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Electromagnetic wave scattering by a small impedance particle of arbitrary shape

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ABSTRACT

Scattering of electromagnetic (EM) waves by one small ($ka \ll 1$) impedance particle (body) D of arbitrary shape, embedded in a homogeneous medium, is studied. Physical properties of the particle are described by its boundary impedance. The problem is of interest because scattering of light by colloidal particles, or by dust in the air is an example of the scattering theory discussed in this paper. An analytic formula is obtained for the EM field in the far zone without usage of boundary integral equation. If a monochromatic incident field of frequency ω is $E_0(x, \omega)$, then the scattered field v in the zone $r := |x| \gg a$, where $a = 0.5 diamD$ is the characteristic size of D, is calculated by the formula $v = \left[\nabla \frac{e^{ikr}}{4\pi r}, Q\right]$, where $[A, B]$ is the cross product of two vectors, (Q, e_j) is the dot product, e_j, $1 \le j \le 3$, are orthonormal basis vectors in $\mathbb{R}^3$, $Qj := (Q, e_j) = -\frac{i\zeta|S|}{\omega\mu_0}\tau_{jp}(\nabla\times E_0(O))_p$, over the repeated index p summation is understood from Eqs. (1) to (3), ζ is the boundary impedance and $|S|$ is the surface area of the particle, $O \in D$ is the origin, the tensor $\tau_{jp} := \delta_{jp} - |S|^{-1}\int_S N_j(s)N_p(s)ds$, where $N_j(s)$ is the j-th component of the unit normal $N(s)$ to the surface S at the point $s \in S$, $k = \omega(\epsilon_0\mu_0)^{1/2}$ is the wave number.

1. Introduction

In this paper we propose a theory of electromagnetic (EM) wave scattering by one small ($ka \ll 1$, $a = 0.5 diamD$) impedance particle (body) D, embedded in a homogeneous medium which is described by the constant permittivity $\varepsilon_0 > 0$, permeability $\mu_0 > 0$ and, possibly, constant conductivity $\sigma_0 \ge 0$. Although scattering of EM waves by small bodies has a long history, going back to Rayleigh (1871), see [1], [16], the result of this paper is new. It might be useful in applications because light scattering by colloidal particles in a solution, and light scattering by small dust particles in the air are examples of the problems to which our theory is applicable. The Mie theory deals with scattering by a sphere, not necessarily small, and gives the solution to the scattering problem in terms of the series in spherical harmonics. If the sphere is small, $ka \ll 1$, then the first term in the Mie series yields the main part of the solution. Our theory is applicable only to small particles, which can be of arbitrary shapes, and gives the solution explicitly, not in the form of series in special functions, as in Mie theory.

Wave scattering problems can be studied theoretically only in the limiting cases of scattering by small particles, $ka \ll 1$, or large bodies, $ka \gg 1$, in which case geometrical optics is applicable. This paper deals with the case $ka \ll 1$. Rayleigh (1871) understood that the scattering by a small body is given mainly by the dipole radiation. For a small body of arbitrary shape this dipole radiation is determined by the polarization moment, which is defined by the polarizability tensor. For homogeneous bodies of arbitrary shapes analytical formulas, which allow one to calculate this tensor with any desired accuracy, were derived in [16]. These bodies were assumed dielectric or conducting in [16].

In this paper we want to study wave scattering by small impedance particles. The reason is: we wish to consider subsequently the EM wave scattering by many small impedance particles with the objective to develop a method for creating materials with a desired refraction coefficient by embedding many small impedance particles into a given material. Such a theory has been developed by the author for scalar wave scattering, e.g., acoustic wave scattering, in a series of papers [3–15,17]. The novel physical idea is to reduce solving the scattering problem to finding some constant vector Q (see formula (26)), rather than a vector function σ (see formula (11)) on the surface of the scatterer. The vector Q is analogous to the total charge on the surface of the scatterer D, while the function σ is analogous to the surface current density. We assume for simplicity that the impedance ζ (see formula (5)) is a constant given in Eq. (24). The reason for this assumption comes from paper [4], where this assumption was used in scalar wave scattering theory. The result of this theory was a recipe for creating materials with a desired refraction coefficient in acoustics ([12], [13]). The key point is: the boundary impedance in Eq. (24) grows as $a \to 0$, and allows one to pass to the limit in the equation for the effective (self-consistent) field in the medium, obtained by embedding many small impedance particles into a given medium. Such a theory is briefly summarized in paper [12], where the equation for the limiting field in the medium is given. Our aim in this paper is to prepare a way for developing a similar theory for EM wave scattering by many small impedance particles embedded in a given material.

