



# On the ladder operators and nonclassicality of generalized coherent state associated with a particle in an infinite square well

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## ABSTRACT

In this paper the factorization method is used in order to obtain the eigenvalues and eigenfunctions of a quantum particle confined in a one-dimensional infinite well. The output results from the mentioned approach allow us to explore an appropriate new pair of raising and lowering operators corresponding to the physical system under consideration. From the symmetrical considerations, the connection between the obtained ladder operators with  $su(1,1)$  Lie algebra is explicitly established. Next, after the construction of Barut–Girardello and Gilmore–Perelomov representations of coherent states associated with the considered system, some of their important properties like the resolution of the identity including a few nonclassical features are illustrated in detail. Finally, a theoretical scheme for generation of the Gilmore–Perelomov type of coherent state via a generalized Jaynes–Cummings model is proposed.

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## 1. Introduction

The time-independent form of Schrödinger equation is written as

$$H\psi = E\psi, \quad (1)$$

where  $\psi$  and  $E$  are the eigenfunction and eigenenergy and  $H$  represents the Hamiltonian of the considered system defined as

$$H = \frac{p^2}{2m} + V(x), \quad p = \frac{\hbar}{i} \frac{d}{dx}. \quad (2)$$

As the first pedagogical examples which may be found in any text book of quantum mechanics, which has also enough applications in various fields of physics, one may refer to a particle in an infinite well (in parallel with harmonic oscillator potential). But, while the creation and annihilation (ladder) operators associated with harmonic oscillator potential is a well-known matter in quantum mechanics, for the infinite potential well this is not explicitly known as is known for harmonic oscillator. This is our first purpose of the present paper. For a particle confined in

one-dimensional box, the potential is defined as

$$V(x) = \begin{cases} 0, & 0 \leq x \leq L, \\ \infty, & x < 0, x > L. \end{cases} \quad (3)$$

The second-order differential equation in (1) for above  $V(x)$  has been traditionally solved by using the power series method, and after applying the boundary conditions, i.e.,  $\psi(0) = \psi(L) = 0$ , one can obtain the exact form of the eigenenergies and eigenfunctions. In this paper, we intend to solve the same equation by the factorization method through which, with similar spirit of obtaining the ladder operators for the harmonic oscillator, we will be able to find out the generalized creation and annihilation operators for the system under consideration. It is worth to mention that in contrast to the simple harmonic oscillator, the energy levels of a particle in one-dimensional box are not equally spaced. This observation will make the problem more complicated. Anyway, we will try to find the appropriate pair of annihilation–creation operators corresponding to the mentioned system, by which we will be able to construct two distinct classes of coherent states. In this respect, as is well-known, the increasing interest in the coherent states [1–3] is subject to the fact that they are the main source of various nonclassical properties. Nonclassical states have found their appropriate applications in various fields of physics in recent decades (for a useful and review text in coherent states see [4] and references therein). So, after exploring the appropriate creation and annihilation operators we are lead to the construction of the corresponding coherent states by two

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well-known manners: algebraic and group theoretic method. At last, after demonstrating some of important properties of the introduced states, we investigate the nonclassicality features of them for completing our presentation. At last, by using a generalized Jaynes–Cummings model, a theoretical proposal for the generation of the Gilmore–Perelomov type of coherent state is presented.

We end this introduction by recalling the fact that this potential, in a sense, has received a lot of attention in recent years. Indeed, as an equivalent approach which is used for the infinite square well potential, we may recall the well-known SUSY quantum mechanics technique (for instance see Ref. [5]). Also, the algebraic treatment for the system in hand, can be found in [6], and at last in the context of the construction of coherent states corresponding to such system one may refer to Ref. [7]. Altogether, it may clearly be seen that these works are all essentially different from the one has been presented here, either from the way to approach the problem or from the final obtained results and presented discussions. Thus, to the best of our knowledge, the approach to the treatment of the problem, the corresponding ladder operators, the associated coherent state of the type which is obtained in this paper, in addition to the discussion on the resolution of the identity and the related nonclassicality signs may not be found in the literature.

In the continuation, at first, we briefly review the factorization method. In Section 3, the method is used to solve the Schrödinger equation for a particle in one-dimensional infinite square potential well. Based on the obtained results, we construct the creation and annihilation operators in Section 4, by which we study the dynamical group corresponding to this system. In Section 5, the Barut–Girardello and Gilmore–Perelomov coherent states associated with a particle in the potential well are constructed. Then, we demonstrate (in Section 6) some of the nonclassical properties of the introduced states like squeezing (first and second-order) and sub-Poissonian statistics, in addition to establishing the resolution of the identity corresponding to both classes of introduced states. A theoretical scheme for generation of the Gilmore–Perelomov type of coherent state is presented in Section 7. Finally, Section 8 deals with a summary and concluding remarks.

## 2. The factorization method: a brief review

The factorization method was firstly introduced by Schrödinger [8–10] and Dirac [11] to avoid the use of cumbersome and lengthy mathematical calculations. It has been successfully applied to solve essentially the problems whose their exact solutions are exist. The formalism further developed by Infeld and Hull [12] and Green [13]. The interest in the factorization method comes from the fact that we may discover the hidden symmetry of the quantum system through the establishment of an appropriate Lie algebra, which can be realized by the constructed ladder operators.

