^+) \end{aligned}$$

by

$$J_2((x_i^+, x_i^-), (y_i^+, y_i^-))_i = (x_i^+, x_i^-, y_i^+ - y_i^-).$$

Since $\mathbb{C}^{4k} = \mathbb{C}T(\theta_i)Z_1(0) \oplus \mathbb{C}T(\theta_i)T'(0)Q(0) \oplus T(\theta_i)Y^- \oplus T(\theta_i)Y^+$, J_2 is an isomorphism. Using this and the fact that $(b_i^-, b_i^+) = (x_i^- + y_i^-, x_i^+ + y_i^+)$, we can write (130) as

$$J_2((x_i^+, x_i^-), (y_i^+, y_i^-))_i + L_4(\lambda)_i(b, 0) + L_4(\lambda)_i(0, d) = 0. \tag{131}$$

Consider the map

$$K_2(b)_i = J_2((x_i^+, x_i^-), (y_i^+, y_i^-))_i + L_4(\lambda)_i(b, 0).$$

Substituting this in (131), we have

$$K_2(b) = -L_4(\lambda)(0, d).$$

For sufficiently small δ, the operator $K_2(b)$ is invertible. Thus we can solve for b to get

$$b = B_1(\lambda)(d) = -K_2^{-1}\left(L_4(\lambda)(0,d)\right), \tag{132}$$

where we have the uniform bound on B_1

$$|B_1(\lambda)(d)| \leq C\left((r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2|d|\right). \tag{133}$$

We can plug this into A_1, W_2, and A_2 to get operators A_3, W_3, and A_4 with bounds

$$\begin{aligned} |A_3(\lambda)(d)| &\leq C\left(|\lambda|^2 + |D|\right)|d| \\ \|W_3(\lambda)(d)\| &\leq C\left(|\lambda|^2 + |D|\right)|d| \\ |A_4(\lambda)(d)| &\leq C\left((r^{-N} + \|G\| + |\lambda|)|D||d| + |\lambda|^2|d|\right). \quad \square \end{aligned}$$

7.4. Jump conditions

Given λ and d, we have used Lin's method to find a unique solution to Eqs. (112), (113), (114), and (115), which is given by $W = W_3(\lambda)(d)$. Such a solution will generically have m jumps in the direction of $T(\theta_i)Z_1(0)$, which are given by

$$\xi_i = \langle T(\theta_i)Z_1(0), W_i^+(0) - W_i^-(0)\rangle. \tag{134}$$

In the next lemma, we derive formulas for these jumps.

Lemma 11. *$W_i^+(0) = W_i^-(0)$ for $i = 1, \ldots, m$ if and only if the m jump conditions*

$$\xi_i = \langle T(\theta_i)Z_1(0), W_i^+(0) - W_i^-(0)\rangle = 0 \tag{135}$$

are satisfied. The jumps ξ_i can be written as

$$\begin{aligned} \xi_i &= \langle T(\theta_i)Z_1(N_i^+), P_0^u(\theta_i)D_i d\rangle + \langle T(\theta_i)Z_1(-N_{i-1}^-), P_0^s(\theta_{i-1})D_{i-1}d\rangle \\ &\quad - \sum_{j=-\infty}^{\infty} \langle Z_1(j+1), B\partial_\omega Q(j)\rangle + R(\lambda)_i(d), \end{aligned} \tag{136}$$

where the remainder term $R(\lambda)(d)$ has bound

$$|R(\lambda)(d)| \leq C(r^{-N} + \|G\| + |\lambda|)\left((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d|. \tag{137}$$

Proof. From the previous lemma, the fixed point equations at $n=0$ are given by

$$\begin{aligned} W_i^-(0) &= b_i^- + \Phi_s^-(0, -N_{i-1}^-; \theta_i)a_{i-1}^- \\ &\quad + \sum_{j=-N_{i-1}^-}^{-1} \Phi_s^-(0, j+1; \theta_i)[(G_i^-(j) + \lambda B)W_i^-(j) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^-(j)] \\ W_i^+(0) &= b_i^+ + \Phi_u^+(0, N_i^+; \theta_i)a_i^+ \\ &\quad - \sum_{j=0}^{N_i^+-1} \Phi_u^+(0, j+1; \theta_i)[(G_i^+(j) + \lambda B)W_i^+(j) \\ &\quad + \lambda^2 d_i B\tilde{H}_i^+(j)]. \end{aligned} \tag{138}$$

