



ELSEVIER

15 November 2000

OPTICS
COMMUNICATIONS

Optics Communications 185 (2000) 479–485

www.elsevier.com/locate/optcom

Robustness of quadratic solitons with periodic gain

Lluís Torner^{a,*}, Juan P. Torres^a, Ole Bang^b

^a *Laboratory of Photonics, Department of Signal Theory and Communications, Universitat Politècnica de Catalunya, Gran Capitan UPC-D3, Barcelona ES 08034, Spain*

^b *Department of Mathematical Modelling, Technical University of Denmark, DK-2800 Lyngby, Denmark*

Received 20 July 2000; accepted 21 September 2000

Abstract

We address the robustness of quadratic solitons with periodic non-conservative perturbations. We find the evolution equations for guiding-center solitons under conditions for second-harmonic generation in the presence of periodic multi-band loss and gain. Under proper conditions, a robust guiding-center soliton formation is revealed. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Solitons; Second-harmonic generation; Self-frequency doubling lasers; Cascading; Optical amplifiers

Multi-color optical soliton formation mediated by cascading of quadratic non-linearities has been demonstrated experimentally during the last few years in a variety of geometries and frequency-mixing processes, in settings for spatial, temporal and spatio-temporal trapping of light [1–4]. Soliton signals exist in particular in the process of second-harmonic generation (SHG) that is addressed here, where solitons form in waveguides and in bulk crystals by the mutual trapping between the fundamental frequency (FF) and second-harmonic waves. Multi-dimensional soliton families exist above a threshold light intensity for all values of the phase mismatch between the waves, and most of such solitons have been shown to be dynamically stable under propagation with the equations that model the ideal evolution under conditions of focused and pulsed SHG. Adiabatic soliton decay

and amplification in the presence of weak loss or gain, have been also studied [5–9].

In this letter we address the robustness of quadratic solitons against strong, but periodic non-conservative perturbations. To start the program we consider soliton formation in the presence of multi-frequency losses and large, but rapidly varying periodic gain. Our goal is to derive the corresponding guiding-center evolution equations and to expose the robustness of the existing solitons under proper conditions. We believe that the results reported bear a generic fundamental interest to the robustness of quadratic soliton formation in structures with periodic non-conservative perturbations. Moreover, they might find direct applications in reduced models of multi-color laser systems with intracavity frequency generation, including self-frequency doubling schemes, operating in the solitonic regime [10–13].

Here the focus is on solitons formed in one-dimensional structures under conditions for non-critical type I SHG, but the analysis can be

* Corresponding author. Fax: +34-93-401-72-32.
E-mail address: torner@tsc.upc.es (L. Torner).

extended to different physical settings. The evolution of the slowly varying envelopes of the light waves in the presence of multi-frequency band loss and periodic gain can be described by the reduced equations [14]

$$i \frac{\partial a_1}{\partial \xi} - \frac{\alpha_1}{2} \frac{\partial^2 a_1}{\partial s^2} + a_1^* a_2 \exp(-i\beta\xi) = i\Gamma_1(\xi)a_1, \quad (1)$$

$$i \frac{\partial a_2}{\partial \xi} - \frac{\alpha_2}{2} \frac{\partial^2 a_2}{\partial s^2} + a_1^2 \exp(i\beta\xi) = i\Gamma_2(\xi)a_2, \quad (2)$$

where a_1 and a_2 are the normalized amplitudes of the FF and second-harmonic (SH) waves. In the case of spatial solitons, $\alpha_1 = -1$, and $\alpha_2 = -k_1/k_2 \simeq -0.5$, where k_v , with $v = 1, 2$ are the linear wave numbers at both frequencies. In the case of temporal solitons, α_v stand for the ratio between the group-velocity dispersions existing at both frequencies. The transverse and longitudinal coordinates are normalized to the beam or pulse width and to the diffraction or dispersion length at the FF, respectively. Under typical experimental conditions suitable for quadratic soliton generation, with diffraction or dispersion lengths of a few mm, ξ in the range 0–10 corresponds to a few cm. The parameter β is the scaled phase mismatch, and $\Gamma_v(\xi)$ stand for periodic gain and loss. Let $\Gamma_v(\xi) = \Gamma_v^{(0)} + R_v(\xi)$, where $\Gamma_v^{(0)}$ are the average gain or loss at the FF and SH frequencies, and $R_v(\xi)$ are periodic functions, with period $L \ll 1$, and zero mean.

