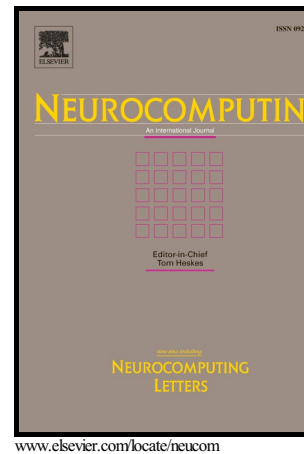


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Delay-dependent Stability for Neural Networks with Time-varying Delays via a Novel Partitioning Method

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Abstract: In this brief, a novel partitioning method for the conditions on bounding the activation function in the stability analysis of neural networks systems with time-varying delays is presented. Certain further improved delay-dependent stability conditions, which are expressed in terms of linear matrix inequalities (LMIs), are derived by employing a suitable Lyapunov-Krasovskii functional (LKF) and utilizing the Wirtinger integral inequality. Two well-known examples are investigated in a comparison mode with results to show the effectiveness and improvements achieved by the new results proposed.

Keywords: neural networks, time-varying delay, stability, Lyapunov-Krasovskii functional, Wirtinger integral inequality

1. Introduction

In recent years, an increasing number of research studies on the neural networks dynamics became apparent. Largely, this trend is due to their many successful applications in areas of pattern recognition, image processing, associative memories [1], optimization problems and even mechanics of structures and materials [2]. It should be noted, due to the finite switching speed of electronics involved and the inherent communication time between the neurons, inevitably time-delay exists regardless how small it may be. Precisely the time-delay is a main factor that can cause performance degradation and/or the instability of neural networks. It is therefore that the stability problem of neural networks with time delays has attracted considerable attention of many researchers in the last few decades and considerable stability research results have emerged. The existing stability criteria may well be grouped into delay-independent and delay-dependent types of criteria. In general, the delay-dependent stability criteria are less conservative than the delay-independent ones. For the delay-dependent stability criteria, the maximum delay bound is an important index for checking and evaluating the conservatism of the criteria. In turn, rather significant research efforts [3-16] have been devoted to the reduction of conservatism of the delay-dependent stability criteria for neural networks with time delays even when these are fairly small.

As known from Lyapunov stability theory, there are two effective ways to reduce the conservatism in stability analysis. One is the choice of suitable LKF and the other is the estimation of its time derivative. Recently, some new techniques of the construction of a suitable LKF and the estimation of its derivative for time-delay systems have been presented as seen from [3-39]. Methods for constructing a delicate LKF include delay-partitioning ideas, triple integral terms, augmented vectors, and involving more information of activation functions. Methods for estimating the time-derivative of LKF include P.Park's inequality, Jensen's inequality, free-weighting matrices, reciprocally convex optimization, quadratic convex combination method, and so on. Since it was noted Jensen's inequality to introduce an undesirable conservatism in the stability conditions, A. Seuret and F. Gouaisbaut [32] introduced certain Wirtinger inequalities and overcame that conservatism. An originally developed free-matrix-based inequality, which encompasses the Wirtinger-based inequality and was more tighter than existing ones, was presented by Zeng[40,41]. The developments of the mentioned methods appeared very useful in the investigation of the stability problems for neural networks with time delays.

For instance, here we refer to several of the recent developments. Authors of [22, 23] used the property of first order convex combination and derived some less conservative criteria. J. H. Kim proposed quadratic convex function for linear system [24] and this quadratic convex combination method was developed further in [30]. The approach of free-weighting matrices was used in [25-26]. Authors of [27-28] took a new augmented vector. Delay-partitioning methodology has been used in [25-28] and, in turn, the conservatism of delay-dependent stability criteria reduced. However, as the partitioning number of delay increases, the matrix formulation becomes more complex and the dimension of the stability criterion grows bigger, and thus the computational burden and time consumption growth become a tangible problem. In contrast to

the delay-partitioning method, recently the approach of activation function dividing was proposed in [13], and new improved delay-dependent criteria for neural networks with time delays were established. Work [35] presented an improved delay-dependent stability criterion for recurrent neural networks with time-varying delays by adopting a more general method of activation function dividing.

