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# On calculation of ohmic losses at metallic surface with sharp edges

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## Abstract

We discuss the applicability of the perturbation theory in electrodynamic problems where the local impedance boundary conditions are used to calculate the ohmic losses at the metallic surface. As an example, we examine a periodic grating formed from semi-infinite rectangular plates exposed to the s-polarized electromagnetic wave. Two different ways of calculation are presented: (i) the calculation of the reflection coefficient obtained with the aid of the perturbation theory (the impedance is the small parameter) and (ii) the direct calculation of the energy flux through the metallic surface. In the case (ii) to get the answer only the tangential magnetic field at the surface of a perfect conductor of the same geometry has to be known. The results (i) and (ii) differ noticeably. The same difficulty is inherent in all the problems where the metallic surface has rectangular grooves. We show that in this problem the standard first-order perturbation theory is not applicable. The point is that beginning from a number  $n$  the first corrections to the modal functions  $\phi_n$  used to calculate the fields, are of the same order as the zero-order modal functions (the impedance is equal to zero). Basing on the energy conservation law we show that the accurate value for the ohmic losses is obtained with the aid of the approach (ii). © 2002 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

In the present publication we would like to fix the readers attention on an unexpected situation arising when the perturbation theory is used for

calculation of the ohmic losses at the highly conducting metallic surfaces with sharp edges. All the calculations are performed in the framework of the local impedance boundary conditions applicability [1]. In other words, we assume that the penetration depth of an electromagnetic wave into the metal  $\delta$  is much less than the characteristic size of the surface contour, and the last is much less than the vacuum wavelength  $\lambda = 2\pi c/\omega$ ,  $\omega$  is the frequency of the incident wave.

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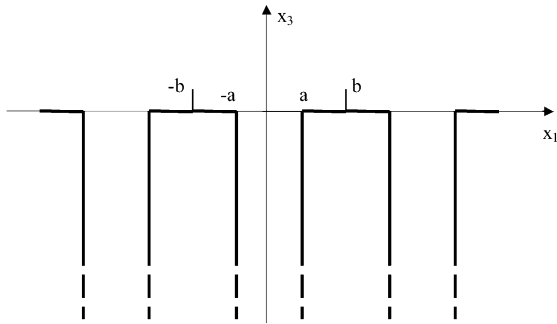


Fig. 1. The grating configuration;  $2b$  is the period,  $2a$  is the width of the grooves throats.

As an example we examine the one-dimensional metallic periodic grating ( $2b$  is the period) formed from semi-infinite rectangular plates (Fig. 1). The grating is illuminated by s-polarized light. In our example

$$\delta \ll 2b \ll \lambda, \quad (1)$$

This allows us to restrict ourselves to solution of the Maxwell equations in the region external with respect to a metal. At the metallic surface the boundary conditions for the fields are

$$\mathbf{E}_t = \hat{\zeta}[\mathbf{n}, \mathbf{H}_t], \quad (2)$$

$\mathbf{E}_t$  and  $\mathbf{H}_t$  are tangential components of electric and magnetic vectors at the metallic surface,  $\mathbf{n}$  is the unit external normal vector to the surface. The two-dimensional tensor  $\hat{\zeta}$  is the surface impedance tensor. The real and imaginary parts of the impedance define the dissipated energy and the phase shift of the reflected electromagnetic wave, respectively (see, for example, [1]). In what follows we suppose that in Eq. (2) the impedance  $\hat{\zeta}$  is an ordinary multiplying operator, and it is the same as the impedance of a perfectly flat surface of the same metal. This form of the boundary conditions is due originally to Leontovich [2]. Often these boundary conditions are called the Leontovich boundary conditions (LBC).

Of course, LBC are not rigorous ones. In general the relation between  $\mathbf{E}_t$  and  $\mathbf{H}_t$  is nonlocal. Moreover, when the metallic surface is not flat, even the local terms in the expression for the components of the surface impedance tensor have corrections depending on the shape of the metallic

surface (see, for example, [3,4].) However, when the inequalities (1) are fulfilled, the leading term in the expression for  $\hat{\zeta}$  is equal to the impedance of the perfectly flat metallic surface. Having in mind to calculate the leading term in the expression for the ohmic losses we restrict ourselves to this approximation only.

In what follows we assume that the metal is an isotropic one. Then the surface impedance tensor is  $\zeta_{\alpha\beta} = \zeta \delta_{\alpha\beta}$ . Under the conditions of normal skin effect [1]  $\zeta = \sqrt{\omega/8\pi\sigma}(1-i)$ , where  $\sigma$  is the conductivity of the metal. Under the conditions of anomalous skin effect the expression for the surface impedance is more complex (see, for example, [5]). We do not specify the character of skin effect. We only take into account that in the order of magnitude  $|\zeta| \sim \delta/\lambda$ . For good metals this ratio is extremely small.

Note, LBC are valid up to the terms of the order of  $|\zeta|$  only. Consequently, when calculating the fields the terms of the higher order (for example, of the order  $b|\zeta|/\lambda$ ,  $|\zeta|^2$  and so on) must be omitted as far as they exceed the accuracy allowed by the boundary conditions.

Strictly speaking, inequalities (1) provide the possibility to use LBC if the surface of the metal is rather smooth. The presence of sharp edges makes this possibility questionable. Of course, in real life all the sharp edges are rounded ones. Undoubtedly, the boundary conditions (2) can be used only if the smoothing characteristic size is greater than  $\delta$  too. However, it is rather difficult to take the smoothing regions into account accurately. Therefore rather often when making use of Eq. (2), one assumes that a smoothing changes the results only slightly.

It must be noted also that the grating depicted in Fig. 1 is infinitely long in the  $x_3$  direction. At first it seems that the boundary conditions (2) are not valid in this case. To justify ourselves we take into account that in the s-polarization state the fields decay exponentially inside the grooves when the distance from the throat of the groove increases. As a result the fields in the vicinity of infinitely deep grooves and the grooves of finite, but rather large depth  $h$  ( $h > 2a$ ) are practically the same. For such grooves the smoothing of the contour allowing us to use the boundary conditions (2) can be done rather easily.

After all these preliminary remarks we are faced with a rigorously set mathematical problem: suppose that a plane s-polarized wave  $E^i = E_0 e^{-i(kx_3 + \omega t)}$  ( $k = \omega/c$ ) is normally incident upon the surface depicted in Fig. 1. At the metallic surface we have the boundary conditions (2) that are valid up to the terms of the order  $|\zeta|$ . We impose an outgoing wave or exponentially decaying boundary conditions on the reflected fields. Our problem is to define the ohmic losses. In other words, we need to calculate the energy flux through the surface of the metal.