To evaluate (134), we will compute the inner product of each of the terms in (138) with $T(\theta_i)Z_1(0)$. The $b_i^\pm$ terms will vanish since they lie in spaces orthogonal to $T(\theta_i)Z_1(0)$. We will evaluate the remaining terms in turn. For the terms involving a, we substitute A_4 from Lemma 10 to get

$$\begin{aligned} &\langle T(\theta_i)Z_1(0), \Phi_s^-(0, -N_{i-1}^-; \theta_i)a_{i-1}^-\rangle \\ &\quad = -\langle T(\theta_i)Z_1(-N_{i-1}^-), P_0^s(\theta_{i-1})D_{i-1}d\rangle \\ &\qquad + \mathcal{O}\left(r^{-N}((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2)|d|\right) \\ &\langle T(\theta_i)Z_1(0), \Phi_u^+(0, N_i^+; \theta_i)a_i^+\rangle \\ &\quad = \langle T(\theta_i)Z_1(N_i^+), P_0^u(\theta_i)D_i d\rangle \\ &\qquad + \mathcal{O}\left(r^{-N}((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2)|d|\right). \end{aligned}$$

The sums involving $\tilde{H}$ give us a Melnikov-type sum.

$$\begin{aligned}
&\Bigg\langle T(\theta_i)Z_1(0), \sum_{j=-N_{i-1}^-}^{-1} \Phi_s^-(0,j+1;\theta_i)B\tilde{H}_i^-(j) \\
&+ \sum_{j=0}^{N_i^+-1} \Phi_u^+(0,j+1;\theta_i)B\tilde{H}_i^+(j)\Bigg\rangle \\
&= \sum_{j=-N_{i-1}^-}^{-1} \langle T(\theta_i)Z_1(j+1), BT(\theta_i)\partial_\omega Q(j)\rangle \\
&\quad + \sum_{j=0}^{N_i^+-1} \langle T(\theta_i)Z_1(j+1), BT(\theta_i)\partial_\omega Q(j)\rangle + \mathcal{O}(r^{-N}) \\
&= \sum_{j=-\infty}^{\infty} \langle T(\theta_i)Z_1(j+1), BT(\theta_i)\partial_\omega Q(j)\rangle + \mathcal{O}(r^{-N}) \\
&= \sum_{j=-\infty}^{\infty} \langle Z_1(j+1), B\partial_\omega Q(j)\rangle + \mathcal{O}(r^{-N}),
\end{aligned}$$

where in the last line we used the fact that $T(\theta)$ is unitary and commutes with B.

Finally, we need to obtain a suitable bound for the sum involving W. To do this, as in [2], we will need an improved bound for W. Plugging in the bounds for A_3, W_3, and B_1 into the fixed point equations (117), we have piecewise bounds

$$\begin{aligned}
|W_i^-(n)| &\leq C\left(r^{-(N_{i-1}^-+n)}|D| + (r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d| \\
|W_i^+(n)| &\leq C\left(r^{-(N_i^+-n)}|D| + (r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d|.
\end{aligned}$$

It follows from the definition (42) of $Z_1(n)$ and Eq. (40) that $Z_1(n) \leq Cr^{-|n|}$. Since $DF(0)$ is hyperbolic, we can find a constant $\tilde{r} > r$ such that $|Z_1(n)| \leq C\tilde{r}^{-n}$. The price to pay is a larger constant C. Using this bound, the sum involving W becomes

$$\begin{aligned}
&\left|\sum_{j=-N_{i-1}^-}^{-1} \langle Z_1(j+1), (G_i^-(j) + \lambda B)W_i^-(j)\rangle\right| \\
&\leq C(\|G\| + |\lambda|) \sum_{j=-N_{i-1}^-}^{-1} \tilde{r}^{-|j+1|} r^{-(N_{i-1}^-+j)}|D||d| \\
&\quad + C(\|G\| + |\lambda|)\left((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d| \\
&\leq C|D|r^{-N}(\|G\| + |\lambda|)|d| \sum_{j=1}^{\infty}\left(\frac{r}{\tilde{r}}\right)^j \\
&\quad + C(\|G\| + |\lambda|)\left((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d| \\
&\leq C(\|G\| + |\lambda|)\left((r^{-N} + \|G\| + |\lambda|)|D| + |\lambda|^2\right)|d|.
\end{aligned}$$

The infinite sum is convergent by our choice of $\tilde{r}$. We have a similar bound for the other sum. Putting this all together, we obtain the jump equations (136) and the remainder bound (137). □

