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Parametric competition in non-autonomous Hamiltonian systems

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ABSTRACT

In this work we use the formalism of chord functions (*i.e.* characteristic functions) to analytically solve quadratic non-autonomous Hamiltonians coupled to a reservoir composed by an infinity set of oscillators, with Gaussian initial state. We analytically obtain a solution for the characteristic function under dissipation, and therefore for the determinant of the covariance matrix and the von Neumann entropy, where the latter is the physical quantity of interest. We study in details two examples that are known to show dynamical squeezing and instability effects: the inverted harmonic oscillator and an oscillator with time dependent frequency. We show that it will appear in both cases a clear competition between instability and dissipation. If the dissipation is small when compared to the instability, the squeezing generation is dominant and one can see an increasing in the von Neumann entropy. When the dissipation is large enough, the dynamical squeezing generation in one of the quadratures is retained, thence the growth in the von Neumann entropy is contained.

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1. Introduction

The dynamics of quantum open systems has raised increasing interest of physicists specially in the last decades [1,2]: it can be directly connected to the non-observance of quantum phenomena in the classical world. The same phenomena which led Schroedinger to discredit his own theory is directly connected to the linear structure of the Hilbert space. Most of them have nowadays been observed and the usual approach as to why they are not present in our everyday life is to consider that quantum mechanics was first conceived for closed systems and effects of the surrounding environment, when included, tend to wash out quantum properties [1].

This problem is however far from being a closed issue and the classical limit of quantum mechanics is still a matter of enthusiastic debates [1–4]. In particular, the questions on quantum-to-classical transition acquire a singular aspect in the case of quantum systems with nonlinear or chaotic classical counterparts. If dissipation is absent, it is expected that instabilities yield the fast spreading of the wave function throughout the phase space for such systems, especially the macroscopic ones. Thus, an initially well localized wave packet will soon be fragmented throughout available regions of the phase space, and coherent superpositions will appear between the fragments, leading to a rapid breakdown of the correspondence between classical

and quantum descriptions. Some authors [5–9] advocate that the unavoidable interaction of a macroscopic system with its environment is essential to prevent the appearing of these quantum signatures yielded by inherent instabilities exhibited by the unitary evolution. Notwithstanding, other authors [10] sustain that the coupling with an environment is not necessary because such quantum effects are so tiny that they are not measurable, especially in the case of macroscopic objects. This controversy only stresses the importance of the study of the role played by instabilities in the question of quantum-to-classical transition.

In the present contribution, we are concerned to the questions: what happens if the unitary evolution, *i.e.* the Hamiltonian of the problem, may lead to instability? What role this instability effects does play? Examples of application of non-autonomous Hamiltonian systems can be found in a huge range of areas of physics, in particular: in quantum optics, where a harmonic oscillator with time dependent frequency is shown to generate squeezing [11,12], tunneling [13], exact solutions for mathematical problems and toy models [14], parametric amplification [15], quantum Brownian motion [16]. Most of these works employs the model of the harmonic oscillator with time dependent frequency. It is worth to mention that this model is largely studied both in classical and quantum physics and, as a merit, is amenable to analytical treatment. In fact, the time independent Schroedinger equation for the harmonic oscillator with time dependent frequency assumes the form of Hill differential equation, which, in turn, is a particular form of Pinney equation. Examples of Hill or Pinney equation in physics can be found in studies on synchrotron accelerators [17], anisotropic

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Bose–Einstein condensates [18,19], Paul traps [20], and cosmological models of particle creation [21]. Further, one of the first approaches to include dissipation in quantum physics employed a class of time dependent Hamiltonians, known as Caldirola–Kanai Hamiltonians [16,22]. Even in cosmology, in the inflationary era, when quantum effects are supposedly important, studies using non-autonomous Hamiltonians, leading to instabilities and squeezing effects are found [23]. One then frequently uses non-autonomous unitary evolutions of the same type, now modeling transitions between harmonic oscillators which give rise to particle formation [11]. Quantum chaos and instabilities also arise in recent experiments and theoretical models [24], rendering new perspectives to this interesting area in physics. It is interesting to note that, due the features shared by both models, some authors propose the Bose–Einstein condensates as a test bench of some cosmological scenarios [19].