The factorization method which is used in this paper is equivalent to the SUSY treatment [14]. Briefly, in SUSY one concerns with two Hamiltonians  $H_-$  and  $H_+$  which are known as SUSY partner Hamiltonians. The SUSY analysis allows one to generate a hierarchy of Hamiltonians [15]. In this method, the second-order differential operator  $H$  in (2) is rewritten as the product of two first-order differential operators  $a_1^\dagger$  and  $a_1$  plus a constant  $E_1$ , i.e.,

$$H_1 = a_1^\dagger a_1 + E_1. \quad (4)$$

In other words, rewriting the SUSY Hamiltonians  $H_-$  and  $H_+$  respectively as  $H_1$  and  $H_2$ , and changing the subscripts in a

suitable manner, we have (4) with its SUSY partner as

$$H_2 = a_1 a_1^\dagger + E_1.$$

Next, by factorizing  $H_2$  one has

$$H_2 = a_2^\dagger a_2 + E_2,$$

with its SUSY partner as

$$H_3 = a_2 a_2^\dagger + E_2.$$

Continuing the above procedure leads to the generation of a hierarchy of Hamiltonians given by

$$H_{j+1} = a_{j+1}^\dagger a_{j+1} + E_{j+1} = a_j a_j^\dagger + E_j, \quad (5)$$

where  $j = 1, 2, 3, \dots$ . In general, these equations can have many solutions, but  $a_j$  should be chosen in such a way that one gets the maximum value for  $E_j$  (we will explain this issue in the continuation). In that case, the solution of Eqs. (4) and (5) will be unique. Also, recall that,  $j$  is a counter indicator and so there is no indispensable reason for beginning it from 1. Indeed, it can begin from 0, too, and we will be able to continue without any problem. However, due to the outlined example which we will work with it in the next sections of the paper (a particle in an infinite square well) it seems to be reasonable to start with 1 (notice that  $\psi_j(y) \sim \sin(jy)$ ).

Now, upon using the following theorem we can find eigenfunctions and eigenvalues of the system Hamiltonian [16].

**Theorem.** Assume that Eqs. (4) and (5) with real constants  $E_j$  are satisfied. Also, suppose that there exists a null eigenfunction  $|\xi_j\rangle$  with zero eigenvalue for each  $a_j$ , i.e.,

$$a_j |\xi_j\rangle = 0. \quad (6)$$

Then,

- (a)  $E_j$  is the  $j$ th eigenvalue of the Hamiltonian.
- (b) Corresponding eigenfunction of the Hamiltonian  $H$  is of the form

$$|\psi_j\rangle = c_j a_1^\dagger a_2^\dagger \dots a_{j-1}^\dagger |\xi_j\rangle, \quad (7)$$

where  $c_j$  is a normalization constant may be determined as

$$|c_j| = [(E_j - E_{j-1})(E_j - E_{j-2}) \dots (E_j - E_1)]^{-1/2}. \quad (8)$$

There are three important points in the theorem:

- (i) This method applies only to the discrete energy spectra.
- (ii) In the selection of  $a_j$ , one should choose it in such a way that  $E_j$  gets the maximum value [16]. Indeed, it is an obvious fact which comes from the assumption  $a_j |\xi_j\rangle = 0$ . For more detail, if one supposes that  $E_j$  and  $a_j$  are changed, whereas  $E_{j-1}$  and  $a_{j-1}$  remain fixed, then

$$\begin{aligned} \langle \xi_j | \delta E_j | \xi_j \rangle &= -\langle \xi_j | \delta(a_j^\dagger a_j) | \xi_j \rangle = \langle \xi_j | (-\delta a_j^\dagger a_j - a_j^\dagger \delta a_j) | \xi_j \rangle, \\ &= -\langle \xi_j | \delta a_j^\dagger (a_j | \xi_j \rangle) - (\langle \xi_j | a_j^\dagger) \delta a_j | \xi_j \rangle = 0. \end{aligned}$$

- (iii) The theorem does not give any information about the form of  $a_j$ . Moreover, evidently the form of  $a_j$  depends explicitly on the particular potential  $V(x)$  in entered in (2). Anyway, whereas  $a_1^\dagger a_1$  is equal to  $H$ , except for a constant,  $a_1$  must contain a linear momentum term to be consistent with the kinetic energy part of the Hamiltonian. Accordingly,  $a_j$  can be expressed as

$$a_j = \frac{1}{\sqrt{2m}}(p + if_j(x)), \quad (9)$$

where  $p$  is the momentum operator and  $f_j(x)$  is a real and differentiable unknown function of  $x$  by which Eq. (4) is satisfied. These operators are not hermitian ( $a_j^\dagger = (1/\sqrt{2m})(p - if_j(x)) \neq a_j$ ), however using the commutation relation  $[f_j(x), p] = i\hbar(df_j(x)/dx)$ , their hermitian product  $a_j^\dagger a_j$  can be written as follows:

$$a_j^\dagger a_j = \frac{p^2}{2m} + g_j(x), \quad (10)$$

where the real function  $g_j(x)$  is defined as  $g_j(x) = f_j^2(x)/2m + (\hbar/2m)(df_j(x)/dx)$ .