7.5. Proof of Theorem 2

Using the estimates (47), we have

$$\begin{aligned}
T'(0)\tilde{Q}_{i+1}^-(-N_i^-) &= T(\theta_i)T'(0)Q(N_i^+) + \mathcal{O}(r^{-2N}) \\
T'(0)\tilde{Q}_i^+(N_i^+) &= T(\theta_{i+1})T'(0)Q(-N_i^-) + \mathcal{O}(r^{-2N}),
\end{aligned}$$

since the infinitesimal generator of a group commutes with the group elements. Substituting these into Eq. (110) and simplifying, we have

$$\begin{aligned}
D_i d = {} & [T(\theta_{i+1})T'(0)Q(-N_i^-) + T(\theta_i)T'(0)Q(N_i^+)]d_{i+1} \\
& - [T(\theta_i)T'(0)Q(N_i^+) + T(\theta_{i+1})T'(0)Q(-N_i^-)]d_i \\
& + \mathcal{O}\left(r^{-N}(|\lambda| + r^{-N})\right).
\end{aligned} \tag{139}$$

Next, we substitute Eq. (139) into the jump expressions ξ_i from Lemma 11. For the inner product term $\langle T(\theta_i)Z_1(N_i^+), P_0^u(\theta_i)D_i d\rangle$, we use Eq. (73) to get

$$\begin{aligned}
&\langle T(\theta_i)Z_1(N_i^+), P_0^u(\theta_i)D_i d\rangle \\
&= \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})T'(0)Q(-N_i^-)\rangle(d_{i+1} - d_i) + \mathcal{O}(r^{-3N}),
\end{aligned}$$

since $T(\theta)$ is unitary and $\langle Z_1(n), T'(0)Q(n)\rangle = 0$ for all n. Similarly, we have

$$\begin{aligned}
&\langle T(\theta_i)Z_1(-N_{i-1}^-), P_0^s D_{i-1} d\rangle \\
&= \langle T(\theta_i)Z_1(-N_{i-1}^-), T(\theta_{i-1})T'(0)Q(N_{i-1}^+)\rangle(d_i - d_{i-1}).
\end{aligned}$$

For the Melnikov sum, we use Eq. (42) for $Z_1(j)$ to get

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \langle Z_1(j+1), B\partial_\omega Q(j)\rangle &= \sum_{j=-\infty}^{\infty} \left\langle \begin{pmatrix} -R'(0)q(j) \\ R'(0)q(j+1)\end{pmatrix}, \frac{1}{d}\begin{pmatrix} J\partial_\omega q(j) \\ 0 \end{pmatrix}\right\rangle \\
&= -\frac{1}{d}\sum_{j=-\infty}^{\infty} \langle R'(0)q(j), J\partial_\omega q(j)\rangle \\
&= -\frac{1}{d}\sum_{j=-\infty}^{\infty} \langle J^{-1}R'(0)q(j), \partial_\omega q(j)\rangle \\
&= \frac{1}{d}M,
\end{aligned}$$

where M is defined in (26).

Substituting these into the jump equations, we obtain the jump conditions

$$\begin{aligned}
\xi_i = {} & \langle T(\theta_i)Z_1(N_i^+), T(\theta_{i+1})T'(0)Q(-N_i^-)\rangle(d_{i+1} - d_i) \\
& + \langle T(\theta_i)Z_1(-N_{i-1}^-), T(\theta_{i-1})T'(0)Q(N_{i-1}^+)\rangle(d_i - d_{i-1}) \\
& - \frac{1}{d}M + R(\lambda)_i(d).
\end{aligned} \tag{140}$$

For the remainder term, we substitute $|D|, \|G\| = \mathcal{O}(r^{-N})$ into the remainder term in Lemma 11 to get

$$|R(\lambda)(d)| \leq C\left((r^{-N} + |\lambda|)^3\right).$$

The result follows by writing the jump conditions (140) in matrix form as in [2].