Another interesting problem was raised by Zurek and coworkers [25] as to the rate of entropy increase when the system of interest is coupled not to a reservoir but to an unstable, two degrees of freedom system. In Ref. [14] the authors analytically showed that, in fact, entropy grows faster, but for that, chaos is not necessary (although sufficient). Instability alone already reflects this physics. Also, more realistically, as discussed in [14], the potential modelling Paul–Penning traps [26] has instability points which can be, to a certain degree, approximated by an inverted oscillator. What happens to the well known physics described, if an environment is added to the non-autonomous unitary dynamics? Can dissipation stop the inevitable acceleration caused by instabilities?

A word about the formal mathematical approach to the problem is in order: for autonomous systems, there are several possibilities to solve a master equation. One of the frequently used and powerful tools is that of Lie algebras of superoperators. Perhaps that is the reason why there is not so much work devoted to the question of non-autonomous systems evolving under nonunitary dynamics. As discussed above, however, several interesting issues may be cleared, if one manages to formulate the problem in appropriate language. In the present case, we will be considering single-mode Gaussian states. For these states, all we need are the second statistical moments or the covariance matrix, which can be gotten very simply as derivatives of the characteristic function (the Fourier transform of the Wigner function), by taking the derivatives of this function at the origin [27,28]. Moreover, a very elegant theoretical method for Wigner functions and nonunitary quadratic evolutions is given in Ref. [28]. It involves several classical elements, rendering the physics of the problem very transparent and the inclusion of nonunitary terms is natural.

In Section 2 we present an analytical solution for the characteristic function, using the most general bilinear Lindbladian (for dissipative reservoirs). We show our results for the inverted harmonic oscillator (IHO) and for a non-autonomous harmonic oscillator (NAHO) with frequency $\omega(t) = \omega_0 \sqrt{1 + \gamma t}$ in Section 3 and in the last section we make our final remarks.

2. Analytic solution for the Wigner and characteristic function

In this section we review some aspects concerning the evolution of single-mode Gaussian states under dissipation. The literature is plenty of references on this subject (theory and applications) [11,27–36]. To obtain our main result – analytical solutions for non-autonomous Hamiltonians – this section is, although straightforward, useful.

2.1. Unitary dynamics of single mode Gaussian states

We can define a general form of the Hamiltonian part of the equations of motion for both models studied in this work, namely,

the inverted harmonic oscillator (IHO) and the non-autonomous harmonic oscillator. The Hamiltonian reads

$$\hat{H}(\hat{q}, \hat{p}, t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{q}^2, \quad (1)$$

where \hat{q} and \hat{p} are position and linear momentum operators, respectively, m is the mass of the oscillator and $\omega(t)$ is a time-dependent frequency. If we take $\omega_0 = |\omega(0)|$, the annihilation and creation operators for $t=0$, \hat{a} and \hat{a}^\dagger , are given by $\hat{a} = \sqrt{m\omega_0/2\hbar}(\hat{q} + i(\hat{p}/m\omega_0))$ and $\hat{a}^\dagger = \sqrt{m\omega_0/2\hbar}(\hat{q} - i(\hat{p}/m\omega_0))$. The Hamiltonian above can be written as [35,36]

$$\hat{H}(\hat{a}, \hat{a}^\dagger, t) = \hbar[f_1(t)(\hat{a}^\dagger \hat{a} + \frac{1}{2}) + f_2(t)(\hat{a}^{\dagger 2} + \hat{a}^2)], \quad (2)$$

where $f_1(t) = \omega_0/2[(\omega(t)/\omega_0)^2 + 1]$ and $f_2(t) = \omega_0/4[(\omega(t)/\omega_0)^2 - 1]$.

In order to establish the notation, we will first present single-mode Gaussian states and its parameters, well known in the literature by several methods. The initial state is

$$\hat{\rho}(0) = \hat{D}(\alpha(0))\hat{S}(r(0), \phi(0))\hat{\rho}(\nu(0))\hat{S}^\dagger(r(0), \phi(0))\hat{D}^\dagger(\alpha(0)), \quad (3)$$

where all the parameters are given by the first and second moments:

$$\begin{aligned} \alpha &= \langle \hat{a} \rangle \\ \alpha^* &= \langle \hat{a}^\dagger \rangle \\ e^{i\phi} &= \sqrt{\frac{\sigma_{\hat{a}\hat{a}}}{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}}} \\ \nu &= \sqrt{\left(\sigma_{\hat{a}^\dagger\hat{a}} - \frac{1}{2}\right)^2 - \sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}} - \frac{1}{2}} \\ r &= \frac{1}{4} \ln \left(\frac{\sigma_{\hat{a}^\dagger\hat{a}} - \frac{1}{2} + \sqrt{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}}}}{\sigma_{\hat{a}^\dagger\hat{a}} - \frac{1}{2} - \sqrt{\sigma_{\hat{a}^\dagger\hat{a}^\dagger}\sigma_{\hat{a}\hat{a}}}} \right). \end{aligned} \quad (4)$$

In the equations above $\sigma_{\hat{a}^\dagger\hat{a}^\dagger} = \langle (\hat{a}^\dagger)^2 \rangle - \langle \hat{a}^\dagger \rangle^2$, $\sigma_{\hat{a}\hat{a}} = \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2$, $\sigma_{\hat{a}^\dagger\hat{a}} = \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + 1$. Those parameters are related to displacement (α), squeezing (r, ϕ) and “impurity” (ν) of the state. In our study the initial state will always be in this general single-mode Gaussian form and, since the dynamics is quadratic, the state will evolve as a single-mode Gaussian state [28].

One can study the state by analyzing the evolution of the parameters above, or the covariance matrix (CM):

$$\sigma = \begin{pmatrix} \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 & \frac{1}{2} \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle \\ \frac{1}{2} \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle & \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \end{pmatrix}. \quad (5)$$

2.2. Wigner and characteristic functions – dissipationless case

The Wigner function is defined as [32]

$$W(\vec{x}) = \frac{1}{2\pi\hbar} \int dq' \left\langle q + \frac{q'}{2} \left| \rho \right| q - \frac{q'}{2} \right\rangle \exp\left(-i \frac{pq'}{\hbar}\right), \quad (6)$$

where $\vec{x} = (p, q)$. It propagates “classically” for up to quadratic dynamics [28]:

$$\frac{\partial}{\partial t} W_t(\vec{x}) = \{H(\vec{x}), W_t(\vec{x})\}, \quad (7)$$

where $\{f, g\} = (\partial f/\partial q)(\partial g/\partial p) - (\partial f/\partial p)(\partial g/\partial q)$ is the classical Poisson bracket, and $H(\vec{x}) = \vec{x} \cdot \hat{H} \vec{x}$.

One can write the propagated Wigner functions as [37]

$$W_t(\vec{x}) = W_0(\mathbf{R}_t \vec{x}), \quad (8)$$

where

$$\mathbf{R}_t = \exp(2\Omega \hat{H} t), \quad (9)$$

and Ω is the symplectic form:

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (10)$$

Note that (9) reflects the evolution of a classical Hamiltonian for points in phase space, i.e., given $H(\vec{x})$, the time evolution of the variables is $\vec{x}_t = \mathbf{R}_t \vec{x}$, or, explicitly

$$\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} \dot{v}_t & \dot{u}_t \\ v_t & u_t \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \quad (11)$$

This “classical” feature will become clear in the two models we studied in this work. The quantum behavior, for instance the dynamical squeezing generation due to the non-autonomous Hamiltonian, will be sustained by the non-unitary dynamics in some conditions, since the “classicality” will prevail through the “quantumness” for these cases.

For the Hamiltonian given by Eq. (1), both functions u_t and v_t obey the following equation [11]:

$$\ddot{\phi}_t + \omega^2(t)\phi_t = 0. \quad (12)$$

Here ϕ_t represents both u_t or v_t . Two initial value problems are defined with the above equation, provided the initial conditions $u(0) = 1$, $\dot{u}(0) = 0$ or $v(0) = 0$, $\dot{v}(0) = 1$. From Eq. (11) one can see that

$$\mathbf{R}_t = \begin{pmatrix} \dot{v}_t & \dot{u}_t \\ v_t & u_t \end{pmatrix}. \quad (13)$$

The characteristic function is the Fourier transform of the Wigner function (6) and it is given by

$$\chi(\vec{\xi}) = \frac{1}{2\hbar\pi} \int d\vec{x} \exp\left(-\frac{i}{\hbar} \vec{\xi} \wedge \vec{x}\right) W(\vec{x}). \quad (14)$$