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An analytic formula for the electromagnetic field in the region $r := |x| \gg a$, is derived:

$$E(x) = E_0(x) + \left[\nabla \frac{e^{ikr}}{4\pi r}, Q\right], \qquad r \gg a, \tag{1}$$

where E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D, $[A, B] = A \times B$ is the cross product of two vectors, $(Q, e_j) = Q \cdot e_j$ is the dot product, $\{e_j\}_{j=1}^3$ is the orthonormal basis in $\mathbb{R}^3$,

$$Q_j := \left(Q, e_j\right) = -\frac{i\zeta|S|}{\omega\mu_0}\tau_{jp}(\nabla \times E_0(O))_p, \tag{2}$$

over the repeated index p summation is understood from Eqs. (1) to (3), ζ is the boundary impedance, $|S|$ is the surface area of the particle, the tensor τ_{jp} is defined by the formula

$$\tau_{jp} := \delta_{jp} - |S|^{-1}\int_S N_j(s)N_p(s)ds, \tag{3}$$

where $N_j(s)$ is the j-th component of the unit normal $N(s)$ to the surface S at the point $s \in S$, $k = \omega(\varepsilon_0\mu_0)^{1/2}$ is the wave number, and $O \in D$ is the origin. By S^2 we denote the unit sphere in $\mathbb{R}^3$. The boundary S of the small body D we assume smooth: it is sufficient to assume that in local coordinates the equation of S is given as $x_3 = \phi(x_1, x_2)$, where the function ϕ has first derivative satisfying a Hölder condition.

Formulas (1)–(3) are the main results of this paper. The paper is addressed to experimentalists by the following reason: the assumption, made in the derivations, that the surface divergence of E vanishes, is not justified. However, in practice it may be that the formulas (1)–(3) will be useful. Experimental verification of these formulas is of interest.

The scattering problem is formulated and studied in the next two Sections and in the Appendix. In this paper we do not try to solve the boundary integral equation to which the scattering problem is reduced, but rather find asymptotically exact analytic expression for the vector Q which defines the behavior of the field at distances much greater than the size a of the small scatterer. In fact, these distances d can be very close to the scatterer: if a is sufficiently small then d can be less than the wavelength $\lambda = \frac{2\pi}{k}$.

2. EM wave scattering by one small impedance particle

Let a small body D, $ka \ll 1$, $a = 0.5 diam D$, $k > 0$ is a wavenumber, $k = \frac{2\pi}{\lambda}$, λ is the wavelength of the incident EM wave, be embedded in a homogeneous medium with constant parameters ϵ_0, μ_0. Let $k^2 = \omega^2\epsilon_0\mu_0$, where ω is the frequency. Our arguments remain valid if one assumes that the medium has a constant conductivity $\sigma_0 > 0$. In this case ϵ_0 is replaced by $\epsilon_0 + i\frac{\sigma_0}{\omega}$. Denote by S the boundary of D, by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors.

Electromagnetic (EM) wave scattering problem consists of finding vectors E and H satisfying the Maxwell's equations:

$$\nabla \times E = i\omega\mu_0 H, \ \nabla \times H = -i\omega\epsilon_0 E \quad \text{in } D' := \mathbb{R}^3 \setminus D, \tag{4}$$

the impedance boundary condition:

$$[N, [E, N]] = \zeta[H, N] \text{on } S \tag{5}$$

and the radiation condition:

$$E = E_0 + v_E, \quad H = H_0 + v_H, \tag{6}$$

where ζ is the boundary impedance of the particle, N is the unit normal to S pointing out of D, E_0, H_0 are the incident fields satisfying Eq. (4) in all of $\mathbb{R}^3$, $v_E = v$ and v_H are the scattered fields. One often assumes that the incident wave is a plane wave, i.e., $E_0 = \varepsilon e^{ik\alpha\cdot x}$, ε is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \varepsilon = 0$, v_E and v_H satisfy the radiation condition: $r\left(\frac{\partial v}{\partial r} - ikv\right) = o(1)$ as $r := |x| \to \infty$.