### 3. A particle in one-dimensional infinite potential well

Now, we are going to search for the eigenenergies and eigenfunctions of our system, i.e. a particle in an infinite well potential by the factorization method. At first, with considering Eq. (10) for  $j=1$  and using (4) we arrive at the following equation for  $f_1(x)$ :

$$\frac{df_1(x)}{dx} = -\frac{f_1^2(x)}{\hbar} - \frac{2mE_1}{\hbar}, \quad (11)$$

which is a simple Riccati differential equation. The above equation can be easily solved using the change of variable  $f_1(x) = \sqrt{2mE_1} \cot(y)$ , from which we finally have

$$f_1(x) = \sqrt{2mE_1} \cot\left(\frac{\sqrt{2mE_1}}{\hbar}(x-a)\right), \quad (12)$$

where  $a$  is an integration constant. Possible selections for  $E_1$  and  $a$  may be restricted by cotangent function and since  $0 \leq x \leq L$ , the cotangent function must be finite in this allowed range. Of course we should get the maximum value of  $E_1$  and it will correctly be obtained if one of the cotangent singularities be in  $x=0$  and the other in  $x=L$ . Hence, we choose the point  $x=0$  as the singularity point of  $f_1(x)$ ,

$$f_1(0) = \infty \rightarrow a = 0, \quad (13)$$

with substituting  $a=0$  in (12) and choosing  $x=L$  as the other singularity of  $f_1(x)$

$$f_1(L) = \infty \rightarrow \frac{\sqrt{2mE_1}}{\hbar}L = \pi. \quad (14)$$

As a result, with substituting (13) and (14) in (12) and using Eq. (9),  $a_1$  can be determined as follows:

$$a_1 = \frac{1}{\sqrt{2m}} \left( p + i \frac{\pi\hbar}{L} \cot\left(\frac{\pi}{L}x\right) \right) \quad (15)$$

and from (14) the ground state energy will be achieved as

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}. \quad (16)$$

Now, after finding the form of the function  $f_1(x)$  as the ground state, in order to obtain all of the eigenstates, we consider

$$f_j(x) = c_j \cot(b_j x) \quad (17)$$

and make use of recurrence relation (5) with the restriction  $0 \leq b_j \leq \pi/L$ , then one has

$$\begin{aligned} \frac{1}{2m} [p^2 - c_{j+1}b_{j+1}\hbar + c_{j+1}(c_{j+1} - b_{j+1}\hbar) \cot^2(b_{j+1}x)] + E_{j+1} \\ = \frac{1}{2m} [p^2 + c_j b_j \hbar + c_j(c_j + b_j \hbar) \cot^2(b_j x)] + E_j \end{aligned} \quad (18)$$

Eq. (18) will be true, if

$$b_{j+1} = b_j \rightarrow b_j = b_1 = \frac{\pi}{L} \quad (19)$$

$$c_{j+1}(c_{j+1} - b_{j+1}\hbar) = c_j(c_j + b_j\hbar) \rightarrow c_{j+1} = \begin{cases} -c_j, \\ c_j + \frac{\pi\hbar}{L}, \end{cases} \quad (20)$$

$$2mE_{j+1} - c_{j+1}b_{j+1}\hbar = 2mE_j + c_j b_j \hbar \rightarrow E_j = \frac{c_j^2}{2m}. \quad (21)$$

From the solutions of (20) the second answer namely  $c_{j+1} = c_j + \pi\hbar/L$  provides the maximum value of  $E_{j+1}$ , as we require, from which one obtains

$$c_j = \frac{j\pi\hbar}{L}. \quad (22)$$

Finally, from (21) and (22) we can derive the eigenvalues as

$$E_j = \frac{j^2\pi^2\hbar^2}{2mL^2}, \quad j = 1, 2, 3, \dots \quad (23)$$

With substituting  $c_j$  and  $b_j$ , respectively from (22) and (19) in (17) and using the obtained result in (9),  $a_j(x)$  will be explicitly obtained

$$a_j = \frac{1}{\sqrt{2m}} \left( p + i \frac{j\pi\hbar}{L} \cot\left(\frac{\pi}{L}x\right) \right). \quad (24)$$

Eq. (23) gives all the eigenvalues if there was one answer for arbitrary  $j$  in (6). Thus, we require to solve

$$a_j \xi_j(x) = 0 \quad (25)$$

from which one obtains

$$\frac{d\xi_j(x)}{dx} = j \frac{\pi}{L} \cot\left(\frac{\pi}{L}x\right) \xi_j(x). \quad (26)$$

Solving the above equation for  $\xi_j(x)$  one easily reads

$$\xi_j(x) = \left( \sin\left(\frac{\pi}{L}x\right) \right)^j \quad (27)$$

and finally by using (7) the normalized eigenfunctions of a particle in box will be obtained as

$$\psi_j(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{j\pi}{L}x\right). \quad (28)$$

Therefore, the evaluated results in (23) and (28) are our final solutions of the considered eigenvalue problem.