The “wedge” product in the above equation is defined by $\vec{\xi} \wedge \vec{x} = \xi_p q - \xi_q p$. Since we will be working with single-mode Gaussian states, it is easy to compute its initial ($t=0$) Wigner function as

$$W_0(V) = \frac{\exp\left(-\frac{1}{2} V \sigma^{-1} V^T\right)}{\pi \sqrt{\det \sigma}}, \quad (15)$$

where $V = (q, p)$ and σ is the covariance matrix. Thus, the characteristic function for the initial state is

$$\chi_0(\vec{\xi}) = \frac{1}{2\pi} \int dV \exp[-i(\xi_p q - \xi_q p)] W_0(V). \quad (16)$$

Evolving the Wigner function with Eq. (8), one can find the general solution of the characteristic function for the *dissipationless case*. Note the classical ingredient introduced by \mathbf{R}_t and Eq. (11) in the solution.

2.3. Wigner and characteristic functions – dissipative case

In this section we introduce nonunitary terms to the non-autonomous dynamics considered above. In order to do this, we suppose the system of interest coupled to a thermal bath at temperature T . The nonunitary contribution is given by

$$\dot{\hat{\rho}} = \mathcal{L}\hat{\rho}, \quad (17)$$

with

$$\begin{aligned} \mathcal{L} \cdot = & -i[\hat{H}(t), \cdot] + k(\bar{n}_B + 1)(2\hat{a} \cdot \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \cdot - \cdot \hat{a}^\dagger \hat{a}) \\ & + k\bar{n}_B(2\hat{a}^\dagger \cdot \hat{a} - \hat{a} \hat{a}^\dagger \cdot - \cdot \hat{a} \hat{a}^\dagger). \end{aligned} \quad (18)$$

Here, $\hat{H}(t)$ is the non-autonomous Hamiltonian presented in Eq. (1), k is a dissipation constant and \bar{n}_B is the average number of thermal excitations of the bath.

It is relatively simple, using the approach presented in this section, plus Gaussian states and quadratic Hamiltonians, to obtain the characteristic function [28]:

$$\chi_t(\vec{\xi}) = \chi_0(\vec{\xi}_{-t}) \exp\left(-\frac{1}{2} \vec{\xi} \cdot \mathbf{M}(t) \vec{\xi}\right), \quad (19)$$

where

$$\mathbf{M}(t) = \sum_j \int_0^t dt' e^{2k(t'-t)} \mathbf{R}_{t-t'}^T (\mathbf{I}_j^T + \mathbf{I}_j^{\prime T}) \mathbf{R}_{t-t'}, \quad (20)$$

with $j=1,2$. The variables $\vec{\xi}$ evolve in time as

$$\vec{\xi}_t = e^{kt} \mathbf{R}_t \vec{\xi}. \quad (21)$$

For the specific case of Eq. (18) we have found

$$\mathbf{I}_1 = \begin{pmatrix} 0 \\ \sqrt{k(\bar{n}_B + 1)} \\ 0 \end{pmatrix}, \quad \mathbf{I}_1' = \begin{pmatrix} \sqrt{k(\bar{n}_B + 1)} \\ 0 \end{pmatrix},$$

$$\mathbf{I}_2 = \begin{pmatrix} 0 \\ \sqrt{k\bar{n}_B} \\ 0 \end{pmatrix}, \quad \mathbf{I}_2' = \begin{pmatrix} -\sqrt{k\bar{n}_B} \\ 0 \end{pmatrix},$$

so that, in matrix (20): $\mathbf{I}_1 \mathbf{I}_1^T + \mathbf{I}_1' \mathbf{I}_1'^T = k(\bar{n}_B + 1) \mathbf{1}$ e $\mathbf{I}_2 \mathbf{I}_2^T + \mathbf{I}_2' \mathbf{I}_2'^T = k\bar{n}_B \mathbf{1}$.

Notice that the apparent *form* of the characteristic function is very similar to the free case. The influence of the nonunitary dynamics is contained in the matrix $\mathbf{M}(t)$ and in the evolution of the variable $\vec{\xi}$ (21). The term \mathbf{R}_t contain the classical evolution. In order to obtain dissipative effects, we have computed analytically $\mathbf{M}(t)$ for both studied cases.

