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Phase-space inequality for partially coherent optical beams

Taco D. Visser [a], Ari T. Friberg [b], Emil Wolf [c,*]

[a] Department of Physics and Astronomy, Free University, De Boelelaan 1081, NL-1081 HV Amsterdam, Netherlands
[b] Department of Physics – Optics, Royal Institute of Technology, SE-10044 Stockholm, Sweden
[c] Department of Physics and Astronomy, Rochester Theory Center for Optical Science and Engineering, University of Rochester, River Campus Station, Rochester, NY 14627-0171, USA

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Abstract

A phase-space inequality is derived for beams of arbitrary state of spatial coherence. It applies to the product of a factor which expresses the effective coherence area of the source that generates the beam and the effective angular spread of the beam; and, by analogy with coherent beams, it may be regarded as a measure of the beam quality. It is found that the factor attains a minimum for the entire class of Gaussian Schell-model beams (which include the Hermite Gaussian laser mode).

Keywords: Optical beams; Coherence theory; Beam quality

1. Introduction

Since Siegman [1] introduced in a well-known paper a measure of the quality of laser beams there have been numerous publications dealing with such measures for other types of beams. However, most of the investigations were concerned with fully coherent beams. In this connection it should be noted that the usual reciprocity relations (often referred to as uncertainty relations because of formal analogy with the quantum mechanical uncertainty principle) are derived for sources and fields which are fully spatially coherent. This is so, because in such case one can utilize a well-known inequality which involves the product of second moments of both the squared modulus of a function and the squared modulus of its Fourier transform.

In recent years there has also been a good deal of interest in partially coherent beams, partly because high-power lasers and diode lasers give rise to such beams and also because such beams have found useful applications, for example to suppress disturbing effects of speckles [2, pp. 259–260], reducing the influence of atmospheric turbulence [3], and in connection with laser fusion [4,5]. A number of papers deal with measures of quality of beams which are partially coherent (see, for example Refs. [6–11]). In particular, Bastiaans [12] derived reciprocity relations ("uncertainty relations") for coherent as well as for partially coherent light. The lower bound is expressed in terms of the eigenvalues of an integral equation whose kernel is the mutual intensity.

In the present paper we introduce a new phase-space product for beams of arbitrary state of

* Corresponding author. Tel.: +1-716-275-4397; fax: +1-716-473-0687.
E-mail address: ewlupus@pas.rochester.edu (E. Wolf).

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coherence. It is the product of a factor which is related to the effective coherence area of the source that generates the beam and the effective angular spread of the beam. This factor is shown to attain a minimum for the so-called Gaussian Schell-model (GSM) beams, which includes, but is not restricted to, the lowest order, fully coherent Hermite Gaussian laser mode.

2. A reciprocity relation for beams of arbitrary state of spatial coherence

Let us consider a planar, secondary source of any state of spatial coherence, located in the plane $z = 0$ and radiating into the half-space $z > 0$. We assume that the source is statistically stationary and we will characterize its second-order coherence properties by the cross-spectral density function $W^{(2)}(\boldsymbol{\rho}, \boldsymbol{\rho}', \nu)$ [2, Section 4.3.2] where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ denote the position vectors of any two points in the source plane (see Fig. 1), and ν is the frequency.

It was shown in Ref. [13] (see also Ref. [2, Eq. (5.3–10)]), that the radiant intensity of the field generated by the source, in the direction specified by a unit vector $\mathbf{s}$ may be expressed in the form

$$J(\mathbf{s}, \nu) = k^2 A \widetilde{C}^{(0)}(k\mathbf{s}_\perp, \nu)\cos^2\theta. \tag{1}$$

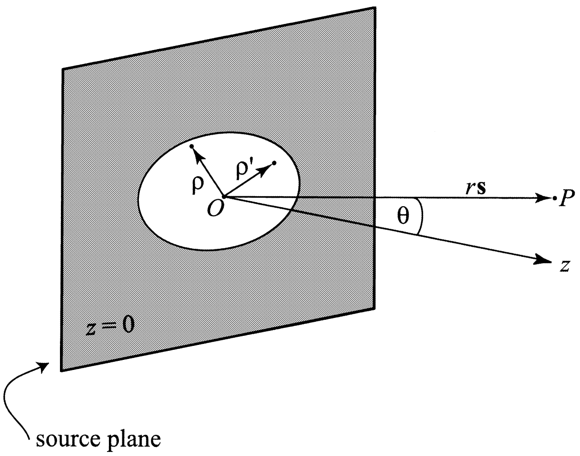

Fig. 1. Illustrating the notation. P is a point in the far zone, and $\mathbf{s}$ is a unit vector in the direction of OP.