For simplicity, we assume in this paper that the impedance ζ is a constant, Re $\zeta \geq 0$. One could assume that ζ is a matrix function 2×2 acting on the tangential to S vector fields, such that

$$\mathrm{Re}\left(\zeta E^t, E^t\right) \geq 0 \quad \forall E^t \in T, \tag{7}$$

where T is the set of all tangential to S continuous vector fields such that Div$E^t = 0$, where Div is the surface divergence, and E^t is the tangential component of E. By the tangential to S component E^t of a vector field E the following is understood in this paper:

$$E^t = E - N(E, N) = [N, [E, N]]. \tag{8}$$

This definition differs from the one used often in the literature, namely, from the definition $E^t = [N, E]$. Our definition (8) corresponds to the geometrical meaning of the tangential component of E and, therefore, should be used. The impedance boundary condition is written usually as

$$E^t = \zeta\left[H^t, N\right],$$

where ζ is the boundary impedance. If one uses definition (8), then this condition reduces to Eq. (5), because $[[N,[H, N]], N] = [H, N]$. The assumption Re$\zeta \geq 0$ is physically justified by the fact that this assumption guarantees the uniqueness of the solution to the boundary problem in Eqs. (4)–(7).

Lemma 1. *Problem in Eqs. (4)–(7) has at most one solution.*

Lemma 1 is proved in the next Section.

Let us note that problem in Eqs. (4)–(7) is equivalent to the problems in Eq. (9), (10), (6), and (7), where

$$\nabla \times \nabla \times E = k^2 E \text{ in } D', \quad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{9}$$

$$[N, [E, N]] = \frac{\zeta}{i\omega\mu_0}[\nabla \times E, N] \text{on } S. \tag{10}$$

Thus, we have reduced our problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla \times E}{i\omega\mu_0}$, and the pair E and H solves the Maxwell's equations and satisfies the impedance boundary condition.

Let us look for E of the form

$$E = E_0 + \nabla \times \int_S g(x, t)\sigma(t)dt, \qquad g(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{11}$$

where E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D, t is a point on the surface S, $t \in S$, dt is an element of the area of S, and $\sigma(t)$ is an unknown vector-function on S, which is tangential to S, i.e., $N(t) \cdot \sigma(t) = 0$, where $N(t)$ is the unit normal to S at the point $t \in S$. This E solves Eq. (9) in D for any continuous $\sigma(t)$, because E_0 solves Eq. (9) and

$$\begin{aligned} \nabla \times \nabla \times \nabla \times \int_S g(x, t)\sigma(t)dt &= \nabla\nabla\cdot\nabla \times \int_S g(x, t)\sigma(t)dt \\ -\nabla^2\nabla \times \int_S g(x, t)\sigma(t)dt &= k^2\nabla \times \int_S g(x, t)\sigma(t)dt, \quad x \in D'. \end{aligned} \tag{12}$$

Here we have used the known identity div $curl E = 0$, valid for any smooth vector field E, and the known formula

$$-\nabla^2 g(x, y) = k^2 g(x, y) + \delta(x-y). \tag{13}$$

The integral $\int_S g(x, t)\sigma(t)dt$ satisfies the radiation condition. Thus, formula (11) solves problem in Eqs. (9), (10), (6), and (7), if $\sigma(t)$ are chosen so that boundary condition (10) is satisfied.