To derive the evolution equations of the guiding-center solitons in the presence of the periodic gain, we first use the approach originally introduced by Mollenauer et al. for the case of Kerr solitons propagating in optical fibers [15]. Let $A_1 = a_1$ and $A_2 = a_2 \exp(-i\beta\xi)$. The explicit periodic gain can be removed from Eqs. (1) and (2) by making the transformations $A_v(\xi, s) = U_v(\xi, s) \exp[\int_0^\xi R_v(\xi') d\xi']$. Substitution in Eqs. (1) and (2) leads to the evolution equations

$$i \frac{\partial U_1}{\partial \xi} - \frac{\alpha_1}{2} \frac{\partial^2 U_1}{\partial s^2} - i\Gamma_1^{(0)} U_1 + \epsilon_2(\xi) U_1^* U_2 = 0, \quad (3)$$

$$i \frac{\partial U_2}{\partial \xi} - \frac{\alpha_2}{2} \frac{\partial^2 U_2}{\partial s^2} - i\Gamma_2^{(0)} U_2 - \beta U_2 + \epsilon_3(\xi) U_1^2 = 0, \quad (4)$$

where the resulting longitudinally varying non-linear coefficients write

$$\epsilon_2(\xi) = \exp \left[\int_0^\xi R_2(\xi') d\xi' \right], \quad (5)$$

$$\epsilon_3(\xi) = \exp \left[\int_0^\xi (2R_1(\xi') - R_2(\xi')) d\xi' \right]. \quad (6)$$

To proceed further one now assumes that the wave evolution over a period of the map is mostly dictated by the gain and loss, which are responsible for fast amplitude oscillations of the fields A_v , in addition to the residual effects induced by the non-linearity. Therefore, U_v vary slowly over a period of the map, so that one can average Eqs. (3) and (4) to approximately get

$$i \frac{\partial U_1}{\partial \xi} - \frac{\alpha_1}{2} \frac{\partial^2 U_1}{\partial s^2} - i\Gamma_1^{(0)} U_1 + \eta_2 U_1^* U_2 \simeq 0, \quad (7)$$

$$i \frac{\partial U_2}{\partial \xi} - \frac{\alpha_2}{2} \frac{\partial^2 U_2}{\partial s^2} - i\Gamma_2^{(0)} U_2 - \beta U_2 + \eta_3 U_1^2 \simeq 0, \quad (8)$$

where

$$\eta_{2,3} = \langle \epsilon_{2,3}(\xi) \rangle = \frac{1}{L} \int_0^L \epsilon_{2,3}(\xi) d\xi, \quad (9)$$

are the averaged non-linear coefficients over a period of the map. Therefore, the evolution equations for the slowly varying averaged fields $U_{v,0}(\xi, s)$, defined as

$$U_{v,0}(\xi, s) = \left\langle \exp \left[\int_0^\xi R_v(\xi') d\xi' \right] \right\rangle U_v(\xi, s), \quad (10)$$

write

$$i \frac{\partial U_{1,0}}{\partial \xi} - \frac{\alpha_1}{2} \frac{\partial^2 U_{1,0}}{\partial s^2} - i\Gamma_1^{(0)} U_{1,0} + U_{1,0}^* U_{2,0} \simeq 0, \quad (11)$$

$$i \frac{\partial U_{2,0}}{\partial \xi} - \frac{\alpha_2}{2} \frac{\partial^2 U_{2,0}}{\partial s^2} - i\Gamma_2^{(0)} U_{2,0} - \beta U_{2,0} + \eta U_{1,0}^2 \simeq 0, \quad (12)$$

where $\eta = \eta_2 \eta_3 / \eta_1^2$, with $\eta_1 = \langle \epsilon_1(\xi) \rangle$, and

$$\epsilon_1(\xi) = \exp \left[\int_0^\xi R_1(\xi') d\xi' \right]. \quad (13)$$

One thus concludes that under the conditions where the approximations used to derive Eqs. (11)

and (12) hold, the guiding-center solitons are given by those existing without gain and loss when $\Gamma_v^{(0)} = 0$ [16,17], or otherwise with compensated gain and loss [18,19], but with properly renormalized amplitudes. To expose the value and dependencies of the renormalization coefficient η , consider the illustrative map $R_v(\xi) = G_v H(\xi)$, where $H(\xi)$ is the Heaviside function

$$H(\xi) = \begin{cases} -1, & nL < \xi < (n + 1/2)L, \\ 1, & (n + 1/2)L < \xi < (n + 1)L. \end{cases} \quad (14)$$