In the region outside the metal let us introduce the time-averaged Poynting vector that is the density of the time-averaged energy flux

$$\bar{\mathbf{S}} = (c/8\pi) \operatorname{Re}[\mathbf{E}\mathbf{H}^*] \quad (3)$$

(the star means the complex conjugate). According to the energy conservation law [1]

$$\operatorname{div} \bar{\mathbf{S}} = 0. \quad (4)$$

The consequence of the last equation is the equality of the integrals

$$\int_{S_\infty} \bar{\mathbf{S}} d\mathbf{f} = \int_{S_m} (\bar{\mathbf{S}} \mathbf{n}) d\mathbf{f}, \quad (5)$$

where the integral in the left-hand side is taken over the surface infinitely distant from the metal. The integral in the right-hand side is taken over the surface of the metal. This integral defines the energy flux penetrating through the metallic surface.

Infinitely far from the metallic surface ( $x_3 \gg \lambda$ ) the only nonzero component of the Poynting vector is expressed in terms of the reflection coefficient  $R$ :  $\bar{S}_3 = -(c/8\pi)|\mathbf{E}_0|^2(1 - R)$ . Consequently, when the reflection coefficient is known, we can calculate the ohmic losses with the aid of the left-hand side of Eq. (5).

On the other hand we can calculate the energy flux through the surface of the metal directly. With regard to the boundary conditions (2) at the metallic surface the normal component of the time-averaged Poynting vector is  $\bar{S}_n = (c/8\pi) \operatorname{Re} \zeta |\mathbf{H}_t|^2$ . The last expression can be simplified.

Since  $|\zeta| \ll 1$ , the magnetic vector  $\mathbf{H}_t$  at the surface of a good metal is approximately the same as the magnetic vector  $\mathbf{H}_t^{(0)}$  at the surface of the perfect conductor (the conductivity is infinitely

large and, consequently, the impedance  $\zeta = 0$ ) of the same geometry [1]. Then within the allowed accuracy we must rewrite  $\bar{S}_n$  as  $\bar{S}_n = (c/8\pi) \operatorname{Re} \zeta |\mathbf{H}_t^{(0)}|^2$ .

Let us introduce the energy  $Q_d$  dissipating in the metal per unit length in the  $x_2$  direction and per one period. According to the previous argumentation and making use of Eq. (5) we can write two equivalent expressions for  $Q_d$ :

$$Q_d = \frac{c}{8\pi} \operatorname{Re} \zeta \int_L |\mathbf{H}_t^{(0)}|^2 dL, \quad (6a)$$

the integral is taken over the contour of the surface relating to one period, and

$$Q_d = 2b|\mathbf{E}_0|^2 \frac{c}{8\pi} [1 - R]. \quad (6b)$$

Note, to calculate  $Q_d$  with the aid of Eq. (6a) we need to know the magnetic vector  $\mathbf{H}_t^{(0)}$  at the surface of the perfect conductor only. On the other hand, when we make use of Eq. (6b) we need to know the reflection coefficient  $R$ . Since for a perfect conductor  $R = 1$ , to calculate  $R$  we need the solution of the complete electrodynamic problem in the region over the real metal.

A lot of authors calculated the fields above perfectly conducting gratings (see, for example, [6]). It can be done with the aid of different methods [7]. For example, one can present the fields inside the grooves as series of modal functions that are the solutions of the Maxwell equations. Above the grooves the usual Rayleigh plane-wave representation is adopted. Then applying the boundary conditions for perfect conductors on the surface and the continuity conditions across the central slit results in a matrix equation for the modal amplitudes. This method makes use of the standard Fourier analysis. There are no doubts with respect to the results obtained with the aid of this method. The other way is based on the solution of the Wiener–Hopf equation. Both methods give the same result.

In the framework of LBC we can use the same methods when calculating the fields above highly conducting metallic gratings. However, to be sure that we do not fall outside the limits allowed by the boundary conditions (2), we have to exploit the perturbation theory based on the inequalities (1).

In what follows we show that in this problem the standard perturbation theory is not applicable. More accurately, we show that the dissipated energy  $Q_d$  calculated with the aid of Eq. (6a) differs from the result obtained with the aid of Eq. (6b) after calculation of the reflection coefficient  $R$  in the framework of the perturbation theory. This discrepancy leads to violation of the energy conservation law.

We would like to stress once more, the field  $\mathbf{H}_t^{(0)}$  entering Eq. (6a) can be calculated with high accuracy. Then, in the framework of our approximation Eq. (6a) provides the correct result for the ohmic losses. Next, as it is shown below, the method of calculation of the fields above the real metal (the impedance  $\zeta \neq 0$ ) is much alike the one used when calculating the fields above the perfect grating. However, to calculate the difference  $1 - R$  up to the terms of the order of  $\zeta$ , we have to use the perturbation theory with respect to the small parameters of the problem. Consequently, the source of the obtained discrepancy can be the perturbation theory only.

We faced this problem when calculating the ohmic losses on the lamellar grating with finite depth of the grooves exposed to the s-polarized light [8]. In [8] we used Eq. (6a). We found out that for rather deep grooves our result was approximately 20% less than the result obtained by Vainshtein et al. [9]. In the same frequency region defined by Eq. (1) these authors used LBC to examine the one-dimensional grating with infinitely deep grooves and very thin plates ( $(b - a)/b \ll 1$ ). They based their solution on the application of the Wiener–Hopf equation and calculated the reflection coefficient. Then taking into account that according to Eq. (1) the parameter  $|\zeta|/kb \ll 1$ , they used the perturbation theory and calculated the difference  $1 - R$  entering Eq. (6b) up to the terms of the order of  $\zeta$ .

In the s-polarization state there are no physical reasons of different results for very deep and infinitely deep grooves. Therefore, trying out to verify our approach, we examined the grating with infinitely deep grooves. We reproduced the perturbation theory calculation of Vainshtein et al. [9] with the aid of the modal function method adopted in [8]. The results obtained with the aid of the per-

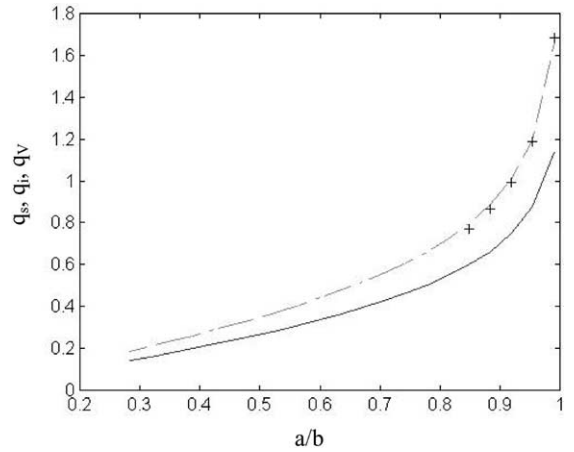


Fig. 2. The dimensionless values  $q_s = (\pi Q_d^{(\text{sur})}/cb)|E_0|^2$  (solid line) and  $q_i = (\pi Q_d^{(\infty)}/cb)|E_0|^2$  (dashed line) as functions of the ratio  $a/b$ . The crosses show the values of  $q_v = (\pi Q_d^{(V)}/cb)|E_0|^2$  calculated with the aid of Eq. (26).

turbation theory were the same as in the work of Vainshtein and his co-authors. Then we repeated the calculation with the aid of Eq. (6a). Comparing the results we found the same twenty percents difference (see Fig. 2).