Let $O \in D$ be a point inside D. To derive an integral equation for $\sigma = \sigma(t)$, substitute $E(x)$ from Eq. (11) into impedance boundary condition (10), use the known formula (see, e.g., [2]):

$$\left[N, \nabla \times \int_S g(x,t)\sigma(t)dt\right]_{\mp} = \int_S [N_s, [\nabla_x g(x,t)|_{x=s}, \sigma(t)]]dt \pm \frac{\sigma(s)}{2}, \quad (14)$$

where the $\pm$ signs denote the limiting values of the left-hand side of Eq. (14) as $x \to s$ from D, respectively from D', and get the following equation:

$$\sigma(t) = A\sigma + f, \qquad A\sigma = -2[N_s, B\sigma]. \quad (15)$$

Here A is a linear Fredholm-type integral operator, where the operator B is defined by formula (21), and f is a continuously differentiable vector function, defined by formula (16).

Let us find formulas for A and f. Eq. (14) is derived in Appendix and there the formulas for f and A are obtained.

One has:

$$f := 2[f_e(s), N_s], \quad f_e(s) := [N_s, [E_0(s), N_s]] - \frac{\zeta}{i\omega\mu_0}[\nabla \times E_0, N_s]. \quad (16)$$

Boundary condition (10) and formula (14) yield

$$\begin{aligned} f_e(s) + \frac{1}{2}[\sigma(s), N_s] + \left[\int_S [N_s, [\nabla_s g(s,t), \sigma(t)]]dt, N_s\right] \\ - \frac{\zeta}{i\omega\mu_0}\left[\nabla \times \nabla \times \int_S g(x,t)\sigma(t)dt, N_s\right]\Big|_{x \to s} = 0. \end{aligned} \quad (17)$$

Using the known formula $\nabla \times \nabla \times = graddiv - \nabla^2$, the relation

$$\begin{aligned} \nabla_x \nabla_x \cdot \int_S g(x,t)\sigma(t)dt &= \nabla_x \int_S (-\nabla_t g(x,t), \sigma(t))dt \\ &= \nabla_x \int_S g(x,t) \mathrm{Div}\sigma(t)dt = 0, \end{aligned} \quad (18)$$

where Div is the surface divergence, and the formula

$$-\nabla_x^2 \int_S g(x,t)\sigma(t)dt = k^2 \int_S g(x,t)\sigma(t)dt, \quad x \in D, \quad (19)$$

where Eq. (12) was used, one gets from Eq. (17) the following equation

$$-[N_s, \sigma(s)] + 2f_e(s) + 2B\sigma = 0. \quad (20)$$

Here

$$B\sigma := \left[\int_S [N_s, [\nabla_s g(s,t), \sigma(t)]]dt, N_s\right] + \zeta i\omega\epsilon_0 \left[\int_S g(s,t)\sigma(t)dt, N_s\right]. \quad (21)$$

Take cross product of N_s with the left-hand side of Eq. (20) and use the formulas $N_s \cdot \sigma(s) = 0$, $f := f(s) := 2[f_e(s), N_s]$, and

$$[N_s, [N_s, \sigma(s)]] = -\sigma(s), \quad (22)$$

to get from Eq. (20) Eq. (15):

$$\sigma(s) = 2[f_e(s), N_s] - 2[N_s, B\sigma] := A\sigma + f, \quad (23)$$

where $A\sigma = -2[N_s, B\sigma]$. The operator A is linear and compact in the space $C(S)$, so that Eq. (23) is of Fredholm type. Therefore, Eq. (23) is solvable for any $f \in T$ if the homogeneous version of Eq. (23) has only the trivial solution $\sigma = 0$. In this case the solution σ to Eq. (23) is of the order of the right-hand side f, that is, $O(a^{-\kappa})$ as $a \to 0$, see formula (16). Moreover, it follows from Eq. (23) that the main term of the asymptotics of σ as $a \to 0$ does not depend on $s \in S$. The role of the assumption concerning the surface divergence-free vector field σ is interesting to verify by numerical simulation of the theory, proposed in this paper.

Lemma 2. *Assume that σ is a smooth tangential vector-field on S, and $\sigma(s) = A\sigma$. Then $\sigma = 0$.*

Lemma 2 is proved in the next Section.

We assume that

$$\zeta = \frac{h}{a^{\kappa}}, \quad (24)$$

where Re $h \geq 0$ and $\kappa \in [0, 1)$ is a constant.