8. Proofs of results from Section 4

8.1. Proof of Theorem 4

First, we will look for real-valued solutions to (1). In this case, the stationary equation (2) reduces to

$$d(u_{n+1} - 2u_n + u_{n-1}) - \omega u_n + u_n^3 = 0.$$

For $d \neq 0$, this is equivalent to the first order difference equation $U(n+1) = F(U(n))$, where $U(n) = (u_n, \tilde{u}_n) \in \mathbb{R}^2$, $\tilde{u}_n = u_{n-1}$, and

$$F(U) = \begin{pmatrix} \frac{\omega}{d} + 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ \tilde{u} \end{pmatrix} - \frac{1}{d}\begin{pmatrix} u^3 \\ 0 \end{pmatrix}. \tag{141}$$

The symmetry group $G = \{1, -1\}$ acts on $\mathbb{R}^2$ via $T(\theta) = \theta I$. For $d, \omega > 0$, $DF(0)$ has a pair of real eigenvalues $\{r, 1/r\}$, where r depends on both d and ω, and is given by Eq. (57). As $d \to \infty$, $r \to 1$, thus the spectral gap decreases with increasing d. As $d \to 0$, $r \to \infty$.

It follows that 0 is a hyperbolic equilibrium point with 1-dimensional stable and unstable manifolds. Let q_n be the symmetric, real-valued, on-site soliton solution to DNLS, and let $Q(n) = (q_n, \tilde{q}_n)$ be the primary pulse solution, where $\tilde{q}_n = q_{n-1}$. Since the variational equation does not have a bounded solution, the stable and unstable manifolds intersect transversely. Thus we have satisfied Hypothesis 5. Using Theorem 3, for sufficiently large N (which depends on r, thus ω and d) there exist $m-$pulse solutions for any $\theta_i = \pm 1$ and lengths $N_i \geq N$. These correspond to phase differences of 0 and π.

We will now show that there are no multi-pulse solutions with phase differences other than 0 and π. For this, we write the DNLS equation (8) as the first order system (28) in $\mathbb{R}^4$. In this formulation, the primary pulse solution is given by $Q(n) = (q_n, 0, \tilde{q}_n, 0)$. The unique bounded solutions to the variational equation (79) and the adjoint variational equation (80) are

$$T'(0)Q(n) = (0, q_n, 0, \tilde{q}_n)$$
$$Z_1(n) = (0, -\tilde{q}_n, 0, q_n).$$

Using Theorem 1, for sufficiently large N (which depends on r, thus ω and d) there exist $m-$pulse solutions with lengths $N_i^{\pm}$ and phase parameters θ_i if and only if the jump conditions (46) are satisfied. Since the symmetry group $T(\theta)$ is unitary, we can rewrite the jump conditions in terms of the phase differences $\Delta\theta_i = \theta_{i+1} - \theta_i$ to get the jump conditions

$$\xi_i = \langle T(-\Delta\theta_i)Z_1(N_i^+), Q(-N_i^-)\rangle - \langle T(\Delta\theta_{i-1})Z_1(-N_{i-1}^-), Q(N_{i-1}^+)\rangle + R_i, \tag{142}$$

where we take $\Delta\theta_0 = \Delta\theta_m = 0$. The inner product terms in Eq. (142) are

$$\begin{aligned} \langle T(-\Delta\theta_i)Z_1(N_i^+), Q(-N_i^-)\rangle &= -b_i \sin(\Delta\theta_i) \\ \langle T(\Delta\theta_{i-1})Z_1(-N_{i-1}^-), Q(N_{i-1}^+)\rangle &= -b_{i-1} \sin(\Delta\theta_{i-1}), \end{aligned} \tag{143}$$

where

$$b_i = q(N_i^+ - 1)q(N_i^-) - q(N_i^+)q(N_i^- + 1).$$

Since the single pulse $q(n)$ is an even function, the b_i are given by Eq. (61). Since $q(n)$ is non-negative, unimodal, and exponentially decaying [20, Theorem 1], $q(n)$ is strictly decreasing as n moves away from 0, thus $b_i < 0$ for all i.

Substituting Eqs. (143) into (142), and letting $s_i = \sin\Delta\theta_i$, the jump conditions become

$$\begin{aligned} \xi_1 &= -b_1 s_1 + R_1 \\ \xi_i &= b_{i-1}s_{i-1} - b_i s_i + R_i \quad \text{for } i = 2, \dots, m-1 \\ \xi_m &= b_{m-1}s_{m-1} + R_m. \end{aligned} \tag{144}$$

Since $b_i = \mathcal{O}(r^{-2N})$ and $R_i = \mathcal{O}(r^{-3N})$, the jump conditions can only be satisfied if $s_i = \mathcal{O}(r^{-N})$. Thus we only have to consider that case from here on. Since the steady state equation (8) has a conserved quantity (10), we can eliminate the final equation in (144) as is done in [24]. We write the $m-1$ remaining jump conditions in matrix form as $Hs + R = 0$, where $s = (s_1, \dots, s_{m-1})$ and H is the $(m-1) \times (m-1)$ matrix

$$H = \begin{pmatrix} -b_1 & & & & \\ b_1 & -b_2 & & & \\ & b_2 & -b_3 & & \\ & & \ddots & \ddots & \\ & & & b_{m-2} & -b_{m-1} \end{pmatrix}.$$

Since H is lower triangular and all the b_i are nonzero, B is invertible, thus $s = B^{-1}R$ is the unique value of s for which all the jump conditions are satisfied.