The result of this analysis is presented in this publication. From our point of view our calculations show that in this problem the application of the standard perturbation theory provide a regular error when calculating the fields above the grating. In what follows we show the place where the standard perturbation theory failed.

It was not the goal of this work to develop a regular perturbation theory allowing us to calculate the fields above lamellar conducting gratings correctly. The main results of our investigation can be formulated as follows:

- (i) For highly conducting lamellar gratings the correct result for ohmic losses  $Q_d$  is given by Eq. (6a).
- (ii) The coincidence of the result for  $Q_d$  obtained with the aid of Eqs. (6a) and (6b) can be used as a proof of correct calculation of the fields above the metallic surface.

The organization of this paper is as follows. In Section 2 we present the modal function approach allowing us to calculate the fields above the metallic grating. In Section 3 we describe the results obtained in the framework of the standard

perturbation theory. In Section 4 we calculate the ohmic losses and discuss the reasons why the standard perturbation theory fails. Concluding remarks are given in Section 5. In Appendix A we show that for an arbitrary value of  $kb < 1$ , the formulae used when calculating the fields above the infinitely conducting grating provide the total reflection of the incident wave (the reflection coefficient  $R = 1$ ).

## 2. Calculation of the fields

Let the one-dimensional periodic metallic grating depicted in Fig. 1 be exposed to the s-polarized normally incident electromagnetic wave  $\mathbf{E}^i = (0, E_0, 0) \exp[-ikx_3]$ . We assume that the first period corresponds to  $-b < x_1 < b$ . For simplicity we assume that on the flat parts  $x_3 = 0$  of the surface (for the first period they are  $-b < x_1 < -a$  and  $a < x_1 < b$ ) the local impedance is equal to zero. This simplification, being not very significant, only allows us to shorten the calculations that are rather lengthy and tedious. On the inner surfaces of the semi-infinite grooves the boundary conditions (2) are fulfilled.

The inequalities (1) provide us with three small parameters. Namely,

$$|\zeta| \sim \frac{\delta}{\lambda} \ll 1, \quad kb \ll 1 \quad \text{and} \quad \frac{\delta}{b} \sim \frac{|\zeta|}{kb} \ll 1. \quad (7)$$

At first, we will not pay attention to inequalities (7). In what follows we obtain the solution of the problem without using the perturbation theory. We assume only that  $kb < 1$ .

Let the superscripts  $+$  and  $-$  denote the fields in the half-space  $x_3 > 0$  and in the central groove ( $-a < x_1 < a$ ), respectively. With regard to the surface periodicity we adopt the representation

$$E^+(x_1, x_3) = E_0 \left\{ e^{-ikx_3} + \sum_{q=-\infty}^{\infty} e_q^+ e^{i(\pi q x_1/b + \alpha_q x_3)} \right\}, \quad (8)$$

for the electric field above the metal. In Eq. (8)  $\alpha_0 = k$  and  $\alpha_q = i\sqrt{(\pi q/b)^2 - k^2}$ , if  $q \neq 0$ . Since on the plain parts  $x_3 = 0$  of the surface the local impedance is equal to zero,  $E_2^+(x_1, 0) = 0$ , when  $a < |x_1| < b$ .

Inside the central groove ( $|x_1| \leq a$ ,  $x_3 < 0$ ) we present  $E^-(x_1, x_3)$  as a series of modal functions  $\phi_n(x_1, x_3)$  that are the solutions of the Maxwell equations incorporating the boundary conditions (2) on the vertical sides of the groove.<sup>1</sup> Let  $B_n$  be the coefficients of this series.

Separating the variables we write the modal functions  $\phi_n(x_1, x_3)$  as

$$\begin{aligned} \phi_n(x_1, x_3) &= \psi_n(x_3) \varphi_n(x_1), \quad \psi_n(x_3) \\ &= \exp \left[ \frac{x_3}{a} \sqrt{\gamma_n^2 - (ka)^2} \right]. \end{aligned} \quad (9a)$$

It is easy to see that the functions  $\varphi_n(x_1)$  are given by different expressions for even and odd numbers  $n$ :

$$\begin{aligned} \varphi_{2n}(x_1) &= \frac{\sin(\gamma_{2n} x_1/a)}{\cos \gamma_{2n}}, \\ \varphi_{2n-1}(x_1) &= -\frac{\cos(\gamma_{2n-1} x_1/a)}{\sin \gamma_{2n-1}}, \end{aligned} \quad (9b)$$

with  $\gamma_{2n}$  and  $\gamma_{2n-1}$  being the solutions of dispersion equations

$$\frac{\tan \gamma_{2n}}{\gamma_{2n}} = -\frac{i\zeta}{ka}, \quad \frac{\cot \gamma_{2n-1}}{\gamma_{2n-1}} = \frac{i\zeta}{ka}. \quad (9c)$$

The set of modal functions  $\phi_n(x_1, x_3)$  has a lot of interesting properties. We do not discuss them here, but refer the reader to the review work of Lyubarskii [10]. Here let us note only that the functions  $\varphi_n(x_1)$  are not orthogonal ones: their scalar product  $(\varphi_m, \varphi_n) = (1/2a) \int_{-a}^a dx_1 \varphi_m^*(x_1) \varphi_n(x_1)$  (the star means the complex conjugate) does not equal to zero when  $m \neq n$ . However, the functions  $\varphi_n(x_1)$  with different subscripts are orthogonal:

$$\frac{1}{2b} \int_{-a}^a dx_1 \varphi_m(x_1) \varphi_n(x_1) = I_{mm} \delta_{mn}, \quad (10a)$$

where

$$\begin{aligned} I_{2m,2m} &= \frac{a}{2b} \left\{ \frac{1}{\cos^2 \gamma_{2m}} + \frac{i\zeta}{ka} \right\}, \\ I_{2m-1,2m-1} &= \frac{a}{2b} \left\{ \frac{1}{\sin^2 \gamma_{2m-1}} + \frac{i\zeta}{ka} \right\}. \end{aligned} \quad (10b)$$

<sup>1</sup> An analogous method was repeatedly used when calculating the fields above infinitely conducting rectangular gratings (see, for example, [6]).

We use Eqs. (10a) and (10b) when deriving the matrix equation for the modal coefficients (see below Eq. (13)).