Substitution in the above expressions and performing all the averages yields

$$\eta = \frac{F[G_2L/2]F[(2G_1 - G_2)L/2]}{F^2[G_1L/2]}, \quad (15)$$

where

$$F[\mu] = \frac{1 - \exp(-\mu)}{\mu}. \quad (16)$$

Of particular interest are the weak maps with $\mu \ll 1$, corresponding to $G_vL \ll 1$. In such cases, one might expand Eq. (15) in powers of the corresponding parameters to get, at order L^2

$$\eta \simeq 1 + \frac{(G_1 - G_2)^2 L^2}{48}. \quad (17)$$

Note that when $G_1 = G_2$, Eq. (15) gives $\eta = 1$, regardless the value of the map period L . However, this does not mean that the guiding-center evolution is given by Eqs. (11) and (12) with $\eta = 1$ at all orders of L , because the derivation of Eqs. (11) and (12), hence the governing equations themselves, are only intended to hold when $L \ll 1$. To elucidate the applicability limits of Eqs. (11) and (12), it is worth deriving the guiding-center evolution equations using more mathematically systematic approaches [20,21]. Next we outline the outcome of the asymptotic expansion method developed by Kivshar and coworkers for similar problems but for Kerr solitons [21]. Such approach has been recently employed to obtain guiding-center evolution equations of light signals propagating in quasi-phase-matched quadratic structures [22,23].

Assuming perfect periodicity of the functions $R_v(\xi)$ as above, one can express the periodic gain and all fields as Fourier series of the form

$$R_v(\xi) = \sum_{n \neq 0} d_{v,n} \exp(in\lambda\xi), \quad (18)$$

$$A_v(\xi, s) = \sum_n U_{v,n}(\xi, s) \exp(in\lambda\xi), \quad (19)$$

where $\lambda = 2\pi/L$, and the Fourier coefficients $U_{v,n}(\xi, s)$ vary with ξ much slower than the corresponding carrier $\exp(in\lambda\xi)$. For the map (14) considered here one has

$$d_{v,2n} = 0, \quad d_{v,(2n+1)} = \frac{2G_v}{i\pi(2n+1)}. \quad (20)$$

Assuming the amplification maps to vary rapidly over a diffraction/dispersion length, i.e. $L \ll 1$, so that the spatial frequency λ is large enough, one can expand the harmonic amplitudes $U_{v,n \neq 0}$, as power series of $1/\lambda$. Namely,

$$U_{v,n \neq 0} = \sum_{m>0} \frac{F_{v,n}^{(m)}}{\lambda^m}. \quad (21)$$

Substitution in the governing equations and matching leading-order contributions leads to

$$F_{v,n}^{(1)} = -i \frac{d_{v,n}}{n} U_{v,0}. \quad (22)$$

The guiding-center evolution is obtained by solving recursively for the higher-order contributions $F_{v,n}^{(m>1)}$ and substituting the corresponding expressions in the evolution equations for the average fields $U_{v,0}$. At second order one gets

$$nF_{1,n}^{(2)} = i \frac{\partial F_{1,n}^{(1)}}{\partial \xi} - \frac{\alpha_1}{2} \frac{\partial^2 F_{1,n}^{(1)}}{\partial s^2} + U_{1,0}^* F_{2,n}^{(1)} + U_{2,0} F_{1,-n}^{(1)*} - i \sum_{m \neq 0, n} d_{1,n-m} F_{1,m}^{(1)}, \quad (23)$$

$$nF_{2,n}^{(2)} = i \frac{\partial F_{2,n}^{(1)}}{\partial \xi} - \frac{\alpha_2}{2} \frac{\partial^2 F_{2,n}^{(1)}}{\partial s^2} + 2U_{1,0} F_{1,n}^{(1)} - \beta F_{2,n}^{(1)} - i \sum_{m \neq 0, n} d_{2,n-m} F_{2,m}^{(1)}. \quad (24)$$