Let us write Eq. (11) as

$$E(x) = E_0(x) + [\nabla_x g(x,O), Q] + \nabla \times \int_S (g(x,t) - g(x,O))\sigma(t)dt, \quad (25)$$

where

$$Q := \int_S \sigma(t)dt. \quad (26)$$

The central physical idea of the theory, developed in this paper, is simple: one can neglect the second sum in Eq. (25) compared with the first sum, if $ka \ll 1$. Consequently, the scattering problem is solved if one vector Q is found, rather than an unknown function $\sigma(t)$, which is usually found numerically by the boundary integral equations (BIE) method. The reason for the second sum in Eq. (25) to be negligible, compared with the first one, is explained by the estimates, given below. In these estimates the smallness of the body is used essentially: even if one is in the far zone, i.e., $\frac{a}{d} \ll 1$, one cannot conclude that estimate (29) holds unless one assumes that $ka \ll 1$. Thus, the second sum in Eq. (25) cannot be neglected in the far zone if the condition $ka \ll 1$ does not hold.

Since $\sigma = O(a^{-\kappa})$, one has $Q = O(a^{2-\kappa})$. We want to prove that the second sum in Eq. (25) is negligible compared with the first one. This proof is based on several estimates.

We assume in these estimates that $a \to 0$, and $d := |x - O| \gg a$. Under these assumptions one has

$$j_1 := |[\nabla_x g(x,O), Q]| \leq O\left(\max\left\{\frac{1}{d^2}, \frac{k}{d}\right\}\right) O\left(a^{2-\kappa}\right), \quad (27)$$

$$j_2 := |\nabla \times \int_S (g(x,t) - g(x,O))\sigma(t)dt| \leq aO\left(\max\left\{\frac{1}{d^3}, \frac{k^2}{d}\right\}\right) O\left(a^{2-\kappa}\right), \quad (28)$$

and

$$\left|\frac{j_2}{j_1}\right| = O\left(\max\left\{\frac{a}{d}, ka\right\}\right) \to 0, \qquad \frac{a}{d} = o(1), \qquad a \to 0. \quad (29)$$

These estimates show that one may neglect the second sum in Eq. (25), and write

$$E(x) = E_0(x) + [\nabla_x g(x,O), Q] \quad (30)$$

with an error that tends to zero as $a \to 0$ under our assumptions.

Note that the assumption $|x| \gg ka^2$, describing far zone, is satisfied for $d = O(a)$ if $ka \ll 1$. Thus, formula (30) is applicable in a wide region.

Let us estimate Q asymptotically, as $a \to 0$.

Integrate Eq. (22) over S to get

$$Q = 2\int_S [f_e(s), N_s]ds - 2\int_S [N_s, B\sigma]ds. \quad (31)$$

We will show in the Appendix that the second term in the right-hand side of the above equation is equal to $-Q$ plus terms negligible compared with $|Q|$ as $a \to 0$. Thus,

$$Q = \int_S [f_e(s), N_s]ds, \qquad a \to 0. \quad (32)$$

Let us estimate the integral in the right-hand side of Eq. (32).

It follows from Eq. (16) that

$$[N_s,f_e] = [N_s,E_0] - \frac{\zeta}{i\omega\mu_0}[N_s,[\nabla\times E_0,N_s]]. \tag{33}$$

If E_0 tends to a finite limit as $a\to 0$, then formula (33) implies that

$$[N_s,f_e] = O(\zeta) = O\left(\frac{1}{a^{\kappa}}\right), \quad a\to 0. \tag{34}$$

By Lemma 2, the operator $(I-A)^{-1}$ is bounded, so $\sigma = O\left(\frac{1}{a^{\kappa}}\right)$, and

$$Q = O\left(a^{2-\kappa}\right), \quad a\to 0, \tag{35}$$

because the integration over S adds factor $O(a^2)$. It will follow from our arguments that Q does not vanish at almost all points.

The Q can be expressed in terms of E_0. If S is a sphere of radius a then

$$Q = -\frac{8\pi i a^{2-\kappa}}{3\omega\mu_0}h(\nabla\times E_0(O)). \tag{36}$$

This important formula is derived in Appendix.