Now with regard to Eqs. (9a)–(9c) we write the field  $E^-(x_1, x_3)$  ( $|x_1| < a$ ,  $x_3 < 0$ ) as

$$E_2^-(x_1, x_3) = E_0 \left\{ \sum_{n=0}^{\infty} B_{2n} \psi_{2n}(x_3) \varphi_{2n}(x_1) + \sum_{n=1}^{\infty} B_{2n-1} \psi_{2n-1}(x_3) \varphi_{2n-1}(x_1) \right\}. \quad (11)$$

Applying the boundary conditions on the plateaus  $x_3 = 0$  and the continuity conditions across the central slit, we obtain two linked matrix equations allowing us to determine both the amplitudes  $e_q^+$  and the modal amplitudes  $B_n$ . It is easy to see that all the amplitudes  $B_{2n}$  are equal to zero. In place of the amplitudes  $B_{2n-1}$  it is convenient to introduce the coefficients  $X_{2n-1}$ ,

$$i(kb)X_{2n-1} = \frac{2\gamma_{2n-1}}{\pi} B_{2n-1}. \quad (12)$$

Now we eliminate the amplitudes  $e_q^{(+)}$  from the set of matrix equations to obtain the matrix equation for the coefficients  $X_{2n-1}$ . Our calculations results in

$$\sum_{m=1}^{\infty} X_{2m-1} A_{2m-1, 2l-1} + \left( \frac{\pi b}{2a} \right)^2 \left( \frac{\pi}{2\gamma_{2l-1}} \right) \times \sqrt{1 - (ka/\gamma_{2l-1})^2} I_{2l-1, 2l-1} X_{2l-1} = 2W_{2l-1}. \quad (13)$$

If we set  $\mu = 2a/b$ , the matrix  $A_{2m-1, 2l-1}$  in Eq. (13) is

$$A_{2m-1, 2l-1} = \sum_{q=1}^{\infty} q\mu \sqrt{1 - (kb/\pi q)^2} \times D_{2m-1, 2l-1}(q) - i \frac{\mu}{\pi} (kb) W_{2l-1} W_{2m-1}, \quad (14a)$$

$$D_{2m-1, 2l-1}(q) = 2 \left( \frac{\pi}{\mu} \right)^2 C_{2l-1, q} C_{2m-1, q}, \quad (14b)$$

$$W_{2l-1} = \left( \frac{\pi}{2\gamma_{2l-1}} \right)^2, \quad (14b)$$

$C_{2l-1, q}$  stands for the integral

$$\left( \frac{2\gamma_{2l-1}}{\pi} \right) C_{2l-1, q} = \frac{1}{2b} \int_{-a}^a dx_1 e^{-i\pi q x_1/b} \varphi_{2l-1}(x_1). \quad (14c)$$

The explicit form of  $C_{2l-1, q}$  is

$$C_{2l-1, q} = \left( \frac{\mu\pi}{4} \right) \frac{\cos \omega_q - \omega_q \sin[\omega_q \cot \gamma_{2l-1}/\gamma_{2l-1}]}{\omega_q^2 - \gamma_{2l-1}^2}, \quad (14d)$$

$$\omega_q = \frac{\pi q \mu}{2},$$

and  $I_{2l-1, 2l-1}$  is defined by Eq. (10b).

We can calculate the coefficients  $X_{2n-1}$  solving numerically the infinite set of Eq. (13). To calculate the reflection coefficient  $R$  we take into account that since  $kb < 1$ , there is only one reflected wave. In accordance with Eq. (8) its dimensionless amplitude is  $e_0^+$  and  $e_q^+$  with  $q \neq 0$  represent evanescent waves. Omitting intermediate calculations, we present the expression for  $e_0^+$  in terms of the coefficients  $X_{2l-1}$ :

$$e_0^+ = -1 - i(kb) \frac{\mu}{\pi} \sum_{l=1}^{\infty} X_{2l-1} W_{2l-1}. \quad (15)$$

Then  $R = |e_0^+|^2$ .

It seems that in such a way we solve the problem and can calculate the ohmic losses substituting the found reflection coefficient into Eq. (6b). However, as it was aforementioned, LBC are true up to the terms of the order of  $\zeta$  only. Consequently, to obtain a reliable answer for the ohmic losses we need to extract the terms of the order of  $\zeta$  from the expression for  $1 - R$ . And this is just the place where the perturbation theory has to be used.

### 3. The result of the first-order perturbation theory

The zero-order term of our perturbation theory corresponds to  $\zeta = 0$ . To denote it we use the superscript (0). Then in accordance with Eqs. (9a)–(9c) we have

$$\gamma_{2n-1}^{(0)} = \frac{\pi(2n-1)}{2}, \quad (16)$$

$$\varphi_{2n-1}^{(0)}(x_1) = (-1)^n \cos(\pi(2n-1)x_1/2a).$$

The functions  $\varphi_{2n-1}^{(0)}(x_1)$  are cosines that can be used as basic functions for the standard Fourier analysis.

Next, from Eq. (9c) it follows that the eigenvalues  $\gamma_{2l-1}$  depend not on  $\zeta$  itself, but on the parameter  $\varepsilon = \zeta/(ka)$ . Since  $ka \ll 1$ , it is evident that

$|\zeta| \ll |\varepsilon| \ll 1$ . With regard to Eq. (9c) we obtain that up to the terms of the order of  $\varepsilon$ :

$$\gamma_{2l-1} = \gamma_{2l-1}^{(0)}(1 - i\varepsilon). \quad (17)$$

Moreover, the impedance  $\zeta$  enters the matrix Eq. (13) only through their dependence on  $\gamma_{2l-1}$ . Thus, from Eq. (13) it follows that the coefficients  $X_{2n-1}$  depend not on all the three small parameters listed in Eq. (7), but on the two small parameters only which are  $kb$  and  $\varepsilon$ . When  $\varepsilon = 0$  the coefficients  $X_{2n-1}^{(0)}(kb)$  define the fields above the infinitely conducting grating. With regard to Eq. (13) and Eqs. (14a)–(14d) it is easy to see that the matrix equation for  $X_{2n-1}^{(0)}(kb)$  is

$$\sum_{m=1}^{\infty} X_{2m-1}^{(0)} A_{2m-1,2l-1}^{(0)} + \frac{1}{\mu} \left( \frac{\pi}{2} \right)^2 \times \sqrt{1 - (kb\mu/\pi(2l-1))^2} \frac{X_{2l-1}^{(0)}}{2l-1} = \frac{2}{(2l-1)^2}, \quad (18a)$$

where

$$A_{2m-1,2l-1}^{(0)}(kb) = \sum_{q=1}^{\infty} q\mu \sqrt{1 - (kb/\pi q)^2} D_{2m-1,2l-1}^{(0)}(q) - i \frac{\mu}{\pi} (kb) \frac{1}{(2m-1)^2(2l-1)^2}, \quad (18b)$$

and

$$D_{2m-1,2l-1}^{(0)}(q) = \frac{1 + \cos 2\omega_q}{d_m(q)d_l(q)}, \quad (18c)$$

$$d_m(q) = (q\mu)^2 - (2m-1)^2,$$

$\omega_q$  is defined in Eq. (14d).

Suppose that we know the solution of Eq. (18a). To take into account the finite conductivity of the metal, we need to make use of the perturbation theory when solving the matrix Eq. (13) and calculate the first corrections to  $X_{2n-1}^{(0)}(kb)$  in the small parameter  $\varepsilon$ .

According to Eq. (11) and the definition of the coefficients  $X_{2n-1}$ , Eq. (12), the same corrections provide the electric field  $E^-$  up to the terms of the order of  $\zeta$ . Since LBC do not allow us to calculate the fields up to the terms higher than  $|\zeta|$ , when calculating the coefficients  $X_{2n-1}$  the terms of the order higher than  $\varepsilon$  (for example, of the order of  $\zeta$ ) must be omitted.