### 4. Ladder operators associated with infinite potential well

Notice that  $a_j$  operators in (24) and their corresponding conjugates do not play the role of lowering and raising operators for the energy levels of a particle in a box, i.e., they are not our required ladder operators. But, as we will see in the continuation of the paper, we will still be able to construct a pair of annihilation-creation operators after a change of variable as  $y = (\pi/L)x$  in the above results. Indeed, after some tries, we acquire the following operators, yielding our above requirement:

$$A_j \equiv \frac{d}{dy} + j \cot(y) \quad (29)$$

and  $A_j^\dagger$

$$A_j^\dagger \equiv -\frac{d}{dy} + j \cot(y). \quad (30)$$

By acting the operator  $A_j^\dagger$  on  $\psi_j(y)$  we have

$$\begin{aligned} A_j^\dagger(\psi_j(y)) &= \sqrt{\frac{2}{L}} [j \cos(jy) + j \cot(y) \sin(jy)] \\ &= \frac{j}{\sin(y)} \sqrt{\frac{2}{L}} \sin[(j+1)y] = \frac{j}{\sin(y)} \psi_{j+1}(y). \end{aligned} \quad (31)$$

Consequently

$$\sin(y)A_j^\dagger\psi_j(y)=j\psi_{j+1}(y). \quad (32)$$

Similarly, by applying  $A_j$  on  $\psi_j(y)$  we obtain

$$\sin(y)A_j\psi_j(y)=j\psi_{j-1}(y). \quad (33)$$

By the way, keeping in mind the above relations, at this stage it is possible to introduce the ladder operators which can really raise and lower any eigenfunction (or corresponding energy level) of our physical system

$$K_\pm = \pm \sin(y) \frac{d}{dy} + \cos(y) K_0, \quad (34)$$

where we have defined the (Hermitian) number operator  $K_0$  that satisfies the eigenvalue equation

$$K_0|j\rangle = j|j\rangle, \quad \psi_j(y) = \langle y|j\rangle. \quad (35)$$

Briefly, it is easy to check that the generalized ladder operators  $K_\pm$  in (34) possess the following required properties

$$K_\pm|j\rangle = j|j \pm 1\rangle, \quad j = 1, 2, 3, \dots, \quad K_\pm^\dagger = K_\mp. \quad (36)$$

A word seems to be necessary about the ladder operators for arbitrary solvable quantum system (and corresponding generalized coherent states). In fact, the number operator dependent of the ladder operators are usual in this typical treatments such as SUSYM quantum mechanics, etc. (for instance see [17]). The operators  $K_\pm$  and  $K_0$  justify the following commutation relations:

$$[K_-, K_+] = 2K_0, \quad [K_0, K_\pm] = \pm K_\pm. \quad (37)$$

As is well known, according to (37), in the infinite dimensional Hilbert space spanned by  $|j\rangle, j = 1, 2, 3, \dots$ , the set of operators  $\{K_-, K_+, K_0\}$  are the generators of  $su(1,1)$  Lie algebra. In addition, the system Hamiltonian can be simply expressed as

$$H = \frac{\pi^2 \hbar^2}{2mL^2} (K_+ K_- + K_0) = \frac{\pi^2 \hbar^2}{2mL^2} (K_- K_+ - K_0) = \frac{\pi^2 \hbar^2}{4mL^2} \{K_+, K_-\}, \quad (38)$$

where  $\{.,.\}$  stands for the anticommutation relation. All of the expressed forms for the Hamiltonian in (38) satisfy the eigenvalue equation  $H|j\rangle = E_j|j\rangle$  according to (23). The operator equation (38) may also be considered as an explicit definition for  $K_0$ -operator. It is worth to notice that the last relation in (38) seems to be similar to that of the harmonic oscillator, i.e.,  $H_{HO} = \hbar\omega\{b, b^\dagger\}$ , where  $b, b^\dagger$  are the standard bosonic annihilation and creation operators which together with number operator  $b^\dagger b$  construct the Weyl–Heisenberg Lie algebra.

## 5. The construction of generalized coherent states

Due to the recently increasing interest in the coherent states and nonclassical states in quantum optics, we are now in the position to produce the coherent states for our considered system. Generally there are three main approaches for the construction of coherent states, i.e., annihilation operator eigenstate (algebraic consideration), displacement operator (group theoretical consideration) and dynamical consideration. Only for harmonic oscillator all of the three manners lead to a unique state known as standard or canonical coherent state  $|z\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} (z^n / \sqrt{n!}) |n\rangle$ . After finding the physical realization of coherent state through the discovery of the direct relation of laser and coherent state, all of the mentioned approaches are vastly generalized in literature. Henceforth, now which we are being equipped with the ladder operators of the infinite well potential we are able to produce the generalized coherent states associated

with the system by generalization of the annihilation operator eigenstate and displacement operator manners.

