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Modulational instability of matter waves under strong nonlinearity management

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ABSTRACT

We study modulational instability of matter-waves in Bose–Einstein condensates (BEC) under strong temporal nonlinearity-management. Both BEC in an optical lattice and homogeneous BEC are considered in the framework of the Gross–Pitaevskii equation, averaged over rapid time modulations. For a BEC in an optical lattice, it is shown that the loop formed on a dispersion curve undergoes transformation due to the nonlinearity-management. A critical strength for the nonlinearity-management strength is obtained that changes the character of instability of an attractive condensate. MI is shown to occur below (above) the threshold for the positive (negative) effective mass. The enhancement of number of atoms in the nonlinearity-managed gap soliton is revealed.

1. Introduction

The phenomenon of modulational instability (MI) of nonlinear plane waves under different types of management of the system parameters has been the subject of intensive research over the last years [1]. Main emphasis was given to dispersion-management and nonlinearity-management. In nonlinear optics strong and rapid modulations of the fiber dispersion is achieved by periodic arrangement of fiber spans with alternating sign of the dispersion. Dispersion-managed solitons supported by such a system have essential advantages over conventional optical solitons for long distance communication purposes [2–4]. Modulations of the nonlinearity is a challenging problem also in fiber ring lasers and in generation of Faraday waves in Bose–Einstein condensates (BEC) [5–9]. MI in the form of Faraday waves can be observed both in *attractive* and *repulsive* condensates. Recent observation of the MI in optical media resulted from the periodic modulation of the nonlinearity in the evolution variable, confirms the existence of parametric resonances in the MI growth rate [6,10,11]. Faraday waves (parametrically excited waves) in a BEC emerging from temporal periodic variation of the atomic scattering length have been studied in [9]. Such type of modulations can be achieved by variation of the external magnetic field near Feshbach resonances (FR). The corresponding technique is known as FR management. In the Gross–Pitaevskii equation this corresponds to a temporal variation of the mean-field nonlinearity, i.e. to the nonlinearity-management. MI in a harmonically trapped BEC under FR management has been investigated in [12].

Recently the strong dispersion-management has been applied to the dynamics of nonlinear periodic waves, namely cnoidal waves, in optical fibers [13,14]. In these works the existence of dispersion-managed cnoidal waves and strong deviation of the stability borders of these waves from the ones of standard cnoidal wave solutions of the nonlinear Schrödinger equation (NLSE) have been established. Extension of the stability regions of some types of nonlinear periodic waves can be due to the different scenarios for the onset of MI of the background plane waves. Adiabatic FR management for cnoidal waves in optical lattices has been considered in [15,16]. The case of strong nonlinearity-management remains unexplored.

The strong nonlinearity-management may be an effective tool for stabilization of matter-wave solitons in multi-dimensional attractive BEC [17–26]. In the context of nonlinear optics such stabilization mechanism was first discussed in [27,28]. The phenomenon of MI is particularly important for generation of soliton trains in BEC with controlled spatial arrangement (repetition rate). MI of BEC in linear and nonlinear optical lattices in the absence of time-periodic nonlinearity-management has been investigated in our recent work [29]. Here we consider both the MI of a homogeneous BEC and MI of a BEC loaded in an optical lattice under FR management. The gap soliton structure existing in a BEC with the zero background scattering length ($a_{sb} = 0$) has been investigated in Ref. [30]. The couple-mode theory can be used to analyze MI of nonlinear plane waves in an optical lattice subject to FR management. In our investigations particular interest will be

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paid to the properties of loop structures emerging in the band gaps (forbidden band).

In the present paper we investigate nonlinear dispersion relations and the process of MI in a BEC under strong temporal nonlinearity management (SNM). The outline of the paper is as follows: The mathematical model is formulated in Section 2; MI in a homogeneous BEC under SNM is considered in Section 3; The nonlinear dispersion relation and loop structures for BEC in an optical lattice under SNM are analyzed in Section 4 using the coupled-mode theory. This section also includes the regions of MI found in different areas of the band structure; The properties of gap solitons are investigated in Section 5; Section 6 is devoted to details of our numerical procedure; In the final Section 7 we summarize our main results.

2. The model

Let us consider a BEC under temporal Feshbach resonance management when the scattering length a_s varies in time. Then an elongated BEC can be described by the quasi-1D GP equation with a periodic potential (optical lattice) and the time-dependent management of the coefficient of nonlinearity

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi - g_{1D}(t)|\psi|^2\psi, \tag{1}$$

where $g_{1D}(t) = 2\hbar a_s(t)\omega_\perp$ is the mean field nonlinearity coefficient, $\omega_\perp$ is the transverse oscillator frequency and $V(x) = V_0\cos^2(kx)$ is an optical lattice potential, $\int_{-\infty}^{\infty} dx|\psi|^2 = N$, N is the number of atoms. In dimensionless units we have

$$x \to kx, \qquad t \to \omega_R t, \qquad \epsilon = \frac{V_0}{2E_R}, \qquad E_R = \frac{\hbar^2 k^2}{2m},$$
$$\omega_R = E_R/\hbar, \qquad u = \sqrt{\frac{2\hbar a_s\omega_\perp}{E_R}}\,\psi e^{-i\epsilon t}.$$