With the aid of Eq. (17) we expand all the coefficients in Eq. (13) up to the terms of the order of  $\varepsilon$ . Then with regard to Eqs. (18a)–(18c) after tedious, but straightforward algebra we obtain that up to the terms of the order of  $\varepsilon$  the solution of Eq. (13) has the form

$$X_{2n-1}(kb; \varepsilon) = X_{2n-1}^{(0)}(kb) + 2i\varepsilon \left( X_{2n-1}^{(0)}(0) - V_{2n-1} \right), \quad (19)$$

where the coefficients  $V_{2n-1}$  satisfy the matrix equation

$$\sum_{m=1}^{\infty} V_{2m-1} A_{2m-1,2l-1}^{(0)}(0) + \frac{1}{\mu} \left( \frac{\pi}{2} \right)^2 \frac{V_{2l-1}}{2l-1} = \sum_{m=1}^{\infty} X_{2m-1}^{(0)}(0) S_{2m-1,2l-1}. \quad (20a)$$

Here the matrix  $A_{2m-1,2l-1}^{(0)}(0)$  is defined by Eq. (18c) when  $kb = 0$ , and the matrix  $S_{2m-1,2l-1}$  is

$$S_{2m-1,2l-1} = \sum_{q=1}^{\infty} (q\mu) T_{2m-1,2l-1}(q) + \frac{1}{\mu} \left( \frac{\pi}{2} \right)^2 \frac{1}{2l-1} \delta_{m,l}, \quad (20b)$$

$$T_{2m-1,2l-1}(q) = -\frac{1}{d_m(q)d_l(q)} \left[ (1 + \cos 2\omega_q) \times \left( \frac{(2m-1)^2}{d_m} + \frac{(2l-1)^2}{d_l} \right) + \omega_q \sin 2\omega_q \right]. \quad (20c)$$

Again  $\omega_q = \pi q\mu/2$  and  $d_m$  is defined in Eq. (18c).

If in consecutive order we solve Eqs. (18a) and (20a), we obtain the coefficients  $X_{2n-1}(kb, \varepsilon)$  up to the terms linear in the parameter  $\varepsilon$ .

#### 4. Calculation of ohmic losses

At first, let us calculate the dissipated energy  $Q_d$  with the aid of Eq. (6a). We mark the result of this calculation with the superscript (sur). We remind that we set  $\zeta = 0$  at the flat parts  $x_3 = 0$  of our surface. Then in Eq. (6a) the integration is performed over the vertical facets of the central groove, where the tangential magnetic vector is

$H_3^{(0)}(\pm a, x_3)$  ( $x_3 < 0$ ). Now we can rewrite Eq. (6a) as

$$Q_d^{(\text{sur})} = \frac{c}{4\pi} \text{Re} \zeta \int_{-\infty}^0 |H_3^{(0)}(a, x_3)|^2 dx_3. \quad (21)$$

We take into account that since the right-hand side of Eq. (21) is proportional to  $\zeta$ , within the accuracy of our approach the field  $H_3^{(0)}(a, x_3)$  in the integrand has to be known only up to the terms independent of the small parameter  $kb$ . Then with regard to the Maxwell equations, Eq. (11) ( $B_{2n} = 0$ ), and Eq. (12) we have

$$H_3^{(0)}(a, x_3)|_{kb=0} = -\frac{\pi}{\mu} E_0 \sum_{l=1}^{\infty} X_{2l-1}^{(0)}(0) \times \exp[\pi(2l-1)x_3/2a], \quad (22)$$

and, consequently,

$$Q_d^{(\text{sur})} = cb|E_0|^2 \frac{\text{Re} \zeta}{8\mu} \sum_{l,m=1}^{\infty} \frac{X_{2l-1}^{(0)}(0)X_{2m-1}^{(0)}(0)}{l+m-1}. \quad (23)$$

Now let us calculate the energy  $Q_d$  with the aid of Eq. (6b). We mark this result with the superscript  $(\infty)$ . Since the reflection coefficient  $R = |e_0^+|^2$ , to calculate  $Q_d^{(\infty)}$  we use Eq. (15). Then

$$Q_d^{(\infty)} = -\frac{cb|E_0|^2}{4\pi} Q, \\ Q = i(kb) \frac{\mu}{\pi} \sum_{l=1}^{\infty} (X_{2l-1} W_{2l-1} - X_{2l-1}^* W_{2l-1}^*) \\ + (kb)^2 \left(\frac{\mu}{\pi}\right)^2 \sum_{l,m=1}^{\infty} X_{2l-1}^* X_{2m-1} W_{2l-1}^* W_{2m-1}. \quad (24)$$

When calculating  $Q_d^{(\infty)}$  it is necessary to have in mind, that the dissipated energy has to be equal to zero for a perfectly conducting surface. In other words, when  $\zeta = 0$  and, consequently,  $X_{2l-1} = X_{2l-1}^{(0)}(kb)$ ,  $Q_d^{(\infty)}$  vanishes for an arbitrary value of  $kb < 1$ . In Appendix A we show that this equality is fulfilled for  $X_{2l-1}^{(0)}(kb)$  that are solutions of Eq. (18a).

As a result, substituting the expression (19) in Eq. (24) we obtain  $Q_d^{(\infty)}$  in terms of the coefficients  $X_{2l-1}^{(0)}(0)$  and  $V_{2l-1}$ .

$$Q_d^{(\infty)} = \frac{2cb|E_0|^2}{\pi^2} \text{Re} \zeta \sum_{l=1}^{\infty} \frac{2X_{2l-1}^{(0)}(0) - V_{2l-1}}{(2l-1)^2}. \quad (25)$$

This equation is valid up to the terms of the order of  $\zeta$ .

Our next step was numerical calculation of the coefficients<sup>2</sup>  $X_{2l-1}^{(0)}(0)$ . With this result in hand we evaluated the right-hand side of the matrix Eq. (20a) and calculated the coefficients  $V_{2l-1}$ . Finally, we calculated numerically  $Q_d^{(\text{sur})}$  and  $Q_d^{(\infty)}$  with the aid of Eqs. (23) and (25), respectively.

In Fig. 2 we plot the calculated dimensionless values  $q_i = (\pi Q_d^{(\infty)}/cb|E_0|^2) \text{Re} \zeta$  and  $q_s = (\pi Q_d^{(\text{sur})}/cb|E_0|^2) \text{Re} \zeta$  as functions of the ratio  $a/b$ . We see that for all  $a/b$  the results for  $q_i$  and  $q_s$  differ noticeably.