Substituting the expressions (22)–(24) into the evolution equations for the average fields $U_{v,0}$, and using the values of the Fourier coefficients given by Eq. (20) to sum up all the resulting numerical series, one finds that all terms of order $1/\lambda$ vanish because of the assumed perfect periodicity of the

map, and that at order $1/\lambda^2$ one exactly finds Eqs. (11) and (12), with Eq. (17).

Eqs. (11), (12) and (17) are the central result of this paper. First, they reveal that when $G_1 = G_2$, at order L^2 the guiding-center evolution equations are identical to those without the periodic gain, regardless the value of the amplitudes G_v . This is a remarkable result, that emphasizes the robustness of quadratic solitons with the type of perturbations considered. Second, the typical values of the gain coefficient that is obtained in erbium-doped lithium niobate or potassium titanyl phosphate around the third telecommunication window centered at $1.55 \mu\text{m}$ fall in the range 0–2 dB/cm [10–12,24,25]. In the case of spatial solitons, this yields values of the normalized gain coefficient of the order of $G_v \sim 10^{-1}$ [8]. With such values and letting $L \sim 10^{-1}$, one always obtains negligible corrections of the order of $|\eta - 1| \sim 10^{-5}$.

To confirm that under proper conditions (11) and (12) hold, we solved Eqs. (1) and (2) numerically using a standard split step Fourier scheme, for a variety of different input conditions and amplification maps. Fig. 1 shows typical examples of the outcome, when $\Gamma_v^{(0)} = 0$. The plots corre-

spond to the phase mismatch $\beta = 3$, but analogous results were obtained for other values. To emphasize that guiding-center solitons form with gigantic values of the gain–loss amplitude, provided that the period of the map is small enough so that the guiding-center approach is justified, we display results for a map with $G_1 = 20$, $G_2 = 0$ and $L = 0.2$. Fig. 1(a) and (b) shows the propagation of a guiding-center soliton excited by the input $A_v(0) = \tilde{A}_v(0)/\rho_v$, where $\tilde{A}_v(0)$ are the corresponding stationary solitons existing without gain and loss with energy flow $\tilde{I} = \tilde{I}_1 + \tilde{I}_2 = \int (|\tilde{A}_1|^2 + |\tilde{A}_2|^2) ds = 30$ [16,17], while $\rho_1 = (\eta_2\eta_3)^{1/2}$, and $\rho_2 = \eta_2$, are the renormalization factors dictated by Eqs. (7) and (8). For the map considered one has $\rho_1 \sim 0.5$, so that in the renormalized input the FF energy is enhanced by a factor of four. Fig. 1(c) and (d) shows the excitation of a guiding-center soliton in the same map but with only FF input light carrying the same energy flow as above in the Gaussian shape $A_1(0) = (2/\pi)^{1/4} I_1^{1/2} \exp(-s^2)$. Here $I_1 = \tilde{I}_1/\rho_1^2$. Fig. 2 shows the detailed evolution of the field amplitudes in the case shown in Fig. 1(a) and (b). Because of the large excursions of the amplitudes, the actual evolution differs