The physics about the system is contained in the elements of the covariance matrix (CM), since the CM completely define any Gaussian state. Dissipative effects are related to the determinant of the covariance matrix, $D(t) = \det \sigma$, which can be analytically calculated from the derivatives of the characteristic function. The von Neumann entropy for single-mode Gaussian states is completely defined by [30,31]

$$S(t) = \left(\sqrt{D(t)} + \frac{1}{2}\right) \ln\left(\sqrt{D(t)} + \frac{1}{2}\right) - \left(\sqrt{D(t)} - \frac{1}{2}\right) \ln\left(\sqrt{D(t)} - \frac{1}{2}\right). \quad (22)$$

In the following section, we will obtain analytical results for the von Neumann entropy of two important examples, showing that is a clear competition between dissipative and instability effects.

3. Results

We are working with quadratic Hamiltonians (1), and as discussed in the introduction, the time dependence in the oscillator frequency can generate squeezing [11,12]. Dissipative effects will be fully reflected by the von Neumann entropy or indirectly by $D(t)$: if the state is isolated, its entropy will be constant; if the state is coupled to a thermal reservoir, the von Neumann entropy will change in time [38]. In this section we show that a competition between the dissipation constant and the non-autonomous frequency appear, and the dissipation can sustain a possible increasing in the von Neumann entropy due to instability effects (as dynamical squeezing generation). The general Hamiltonian (unitary part of the dynamics) in this section is in the form given by (1): $\hat{H}(\hat{q}, \hat{p}, t) = (\hat{p}^2/2m) + \frac{1}{2} m \omega^2(t) \hat{q}^2$.

3.1. The inverted harmonic oscillator (IHO)

Let us consider first the toy model of the inverted harmonic oscillator [25,39] where $\omega(t) = i\omega_0$, with ω_0 a real constant. For the

IHO, the matrix \mathbf{R}_t is (where we have set $\omega_0 = 1$):

$$\mathbf{R}_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{23}$$

The matrix $\mathbf{M}(t)$ is

$$\mathbf{M}(t) = 2k \left(\bar{n}_B + \frac{1}{2} \right) \int_{-t}^0 dx e^{2kx} \left[\cosh 2x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh 2x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]. \tag{24}$$

In Fig. 1 we show the determinant $D(t)$ for three values of the dissipative constant k . Note that when k is large enough, compared with the frequency ω_0 , the determinant goes to a constant value, larger than the initial one. The same behavior is reflected in Fig. 2, where we show the von Neumann entropy for this case.

In Ref. [14], the authors showed that the coupling between a Gaussian state in a harmonic oscillator ($|\psi_A\rangle$) with another Gaussian state in an IHO ($|\psi_B\rangle$) will produce, in the reduced state $\rho_A = \text{tr}_B |\psi_A \psi_B\rangle \langle \psi_A \psi_B|$, a linear increasing in the time evolution of von Neumann entropy. The authors argued that the instability (and therefore the squeezing generation) of the IHO will generate this effect. Here we can see that, if $k/\omega_0 > 1$ the dissipation will sustain the entropy growing, to a limit where $S(t)$ becomes constant. Otherwise, if the dissipation is lesser than the “intensity” of the instability (measured by ω_0), the entropy will increase monotonically, clearly tending to a linear increasing (as in [14]). We argue here that the dissipation will suppress the squeezing generated by the IHO.

One can make a simple classical analogy in this case: suppose a simple rod which can spin up and down, by one of its end. If the rod is close to the bottom, it will ideally oscillates as a harmonic

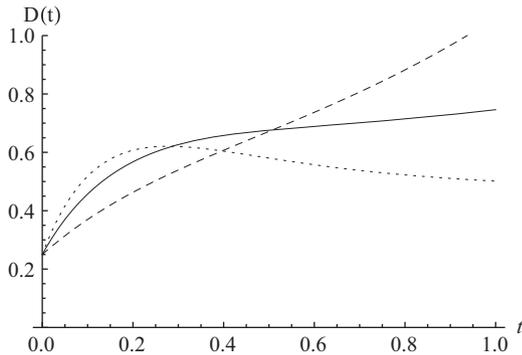


Fig. 1. Determinant of the covariance matrix $D(t)$ as a function of time for the IHO. Parameters: $\omega_0 = 1$, $r_0 = 1$, $\nu_0 = 0$, $\bar{n}_B = 0$. The dissipation constants are $k=0.5$ (dashed), $k=1$ (solid) and $k=1.5$ (dotted).