Eq. (1) takes the form of the NLSE with varying in time mean field nonlinearity coefficient

$$iu_t + u_{xx} + \gamma(t)|u|^2u - 2\epsilon\cos(2x)u = 0, \tag{2}$$

where $\gamma(t)$ describes the strong nonlinearity-management and has the form

$$\gamma(t) = \gamma_0 + \frac{1}{\mu}\gamma_1\left(\frac{t}{\mu}\right), \qquad \int_0^1 \gamma_1(\tau)\,d\tau = 0, \qquad \tau = \frac{t}{\mu}, \qquad \mu \ll 1. \tag{3}$$

This model has been considered in recent papers [15,16,30]. Specifically, in works [15,16] the evolution of nonlinear periodic waves under adiabatic time-variation of the scattering length has been studied and a possibility of generation of a train of solitons by such a management scheme has been shown. Properties of gap solitons under the strong management of nonlinearity were analyzed based on the coupled mode system of equations in [30]. In this work the gap soliton solutions and their stability for the case $\gamma_0 = 0$ were investigated. Here we will study MI of nonlinear plane waves in a BEC (without, and with an optical lattice) under SNM, as well as properties of gap solitons in the model (2) for nonzero value of γ_0. In particular, we will analyze the possibility of enhancement of number of atoms in the gap soliton under SNM.

In deriving averaged equation we follow the works [30,31] and use the transformation

$$u(x,t) = e^{i\gamma_{-1}(t)|v|^2}v(x,t),$$
$$\gamma_{-1}(\tau) = \int_0^1 \gamma(\tau')\,d\tau' - \int_0^1\int_0^\tau \gamma(\tau')\,d\tau'\,d\tau. \tag{4}$$

Supposing the parameter μ to be small (that corresponds to high frequencies of modulation) unknown function v can be expanded in series as

$$v = w + \mu v_1 + \mu^2 v_2 + \cdots, \tag{5}$$

where unknown w is a slowly varying function. Using transformation (4) and expansion (5) in governing Eq. (2) with posterior averaging over the period of rapid modulation, we arrive at the following averaged equation for w [31]

$$iw_t + w_{xx} + \gamma_0|w|^2w - 2\epsilon\cos(2x)w + \sigma^2[2(|w|^2)_{xx}|w|^2 + ((|w|^2)_x)^2]w = 0. \tag{6}$$

Parameter σ is defined as $\sigma^2 = \int_0^1 \gamma_{-1}^2\,d\tau$. For particular case of sinusoidal modulations $\gamma_1 = h\sin(\omega t)$ we have $\sigma^2 = h^2/(2\omega^2) \sim O(1)$ $(\omega = 1/\mu)$. For the step-like modulation with the same amplitude h and frequency ω we have $\sigma^2 = h^2/\omega^2$.

This form of averaged equation can be also obtained for the case of the weak nonlinearity management when $\gamma = \gamma_0 + \gamma_1(t/\mu)$, with $\sigma^2 \ll 1$ [31,32].

3. Modulational instability of nonlinear plane wave in a homogeneous media

Now let us consider the case when the optical lattice is switched off, i.e. $\epsilon = 0$ in Eq. (2). The MI of a nonlinear plane wave $w = A\exp(i(\gamma_0A^2t))$ can be explored using the linear stability analysis, i.e. looking for the solution in the form

$$w = (A + \psi(x,t))\exp[i\gamma_0A^2t], \quad \psi \ll A. \tag{7}$$

We have the following equation for ψ

$$i\psi_t + \psi_{xx} + \gamma_0A^2(\psi + \psi^*) + 2\sigma^2A^4(\psi_{xx} + \psi^*_{xx}) = 0. \tag{8}$$

Representing $\psi = \psi_r + i\psi_i$ and performing Fourier transformation $\psi_r(\psi_i)(x,t) = \int dk\bar{u}(\bar{v})(k,t)\exp(ikx)$ we get the dispersion relation

$$p^2 = k^2[2\gamma_0A^2 - (1 + 4\sigma^2A^4)k^2]. \tag{9}$$

Instability region corresponds to the condition $p^2 > 0$. Thus we obtain

$$k^2 \le \frac{2\gamma_0A^2}{1 + 4\sigma^2A^4}. \tag{10}$$

The maximum of the MI gain is achieved at the value of the wave number

$$k_c = \sqrt{\frac{\gamma_0}{1 + 4\sigma^2A^4}}A. \tag{11}$$

Maximal value of the MI growth rate is

$$p_c = \frac{\gamma_0A^2}{\sqrt{1 + 4\sigma^2A^4}}. \tag{12}$$

Thus we find that under the temporal nonlinearity management the MI growth rate is decreased by a factor of $\sqrt{1 + 4\sigma^2A^4}$. Such decrease of the gain is due to the defocusing effect induced by the nonlinearity management. This observation explains the stabilizing role of the strong nonlinearity management in a higher dimensional attractive BEC [17,18,33,34].