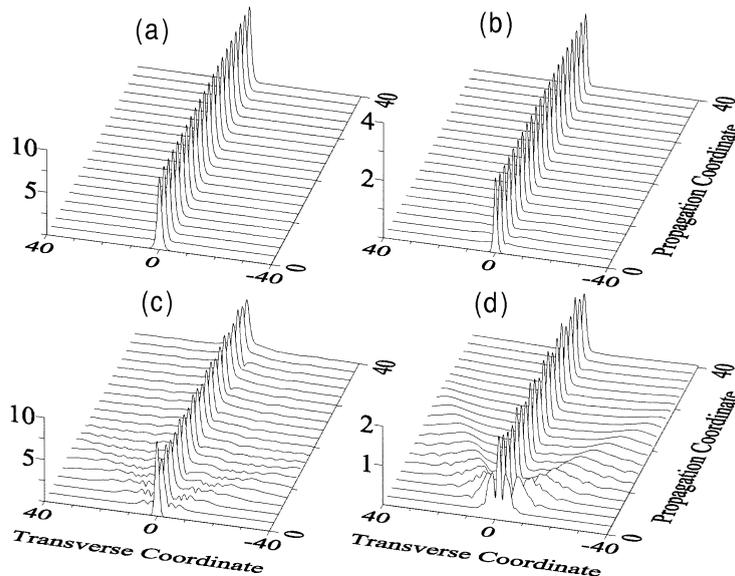


Fig. 1. Typical evolution of guiding-center quadratic solitons in structures with rapidly varying periodic gain. In (a, b): The input is a renormalized stationary soliton solution. In (c, d): The input contains only FF light with a Gaussian shape. (a, c): Evolution of the FF; (b, d): Evolution of the SH. Conditions: $\beta = 3$, $\tilde{I} = 30$, $G_1 = 20$, $G_2 = 0$, $L = 0.2$.

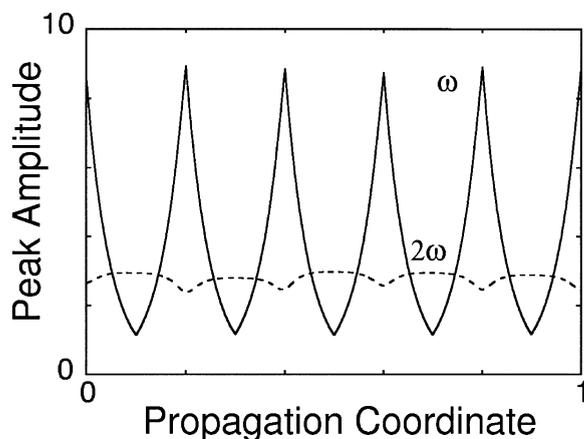


Fig. 2. Detail of the evolution of the peak amplitudes of the signals displayed in Fig. 1(a) and (b), over the first five periods of the map.

slightly from that predicted by Eqs. (11) and (12). Such departures are responsible, e.g., of the small resonance peaks appearing in the Fourier spectra of the FF and SH evolving signals, as shown in Fig. 3.

When $\Gamma_v^{(0)} \neq 0$, but the net loss at one frequency is compensated for by the presence of a net gain at the other frequency band, the system (11) and (12) allows stationary soliton solutions [18,19], which are chirped. However, an interesting result re-

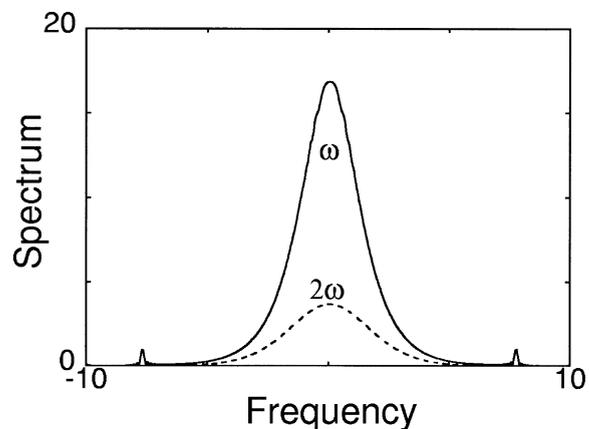


Fig. 3. Fourier spectra of the FF and SH signals displayed in Fig. 1(a) and (b), at $\zeta = 40$.

vealed by the guiding-center evolution Eqs. (11) and (12) is that, in the absence of a net gain or loss (i.e., $\Gamma_v^{(0)} = 0$), on average the guiding-center quadratic solitons are chirpless. Accordingly, in contrast to what is found with solitons in other periodic systems, e.g., with dispersion-managed solitons propagating in optical fibers [26], one concludes that with chirpless input conditions the guiding-center quadratic solitons with periodic gain are best excited when the first domain has the whole nominal length. The outcome of our numerical simulations confirms that such is indeed the case, as shown in Fig. 4. The plot shows the evolution over a period of the map of the quantities

$$C_v(\zeta) = \frac{i}{4} \frac{\int s(A_v A_{v,s}^* - A_v^* A_{v,s}) ds}{\int s^2 |A_v|^2 ds}, \quad (25)$$

where $A_{v,s}$ stands for $\partial A_v / \partial s$. The evolution displayed corresponds to a guiding-center soliton similar to that shown in Fig. 1(a) and (b), but for $G_1 = G_2 = 10$.