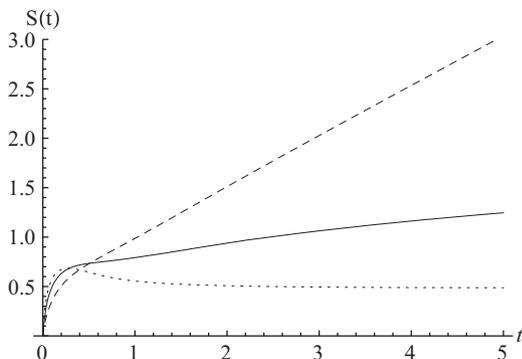


Fig. 2. Von Neumann entropy as a function of time for the IHO. Parameters: $\omega_0 = 1$, $r_0 = 1$, $\nu_0 = 0$, $\bar{n}_B = 0$. The dissipation constants are $k=0.5$ (dashed), $k=1$ (solid) and $k=1.5$ (dotted).

oscillator; but if the rod is in the top, and is released in a not so viscous fluid (as the air), it will oscillates and tend to a configuration totally different to the initial one (the rod in the top). But if the rod is released from the top in a viscous fluid (in the water or in a more exceeded example in a tar pit), the viscosity will retain the rod rotation. This classical behavior in the quantum IHO is pictured in the formalism given in Section 2, specially in the quantities \mathbf{R}_t and \mathbf{M}_t .

3.2. Simple model for dynamical squeezing generation

We consider now the following dynamics: two harmonic oscillators Hamiltonians (autonomous evolution) with different natural frequency ω_i , separated each other by a non-autonomous Hamiltonian with modulated frequency $\omega(t)$. We will be concerned with the part of the time dependence which leads from the first to the second oscillators regime (Fig. 3).

In this model, when the state evolves trough the first to the second harmonic oscillator, one can dynamically produce single-mode squeezing, and therefore $\langle \hat{a}^\dagger \hat{a} \rangle \neq 0$. We have used the following time dependence for the modulated frequency (as proposed in [11]):

$$\omega(t) = \omega_0 \sqrt{1 + \gamma t}. \tag{25}$$

For this time dependence, one can calculate the matrix \mathbf{R}_t (13), where the functions u and v satisfy ($u, v \rightarrow \phi_\tau$)

$$\ddot{\phi}_\tau + \omega_0^2 (1 + \gamma \tau) \phi_\tau = 0, \tag{26}$$

for the following initial conditions: $u(0) = 1$, $\dot{u}(0) = 0$, $v(0) = 0$ and $\dot{v}(0) = 1$. The solution is given by (where we have set $\omega_0 = 1$)

$$\phi_\tau = \text{Ai} \left[-\frac{1 + \gamma \tau}{(-\gamma)^{2/3}} \right] C_1 + \text{Bi} \left[-\frac{1 + \gamma \tau}{(-\gamma)^{2/3}} \right] C_2, \tag{27}$$

where C_1 and C_2 are constants, Ai and Bi are the Airy functions and $\tau = \omega_0 t$.

Since we have the functions u and v , we can write the matrix \mathbf{R}_t and compute $\mathbf{M}(t)$:

$$\mathbf{M}(t) = 2k \left(\bar{n}_B + \frac{1}{2} \right) \int_{-t}^0 dx e^{2kx} \mathbf{R}_x^T \mathbf{R}_x. \tag{29}$$

Now we are able to study the physical quantity of interest, the von Neumann entropy.

For this model, we obtain the results given in Figs. 4 and 5, respectively the von Neumann entropy for a Gaussian state evolving through a unitary dynamics with frequency $\omega(t) = \omega_0 \sqrt{1 + \gamma t}$ for a initially pure state ($\nu_0 = 0$) and for a thermal state

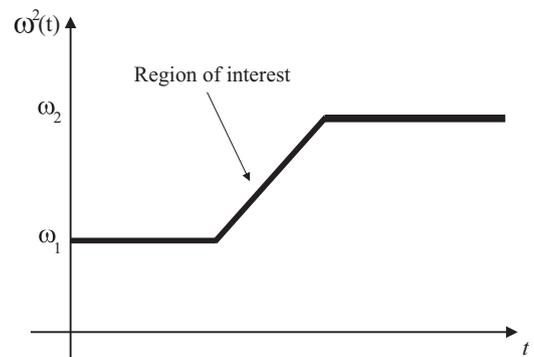


Fig. 3. A pictorial image of the non-autonomous harmonic oscillator model. Our interest is to study the important region between the two harmonic oscillators, due to squeezing generation and instability effects.