In Fig. 2 we also display the results for the ohmic losses obtained by Vainshtein et al. [9]. As it was mentioned in Section 1, they examined the same grating formed from thin plates ( $(b-a)/b \ll 1$ ). In [9] the way of calculation was much alike the one used in the present work when calculating  $Q_d^{(\infty)}$ , however some other modal functions were used, and the problem was reduced to solution of the Wiener–Hopf equation. Supposing that the impedance of the flat parts  $x_3 = 0$  of the metallic surface was equal to zero, the authors of [9] obtained an elegant analytic formulae for the leading term of the ohmic losses

$$Q_d^{(V)} = \frac{cb|E_0|^2}{\pi^2} \text{Re} \zeta \ln \left( \frac{2a}{b-a} \right), \quad (b-a)/b \ll 1. \quad (26)$$

(The superscript  $(V)$  indicates that this is the result of [9].)

We see, that our results for  $Q_d^{(\infty)}$  are in good agreement with the values  $Q_d^{(V)}$ . This confirms our calculations made in the framework of the perturbation theory. However, neither  $Q_d^{(V)}$ , nor  $Q_d^{(\infty)}$  coincide with  $Q_d^{(\text{sur})}$  that is the true energy flux penetrating through metallic surface. Consequently, we conclude that the obtained result for  $Q_d^{(\infty)}$  (the same as  $Q_d^{(V)}$ ) is improper, since it leads to violation of the energy conservation law.

<sup>2</sup> The computational procedure used in the present work when calculating the coefficients  $X_{2l-1}^{(0)}(0)$  is just the same as the one used in [8]. So we refer the readers to the work [8].



#### 4.1. Inefficiency of the standard perturbation theory

We think that the source of this discrepancy is application of the standard perturbation theory. The point is that if  $\zeta \neq 0$  the set of modal functions  $\phi_{2n-1}(x_1, x_3)$  (see Eqs. (9a)–(9c)), used when calculating the reflection coefficient, being complete and minimal still does not form a stable basis. As a result, expansions in this set can be very sensitive to perturbations (see, for example, [10]).

To show this, let us note that according to Eq. (16) the zero-order eigenvalues  $\gamma_{2n-1}^{(0)}$  are equidistant numbers:  $\gamma_{2n+1}^{(0)} - \gamma_{2n-1}^{(0)} = \pi$ . However, the distance between  $\gamma_{2n-1}$  defined by Eq. (17) and  $\gamma_{2n-1}^{(0)}$  increases linearly when  $n$  increases. If, for example,  $\zeta = \zeta_0(1 - i)$  (that is true for homogeneous solids under the conditions of normal skin effect), we have

$$\frac{|\gamma_{2n-1} - \gamma_{2n-1}^{(0)}|}{\beta} = \frac{\pi}{\sqrt{2}}(2n - 1), \quad \beta = \zeta_0/ka. \quad (27)$$

Now let us calculate the eigenfunctions  $\varphi_{2n-1}(x_1)$  substituting  $\gamma_{2n-1}$  from Eq. (17) in Eq. (9b). The result of this calculation is

$$\begin{aligned} \varphi_{2n-1}(x_1) = & (-1)^n \times \left\{ \cosh \beta \gamma_{2n-1}^{(0)} \frac{x_1}{a} \cos(1 + \beta) \gamma_{2n-1}^{(0)} \frac{x_1}{a} \right. \\ & \times \frac{x_1}{a} - i \sinh \beta \gamma_{2n-1}^{(0)} \frac{x_1}{a} \sin(1 + \beta) \gamma_{2n-1}^{(0)} \frac{x_1}{a} \left. \right\} \\ & / \left\{ \cos \beta \gamma_{2n-1}^{(0)} \cosh \beta \gamma_{2n-1}^{(0)} \right. \\ & \left. - i \sin \beta \gamma_{2n-1}^{(0)} \sinh \beta \gamma_{2n-1}^{(0)} \right\}. \quad (28) \end{aligned}$$

We see, that  $\varphi_{2n-1}(x_1)$  is a complex function, whose real and imaginary parts are of the same order. The difference between  $\varphi_{2n-1}(x_1)$  and  $\varphi_{2n-1}^{(0)}(x_1)$  is small only if  $n \ll 1/\pi\beta$ . When  $n \gg 1/\pi\beta$  the eigenfunctions  $\varphi_{2n-1}(x_1)$  do not resemble the non-perturbed eigenfunctions  $\varphi_{2n-1}^{(0)}(x_1)$  at all. In Fig. 3 for  $x_1 = 0$  we plot the ratios  $\text{Re} \varphi_{2n-1}(0)/\varphi_{2n-1}^{(0)}(0)$  and  $\text{Im} \varphi_{2n-1}(0)/\varphi_{2n-1}^{(0)}(0)$  for two values of  $\beta$ :  $\beta = 0.05$  and  $\beta = 0.01$ .

We must also take into account that in the s-polarization state the strength of the magnetic field in the immediate vicinity of a rectangular two-dimensional wedge increases significantly (see, for example, [7]). When the fields are represented by the expressions of the type (11), to describe the

fields near the corner points correctly we need to provide a special asymptotic behavior of the coefficients  $X_{2n-1}$  when  $n \rightarrow \infty$  (see [7]). For  $\zeta = 0$  the coefficients  $X_{2n-1}^0 \sim 1/(2n - 1)^{2/3}$  when  $n \rightarrow \infty$  (see [8]).

With regard to all the aforementioned arguments we take for granted that in the s-polarization state the perturbation theory presented in Section 3 is not applicable when LBC are used to calculate the fields above one-dimensional periodic rectangular metallic gratings.

#### 5. Conclusions

In this section we summarize the results of our analysis. We would like to remind that here we examined the case of the s-polarization only. We put aside the p-polarization state. In the end of this section we'll add some remarks relating to the p-polarization state.

From our point of view, the main result of this publication is the demonstration of the fact that the standard perturbation theory can fail if in the framework of LBC one try to use it when calculating the electromagnetic fields above periodic metallic gratings. We examined the case of rectangular infinitely deep grooves. However, it is quite possible that the same problems arise when the shape of the grooves is, for example, a triangular one, or, more general, when the surface profile has sharp edges.

Our calculations showed that for rough surfaces the results for the reflection coefficients obtained with the aid of the perturbation theory (the impedance  $\zeta$  is a small parameter) are not always reliable. On the other hand, there are no doubts that the dissipated energy calculated with the aid of Eq. (6a) defines the ohmic losses accurately. Thus, we can define the reflection coefficient comparing Eqs. (6a) and (6b). Then

$$R = 1 - \frac{\text{Re} \zeta}{2b|E_0|^2} \int_L |\mathbf{H}_t^{(0)}|^2 dL, \quad (29)$$

where the tangential magnetic field  $\mathbf{H}_t^{(0)}$  is calculated up to the terms independent of  $kb$ . This result is true in the frequency region consistent with the inequalities (1).