In this formula $\widetilde{C}^{(0)}(\mathbf{f}, \nu)$ denotes the two-dimensional spatial Fourier transform

$$\widetilde{C}^{(0)}(\mathbf{f}, \nu) = \frac{1}{(2\pi)^2}\int C^{(0)}(\boldsymbol{\rho}', \nu)\mathrm{e}^{-\mathrm{i}\mathbf{f}\cdot\boldsymbol{\rho}'}\mathrm{d}^2\rho' \tag{2}$$

of the so-called source-averaged correlation function [1]

$$C^{(0)}(\boldsymbol{\rho}', \nu) = \frac{1}{A}\int W^{(0)}\left(\boldsymbol{\rho} - \frac{1}{2}\boldsymbol{\rho}', \boldsymbol{\rho} + \frac{1}{2}\boldsymbol{\rho}'\right)\mathrm{d}^2\rho \tag{3}$$

and A denotes the area of the domain occupied by the source. Further $\mathbf{s}_\perp$ in Eq. (1) is the projection, considered as a two-dimensional vector, of the unit vector $\mathbf{s}$ onto the source plane $z = 0$, θ is the angle which the vector $\mathbf{s}$ makes with the positive z-axis (see Fig. 1) and $k = 2\pi\nu/c$ is the free-space wave number associated with frequency ν.

Since we are considering beams, it is appropriate to use the paraxial approximations $\sin\theta \approx \theta$, $\cos\theta \approx 1$. It then follows from Eq. (1) that the quantity $J(\mathbf{s}, \nu)$ is proportional to the Fourier transform of the source-averaged correlation function $C^{(0)}(\boldsymbol{\rho}', \nu)$. This observation provides a basis for our subsequent analysis which makes it possible to formulate a reciprocity relation between the angular distribution of radiation and a measure of the spatial coherence of the source which generates the radiation field.

Formula (2) shows that $\boldsymbol{\rho}'$ and $\mathbf{f}$ are conjugate variables. Since in Eq. (1) $\mathbf{f} = k\mathbf{s}_\perp$ the Fourier transform relationship suggests that we introduce a phase-space product, which may be regarded as a measure of the quality of the beam, namely the quantity

$$F = (\Delta k\mathbf{s}_\perp)(\Delta\boldsymbol{\rho}), \tag{4}$$

where $(\Delta k\mathbf{s}_\perp)^2$ and $(\Delta\boldsymbol{\rho})^2$ are variances of J^2 and of $C^{(0)^2}$, defined by the formulas [2]

[1] The definition of the source-averaged correlation function given in Refs. [2,9] differs trivially from that given by our Eq. (3).

[2] As is well-known, the variance of the squared modulus of a function and the squared modulus of its Fourier transform satisfy a reciprocity inequality [10, p. 193]. For this reason we use J^2 and $|C^{(0)}|^2$ rather than J and $C^{(0)}$ as weighting factors in Eqs. (5), (5a) and (6), (6a).

$$(\Delta k\mathbf{s}_{\perp})^2 = \frac{1}{N_1}\int (k\mathbf{s}_{\perp})^2[J(\mathbf{s},\nu)]^2\mathrm{d}^2(k\mathbf{s}_{\perp}), \tag{5}$$

and

$$(\Delta\boldsymbol{\rho})^2 = \frac{1}{N_2}\int \boldsymbol{\rho}^2\left|C^{(0)}(\boldsymbol{\rho},\nu)\right|^2\mathrm{d}^2\rho, \tag{6}$$

where

$$N_1 = \int [J(\mathbf{s},\nu)]^2\mathrm{d}^2(k\mathbf{s}_{\perp}) \tag{5a}$$

and

$$N_2 = \int \left|C^{(0)}(\boldsymbol{\rho},\nu)\right|^2\mathrm{d}^2\rho. \tag{6a}$$

The integrations in Eqs. (5) and (5a) extend over the domain $(k\mathbf{s}_{\perp})^2 \leqslant k^2$ and the integrations in Eqs. (6) and (6a) extend over the source area.