Numerical simulations of the 1D GP Eq. (2) with a strong nonlinearity management confirm these predictions. In Fig. 1 we plot the MI gain versus the wave number of modulations k for three different cases with $\gamma_0 = 1$ and $\omega = 10$: (a) when the nonlinearity-management is absent, $\sigma^2 = 0$ and when the management is present (b) $\sigma^2 = 0.125 (h = 5)$, (c) $\sigma^2 = 0.5 (h = 10)$. One

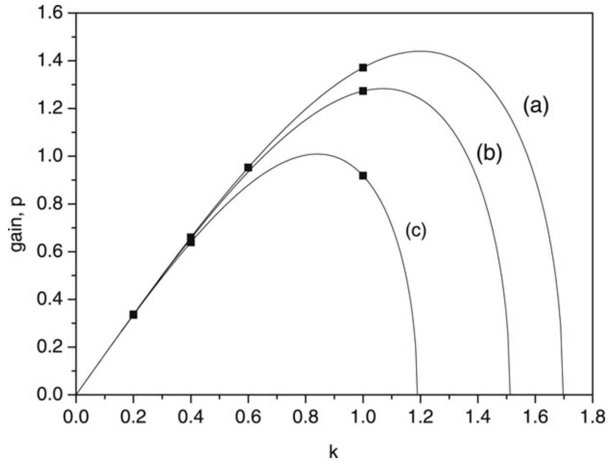

Fig. 1. MI gain p versus the wave number modulations k. Three curves correspond to the cases when: (a) nonlinear management is turned off, $\sigma^2 = 0$; (b), (c) nonlinear management is turned on, with $A = 1.2$, $\sigma^2 = 0.125$ and $\sigma^2 = 0.5$. Filled squares correspond to gains obtained from full PDE simulations.

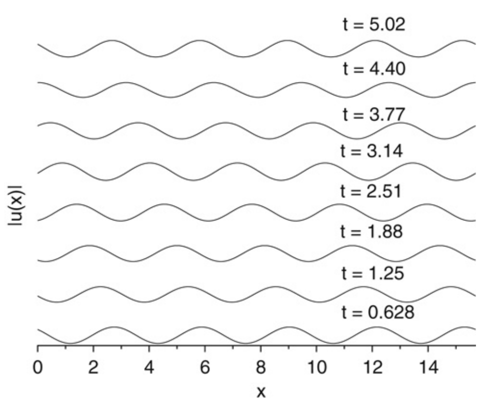

Fig. 2. Evolution of the small spatially periodic perturbation when p and the wave number k are in the region of stability and inequality (10) is not fulfilled. The case with $\sigma^2 = 0.02$, $k = 2$, $A = 1.2$ is presented.

can observe a good agreement between the theory and numerical simulations for the value and the position of the MI gain maximum given by Eqs. (11) and (12). In Fig. 2 we plot the profiles of the field module $|u(x)|$ in the region of stability. Fig. 3 depicts the case of breakdown of the stability caused by increasing the strength of the nonlinearity-management, σ^2.

One can see that modulation in an initial plane wave evolves into a train of solitons when the wave number of the modulation is in the region of instability. As can be seen from Fig. 3(a) even moderate nonlinearity management ($\sigma^2 = 0.125$) causes notable decreasing in the amplitude of solitons.

4. MI in a BEC loaded in an optical lattice and nonlinearity-management

4.1. Nonlinear dispersion relation. The loop structure

The analysis performed in the previous section was relevant to a BEC without optical lattice potential. In the presence of an optical lattice the band structure strongly affects the process of MI [35,36]. Equations of the coupled-mode theory for the GP Eq. (6) with shallow optical lattice have been obtained in [30]. The wave function can be represented in the form of superposition of backward and forward propagating waves

$$w(x,t) = \sqrt{\epsilon}\left(A(X,T)\mathrm{e}^{\mathrm{i}x} + B(X,T)\mathrm{e}^{-\mathrm{i}x}\right)\mathrm{e}^{-\mathrm{i}t}. \quad (13)$$

where $X = \epsilon x$, $T = \epsilon t$ are slow variables. Substituting this into the averaged equation we get the following coupled mode system of equations

$$\begin{aligned} \mathrm{i}A_T + 2\mathrm{i}A_X &= B - \gamma_0(|A|^2 + 2|B|^2)A \\ &\quad + 8\epsilon\sigma^2(2|A|^2 + |B|^2)|B|^2 A, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathrm{i}B_T - 2\mathrm{i}B_X &= A - \gamma_0(2|A|^2 + |B|^2)B \\ &\quad + 8\epsilon\sigma^2(|A|^2 + 2|B|^2)|A|^2 B. \end{aligned} \quad (15)$$

In derivation of this system, the derivatives of the nonlinear terms have been neglected as the terms of the next order of smallness with respect to ϵ. The group velocity varies in the interval $-2 < v < 2$, in physical units that corresponds to $-v_R < v < v_R$, $v_R = \hbar k/m$. This system describes two counter propagating waves, with the cubic self phase modulation term and cubic and quintic cross-phase modulation terms. The quintic cross modulation term describes effect of the Feshbach resonance management. Note that this system has a similarity with the one previously considered for description of MI in the cubic–quintic NLSE with the Bragg grating [37]. However, as distinct from that model, no self-phase modulation quintic terms like $|A|^4A$ and $|B|^4B$ present in our model. The absence of these terms changes significantly the MI process in NM systems in comparison with the standart cubic–quintic NLS model.