The factor $\frac{8\pi}{3}$ appears if D is a ball. Otherwise a tensorial factor τ_{jp} appears:

$$Q_j := \left(Q,e_j\right) = -\frac{i\zeta|S|}{\omega\mu_0}\tau_{jp}(\nabla\times E_0(O))_p, \tag{37}$$

where over repeated index p summation from Eqs. (1) to (3) is assumed, and

$$\tau_{jp} = \delta_{jp} - b_{jp}, \qquad b_{jp} := \frac{1}{|S|}\int_S N_j N_p ds, \tag{38}$$

where δ_{jp} is the Kronecker delta, and b_{jp} depends on the shape of S. If S is a sphere, then $b_{jp} = \frac{1}{3}\delta_{jp}$. In this case one gets formula (36), where ζ is assumed to be as in Eq. (24).

From Eqs. (36) and (37) one obtains

$$E(x) = E_0(x) - \frac{i\zeta|S|}{\omega\mu_0}[\nabla_x g(x,O),\tau\nabla\times E_0(O)]. \tag{39}$$

In the far zone $r:=|x|\to\infty$ one has $\nabla_x g(x,O) = ikg(x,O)x^0 + O(r^{-2})$, where $x^0 := x/r$ is a unit vector in the direction of x. Consequently, for $r\to\infty$ one can rewrite formula (39) as

$$E(x) = E_0(x) - \frac{i\zeta|S|}{\omega\mu_0}ik\frac{e^{ikr}}{r}\left[x^0,\tau\nabla\times E_0(O)\right]. \tag{40}$$

This field is orthogonal to the radius-vector x in the far zone.

Conclusion. *The field $E(x)$ is given by formula (39) in the region $r\gg a$.*

3. Proofs of Lemmas

Proof of Lemma 1. From Eq. (4) one derives (the bar stands for complex conjugate):

$$\int_{D_R}(\overline{H}\cdot\nabla\times E - E\cdot\nabla\times\overline{H})dx = \int_{D_R}\left(i\omega\mu_0|H|^2 - i\omega\epsilon_{\varepsilon 0}|E|^2\right)dx,$$

where $D_R := D\cap B_R$, and $R>0$ is so large that $D\subset B_R := \{x:|x|\le R\}$. Recall that $\nabla\cdot\left[E,\overline{H}\right] = \overline{H}\cdot\nabla\times E - E\cdot\nabla\times\overline{H}$. Applying the divergence theorem, using the radiation condition on the sphere $S_R = \partial B_R$, and taking real part, one gets

$$0 = \mathrm{Re}\int_S\left[E,\overline{H}\right]\cdot N ds = \sum\mathrm{Re}\int_S\overline{\zeta}^{-1}\overline{E}_t^-\cdot E_t^- ds,$$

where E_t^- is the limiting value of E^t on S from D', $E^t = \zeta[H,N]$. This relation and assumption (7) imply $E_t^- = 0$ on S. Thus, $E = H = 0$ in D. Lemma 1 is proved. □

Proof of Lemma 2. If $\sigma = A\sigma$, then the functions

$$H = \frac{\nabla\times E}{i\omega\mu_0}, \qquad E(x) = \nabla\times\int_S g(x,t)\sigma(t)dt$$

solve Eq. (4) in D, E and H satisfy the radiation condition, and, condition (5). Thus, $E = H = 0$ in D.Consequently,

$$\begin{aligned}0 &= \nabla\times\nabla\times\int_S g(x,t)\sigma(t)dt = \left(grad\ div - \nabla^2\right)\int_S g(x,t)\sigma(t)dt\\ &= k^2\int_S g(x,t)\sigma(t)dt, \quad x\in D.\end{aligned}$$

This implies $\sigma(s) = 0$.Lemma 2 is proved. □

Appendix A

Derivation of the basic Eq. (39)

Boundary condition (10) yields

$$\begin{aligned}0 = [N[E_0,N]] &- \frac{\zeta}{i\omega\mu_0}[\nabla\times E_0,N] + \left[N,\left[\nabla\times\int_S g(s,t)\sigma(t)dt,N\right]\right]\\ &- \frac{\zeta}{i\omega\mu_0}\left[\nabla\times\nabla\times\int_S g(x,s)\sigma(t)dt,N\right].\end{aligned}$$