In the special case that the field generated by the source is a beam which propagates closely to the z-axis and is also rotationally symmetric, one has, to a good approximation $\mathbf{s}_{\perp}^2 = \sin^2\theta \approx \theta^2$, so that

$$(\Delta\mathbf{s}_{\perp})^2 \approx (\Delta\theta)^2 \tag{7}$$

and, if we now write $J(\theta)$ rather than $J(\mathbf{s},\nu)$ and use the fact that $\mathrm{d}^2\mathbf{s}_{\perp} = \mathrm{d}s_x\,\mathrm{d}s_y = \cos\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \approx \theta\,\mathrm{d}\theta\,\mathrm{d}\phi$, we obtain from Eq. (5) the formula

$$(\Delta\theta)^2 = \frac{\int\theta^3[J(\theta)]^2\mathrm{d}\theta}{\int\theta[J(\theta)]^2\mathrm{d}\theta}. \tag{8}$$

The phase-space product (4) now becomes

$$F = k(\Delta\theta)(\Delta\rho). \tag{9}$$

This quantity for partially coherent beams is reminiscent of the well-known beam-quality factor, usually denoted by M^2, introduced by Siegman [1] as a measure of the quality of a fully coherent beam. [3]

3. Example: Gaussian Schell-model beams

We will illustrate the main result which we derived in the previous section with reference to a class of Schell-model beams. Such beams are generated by so-called Schell-model sources, i.e. by sources whose spectral degree of coherence $\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\nu)$ [2, Section 4.3.2] depends on the position vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ only through the difference $\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$, in which case we write

$$\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\nu) \equiv g^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1,\nu). \tag{10}$$

When both the spectral intensity and the spectral degree of coherence are Gaussian functions of position, i.e. when they have the form

$$I^{(0)}(\boldsymbol{\rho},\nu) = \begin{cases} B^2(\nu)\mathrm{e}^{-\rho^2/2\sigma_I^2(\nu)} & \text{when } |\rho| < a, \\ 0 & \text{when } |\rho| > a, \end{cases} \tag{11}$$

$$g^{(0)}(\boldsymbol{\rho}',\nu) = \mathrm{e}^{-\rho'^2/2\sigma_g^2(\nu)}, \tag{12}$$

we speak of GSM sources. In these formulas $B(\nu)$, $\sigma_I(\nu)$ and $\sigma_g(\nu)$ are positive quantities. We will assume that the radius a of the sources is much greater than σ_I. The cross-spectral density of a GSM source is given by the expression (cf. Ref. [2, Eqs. (5.6–62)])

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2) = \sqrt{I^{(0)}(\boldsymbol{\rho}_1)}\sqrt{I^{(0)}(\boldsymbol{\rho}_2)}g^{(0)}(\boldsymbol{\rho}_2-\boldsymbol{\rho}_1), \tag{13}$$

with $I^{(0)}$ and $g^{(0)}$ given by expressions (11) and (12) respectively. In this formula (and also in the sequel) we no longer display the explicit dependence of various quantities on the frequency ν.