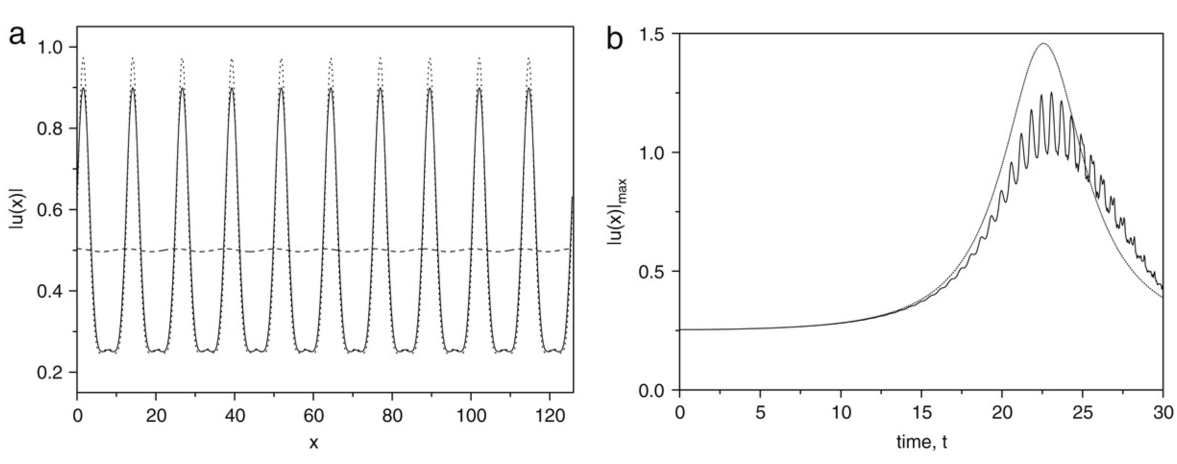

Fig. 3. Development of small spatially periodic perturbations into a soliton train when parameters are in the region of instability. Plot (a) depicts the field profiles $|u(x)|$ at different times. The dashed line stands for initial small modulations at $t = 0$, dotted (solid) line is for the case, when SNM is turned off (on) at $t = 20.1$. Plot (b) depicts time evolution of the maximal value of $|u(x)|$. Dotted (solid) line correspond to turned off (on) SNM. The parameters are $k = 0.5$, $A = 0.5$, $\sigma^2 = 0$ ($\sigma^2 = 0.125$).

The plane wave solutions of Eqs. (14) and (15) are looked for in the form

$$A=\frac{\alpha}{\sqrt{1+f^2}}\mathrm{e}^{\mathrm{i}(QX-\Omega T)},\qquad B=\frac{\alpha f}{\sqrt{1+f^2}}\mathrm{e}^{2\mathrm{i}(QX-\Omega T)},$$

where $\alpha=|A|^2+|B|^2$. The parameter f defines the weight of the forward and backward propagating waves. The case $|f|>1$ corresponds to the domination of the backward wave. Substituting these expressions into the system (14) and (15), we obtain nonlinear dispersion relation

$$\Omega=-\frac{3\gamma_0}{2}\alpha^2+\frac{1}{2}\frac{1+f^2}{f}+\frac{4\epsilon\sigma^2\alpha^4}{(1+f^2)^2}(f^4+4f^2+1), \tag{16}$$

$$Q=\frac{(8\epsilon\sigma^2\alpha^2-\gamma_0)\alpha^2}{4}\frac{1-f^2}{1+f^2}+\frac{1}{4}\frac{1-f^2}{f}. \tag{17}$$

The parameter f determines the position on the dispersion relation in Ω, Q plane. Inspecting the dispersion relation at small α^2 one can observe that $f>0$ corresponds to the upper dispersion curve and $f<0$ to the lower one. The velocity inside the grating is $v=2(1-f^2)/(1+f^2)$ and equals to zero at the edges of the gap $f=\pm 1$.

From Eqs. (16) and (17) one can again see a defocusing role of the strong nonlinearity management. We find that the effect of nonlinearity is cancelled if $|f|=1$ and the density of BEC reaches a threshold value

$$\alpha_c^2=\frac{\gamma_0}{4\epsilon\sigma^2}.$$

Suppression of the mean-field nonlinearity in the lattice leads to enhancement of such an effect as tunnelling between sites. The SNM also introduces changes in the dispersion curves. Indeed, it is well known that the focusing Kerr nonlinearity (attractive BEC) is responsible for appearance of a loop beyond the critical power [38–40] on the upper curve.

Effective *nonlinear dispersion* induced by the nonlinearity-management (the last term in Eq. (6)) will increase the critical power necessary for appearance of the loop. To find this value of critical power let us consider the value of f_c at which Q becomes zero ($|f|\neq 1$). We obtain that

$$f_c=\frac{\alpha^2(\gamma_0-8\epsilon\sigma^2\alpha^2)}{2}\pm\sqrt{\left(\frac{\alpha^2(\gamma_0-8\epsilon\sigma^2\alpha^2)}{2}\right)^2-1}. \tag{18}$$

Let us consider the case of upper curve with $f>0$ and an attractive condensate $\gamma_0>0$. Then a loop appears on the dispersion curve if the power (BEC density)

$$\alpha_2^2<\alpha^2<\alpha_1^2,\ \alpha_{1,2}^2=\frac{\gamma_0}{16\epsilon\sigma^2}\left(1\pm\sqrt{1-\frac{64\epsilon\sigma^2}{\gamma_0^2}}\right). \tag{19}$$

When $\sigma^2=0$ we have a well known result for the critical power [38] $\alpha_c^2=2/\gamma_0$. Fig. 4 depicts two branches of the dispersion relations (16) and (17). The branches in the Ω–Q plane are defined parametrically by Eqs. (16) and (17). Thus, each value of f defines a point in this plane. Two ranges of values $f>0$ and $f<0$ define upper and lower curves correspondingly. In our case the loop structure appears on the upper branch when α is greater than the threshold value α_c^2. One can see from this figure that the loop decreases with increasing of the strength of the management. It should be noted that at the same time the band width (a distance between upper and lower branches) at $Q=0$ does not change.