We showed above that for sufficiently large N, real-valued multi-pulses exist with phase differences which are either 0 or π; in all of those cases, $s = 0$. Since $s = B^{-1}R$ is the unique solution which satisfies the jump conditions, and $s = 0$ is also a solution, we conclude that $s = 0$ must be the unique solution that satisfies the jump conditions. Thus, for sufficiently large N, the jump conditions can only be satisfied if all of the phase differences $\Delta\theta_i$ are either 0 or π. No other phase differences are possible.

8.2. Proof of Theorem 5

To find the interaction eigenvalues for DNLS, we will solve the matrix equation (49) from Theorem 2. For DNLS, the stability criterion M is given by (55), and we are assuming that $M > 0$.

For N sufficiently large, we can find the eigenvalues of (48) using Theorem 2. The matrix A is given by Eq. (60). First, we rescale Eq. (49) by taking

$$A = r^{-2N}\tilde{A}$$
$$\lambda = r^{-N}\tilde{\lambda}$$
$$R(\lambda) = r^{-3N}\tilde{R}(\lambda)$$

and dividing by r^{-2N} to get the equivalent equation

$$\tilde{E}(\lambda) = \det\left(\tilde{A} - \frac{1}{d}M\tilde{\lambda}^2 I + r^{-N}\tilde{R}(\lambda)\right) = 0. \tag{145}$$

To solve $\tilde{E}(\lambda) = 0$, we need to find the eigenvalues of $\tilde{A}$. Since $\tilde{A}$ is symmetric tridiagonal, its eigenvalues are real. Furthermore, $\tilde{A}$ has an eigenvalue at 0 with corresponding eigenvector $(1, 1, \dots, 1)^T$. Let $\{\tilde{\mu}_1, \dots, \tilde{\mu}_{m-1}\}$ be the remaining $m-1$ eigenvalues of $\tilde{A}$. Since $b_i < 0$ for all i, it follows from [2, Lemma 5.4] that the signs of $\{\tilde{\mu}_1, \dots, \tilde{\mu}_{m-1}\}$ are determined by the phase differences $\Delta\theta_i$. Specifically, $\tilde{A}$ has k_π negative real eigenvalues (counting multiplicity), where k_π is the number of $\Delta\theta_i$ which are π, and $\tilde{A}$ has k_0 positive real eigenvalues (counting multiplicity), where k_0 is the number of $\Delta\theta_i$ which are 0.

Next, we show that the eigenvalues of $\tilde{A}$ are distinct. The eigenvalue problem $(\tilde{A} - \tilde{\mu}I)v = 0$ is equivalent to the Sturm–Liouville difference equation with Dirichlet boundary conditions

$$\begin{aligned} \nabla(p_j \Delta d_j) &= \tilde{\mu} d_j \qquad j = 1, \dots, m \\ d_0 &= 0 \\ d_{m+1} &= 0, \end{aligned} \tag{146}$$

where $p_j = \cos(\Delta\theta_j)b_j$, Δ is the forward difference operator $\Delta f(j) = f(j+1) - f(j)$ and ∇ is the backward difference operator $\nabla f(j) = f(j) - f(j-1)$. It follows from [38, Corollary 2.2.7] that the eigenvalues of Eq. (146), thus the eigenvalues of $\tilde{A}$, are distinct.