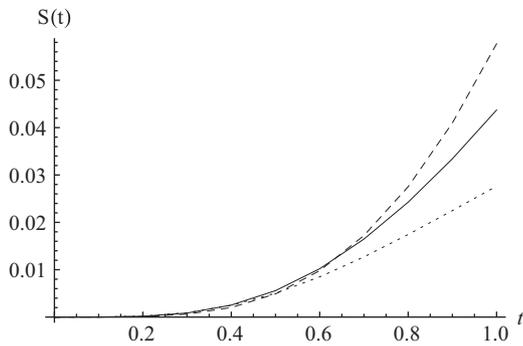


Fig. 4. Time evolution of the von Neumann entropy for a initially pure Gaussian state ($\nu_0 = 0$) evolving in an unitary Hamiltonian with frequency $w(t) = w_0 \sqrt{1 + \gamma t}$ coupled to a thermal reservoir with $\bar{n}_B = 0$. Parameters: $\omega_0 = 1$, $r_0 = 0$, $\gamma = 1$. The dissipation constants are: $k = 0.5$ (dashed), $k = 1.0$ (solid) and $k = 1.5$ (dotted).

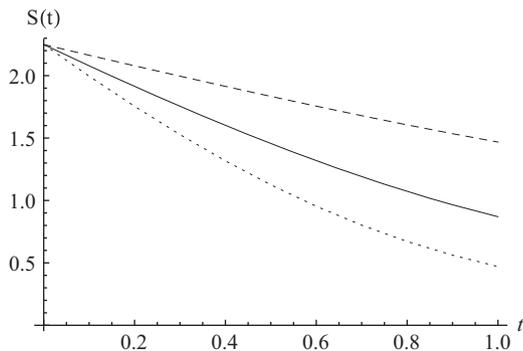


Fig. 5. Time evolution of the von Neumann entropy for a initially non-pure Gaussian state ($\nu_0 = 3$) evolving in an unitary Hamiltonian with frequency $w(t) = w_0 \sqrt{1 + \gamma t}$ coupled to a thermal reservoir with $\bar{n}_B = 0$. Parameters: $\omega_0 = 1$, $r_0 = 0$, $\gamma = 1$. The dissipation constants are $k = 0.5$ (dashed), $k = 1.0$ (solid) and $k = 1.5$ (dotted).

($\nu_0 = 3$), both coupled to a reservoir at zero temperature. We have used those values of ν based on [38], since in this work the authors showed that there is a maximum value of ν for which the state present visible squeezing. Note the competition between unitary and nonunitary effects: the latter will reduce the former always. The amount of the squeezing suppression is governed by k/ω_0 .

This example can be used for various purposes, from Paul traps [20] to cosmological models concerning the initial Universe [23]. Since the dynamical squeezing generation is suppressed by the dissipation, one can conjecture, for example, that the increasing in the average number of particles (also generated dynamically: $\langle \hat{a}^\dagger \hat{a} \rangle \neq 0$) will be contained by the dissipation. This application is direct implications for inflation models in cosmology related to particle and anti-particle generation.

4. Conclusion

In the present contribution, we have shown that the Wigner formalism as constructed in Ref. [28] is most adequate for handling non-autonomous dissipative systems. This calculation facility comes mainly from working with the characteristic function and Gaussian states.

We have been able to show that squeezing generation as observed by time dependent frequencies of the harmonic oscillator may be limited by the process of decoherence. As to instabilities here simulated simple by two examples ($\omega = i\omega_0$ and $\omega(t) = \omega_0 \sqrt{1 + \gamma t}$) a clear competition between instability and dissipation appears. Interestingly enough, for large dissipation (in comparison with the

natural frequency ω_0), the time evolution of the quadratic variances, or the entropy, reaches an asymptotic limit.

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