In the case of the defocusing Kerr nonlinearity $\gamma_0<0$ (repulsive BEC) one could expect formation of the loop on the lower branch of the dispersion curve $f<0$. But from the condition (19) it follows that $\alpha^2<0$. So in this case NM fully suppresses the loop formation.

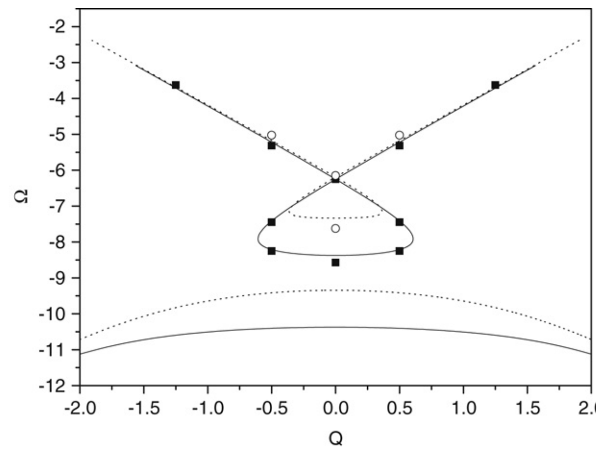

Fig. 4. Loop structure in dispersion relations when $\alpha^2>\alpha_c^2$: solid (dotted) line and full squares (circles) are for the case when nonlinearity management is turned off (on). Scatter points (squares and circles) represent data obtained from numerical simulations. Parameters are: $\sigma^2=0$ and $\sigma^2=0.055(h=4)$ with $\alpha=2.5$, $\epsilon=0.08$, $\omega=12$.

It is also of interest to investigate the loop structure in the case $\gamma_0=0$. This configuration can be realized employing the Feshbach resonance technique. It corresponds to the case of a BEC with the *effective repulsive* nonlinearity in an optical lattice. The loop will be formed on the lower branch of the dispersive curve $f<0$ when the BEC density excesses the value

$$\alpha^2>\frac{1}{2\sqrt{\epsilon}\sigma}.$$

Let us discuss the physical consequences. Existence of a loop at the edge of the Brillouin zone reflects the superfluid character of the BEC, since we have nonzero velocity in the Bragg reflection condition [39,40]. It should be noted that in a linear system of free atoms the Bloch wave at the zone edge has zero velocity. From this point of view a critical value of the SNM strength exists which destroys the superfluid property of the BEC in an optical lattice. Another possible effect is the existence of breakdown of Bloch oscillations due to the tunnelling into the upper band (Landau–Zeener tunnelling). The SNM is expected to suppress this breakdown.

4.2. Modulational instability

To investigate MI of matter waves in an optical lattice under SNM, perturbed plane wave solutions are taken in the form

$$A=\left(\frac{\alpha}{\sqrt{1+f^2}}+\delta A(X,T)\right)\mathrm{e}^{\mathrm{i}(QX-\Omega T)}, \tag{20}$$

$$B=\left(\frac{\alpha f}{\sqrt{1+f^2}}+\delta B(X,T)\right)\mathrm{e}^{\mathrm{i}(QX-\Omega T)},$$

where δA and δB are unknown small perturbations of CW solutions. Substituting these expressions into Eqs. (14) and (15) and using a linear approximation, we get the system of equations for δA and δB

$$\mathrm{i}\delta A_T+2\mathrm{i}\delta A_X+f\delta A-\delta B+\frac{\alpha^2}{1+f^2}\left[\left(\gamma_0-\frac{16\epsilon\sigma^2\alpha^2 f^2}{1+f^2}\right)\right.$$
$$\left.\times\ (\delta A+\delta A^*)+2f(\gamma_0-8\epsilon\sigma^2\alpha^2)(\delta B+\delta B^*)\right]=0, \tag{21}$$

$$\mathrm{i}\delta B_T-2\mathrm{i}\delta B_X+\frac{1}{f}\delta B-\delta A+\frac{\alpha^2 f^2}{1+f^2}\left[\left(\gamma_0-\frac{16\epsilon\sigma^2\alpha^2}{1+f^2}\right)\right.$$
$$\left.\times\ (\delta B+\delta B^*)+\frac{2}{f}(\gamma_0-8\epsilon\sigma^2\alpha^2)(\delta A+\delta A^*)\right]=0. \tag{22}$$