We can now solve Eq. (145) for λ. By Eq. (105), we will always have an eigenvalue at 0 with algebraic multiplicity 2 and geometric multiplicity 1. The remaining eigenvalues result from interaction between the pulses. Let $\eta = r^{-N}$, and rewrite Eq. (145) as

$$K(\tilde{\lambda}; \eta) = \det\left(\tilde{A} - \frac{1}{d}M\tilde{\lambda}^2 I + \eta\tilde{R}(\lambda)\right). \tag{147}$$

For $j = 1, \dots, m-1$, $K\left(\pm\sqrt{d\tilde{\mu}_j/M}; 0\right) = 0$. Since the eigenvalues of $\tilde{A}$ are distinct,

$$\frac{\partial}{\partial\tilde{\lambda}}K(\tilde{\lambda}; 0)\Big|_{\tilde{\lambda} = \pm\sqrt{d\tilde{\mu}_j/M}} \neq 0.$$

Using the implicit function theorem, we can solve for $\tilde{\lambda}$ as a function of η near $(\tilde{\lambda}, \eta) = \left(\pm\sqrt{d\tilde{\mu}_j/M}; 0\right)$. Thus for sufficiently small η, we can find smooth functions $\tilde{\lambda}_j^{\pm}(\eta)$ such that $\tilde{\lambda}_j^{\pm}(0) = \pm\sqrt{d\tilde{\mu}_j/M}$ and $K(\tilde{\lambda}_j^{\pm}(\eta);\eta) = 0$. Expanding $\tilde{\lambda}(\eta)$ in a Taylor series about $\eta = 0$ and taking $\eta = r^{-N}$, we can write $\tilde{\lambda}_j^{\pm}$ as $\tilde{\lambda}_j^{\pm}(N) = \pm\sqrt{d\tilde{\mu}_j/M} + \mathcal{O}(r^{-N})$. Undoing the scaling, the interaction eigenvalues are given by

$$\lambda_j^{\pm} = \pm\sqrt{\frac{d\mu_j}{M}} + \mathcal{O}(r^{-2N}) \qquad \text{for } j = 1, \ldots, m-1,$$

where $\{\mu_1, \ldots, \mu_{m-1}\}$ are the distinct, real, nonzero eigenvalues of A, and $\mu_j = r^{-2N}\tilde{\mu}_j$.

By Hamiltonian symmetry, the eigenvalues of DNLS must come in quartets $\pm\alpha \pm i\beta$. Since the μ_j are distinct and only come in pairs, the eigenvalues $\lambda_j^{\pm}$ must be pairs which are real or purely imaginary. Thus there are $m-1$ pairs of nonzero interaction eigenvalues at $\lambda = \pm\lambda_j$, given by

$$\lambda_j = \sqrt{\frac{d\mu_j}{M}} + \mathcal{O}(r^{-2N}) \qquad \text{for } j = 1, \ldots, m-1.$$

These are either real or purely imaginary, and the remainder term cannot move these off of the real or imaginary axis. Since $M, d > 0$, we conclude that there are k_π pairs of purely imaginary eigenvalues and k_0 pairs of real eigenvalues.

We note that upon variations of d, these interaction eigenvalues may collide with other eigenvalues, including the ones associated with the continuous spectrum, and lead to quartets as, for example, in some of the cases in [8]. We can ensure this will not happen by choosing N sufficiently large.

8.3. *Proof of Corollaries 1 and 2*

First, we prove Corollary 1. For (i), the matrix A in the case of the 2-pulse has a single eigenvalue $\mu_1 = -\cos(\Delta\theta_1)b_1$. For (ii), the matrix A in the case of the 3-pulse with equal pulse distances is given by

$$A = b\begin{pmatrix} -\cos(\Delta\theta_1) & \cos(\Delta\theta_1) & 0 \\ \cos(\Delta\theta_1) & -\cos(\Delta\theta_1) - \cos(\Delta\theta_2) & \cos(\Delta\theta_2) \\ 0 & \cos(\Delta\theta_2) & -\cos(\Delta\theta_2) \end{pmatrix},$$

which has nonzero eigenvalues

$$\mu_{1,2} = \left(\pm\sqrt{\cos(\Delta\theta_1)^2 - \cos(\Delta\theta_1)\cos(\Delta\theta_2) + \cos(\Delta\theta_2)^2} - \cos(\Delta\theta_1) - \cos(\Delta\theta_2)\right) b.$$

For the three distinct 3-pulses, these eigenvalues are

$$\mu_{1,2} = \begin{cases} -3b, -b & (\Delta\theta_1, \Delta\theta_2) = (0,0) \\ \pm\sqrt{3}b & (\Delta\theta_1, \Delta\theta_2) = (0,\pi) \\ 3b, b & (\Delta\theta_1, \Delta\theta_2) = (\pi,\pi). \end{cases}$$