Analogous results than those shown above were obtained for a variety of values of the phase mismatch β , the input energy flow I , the signal shape, and the map amplitude. Naturally, this is so provided that the input conditions, the gain amplitude, and the period L make a guiding-center

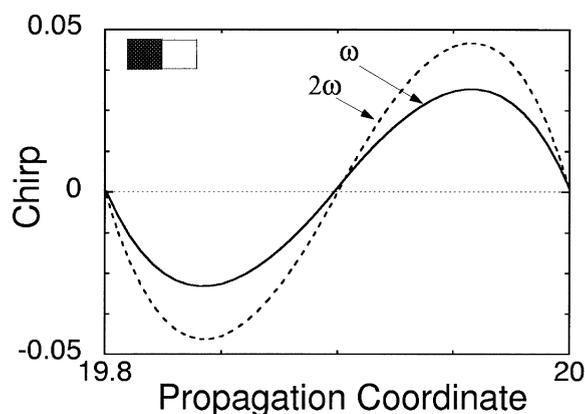


Fig. 4. Typical evolution of the integral chirp of the guiding-center solitons over a period of the map. The plot shows the evolution from $\zeta = 19.8$ to $\zeta = 20$. Conditions: $\beta = 3$, $\tilde{I} = 30$, $G_1 = 10$, $G_2 = 10$, $L = 0.2$. The input is the corresponding renormalized stationary soliton solution.

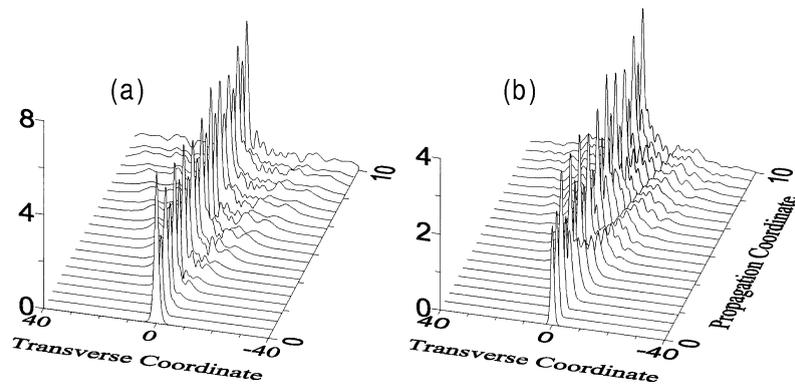


Fig. 5. Unstable evolution when the guiding-center approach fails. The input is a renormalized stationary soliton solution. Same conditions as in Fig. 1(a) and (b), except that here $G_1 = 2$ and $L = 1$. (a) Evolution of the FF; (b) evolution of the SH.

approach justified [20,21]. Otherwise, e.g., when the map period L , the gain rate G_v or the input soliton energy flow I are too large, the guiding-center evolution fails, yielding a totally new scenario. In particular, under such conditions resonance phenomena that make the wave propagation unstable can occur. Fig. 5 shows a typical example. The plot shows the unstable evolution of the renormalized solution with $\tilde{I} = 30$ in a map with $G_1 = 2$, with $L = 1$.

In conclusion, the evolution equations for guiding-center quadratic solitons propagating in structures with multi-frequency losses and rapidly varying periodic gain have been presented. Under proper conditions, robust multi-color quadratic soliton formation has been revealed. Results might find applications to reduced models of multi-color laser systems with intracavity frequency generation, including self-frequency doubling schemes, operating in the solitonic regime. Extension of the analysis to general maps that include the details of the laser structures, including the pump light–matter interaction [10–12,24,25], is worth investigating.

Acknowledgements

This work was supported by the Generalitat de Catalunya. Ole Bang acknowledges support from the Danish Technical Research Council through talent grant no. 9800400.