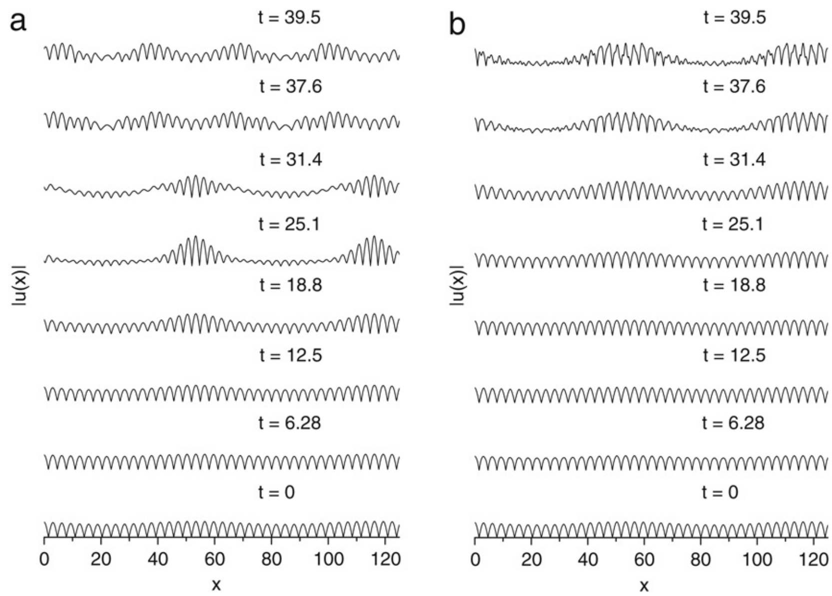

Fig. 5. Evolution of small spatial periodic perturbations when the parameters are in the region of instability with $\gamma_0 = 1$ and $f = 1$ (upper branch of the dispersion relations (16) and (17)). Plot (a) depicts the field profiles $|u(x)|$ at different times when the nonlinearity management is turned off and $\sigma^2 = 0$. Plot (b) depicts the case when the nonlinearity management is turned on and $\sigma^2 = 0.125 (h = 5)$. Other parameters are $\alpha = 0.8$, $Q = 0$, $q = 0.5$, $\omega = 10$. Initial amplitude of modulations is taken to be 0.05.

For $f = \pm 1$ the system coincides with the one considered by de Sterke [35] with renormalized nonlinearity coefficient $\gamma_r = \gamma_0 - 8\epsilon\sigma^2\alpha^2$. One can see that the NM plays essential role in the MI process. When the nonlinearity management is turned off, for the case of attractive condensate the CW wave is unstable if the parameters follow the upper branch of the dispersion curve. On the lower branch the attractive BEC is modulationally stable. The repulsive condensate is modulationally unstable on the lower branch and stable on the upper branch.

In the case of nonlinearity-management there exists a critical value of the management strength σ^2, namely $\sigma_c^2 = \gamma_0/8\epsilon\alpha^2$. If $\sigma^2 > \sigma_c^2$, then the attractive condensate behaves as the repulsive and the modulational instability regions should correspond to the above described picture.

Looking for solutions of Eq. (21) in the form

$$\delta A(B) = C(D)\cos(qX - \omega T) + iE(F)\sin(qX - \omega T)$$

we find the dispersion relation of the form

$$(\omega^2 - 4q^2)^2 - 2(1-N)(\omega^2 - 4q^2) - \frac{1}{f}\left(\frac{1}{f} + P\right)(\omega - 2q)^2 - f(f+M)(\omega + 2q)^2 = 0, \quad (23)$$

where

$$M = \frac{2\alpha^2}{1+f^2}\left(\gamma_0 - \frac{16\epsilon\sigma^2\alpha^2 f^2}{1+f^2}\right),$$

$$N = \frac{4f\alpha^2}{1+f^2}\left(\gamma_0 - 8\epsilon\sigma^2\alpha^2\right),$$

$$P = \frac{2\alpha^2 f^2}{1+f^2}\left(\gamma_0 - \frac{16\epsilon\sigma^2\alpha^2}{1+f^2}\right).$$

Analytical results can be obtained for the particular case $|f| = 1$, corresponding to the edges of the gap. We come to the equation for the frequency ω

$$\omega^2 = 4q^2 + 2 - \tilde{G} \pm \sqrt{16q^2\left(1+\widetilde{G}\right) + \left(2-\widetilde{G}\right)^2}, \quad \widetilde{G} = G/f, \quad f = \pm 1. \quad (24)$$

Evidently, this equation coincides with the one obtained in Ref. [35] where the parameter G is renormalized as $G = (\gamma_0 - 8\epsilon\sigma^2\alpha^2)\alpha^2$.

Let us analyze the condition of MI for different sets of parameters.

(1) The top of the band gap $f = 1$, $\sigma^2 < \sigma_c^2$ ($\widetilde{G} > 0$). The wave is unstable if the wavenumber of modulations is in the interval $-\sqrt{3G/2} < q < \sqrt{3G/2}$. The maximal MI gain occurs at the wavenumber

$$q_m = \sqrt{3G\frac{4+G}{16(1+G)}}. \quad (25)$$

Results of numerical simulations of the Gross–Pitaevskii equation (2) for evolution of the nonlinear plane wave modulations is shown in Fig. 5. The emergence of a train of gap solitons is observed. The reduction of the MI gain when the SNM is applied can be noted.