For (iii), if $b_i = b$ and $\Delta\theta_i = \Delta\theta$ for all i, the eigenvalue problem $(A - \mu I)v = 0$ is equivalent to the difference equation with Neumann boundary conditions

$$v_{n-1} - 2v_n + v_{n+1} - \frac{\mu}{b\cos(\Delta\theta)}v_n = 0$$
$$v_0 = v_1$$
$$v_{m+1} = v_m,$$

which has solutions

$$\mu_j = 2b\left(\cos\frac{\pi j}{m} - 1\right)\cos(\Delta\theta) \qquad \text{for } j = 1, \ldots, m.$$

For Corollary 2, Eq. (64) follows from computing the eigenvalues of A explicitly for the 3-pulse and noting that $(\cos\Delta\theta_i)^2 = 1$ since $\Delta\theta_i \in \{0, \pi\}$. We note that for $N_1 < N_2$, $b_1 > b_2$, thus we can write

$$\sqrt{b_1^2 + b_2^2 - b_1 b_2 \cos\Delta\theta_1 \cos\Delta\theta_2} = b_1\sqrt{1 + \frac{b_2^2}{b_1^2} - \frac{b_2}{b_1}\cos\Delta\theta_1\cos\Delta\theta_2}$$

and expand in a Taylor series to obtain the estimates (65).

CRediT authorship contribution statement

Ross Parker: Conceptualization, Methodology, Software, Formal analysis, Visualization, Writing - original draft. **P.G. Kevrekidis:** Writing - review & editing, Validation, Supervision. **Björn Sandstede:** Writing - review & editing, Visualization, Supervision.

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References

[1] C. Elphick, E. Meron, E.A. Spiegel, Patterns of propagating pulses, SIAM J. Appl. Math. 50 (02) (1990) 490–503.
[2] Björn Sandstede, Stability of multiple-pulse solutions, Trans. Amer. Math. Soc. 350 (02) (1998) 429–473.
[3] J. Knobloch, Lin's method for discrete dynamical systems, J. Difference Equ. Appl. 6 (5) (2000) 577–623.
[4] W.-J. Beyn, V. Thümmler, Freezing solutions of equivariant evolution equations, SIAM J. Appl. Dyn. Syst. 3 (02) (2004) 85–116.
[5] W.-J. Beyn, S. Selle, V. Thümmler, Freezing multipulses and multifronts, SIAM J. Appl. Dyn. Syst. 7 (02) (2008) 577–608.
[6] Wolf-Jürgen Beyn, Denny Otten, Jens Rottmann-Matthes, Freezing traveling and rotating waves in second order evolution equations, in: Pavel Gurevich, Juliette Hell, Björn Sandstede, Arnd Scheel (Eds.), Patterns of Dynamics, Springer International Publishing, Cham, 2017, pp. 215–241.
[7] P.G. Kevrekidis, Multipulses in discrete Hamiltonian nonlinear systems, Phys. Rev. E 64 (2001) 026611.
[8] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, Stability of discrete solitons in nonlinear Schrödinger lattices, Physica D 212 (1–2) (2005) 1–19.
[9] T. Kapitula, P.G. Kevrekidis, B.A. Malomed, Stability of multiple pulses in discrete systems, Phys. Rev. E 63 (3) (2001).
[10] Manoussos Grillakis, Jalal Shatah, Walter Strauss, Stability theory of solitary waves in the presence of symmetry, I, J. Funct. Anal. 74 (1) (1987) 160–197.
[11] Xiao-Biao Lin, Using Melnikov's method to solve Silnikov's problems, Proc. Roy. Soc. Edinburgh Sect. A 116 (3–4) (1990) 295–325.
[12] Xiao-Biao Lin, Lin's method, Scholarpedia 3 (9) (2008) 6972.
[13] Panayotis G. Kevrekidis, The Discrete Nonlinear Schrödinger Equation, Springer Berlin Heidelberg, 2009.
[14] Dmitry E. Pelinovsky, Localization in Periodic Potentials: From Schrödinger Operators to the Gross–Pitaevskii Equation, London Mathematical Society Lecture Note Series, Cambridge University Press, 2011.
[15] J. Cuevas-Maraver, P.G. Kevrekidis, F. Williams, The Sine-Gordon Model and Its Applications, in: Nonlinear Systems and Complexity, Springer-Verlag, 2014.
[16] Ivan Markov, Static multikink solutions in a discrete Frenkel-Kontorova model with anharmonic interactions, Phys. Rev. B 48 (1993) 14016–14019.
[17] Alan Champneys, Yuri S. Kivshar, Origin of multikinks in dispersive nonlinear systems, Phys. Rev. E 61 (2000) 2551–2554.
[18] J. Cuevas, V. Koukouloyannis, P.G. Kevrekidis, J.F.R. Archilla, Multibreather and vortex breather stability in Klein–Gordon lattices: equivalence between two different approaches, Int. J. Bifurcation Chaos 21 (08) (2011) 2161–2177.
[19] G.L. Alfimov, V.A. Brazhnyi, V.V. Konotop, On classification of intrinsic localized modes for the discrete nonlinear Schrödinger equation, Physica D 194 (1) (2004) 127–150.
[20] Michael Herrmann, Homoclinic standing waves in focusing DNLS equations, Discrete Contin. Dyn. Syst. Ser. A 31 (2011) 737.
[21] Magnus Johansson, Serge Aubry, Growth and decay of discrete nonlinear Schrödinger breathers interacting with internal modes or standing-wave phonons, Phys. Rev. E 61 (5) (2000) 5864–5879.