References

- [1] G.I. Stegeman, D.J. Hagan, L. Torner, *Opt. Quant. Electron.* 28 (1996) 1691.
- [2] M. Segev, G.I. Stegeman, *Phys. Today* 51 (1998) 42.
- [3] L. Torner, *Spatial solitons in quadratic nonlinear media*, in: F. Kajzar, R. Reinisch (Eds.), *Beam Shaping and Control with Nonlinear Optics*, Plenum, New York, 1998, pp. 229–258.
- [4] Y.S. Kivshar, in: A.D. Boardman, L. Pavlov, S. Tanev (Eds.), *Advanced Photonics with Second-Order Optically Nonlinear Processes*, Kluwer, Dordrecht, 1998, p. 451.
- [5] K. Hayata, M. Koshiya, *J. Opt. Soc. Am. B* 12 (1995) 2288.
- [6] L. Torner, D. Mihalache, D. Mazilu, N.N. Akhmediev, *Opt. Lett.* 20 (1995) 2183.
- [7] B.A. Malomed, D. Anderson, A. Berntson, M. Florjanczyk, M. Lisak, *Pure Appl. Opt.* 5 (1996) 941.
- [8] L. Torner, *Opt. Commun.* 154 (1998) 59.
- [9] S. Darmany, A. Kobayakov, F. Lederer, *Opt. Lett.* 24 (1999) 1517.
- [10] T.Y. Fan, A. Cordova-Plaza, M.J.F. Digonnet, R.L. Byer, H.J. Shaw, *J. Opt. Soc. Am. B* 3 (1986) 140.
- [11] J. Capmany, E. Montoya, V. Bermúdez, D. Callejo, E. Diéguez, L.E. Bausá, *Appl. Phys. Lett.* 76 (2000) 1374.
- [12] A.V. Kir'yanov, I.V. Mel'nikov, K. Wagner, *Appl. Phys. Lett.* 76 (2000) 2829.
- [13] P.-S. Jin, W.E. Torruellas, M. Haelterman, S. Trillo, U. Peschel, F. Lederer, *Opt. Lett.* 24 (1999) 400.
- [14] C.R. Menyuk, R. Schiek, L. Torner, *J. Opt. Soc. Am. B* 11 (1994) 2434.
- [15] L.F. Mollenauer, S.G. Evangelides, H.A. Haus, *J. Light-wave Technol.* LT-9 (1991) 194.
- [16] A.V. Buryak, Y.S. Kivshar, *Phys. Lett. A* 197 (1995) 407.
- [17] L. Torner, *Opt. Commun.* 114 (1995) 136.
- [18] L.-C. Crasovan, B. Malomed, D. Mihalache, F. Lederer, *Phys. Rev. E* 59 (1999) 7173.
- [19] L. Torner, J. Dörring, J.P. Torres, *IEEE J. Quant. Electron.* QE-34 (1999) 1344.

- [20] A. Hasegawa, Y. Kodama, *Solitons in Optical Communications*, Clarendon, Oxford, 1995.
- [21] Y.S. Kivshar, K. Spatschek, S. Turitsyn, M.L. Quiroga-Teixeiro, *Pure Appl. Opt.* 4 (1995) 281.
- [22] C.B. Clausen, O. Bang, Y.S. Kivshar, *Phys. Rev. Lett.* 78 (1997) 4749.
- [23] O. Bang, C.B. Clausen, P.L. Christiansen, L. Torner, *Opt. Lett.* 24 (1999) 1413.
- [24] C. Becker, T. Oesselke, J. Pandavenes, R. Ricken, K. Rochhausen, G. Schreiber, W. Sohler, H. Suche, R. Wessel, S. Balsamo, I. Montrosset, D. Sciancalepore, *IEEE J. Selected Topics Quant. Electron.* 6 (2000) 101.
- [25] V. Bermúdez, J. Capmany, J. García-Solé, E. Diéguez, *Appl. Phys. Lett.* 73 (1998) 593.
- [26] S.K. Turitsyn, V.K. Mezentsev, E.G. Shapiro, *Opt. Fiber Technol.* 4 (1998) 384.