- **Annihilation operator eigenstate:** The Barut–Girardello coherent state (algebraic method) is defined as the right eigenstate of the annihilation operator, i.e.,

$$K_-|\alpha\rangle_{BG} = \alpha|\alpha\rangle_{BG}. \quad (39)$$

To deduce the explicit form of  $|\alpha\rangle_{BG}$ , with the traditional method we can write

$$|\alpha\rangle_{BG} = \sum_{j=1}^{\infty} c_j |j\rangle. \quad (40)$$

Then, we set it in (39) and upon using (36) finally arrives us at Barut–Girardello type of coherent states as follows:

$$|\alpha\rangle_{BG} = N(|\alpha|^2) \sum_{j=0}^{\infty} \frac{\alpha^j}{(j+1)!} |j+1\rangle, \quad (41)$$

where the factor  $N(|\alpha|^2)$  can be calculated by the normalization condition, i.e.,  $\langle \alpha | \alpha \rangle = 1$  results in

$$N(|\alpha|^2) = \left( \sum_{j=0}^{\infty} \frac{|\alpha|^{2j}}{(j+1)!^2} \right)^{-1/2}. \quad (42)$$

- **Displacement operator coherent state:** On the other hand, keeping in mind the  $su(1,1)$  symmetry algebra of the generators as in (37), we are able to construct the Gilmore–Perelomov coherent states associated with the particle in an infinite potential well as follows [18] (group theoretical method):

$$|\alpha\rangle_{GP} = D(\xi)|1\rangle = \exp(\xi K_+ - \xi^* K_-)|1\rangle \\ = \exp(\alpha K_+)(1 - |\alpha|^2)^{K_0} \exp(-\alpha^* K_-)|1\rangle, \quad (43)$$

with  $\alpha = (\xi/|\xi|)\tanh(\xi)$ . Notice that the disentanglement theorem has been used in obtaining the final result in above equation. Straightforward calculations lead one to the normalized form of Gilmore–Perelomov coherent states, i.e.,

$$|\alpha\rangle_{GP} = \sqrt{1 - |\alpha|^2} \sum_{j=0}^{\infty} \alpha^j |j+1\rangle. \quad (44)$$

Now after finding out the two distinct classes of coherent states, we are ready to investigate some of nonclassical properties which are of most importance in quantum optics studies. But, before paying attention to this subject, we should establish the resolution of the identity for the above introduced states.

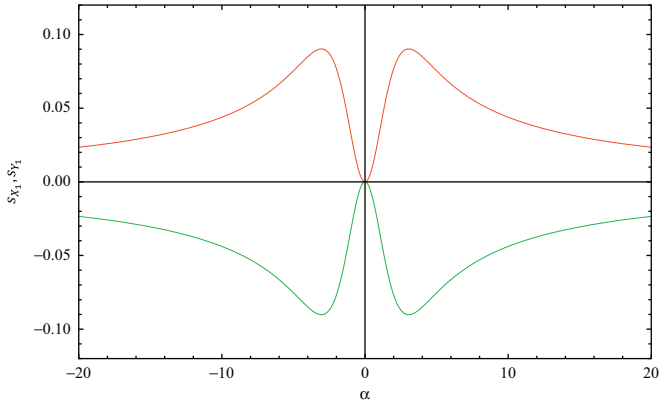
- **Resolution of the identity:** Any generalized coherent state such as  $|\alpha\rangle_{BG}$  should satisfy the resolution of the identity which is defined as follows:

$$\int d^2\alpha w(|\alpha|^2) |\alpha\rangle_{BG} \langle \alpha| = \hat{I} = \sum_{j=0}^{\infty} |j+1\rangle \langle j+1|, \quad (45)$$

where  $w(|\alpha|^2)$  is a suitable weight function should be found. Substituting (41) in (45) and using the change of variables  $\alpha = re^{i\theta}$  and  $r^2 = x$  lead us to the following relation

$$\pi \sum_{j=0}^{\infty} |j+1\rangle \langle j+1| \int_0^\infty w(x) N^2(x) x^j dx = [\Gamma(j+2)]^2, \quad (46)$$

where  $\Gamma(j+2)$  is the gamma function. This is indeed an inverse moment problem which can be solved by the well-known methods such as Mellin transform [19]. It can be easily



**Fig. 1.** Plot of squeezing parameters,  $s_{X_1}$  (up diagram) and  $s_{Y_1}$  (down diagram) against  $\alpha$  for Barut–Girardello coherent states.

checked that the required weight function which can satisfy the above integral equation is of the form

$$w(x) = \frac{2xK_0(2\sqrt{x})}{\pi N^2(x)}, \quad (47)$$

where  $K_0(2\sqrt{x})$  is the Bessel function of second kind.

The Gilmore–Perelomov states according to (44), are formally as harmonious coherent states. The resolution of the identity for such states has been discussed in [20,21].

Therefore, adding the obvious nonorthogonality of the two obtained states in (42) and (44) to the presented discussion in this section, we conclude the over-completeness relation for both classes of introduced coherent states.

## 6. Nonclassical features of the introduced states

- **Normal squeezing:** In order to examine the quantum fluctuations of the quadratures of the field [22], following the well-known generalization of the field quadratures  $x, p$  [23,24], we introduce the hermitian field operators  $X_1 = (K_- + K_+)/2$  and  $Y_1 = (K_- - K_+)/2i$ . The squeezing parameters can be defined as follows:

$$s_\gamma = \frac{(\Delta\gamma)^2}{\sqrt{\frac{1}{4}|\langle[X_1, Y_1]\rangle|^2}} - 1, \quad \gamma = X_1, Y_1, \quad (48)$$