[22] N.G. Vakhitov, A.A. Kolokolov, Stationary solutions of the wave equation in a medium with nonlinearity saturation, Radiophys. Quantum Electron. 16 (7) (1973) 783–789.
[23] Dmitry Pelinovsky, Anton Sakovich, Multi-site breathers in Klein–Gordon lattices: stability, resonances and bifurcations, Nonlinearity 25 (12) (2012) 3423–3451.
[24] B. Sandstede, Instability of localized buckling modes in a one-dimensional strut model, Phil. Trans. R. Soc. A 355 (1732) (1997) 2083–2097.
[25] Scipio Cuccagna, On instability of excited states of the nonlinear Schrödinger equation, Physica D 238 (1) (2009) 38–54.
[26] P.G. Kevrekidis, D.E. Pelinovsky, A. Saxena, When linear stability does not exclude nonlinear instability, Phys. Rev. Lett. 114 (2015) 214101.
[27] Wen-Xin Qin, Xufeng Xiao, Homoclinic orbits and localized solutions in nonlinear Schrödinger lattices, Nonlinearity 20 (10) (2007) 2305–2317.
[28] M. Peyrard, M.D. Kruskal, Kink dynamics in the highly discrete sine-Gordon system, Physica D 14 (1) (1984) 88–102.
[29] C.V. Turner, R.R. Rosales, The small dispersion limit for a nonlinear semidiscrete system of equations, Stud. Appl. Math. 99 (1) (2001) 205–254.
[30] Yasuhiro Ohta, Ryogo Hirota, A discrete KdV equation and its Casorati determinant solution, J. Phys. Soc. Japan 60 (6) (1991) 2095.
[31] T Bountis, H.W. Capel, M Kollmann, J.C. Ross, J.M. Bergamin, J.P. van der Weele, Multibreathers and homoclinic orbits in 1-dimensional nonlinear lattices, Phys. Lett. A 268 (1–2) (2000) 50–60.
[32] J.J. Bramburger, B. Sandstede, Spatially localized structures in lattice dynamical systems, Journal of Nonlinear Science (2019).
[33] P.G. Kevrekidis, B.A. Malomed, A.R. Bishop, Bound states of two-dimensional solitons in the discrete nonlinear Schrödinger equation, J. Phys. A: Math. Gen. 34 (45) (2001) 9615–9629.
[34] Dragomir N. Neshev, Tristram J. Alexander, Elena A. Ostrovskaya, Yuri S. Kivshar, Hector Martin, Igor Makasyuk, Zhigang Chen, Observation of discrete vortex solitons in optically induced photonic lattices, Phys. Rev. Lett. 92 (2004) 123903.
[35] Jason W. Fleischer, Guy Bartal, Oren Cohen, Ofer Manela, Mordechai Segev, Jared Hudock, Demetrios N. Christodoulides, Observation of vortex-ring "discrete" solitons in 2D photonic lattices, Phys. Rev. Lett. 92 (2004) 123904.
[36] B. Sandstede, Convergence estimates for the numerical approximation of homoclinic solutions, IMA J. Numer. Anal. 17 (3) (1997) 437–462.
[37] W.J. Beyn, J.M. Kleinkauf, The numerical computation of homoclinic orbits for maps, SIAM J. Numer. Anal. 34 (3) (1997) 1207–1236.
[38] Alouf Jirari, Second-Order Sturm-Liouville Difference Equations and Orthogonal Polynomials, American Mathematical Society, 1995.