The source-averaged correlation function of a GSM source is obtained on substituting from Eq. (13) (with $I^{(0)}$ and $g^{(0)}$ given by Eqs. (11) and (12) respectively) into formula (3). The result is (see Ref. [2, Section 5.4.2], with σ_I written in place of $\sigma_{\mathbf{s}}$)

$$C^{(0)}(\boldsymbol{\rho}) = \frac{1}{A}2\pi B^2\sigma_I^2\mathrm{e}^{-\rho^2/2\delta^2}, \tag{14}$$

where

$$\frac{1}{\delta^2} = \frac{1}{4\sigma_I^2} + \frac{1}{\sigma_g^2}. \tag{15}$$

Making again use of the assumption that $a \gg \sigma_I$, the two-dimensional Fourier transform $\widetilde{C}^{(0)}(\mathbf{f})$ of expression (14) is readily found to be

[3] A generalization of Siegman's beam quality factor to a class of partially coherent beams, known as Schell-model beams was not long ago introduced in Ref. [7].

$$\widetilde{C}^{(0)}(\mathbf{f}) \approx \frac{1}{A}(B\sigma_I\delta)^2 \mathrm{e}^{-f^2\delta^2/2}. \tag{16}$$

It follows from Eqs. (1) and (16) that one has, within the accuracy of the paraxial approximation,

$$J(\theta) = (kB\sigma_I\delta)^2 \mathrm{e}^{-(k\delta\theta)^2/2}. \tag{17}$$

If we now substitute from Eq. (14) into Eqs. (6a), (6b) and from Eq. (17) into Eq. (8) and recall Eq. (7) we readily obtain (see Appendix A) the following expressions for the "effective coherence r.m.s. width" $\Delta\rho$ of a GSM source and the r.m.s. angular width $\Delta\theta$ of the GSM beam which the source generates:

$$(\Delta\theta)_{\mathrm{GSM}} = \frac{1}{k\delta}, \tag{18}$$

$$(\Delta\rho)_{\mathrm{GSM}} = \delta. \tag{19}$$

The dependence of the angular spread $(\Delta\theta)_{\mathrm{GSM}}$ on the parameters $k\sigma_I$ and $k\sigma_g$ is shown in Fig. 2.

It follows from a well-known theorem on Fourier transform pairs [14] that the product

$$F \equiv k(\Delta\theta)(\Delta\rho) \geqslant 1. \tag{20}$$

The minimum value (unity) is attained when the functions are Gaussian. In the present case this will be so when the source-averaged cross-spectral density and, consequently, also the radiant intensity, have Gaussian forms. In particular, this will be so for GSM sources. Indeed, as we can see from Eqs. (18) and (19) the phase-space product F then has the minimum value unity, irrespective of the values of the source parameters B, σ_I and σ_g. However, the GSM sources are not the only ones which minimize the phase-space product F. For example, Gaussian correlated quasi-homogeneous sources minimize it also. Quasi-homogeneous sources are Schell-model type sources for which the intensity $I^{(0)}(\boldsymbol{\rho})$ varies much more slowly with $\boldsymbol{\rho}$, than the spectral degree of coherence $g^{(0)}(\boldsymbol{\rho}')$ varies with $\boldsymbol{\rho}'$. The cross-spectral density of such a source can evidently be approximated by the expression [2, Section 5.3.2]

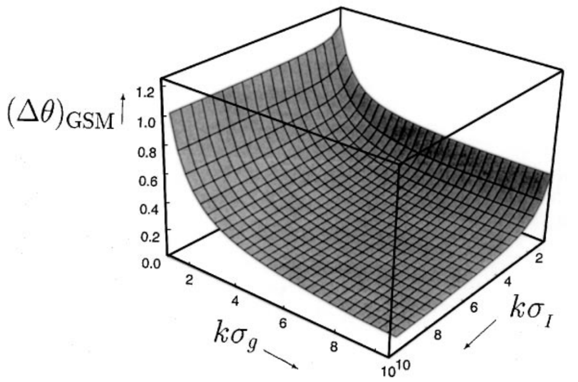

Fig. 2. The angular spread $(\Delta\theta)_{\mathrm{GSM}}$ of a GSM beam as a function of the dimensionless parameters $k\sigma_I$ and $k\sigma_g$.