(2) The bottom of the band gap $f = -1$, $\sigma^2 < \sigma_c^2$ ($\widetilde{G} < 0$). The condensate becomes unstable if $G > 1$ and the wavenumber satisfies the inequality

$$|q| > \frac{2+G}{4}\sqrt{\frac{1}{G-1}}. \quad (26)$$

(3) In the case $\sigma^2 > \sigma_c^2$, an *attractive* condensate behaves like the *repulsive* condensate under the strong nonlinearity management. We can expect modulational instability in the case of $f = -1$, corresponding to the negative effective mass. In this case the condensate is unstable in the region of modulations with the wave numbers $q^2 < 3|G|/2$.

Let us consider separately the case $\gamma_0 = 0$, $G = -8\epsilon\sigma^2\alpha^4$. As it was shown in [30], near the upper edge of the gap, the gap soliton is the solution of the focusing quintic NLSE, while near the bottom of the gap it is a solution of the defocusing quintic NLSE. Fig. 6 depicts the formation of a gap soliton train under strong nonlinearity management. It should be noted that when $\gamma_0 = \sigma^2 = 0$, the soliton does not form.

For $f = -1$ the instability region is $|q| < \sqrt{12\epsilon\sigma^2\alpha^2/2}$. For $|f| \neq 1$ we can perform analytical consideration for the case of vanishing wave numbers of modulations $q = 0$. Then in the ordinary optical lattice the gain of MI turns to be finite and for the

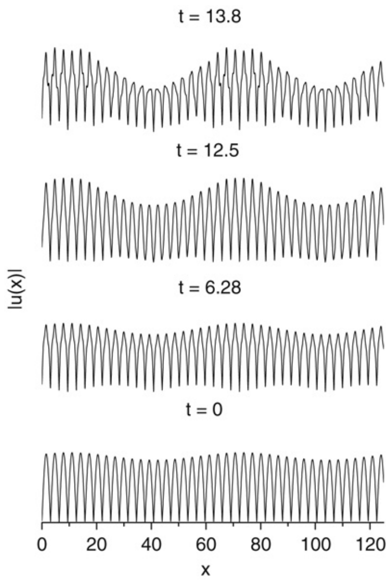

Fig. 6. Evolution of small spatial periodic perturbations when parameters are in the region of instability with $\gamma_0 = 0$ and $f = -1$ (lower branch of the dispersion relations (16) and (17)). The plot depicts the field profiles $|u(x)|$ at different times when the strength of the nonlinearity-management is $\sigma^2 = 0.08(h = 4)$. Other parameters are $\alpha = 1.5$, $Q = 0$, $q = 0.5$, $\omega = 10$. Initial amplitude of modulations is 0.05.

MI in the normal dispersion region there exists a threshold in the power. In the case of the action of a SNM we find from Eq. (23) that the instability occurs if

$$\frac{(1+f^2)^2}{f^2} - \frac{4f(\gamma_0 - 8\epsilon\sigma^2\alpha^4)}{1+f^2} < 0. \tag{27}$$

For example, if $\gamma_0 > 0$ and $f > 0$ the MI is possible only if $\alpha_2^2 < \alpha^2 < \alpha_1^2$, where

$$\alpha_{1,2}^2 = \frac{\gamma_0}{16\epsilon\sigma^2}\left(1 \pm \sqrt{1 - \frac{8(1+f^2)^3\epsilon\sigma^2}{f^3\gamma_0^2}}\right).$$

The MI interval on α^2 for $f < 0$ can be obtained analogously.

5. Gap soliton

Following the works [29,30] let us study the properties of a gap soliton. The solution is sought in the form $A = a(X)\exp(-\mathrm{i}\bar{\Omega}T)$, $B = b(X)\exp(-\mathrm{i}\bar{\Omega}T)$, $a = b^*$, $a = \sqrt{Q(X)}\exp(-\mathrm{i}\theta(X)/2)$. The set of equations for $Q(X)$, $\theta(X)$ is

$$Q_X = Q\sin(\theta), \tag{28}$$

$$\theta_X = -\bar{\Omega} + \cos(\theta) - 3\gamma_0 Q + 24\epsilon\sigma^2 Q^2. \tag{29}$$

The first integral of this set is

$$E = -\bar{\Omega}Q + Q\cos(\theta) - \frac{3}{2}\gamma_0 Q^2 + 8\epsilon\sigma^2 Q^3.$$

Inside the gap $-1 \le \Omega \le 1$. The solution for $\gamma_0 \neq 0$ is difficult to be derived in an explicit form. What we can calculate is the peak value of gap soliton amplitude, the quantity, which is of interest for the experiment. For the soliton peak the condition $Q_x = 0$ is valid. Taking into account that for bright soliton solution $E = 0$, we obtain the following equation for the peak value of the soliton amplitude

$$\pm 1 = \bar{\Omega} + \frac{3}{2}\gamma_0 Q - 8\epsilon\sigma^2 Q^2,$$

where the signs $\pm$ correspond to $\theta = 0$ and $\theta = \pi$ respectively. Peak values, corresponding to the bright soliton solutions, are

$$Q = \frac{3}{32\epsilon\sigma^2}\gamma_0\left[1 - \sqrt{1 + \frac{128\epsilon\sigma^2(\bar{\Omega} \mp 1)}{9\gamma_0^2}}\right]. \tag{30}$$