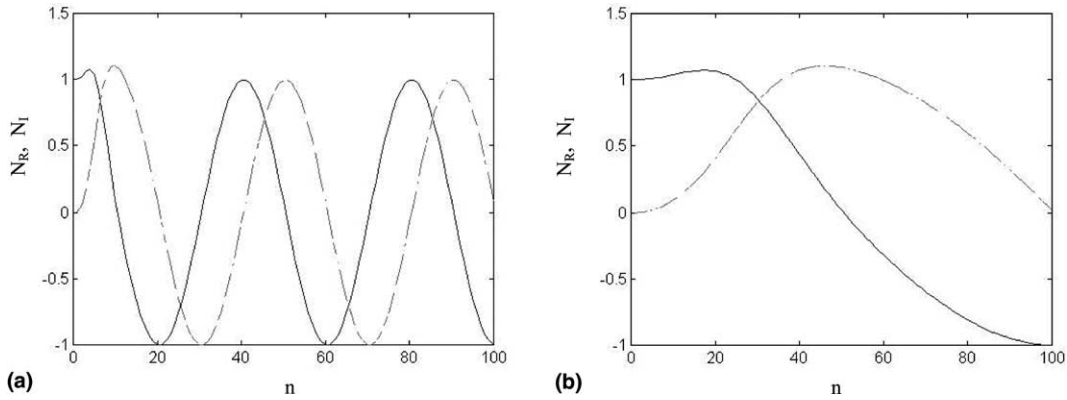


Fig. 3. The ratios  $N_R = \text{Re } \varphi_{2n-1}(0)/\varphi_{2n-1}^{(0)}(0)$  (solid lines) and  $N_I = \text{Im } \varphi_{2n-1}(0)/\varphi_{2n-1}^{(0)}(0)$  (dashed line) in the point  $x_1 = 0$  for two values of  $\beta$ :  $\beta = 0.05$  (a) and  $\beta = 0.01$  (b).

On the other hand, this approach allows us to calculate the reflection coefficient only. The fields in the close vicinity of a rough metallic surface (at the distances  $d$  that are of the order of the period of the surface structure, where the evanescent waves are significant) cannot be calculated, if we know the expression for  $\mathbf{H}_t^{(0)}$  only. To calculate this field one or another version of the perturbation theory must be used. We suggest to use Eq. (29) as a verification that the applied perturbation theory is accurate.

In some sense the calculation of the ohmic losses in the p-polarization state is much more simple. The point is that if the grooves have a finite depth consistent with inequalities (1), the magnetic field penetrates into the grooves on the surface. In this case in the main approximation the magnetic field in the vicinity of the grooves is the same as near a flat surface of the perfect conductor. Thus the entire inner surface of the grooves absorbs the electromagnetic wave. In [8] for an arbitrary one-dimensional periodic surface this argumentation allowed us to obtain rather simple expression for the leading terms of the effective surface impedance  $\zeta_{\text{ef}}^p$  in the p-polarization state:

$$\zeta_{\text{ef}}^p = \text{Re } \zeta \frac{L}{2b} - i \frac{kS}{2b}. \quad (30)$$

In Eq. (30)  $2b$  is the period of the surface structure,  $L$  is the length of the surface profile per one period and  $S$  is the area of the groove in the plane  $x_1x_3$ . If

the effective impedance is known, there are no problems with calculation of the ohmic losses:

$$Q_d^p = \frac{cb}{4\pi} |H_0|^2 \text{Re } \zeta_{\text{ef}}^p. \quad (31)$$

However, some difficulties can arise when the perturbation theory is applied to calculate the evanescent waves near rough surface in the p-polarization state. In the present publication we do not examine this question.

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### Appendix A

Let us demonstrate that the solution  $X_{2l-1}^{(0)}(kb)$  ( $l = 1, 2, \dots$ ) of Eqs. (18a)–(18c) guarantees the fulfillment of the energy conservation law for an arbitrary value of  $kb < 1$ . We also hope that the formulae of the section can be useful when calculating the fields above the infinitely conducting grating for an arbitrary value of  $kb < 1$ .

Eqs. (18a)–(18c) correspond to the infinitely conducting grating ( $\zeta = 0$ ), and, consequently, the

reflection coefficient has to be equal to 1. With regard to our notations this means that after substituting  $X_{2l-1}^{(0)}(kb)$  in Eq. (15), we have to verify that  $1 - R = 1 - |e_0^+|^2 = 0$ . When  $\zeta = 0$ , the explicit form of  $1 - R$  is

$$1 - R = -i(kb) \frac{\mu}{\pi} \sum_{l=1}^{\infty} \frac{(X_{2l-1}^{(0)} - (X_{2l-1}^{(0)})^*)}{(2l-1)^2} - (kb)^2 \left(\frac{\mu}{\pi}\right)^2 \sum_{l,m=1}^{\infty} \frac{(X_{2l-1}^{(0)})^* X_{2m-1}^{(0)}}{(2l-1)^2 (2m-1)^2} \quad (\text{A.1})$$

(compare with Eq. (24);  $\mu = 2a/b$ ). When writing Eq. (A.1) we take into account that  $W_{2l-1}^{(0)} = 1/(2l-1)^2$  (see Eq. (14b)).

We seek  $X_{2m-1}^{(0)}(kb)$  as a power series

$$X_{2m-1}^{(0)} = \sum_{q=0}^{\infty} Y_{2m-1}^{(q)}(kb)^q. \quad (\text{A.2})$$

Then it is convenient to rewrite Eq. (18a) in the form

$$\sum_{r=1}^{\infty} \Phi_{2r-1,2m-1} X_{2r-1}^{(0)} = \frac{1}{(2m-1)} \left\{ 2 + i(kb) \frac{\mu}{\pi} \sum_{r=1}^{\infty} \frac{X_{2r-1}^{(0)}}{(2r-1)^2} \right\}. \quad (\text{A.3a})$$

The elements of the matrix  $\Phi_{2r-1,2m-1}$  depend on  $(kb)^2$  only,

$$\begin{aligned} \Phi_{2r-1,2m-1} &= (2m-1) \sum_{q=1}^{\infty} q\mu \sqrt{1 - (kb/\pi q)^2} D_{2r-1,2m-1}^{(0)} \\ &\quad + \frac{1}{\mu} \left(\frac{\pi}{2}\right)^2 \sqrt{1 - \left(\frac{\mu kb}{\pi(2r-1)}\right)^2} \delta_{rm}, \end{aligned} \quad (\text{A.3b})$$

where  $D_{2r-1,2m-1}^{(0)}$  is defined by Eq. (18c).

Let us also present  $\Phi_{2r-1,2m-1}$  as a series expansion

$$\Phi_{2r-1,2m-1} = \sum_{p=0}^{\infty} \Phi_{2r-1,2m-1}^{(2p)} (kb)^{2p}. \quad (\text{A.4})$$

When we substitute Eq. (A.2) in Eq. (A.3a) and set equal the terms, corresponding to the same powers of  $(kb)$ , we obtain a set of equations relating the

coefficients  $Y_{2m-1}^{(q)}$  with different superscripts. Let us examine the obtained equations in the consecutive order. For  $q = 0$  (the term independent of  $kb$ ) we have

$$\sum_{r=1}^{\infty} \Phi_{2r-1,2m-1}^{(0)} Y_{2r-1}^{(0)} = \frac{2}{2m-1}. \quad (\text{A.5a})$$

Evidently,  $Y_{2r-1}^{(0)} = X_{2r-1}^{(0)}(0)$ . Note, the coefficients  $Y_{2r-1}^{(0)}$  being solutions of the linear set of equations with real coefficients and real right-hand sides, are real numbers.