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = I^{(0)}\left[\tfrac{1}{2}(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)\right] g^{(0)}(\boldsymbol{\rho}_2 - \boldsymbol{\rho}_1). \tag{21}$$

On substituting from Eq. (21) into expression (3) for the source-averaged cross-spectral density, we obtain the formula

$$C^{(0)}(\boldsymbol{\rho}') = g^{(0)}(\boldsymbol{\rho}') \int I^{(0)}(\boldsymbol{\rho})\,\mathrm{d}^2\rho. \tag{22}$$

Since the integral on the right-hand side of Eq. (22) is independent of $\boldsymbol{\rho}'$, it follows that for quasi-homogeneous sources with a Gaussian spectral degree of coherence the source-averaged cross-spectral density is Gaussian, irrespective of the form of the intensity function. We conclude that for such sources – just as for GSM sources – the phase-space product F attains a minimum.

Let us consider the limiting case of a spatially fully coherent source of the GSM class. In this case $\sigma_g \to \infty$, and Eq. (12) shows that $g^{(0)}(\boldsymbol{\rho}'_2 - \boldsymbol{\rho}'_1) \to 1$, with the intensity across the source still being given by Eq. (11). The intensity distribution across the source is then the same as that of the completely coherent lowest-order Gaussian Hermite laser mode, with spot size $w_0 = 2\sigma_I$, and Eq. (15) then gives $\delta = 2\sigma_I$ and the r.m.s. widths (18) and (19) become

$$(\Delta\theta_{\mathrm{GSM}})_{\mathrm{coh}} = \frac{1}{2k\sigma_I}, \tag{23}$$

$$(\Delta\boldsymbol{\rho}_{\mathrm{GSM}})_{\mathrm{coh}} = 2\sigma_I. \tag{24}$$

Formula (23) is the usual expression for the angular spread of the lowest-order Gaussian Hermite laser beam, in spite of somewhat different definitions used in the two cases.

4. Conclusions

By identifying an appropriate Fourier transform relationship, a new measure for characterizing partially coherent beams has been introduced. This measure, which, just as for coherent beams, takes on the form of a phase-space product, relates the radiant intensity of the field to the so-called source-averaged cross-spectral density function. In contrast to the usual phase-space measure, this quantity takes explicitly into account the coherence property of the source, rather than its intensity distribution. For the important class of quasi-homogeneous sources the source-averaged correlation function is proportional to the spectral degree of coherence. A phase-space inequality (reciprocity inequality) which we derive becomes an equality for the entire class of GSM beams and for beams generated by quasi-homogeneous sources with a Gaussian spectral degree of coherence.

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Appendix A. The variances $(\Delta\rho)^2$ and $(\Delta\theta)^2$ for Gaussian Schell-model beams

In this appendix we derive Eqs. (18) and (19). On substituting from Eq. (17) into Eq. (8) we obtain the formulas

$$(\Delta\theta)^2_{\mathrm{GSM}} = \frac{1}{N_3}\int_0^{\pi/2} \theta^3 \mathrm{e}^{-(k\delta\theta)^2}\,\mathrm{d}\theta, \tag{A.1}$$

where

$$N_3 = \int_0^{\pi/2} \theta \mathrm{e}^{-(k\delta\theta)^2}\,\mathrm{d}\theta. \tag{A.2}$$

Since the radiant intensity is sharply peaked at $\theta = 0$ (i.e. $k\delta \gg 1$), we may extend the range of integration in Eqs. (A.1) and (A.2) from $\pi/2$ to infinity without introducing an appreciable error. Then both expressions become standard integrals and we find

$$(\Delta\theta)^2_{\mathrm{GSM}} = \frac{1}{k^2\delta^2}, \tag{A.3}$$

from which Eq. (18) immediately follows.

The r.m.s. width of a GSM source is found by substituting from Eq. (14) into Eq. (6). This gives

$$(\Delta\rho)^2_{\mathrm{GSM}} = \frac{1}{N_4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x^2+y^2)\mathrm{e}^{-(x^2+y^2)/\delta^2}\,\mathrm{d}x\,\mathrm{d}y, \tag{A.4}$$

where

$$N_4 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x^2+y^2)/\delta^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{A.5}$$

Expressions (A.4) and (A.5) are both standard integrals. The result for the r.m.s. source width is

$$(\Delta\rho)^2_{\mathrm{GSM}} = \frac{\pi\delta^4}{\pi\delta^2} = \delta^2, \tag{A.6}$$

from which Eq. (19) follows immediately.

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