Let us denote

$$f_e := [N,[E_0,N]] - \frac{\zeta}{i\omega\mu_0}[\nabla\times E_0,N].$$

One has $\nabla\times\nabla\times = curlcurl = graddiv - \Delta$, and

$$\nabla_x\cdot\int_S g(x,t)\sigma(t)dt = -\int_S(\nabla_t g(x,t),\sigma(t))dt = \int_S g(x,t)\nabla_t\cdot\sigma(t)dt = 0,$$

and

$$-\nabla_x^2\int_S g(x,t)\sigma(t)dt = k^2\int_S g(x,t)\sigma(t)dt,$$

because $-\nabla_x^2 g(x,t) = k^2 g(x,t), x\ne t$, see Eq. (13). Thus, using Eq. (14), one gets:

$$\begin{aligned}0 = f_e &+ \left[\int_S[N_s,[\nabla_s g(s,t),\sigma(t)]]dt,N_s\right] + \frac{1}{2}[\sigma(s),N_s]\\ &+ \frac{\zeta k^2}{i\omega\mu_0}\left[N_s,\int_S g(s,t)\sigma(t)dt\right].\end{aligned}$$

Cross multiply this by N_s from the left and use the relation $N_s\cdot\sigma(s) = 0$, to obtain

$$\begin{aligned}0 = [N_s,f_e] &+ \left[N_s,\left[\int_S[N_s,[\nabla_s g(s,t),\sigma(t)]]dt,N_s\right]\right] + \frac{1}{2}\sigma(s)\\ &- \zeta_m i\omega\epsilon_0\left[N_s,\left[N_s,\int_S g(s,t)\sigma(t)dt\right]\right].\end{aligned}$$

Note that

$$\left[N_s,\left[\int_S [N_s,[\nabla_s g(s,t),\sigma(t)]]dt,N_s\right]\right] = \int_S [N_s,[\nabla_s g(s,t),\sigma(t)]]dt$$
$$-[N_s,N_s]\int_S (N_s,[\nabla_s g(s,t),\sigma(t)]dt)$$
$$= \int_S [N_s,[\nabla_s g(s,t),\sigma(t)]]dt.,$$

Consequently,

$$\sigma(t) = 2[f_e(s),N_s] + 2\zeta i\omega\epsilon_0\left[N_s,\left[N_s,\int_S g(s,t)\sigma(t)dt\right]\right]$$
$$-2\int_S [N_s,[\nabla_s g(s,t),\sigma(t)]]dt := A\sigma + f,$$

which is Eq. (14), and $f := 2[f_e(s), N_s]$, which is Eq. (15).

Denote

$$Q := \int_S \sigma(s)ds.$$

One has

$$\int_S [[N_s,[E_0(s),N_s]],N_s]ds = \int_S [E_0(s),N_s]ds = -\int_D \nabla_x \times E_0 dx.$$

The term $\int_D \nabla_x \times E_0 dx = O(a^3)$ is negligible compared with the terms of order $O(a^2)$. Let us estimate the terms of the order $O(a^2)$. One has

$$\int_S [[\nabla\times E_0,N_s],N_s]ds = -\left(\int_S \nabla\times E_0 ds - \int_S N_s(\nabla\times E_0,N_s)ds\right)$$
$$= -\int_S \nabla\times E_0 ds + \frac{4\pi a^2}{3}\nabla\times E_0(O)$$
$$= -\frac{8\pi a^2}{3}\nabla\times E_0(O), \qquad a\to 0.$$

Here we have used the formulas

$$\int_S \nabla\times E_0 ds = 4\pi a^2\nabla\times E_0(O)(1+o(1)), \qquad a\to 0,$$

and

$$\int_S N_i(s)N_j(s)ds = \frac{4\pi a^2}{3}\delta_{ij},$$

where S is a sphere of radius a, $\{N_i(s)\}_{i=1}^3$ are Cartesian components of the outer unit normal to the sphere S at a point $s\in S$, and $\delta_{ij}=0$ if $i\neq j$, $\delta_{ii}=1$.