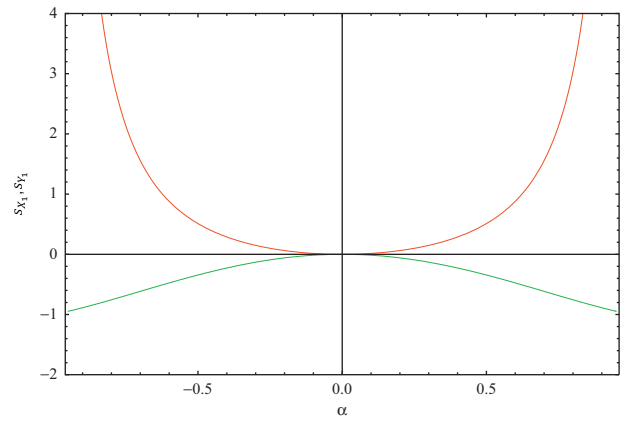
where  $(\Delta\gamma)^2 = \langle\gamma^2\rangle - \langle\gamma\rangle^2$ . A state is squeezed in  $X_1$  or  $Y_1$  if it satisfies the inequalities  $-1 < s_{X_1} < 0$  or  $-1 < s_{Y_1} < 0$ , respectively.

Fig. 1

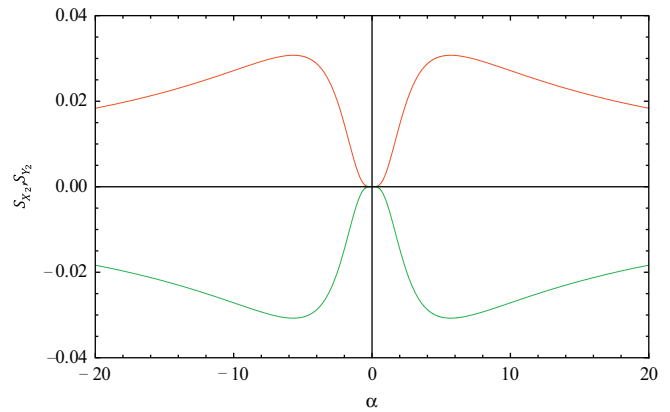
shows that the squeezing for Barut–Girardello coherent states occurs only in  $Y_1$  component in all real space. Similarity, from Fig. 2 we understood that this nonclassical feature for Gilmore–Perelomov are similarly hold in  $Y_1$  quadrature. All figures are symmetrical around  $\alpha = 0$  and the minimum and maximum of these parameters occur in approximately  $\alpha = 0$ .

- **Amplitude-squared squeezing:** This parameter is defined in terms of Hermitian operators  $X_2 = (K_-^2 + K_+^2)/2$  and  $Y_2 = (K_-^2 - K_+^2)/2i$ . The latter generalizations originate from the proposal of Hillery [25] which was firstly introduced in connection with the square of the complex amplitudes of the electromagnetic field. The squeezing conditions in  $X_2$  or  $Y_2$  are respectively given by  $-1 < S_{X_2} < 0$  or  $-1 < S_{Y_2} < 0$ , where  $S_{X_2}$  and  $S_{Y_2}$  defined as follows:

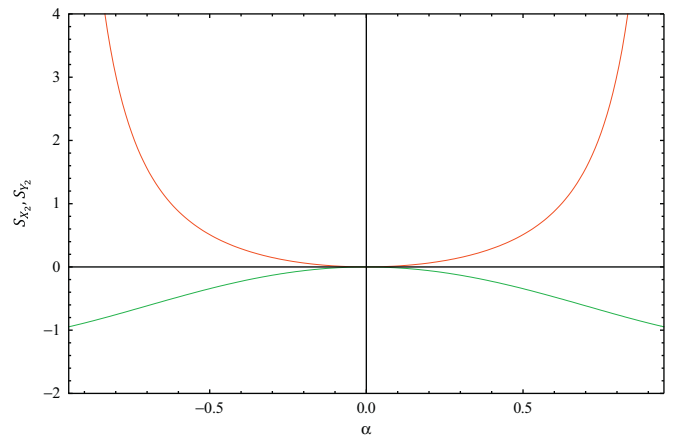
$$S_\ell = \frac{(\Delta\ell)^2}{\sqrt{\frac{1}{4}|\langle[X_2, Y_2]\rangle|^2}} - 1, \quad \ell = X_2, Y_2. \quad (49)$$



**Fig. 2.** Plot of squeezing parameters,  $s_{X_1}$  (up diagram) and  $s_{Y_1}$  (down diagram) against  $\alpha$  for Gilmore–Perelomov coherent states.



**Fig. 3.** Plot of amplitude-squared squeezing parameters,  $S_{X_2}$  (up diagram) and  $S_{Y_2}$  (down diagram) against  $\alpha$  for Barut–Girardello coherent states.



**Fig. 4.** Plot of amplitude-squared squeezing parameters,  $S_{X_2}$  (up diagram) and  $S_{Y_2}$  (down diagram) against  $\alpha$  for Gilmore–Perelomov coherent states.

In Figs. 3 and 4 amplitude-squared squeezing has been plotted against  $\alpha$ , respectively for Barut–Girardello and Gilmore–Perelomov coherent states. According to the displayed figures, it is clear that this parameter for both of introduced coherent states occurs in  $Y_2$  direction, too.