It should be noted that when $\gamma_0 = 0$ we get $Q = \sqrt{(\bar{\Omega} \mp 1)/8\epsilon\sigma^2}$, that coincides with the value obtained in [30]. Existence of two families of gap solitons has similarity with the ones observed in the cubic–quintic NLSE with a periodic potential [41]. From (30) we obtain the restriction

$$\sigma^2 < \frac{9\gamma_0^2}{128\epsilon(1 - \bar{\Omega})}.$$

For the estimations of the experiment with $\epsilon = 0.2$, $\bar{\Omega} = 0.6$, we obtain the restriction $\sigma^2 < 0.6$. The defocusing role of the nonlinearity management leads to the possibility of increasing the number of atoms in the bright gap soliton in comparison with a standard gap soliton. The low nonlinearity requires the larger number of atoms to support soliton solution. Taking $\sigma^2 = 0$ in the low amplitude solution, we obtain for the peak amplitude the value $Q_0 = 2(1 - \bar{\Omega})/3$, that reproduce the standard result for a gap soliton [42]. Expanding the solution (30) in series, we obtain

$$Q \approx Q_0 + \frac{64\epsilon(1 - \bar{\Omega})^2\sigma^2}{27\gamma_0^3}.$$

The number of atoms in the gap soliton is enhanced and the enhancement factor is proportional to the nonlinearity map strength σ^2. For typical values of parameters $V = 0.6E_R(\epsilon = 0.3)$, $h = 3.16\omega$, $f = 33$, $\omega = 10\omega_R(\sigma^2 = h^2/2\omega^2 = 5)$, $\bar{\Omega} = -1$, $\gamma_0 = 1$ we obtain double enhancement in the number of atoms. It means that for the experiment with ^{87}Rb [43] the number of atoms in a gap soliton ($N \sim 600$) can be increased by the nonlinearity management up to $N \sim 1200$. The increasing of number of atoms in the discrete breather of discrete nonlinear Schrödinger equation under weak nonlinearity management has been observed in numerical simulations [32].

6. Numerical simulations

In numerical simulations we proceed from the governing Gross–Pitaevskii equation (2). The problem is discretized in a standard way with the time step Δt and spatial step Δx so that terms u_j^k approximate $u(j\Delta x, k\Delta t)$. More specifically, in the approximation of Eq. (2) we have used the following implicit Crank–Nicholson-type scheme of second order accuracy in space and first order accuracy in time

$$\begin{aligned}\frac{\mathrm{i}(u_j^{k+1} - u_j^k)}{\Delta t} &= -\frac{1}{2\Delta x^2}[(u_{j-1}^{k+1} - 2u_j^{k+1} + u_{j+1}^{k+1}) \\ &\quad + (u_{j-1}^k - 2u_j^k + u_{j+1}^k)] + \epsilon\cos(2x_j)(u_j^k + u_j^{k+1}) \\ &\quad - \frac{1}{2}\gamma(t_k)|u_j^k|^2(u_j^k + u_j^{k+1}),\end{aligned} \tag{31}$$

where the strong nonlinearity management factor $\gamma(t)$ is defined by Eq. (2), $x_k = j\Delta x$ and $t_k = k\Delta t$. In our calculations the second term in Eq. (2) is chosen as $\gamma_1 = h\sin(\omega t)$. For this case $\sigma^2 = h^2/(2\omega^2)$.

Since our problem deals with nonlinear plane waves, periodic boundary conditions are imposed on the governing equation (2). Eq. (31) together with the boundary condition $u_0^{k+1} = u_N^{k+1}$ form a quasi tridiagonal set of equations for unknown u_j^{k+1}, $[j = 0, 1, 2 \ldots N]$ in a lattice of $N + 1$ points. The length of the lattice L is determined by the period of the periodic potential and value of the wave number for which the solution is sought. The set of these algebraic equations is solved by the modified vectorial sweep method. In actual calculations the typical space step Δx ranged from 0.01 to 0.005 and time step Δt from 0.005 to 0.001.

In calculations, the initial wave packet is constructed in the following way. At first slow component of the solution $w(x, t = 0)$ is taken in the form of Eq. (7) or Eq. (13) with Eq. (20), depending on the problem we consider. Then leaving only first term in Eq. (5) and making use of transformation Eq. (4) we obtain actual initial wave function $u(x, t = 0)$ used in computations.

In the simulation of the loop structure (see Fig. 4) and constructing the initial wave function, the position on the loop for given value of the wave number Q is determined by choosing a necessary value of the parameter f, which, in turn, is determined from the dispersion relation Eq. (16).

7. Conclusion

We have investigated the modulational instability and gap soliton formation in the media with Kerr nonlinearity and periodic potential. Such systems appear in the nonlinear optical media with Bragg grating and Bose–Einstein condensates in optical lattices under time-dependent Feshbach resonance management. We considered the case of strong management and showed that in the case of homogeneous Kerr media under NM the gain of MI is strongly suppressed, that explains the defocusing role of the NM and thus the stabilization of 2D and 3D attractive BEC by this method. We have studied the nonlinear dispersion relation in the case of NM and showed that the loop structure is essentially modified by the NM. The critical value of the strength of the NM is shown to exist in the MI regions. In the case of attractive condensate it means that above the threshold an attractive BEC behaves as repulsive. The NM leads to a new effect of enhancement of the number of atoms in the bright gap soliton. The enhancement factor is proportional to the strength of the management σ^2. We confirmed the predictions based on the analysis of the averaged GP equation by direct numerical simulations of the 1D GP equation.

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