Thus, if S is a sphere of radius a, one has

$$Q = 0.5\int_S f(s)ds = -\frac{8\pi i}{3\omega\mu_0}\zeta a^2\nabla\times E_0(O) = O\left(a^{2-\kappa}\right), \qquad a\to 0 \quad (41)$$

provided that $\zeta = \frac{h}{a^\kappa}$, $0<\kappa<1$.

If S is an arbitrary surface, then we define the tensor

$$\tau_{jp} := \delta_{jp}|S|^{-1}\int_S N_j(s)N_p(s)ds, \qquad (42)$$

where $|S|$ is the surface area of S, and formula (41) takes the form

$$Q = 0.5\int_S f(s)ds = -\frac{i\zeta|S|}{\omega\mu_0}\tau(\nabla\times E_0(O)), \qquad (43)$$

or, with $Q_j := (Q, e_j)$,

$$Q_j = -\frac{i\zeta|S|}{\omega\mu_0}\tau_{jp}(\nabla\times E_0(O))_p, \qquad 1\le j\le 3, \qquad (44)$$

where summation is understood over index p.

Let us now show that the term $\int_S A\sigma ds$ contributes the term $-Q$, so

$$Q = 0.5\int_S f(s)ds(1+o(1)), \qquad a\to 0. \qquad (45)$$

This term was not taken into account in [15]. One has

$$-2\int_S ds\int_S [N_s,[\nabla_s g(s,t),\sigma(t)]]dt$$
$$= -2\int_S ds\int_S dt\left(\nabla_s g(s,t)(N_s,\sigma(t)) - \sigma(t)\frac{\partial g(s,t)}{\partial N_s}\right)dt$$
$$= -2\int_S ds\int_S dt\nabla_s g(s,t)(N_s,\sigma(t)) + \int_S \sigma(t)dt 2\int_S ds\frac{\partial g(s,t)}{\partial N_s}.$$

Since

$$2\int_S ds\frac{\partial g(s,t)}{\partial N_s} = -2\int_D dx k^2 g(x,t) - 1,$$

one gets

$$I := \int_S dt\sigma(t)2\int_S ds\frac{\partial g(s,t)}{\partial N_s} = -\int_S \sigma(t)dt - 2k^2\int_S dt\sigma(t)\int_D dx g(x,t).$$

Therefore

$$I := -Q + I_1,$$

where the term I_1 is negligible compared with Q, because

$$\int_D dx g(x,t) = O\left(a^2\right), \qquad a\to 0, \quad x\in D.$$

Consequently, I_1 is negligible compared with I as $a\to 0$.

If $\int_S|\sigma(t)|dt<\infty$ and $Q = \int_S\sigma(t)dt \neq 0$, then

$$\left|\int_S \sigma(t)dt\right| \gg \left|\int_S dt\sigma(t)\int_D dx g(x,t)\right|,$$

because $|\int_D dx g(x,t)| = O(a^2)$ if $x\in D$. If $ka\ll 1$, then the fields E_0 and $\nabla\times E_0$ change negligibly at the distances of order a, and Q is proportional to $a^{2-\kappa}\nabla\times E_0(O)$ on the surface S, and therefore $Q\neq 0$ at all points at which $\nabla\times E_0(O)$ does not vanish.

One has

$$\left|-2\int_S ds\int_S dt\nabla_s g(s,t)(N_s,\sigma(t))\right| \ll \left|\int_S \sigma(t)dt\right| = |Q|,$$

because $|(N_s,\sigma(t))| = O(|s-t|)$ as $|s-t|\to 0$.

Therefore,

$$Q = 0.5\int_S f(t)dt = -\frac{8\pi i}{3\omega\mu_0}\zeta a^2\nabla\times E_0(O), \quad a\to 0. \qquad (46)$$

This yields the following formula, which is a particular case of (39) when S is a sphere:

$$E(x) = E_0(x) - \frac{8\pi i}{3\omega\mu_0}\zeta a^2[\nabla g(x,O),\nabla\times E_0(O)], \qquad a\to 0, \qquad (47)$$

when $|x-O|\gg a$.

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