- **Quantum statistical properties:** For establishing the statistical properties and specially to examine the sub-Poissonian statistics as another nonclassicality sign [26], we calculate the



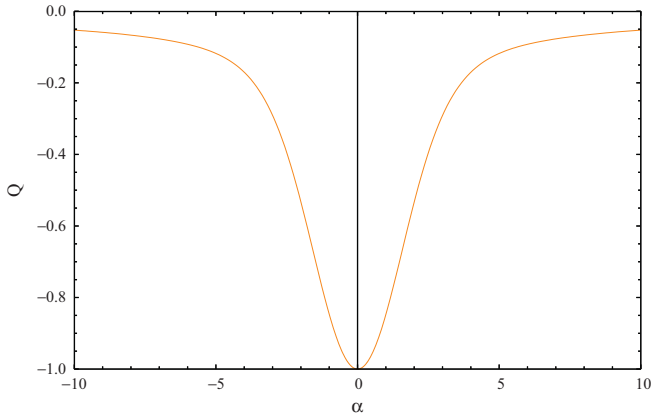


Fig. 5. Plot of Mandel parameter for Barut–Girardello coherent states versus  $\alpha$ .

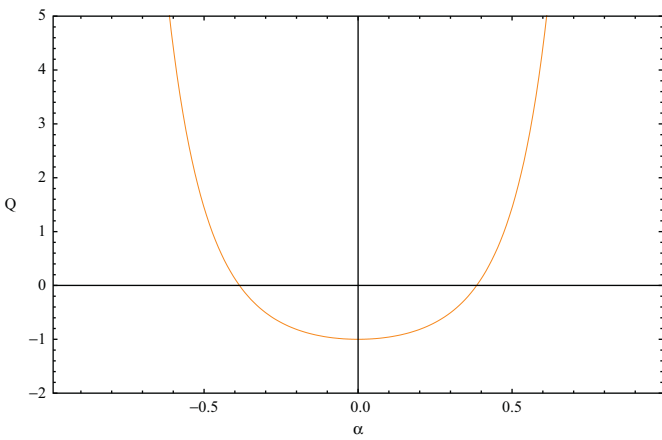


Fig. 6. Plot of Mandel parameter for Gilmore–Perelomov coherent states versus  $\alpha$ .

Mandel parameter [27], which can be generalized as

$$Q = \frac{\langle K_+^2 K_-^2 \rangle - \langle K_+ K_- \rangle^2}{\langle K_+ K_- \rangle} - 1. \quad (50)$$

The state for which  $Q=0$ ,  $Q < 0$  and  $Q > 0$  respectively corresponds to the Poissonian (standard coherent states) sub-Poissonian (nonclassical states) and super-Poissonian (classical states) statistics.

In Figs. 5 and 6 Mandel parameter has been plotted versus  $\alpha$ , respectively for Barut–Girardello and Gilmore–Perelomov coherent states. By comparing the two figures it is deduced that, while Barut–Girardello coherent states show this non-classical feature in all considered regions, the Gilmore–Perelomov coherent states possess this property in a finite region of the allowed space.

## 7. Generation of Gilmore–Perelomov type of coherent state

Recently one of us with his coauthor have proposed a theoretical generation scheme for producing any class of Gilmore–Perelomov type of  $SU(1, 1)$  coherent states [18] (and their superpositions). It is readily found that the proposal can help us for the generation of our GP coherent state, too. To achieve this purpose, we consider the atom–field interaction in which a two-level atom interacts with a quantized single-mode cavity field via an intensity-dependent coupling together with an external classical field. In the resonance condition and under the rotating-wave

approximation, the dynamical evolution of the mentioned system may be appropriately described by the Hamiltonian in the non-linear Jaynes–Cummings regime as follows ( $\hbar = 1$ ):

$$H = \lambda(\sigma_- e^{i\varphi} + \sigma_+ e^{-i\varphi}) + \Lambda(A^\dagger \sigma_- + A \sigma_+), \quad (51)$$

where  $\sigma_- = |g\rangle\langle e|$  and  $\sigma_+ = |e\rangle\langle g|$  are the atomic lowering and raising operators. Also, the parameters  $\lambda$  and  $\Lambda$  (assuming that  $\Lambda \gg \lambda$ ) are respectively the coupling coefficients of the atom with classical and quantized cavity fields,  $A$  and  $A^\dagger$  are the  $f$ -deformed ladder operators and  $\varphi$  is the phase of classical field. In [18] it is assumed that the operators  $A$ ,  $A^\dagger$  and their commutators satisfy the  $su(1, 1)$  Lie algebra (this does not occur for any arbitrary  $f$ -deformed ladder operators). In the present paper we deal really with the generators of such a specific group, therefore we begin with the following Hamiltonian instead:

$$H = \lambda(\sigma_- e^{i\varphi} + \sigma_+ e^{-i\varphi}) + \Lambda(K_+ \sigma_- + K_- \sigma_+), \quad (52)$$

where  $K_+$  and  $K_-$  (with  $K_0$ ) constitute the generators of the  $su(1, 1)$  algebra. It is demonstrated that, in the strong classical field regime ( $\Lambda \gg \lambda$ ), the time evolution operator of the system has been defined by [28]

$$U(t) = R^\dagger T^\dagger(t) U_{eff} T(0) R, \quad (53)$$

where the operators  $R$ ,  $T$  and  $U_{eff}$  are given by

$$R = \exp\left[\frac{\pi}{4}(\sigma_+ - \sigma_-)\right] \exp\left(\frac{i\varphi}{2}\sigma_z\right), \quad (54)$$

$$T(t) = \exp(i\lambda\sigma_z t), \quad (55)$$

$$U_{eff} = \exp\left[-\frac{i\Lambda t}{2}(K_+ e^{-i\varphi} + K_- e^{i\varphi})\sigma_z\right]. \quad (56)$$

Now, demanding the generation of our Gilmore–Perelomov states corresponding to a particle in an infinite well, the initial atom–field state is denoted by

$$|\psi(0)\rangle = \left(\frac{|e\rangle + |g\rangle}{\sqrt{2}}\right) |1\rangle, \quad (57)$$

which means that the cavity is initially prepared in the ground state of the field and the atom in a superposition of excited and ground states with equal weights.