Next, for  $q > 1$  we have

$$\begin{aligned} \sum_{p=0}^{s_q} \sum_{r=1}^{\infty} \Phi_{2r-1,2m-1}^{(2p)} Y_{2r-1}^{(q-2p)} &= \frac{2}{(2m-1)} T_{q-1}, \\ T_q &= i \frac{\mu}{2\pi} \sum_{s=1}^{\infty} \frac{Y_{2s-1}^{(q)}}{(2s-1)^2}. \end{aligned} \quad (\text{A.5b})$$

The first sum in the left-hand side of Eq. (A.5b) is taken over  $p \leq s_q$ , where  $s_q = s-1$ , if  $q = 2s-1$ , and  $s_q = s$ , if  $q = 2s$ .

The structure of the right-hand side of Eq. (A.5b) suggests to seek the coefficients  $Y_{2s-1}^{(q)}$  ( $q > 1$ ) in the form

$$Y_{2s-1}^{(q)} = Y_{2s-1}^{(0)} T_{q-1} + Z_{2s-1}^{(q)}. \quad (\text{A.6})$$

Then for  $q = 1$  we have

$$Z_{2s-1}^{(1)} = 0, \quad Y_{2s-1}^{(1)} = Y_{2s-1}^{(0)} T_0. \quad (\text{A.7a})$$

We see that the coefficients  $Y_{2s-1}^{(1)}$  are imaginary numbers. At the next step ( $q = 2$ ) we obtain the equation for  $Z_{2s-1}^{(2)}$ ,

$$\sum_{r=1}^{\infty} \Phi_{2r-1,2m-1}^{(0)} Z_{2r-1}^{(2)} + \sum_{r=1}^{\infty} \Phi_{2r-1,2m-1}^{(2)} Y_{2r-1}^{(0)} = 0. \quad (\text{A.7b})$$

With regard to Eq. (A.6) and Eqs. (A.7a) and (A.7b) it is evident that both  $Z_{2r-1}^{(2)}$  and  $Y_{2r-1}^{(2)}$  are real numbers. When continuing this procedure, we find out that all the coefficients  $Y_{2r-1}^{(2q-1)}$  are imaginary numbers and all the coefficients  $Y_{2r-1}^{(2q)}$  are real numbers. On the contrary,  $T_q$  is a real number when  $q$  is odd, and  $T_q$  is an imaginary number when  $q$  is even.

In the consecutive order we wrote down equations for  $Z_{2s-1}^{(q)}$ . Examining these equations we obtained formulae relating  $Z_{2s-1}^{(q)}$  with the

coefficients  $Z_{2s-1}^{(r)}$  with  $r \leq q$ . At the last step we eliminated the functions  $Z_{2s-1}^{(r)}$  with the aid of Eq. (A.6).

Here, omitting the intermediate calculations, we present only the recurrence formula for the coefficients  $Y_{2r-1}^{(2q-1)}$  obtained with the aid of the aforementioned procedure

$$Y_{2r-1}^{(2q-1)} = i \frac{\mu}{2\pi} \sum_{s=1}^{\infty} \frac{1}{(2s-1)^2} \sum_{p=0}^{2(q-1)} Y_{2s-1}^{(p)} \left( Y_{2r-1}^{(2(q-1)-p)} \right)^*. \quad (\text{A.8})$$

To obtain Eq. (A.8) we used the definition of the sum  $T_q$  (see Eq. (A.5b)). We do not need the explicit form of  $Y_{2r-1}^{(2q)}$  to prove that the right-hand side of Eq. (A.1) vanishes.

Now let us write the expression for  $1 - R$  as a series in powers of  $kb$  with  $v_r$  ( $r \geq 1$ ) being the coefficients of this series. It is easy to see that since the coefficients  $Y_{2r-1}^{(0)}$  are real numbers,  $v_1 = 0$ . If  $r > 1$ , we have

$$v_r = -i \frac{\mu}{\pi} \sum_{l=1}^{\infty} \frac{Y_{2l-1}^{(r-1)} - \left( Y_{2l-1}^{(r-1)} \right)^*}{(2l-1)^2} - \left( \frac{\mu}{\pi} \right)^2 \sum_{l,m=1}^{\infty} \frac{1}{(2l-1)^2 (2m-1)^2} \times \sum_{q=0}^{r-2} Y_{2m-1}^{(r-q-2)} \left( Y_{2l-1}^{(q)} \right)^*. \quad (\text{A.9})$$

Let us show that  $v_r = 0$  for an arbitrary  $r$ . Indeed, if  $r = 2s - 1$ , taking into account that all the coefficients  $Y_{2l-1}^{(2(s-1))}$  are real numbers, we see that the first term in the right-hand side of Eq. (A.9) is equal to zero. In the second term decomposing the sum over  $q$  into two sums over odd and even values of  $q$ , after a simple transformation we obtain

$$v_{2s-1} = - \left( \frac{\mu}{\pi} \right)^2 \sum_{l,m=1}^{\infty} \frac{1}{(2l-1)^2 (2m-1)^2} \times \sum_{q=0}^{s-2} Y_{2l-1}^{(2q)} \left[ Y_{2m-1}^{(2(s-q)-3)} + \left( Y_{2m-1}^{(2(s-q)-3)} \right)^* \right]. \quad (\text{A.10})$$

and, consequently,  $v_{2s-1} = 0$ .

Next, let us examine the case  $r = 2s$ . If we take into account that all the coefficients  $Y_{2l-1}^{(2s-1)}$  are imaginary numbers, from Eq. (A.9) we obtain

$$v_{2r} = -2i \frac{\mu}{\pi} \sum_{l=1}^{\infty} \frac{Y_{2l-1}^{(2r-1)}}{(2l-1)^2} - \left( \frac{\mu}{\pi} \right)^2 \sum_{l,m=1}^{\infty} \frac{1}{(2l-1)^2 (2m-1)^2} \times \sum_{q=0}^{2(r-1)} Y_{2m-1}^{(2(r-1)-q)} \left( Y_{2l-1}^{(q)} \right)^*. \quad (\text{A.11a})$$

Now we make use of Eq. (A.8). From this equation it follows that

$$\sum_{l=1}^{\infty} \frac{Y_{2l-1}^{(2s-1)}}{(2l-1)^2} = i \frac{\mu}{2\pi} \sum_{l,m=1}^{\infty} \frac{1}{(2l-1)^2 (2m-1)^2} \times \sum_{q=0}^{2(s-1)} Y_{2m-1}^{(q)} \left( Y_{2l-1}^{(2(s-1)-q)} \right)^*. \quad (\text{A.11b})$$

From the last equation we immediately have  $v_{2s} = 0$ .

Thus we have shown that if  $\zeta = 0$  and  $kb < 1$ , the solution of the matrix Eq. (18a) provides the fulfillment of the equality  $1 - R = 0$  in agreement with the energy conservation law.

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