It is a straightforward matter to obtain the final state of the system, following the mentioned procedure of [18], which yields the final state of the atom–field system at time  $t$  given by

$$\begin{aligned} |\Psi(t)\rangle = & \frac{1}{\sqrt{2}} \left( e^{-i\varphi/2} e^{-i\lambda t} \cos \frac{\varphi}{2} |\alpha\rangle_{GP} + i e^{-i\varphi/2} e^{i\lambda t} \sin \frac{\varphi}{2} |-\alpha\rangle_{GP} \right) |e\rangle \\ & + \frac{1}{\sqrt{2}} \left( e^{i\varphi/2} e^{-i\lambda t} \cos \frac{\varphi}{2} |\alpha\rangle_{GP} - i e^{i\varphi/2} e^{i\lambda t} \sin \frac{\varphi}{2} |-\alpha\rangle_{GP} \right) |g\rangle, \end{aligned} \quad (58)$$

where we have set  $\alpha \doteq -(i\Lambda t/2)e^{-i\varphi}$ . The final result  $|\Psi(t)\rangle$  is a general superposition of Gilmore–Perelomov coherent states of  $SU(1, 1)$  group. The state of quantized cavity field may be determined after doing a measurement on the initial state of the atom. For instance, if the atom is detected in excited or ground state, the state of the field will be collapsed respectively to

$$|\Psi^+\rangle = \frac{N^+}{\sqrt{2}} \left( e^{-i\varphi/2} \cos \frac{\varphi}{2} |\alpha\rangle_{GP} + i e^{-i\varphi/2} \sin \frac{\varphi}{2} |-\alpha\rangle_{GP} \right), \quad (59)$$

or

$$|\Psi^-\rangle = \frac{N^-}{\sqrt{2}} \left( e^{i\varphi/2} \cos \frac{\varphi}{2} |\alpha\rangle_{GP} - i e^{i\varphi/2} \sin \frac{\varphi}{2} |-\alpha\rangle_{GP} \right), \quad (60)$$

where  $N^\pm = \sqrt{2}$  and  $\lambda t = 2k\pi$ . Eqs. (59) and (60) show that the cavity field has been arrived at a combination of  $SU(1, 1)$  coherent state.

Now notice that, the expansion coefficients in (59) and (60) are  $\varphi$ -dependent. For instance, in particular, with selecting  $\varphi = 2\pi$  ( $\varphi = \pi$ ), regardless of the atomic detection, the state of the field collapses to  $|\alpha\rangle_{GP}$  ( $|-\alpha\rangle_{GP}$ ), i.e.,  $SU(1, 1)$  coherent states may be generated.

Although out of the scope of the present paper it is worth mentioning the possible generation of even or odd superposition of Gilmore–Perelomov  $SU(1, 1)$  coherent states by mean of the proposal in [18], too.

## 8. Summary and conclusion

In this paper we obtained the eigenvalues and eigenfunctions of a single particle in a one-dimensional infinitely deep square potential well by the factorization method. Then, appropriate generalized creation and annihilation operators are obtained based on the used method and derived results. Then, we have explored a symmetrical realization of the dynamical group; it is demonstrated that these operators can constitute a  $SU(1, 1)$  group. Next, we have constructed the explicit form of two distinct classes of coherent states, i.e., Barut–Girardello and Gilmore–Perelomov coherent states for the system under our consideration. The resolution of the identity and so the over-completeness relation for both states are illustrated in detail. Finally, in view of the importance of nonclassical states in recent decades in both theoretical and experimental aspects, we were succeeded in highlighting some of the nonclassical properties such as first and second-order squeezing and sup-Poissonian statistics for our introduced states, numerically. This is while the standard coherent states of harmonic oscillator possess neither of the nonclassicality features outlined in this paper. At last, via the nonlinear Jaynes–Cummings model, a theoretical scheme for generation of the Gilmore–Perelomov type of the outlined coherent state is proposed. Also, one can go further to produce even and odd superposition of both states in addition to excited coherent states which may be done elsewhere.

We end the concluding remarks with mentioning the fact that there exists a general way for the construction of generalized coherent state associated with any solvable quantum system with nondegenerate discrete spectrum (such as the one considered in this paper) known as Gazeau–Klauder type of coherent states [29,30]. But, while our approach is rather new and novelty, the results (the obtained ladder operators, the corresponding generalized coherent states and their properties) are all essentially different and distinguishable. Moreover, the Gazeau–Klauder coherent state corresponding to a particle in an infinite well is very similar to Pöschl–Teller potential has been adequately studied in [31] and so should not be considered here.

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