In this paper, following the above enlightenment discussion, we present a set of new contributions. Firstly, a new LKF is constructed by taking more information of state and activation functions as augmented vectors, and then reciprocal convex approach and Wirtinger integral inequality is used to handle the integral term of quadratic quantities in the estimation of LKF's derivative. With the new LKF at hand, in Theorem 1, we derive the stability condition in terms of the convex combination with respect to the time-varying delays and its variation, and the delay-dependent stability criterion in which both the upper and lower bounds of delay derivative are available. Secondly, unlike the delay partitioning method, in Theorem 2, a more general dividing approach of the bounding conditions on activation function is employed. The bounding of activation functions of neural networks with time-varying delays is built within two subintervals, which can be either equal or unequal, in order to account for reducing the computing time and for improvement of the feasible region. By using the information of new bounding conditions for the two subintervals of activation functions, a new and not in-so-far proposed LKF constructed for the poof of Theorem 3. Thirdly, by utilizing the results of Theorem 3, when only the upper bound of the delay derivative of the time-varying delay is available, Corollary 1 presents the corresponding results. And when the information about the delay derivative of time-varying delay is unknown, this case can be readily derived from Corollary 1. Finally, our stability analysis method is applied to two well-known examples in the literature and our obtained results are compared with the existing the corresponding results, respectively, to illustrate its effectiveness and demonstrate the improvements obtained.

Throughout this paper the following notation is used: C^T represents the transposition of matrix C . \mathbb{R}^n denotes n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. $P > 0$ means that P is a real symmetric positive-definite matrix. Symbol $*$ represents symmetric term in a symmetric matrix and $\text{diag}\{\dots\}$ denotes a block diagonal matrix. $\text{Sym}(X)$ is defined as $\text{Sym}(X) = X + X^T$.

2. Problem formulation

Consider the following class of neural networks with discrete time-varying delays

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t-h(t))) + J. \quad (1)$$

In representation model (1), symbols denote: $z(t) = [z_1(t), \dots, z_n(t)]^T \in \mathbb{R}^n$ is a real-valued n -vector representing the state variables associated with the neurons in the neural networks; $f(z) = [f_1(z_1), \dots, f_n(z_n)]^T \in \mathbb{R}^n$ is n -vector of the neuron activation functions; $J = [J_1, \dots, J_n]^T \in \mathbb{R}^n$ is a constant input vector; $C = \text{diag}\{c_1, \dots, c_n\} \in \mathbb{R}^{n \times n}$ and A, B are the constant matrices of appropriate dimensions completing the description of this class of neural networks.

The delay $h(t)$ is assumed to be represented by a time-varying continuous function satisfying

$$C1: 0 \leq h(t) \leq h_M, \quad h_D^l \leq \dot{h}(t) \leq h_D^u < 1,$$

$$C2: 0 \leq h(t) \leq h_M, \quad \dot{h}(t) \leq h_D^u,$$

where $h_M > 0$ and h_D^l, h_D^u are known constants.

The activation functions $f_i(z_i(t))$, $i = 1, \dots, n$, are continuous, bounded, and satisfy the inequalities

$$k_i^- \leq \frac{f_i(u) - f_i(v)}{u - v} \leq k_i^+, \quad u, v \in \mathbb{R}, u \neq v, i = 1, \dots, n \quad (2)$$

where k_i^- and k_i^+ are constants.

For simplicity, in the stability analysis of the neural networks (1), we first shift the equilibrium point z^* to the origin by letting $x = z - z^*$, $g(x) = f(x + z^*) - f(z^*)$. Then the network system model (1) can be converted into

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t-h(t))) \quad (3)$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ is the state vector of the transformed system, $g(x(t)) = [g_1(x_1(t)), \dots, g_n(x_n(t))]^T$ with $g_j(x_j(t)) = f_j(x_j(t) + z_j^*) - f_j(z_j^*)$ satisfying $g_j(0) = 0$ ($j = 1, \dots, n$). Notice that functions $g_i(\cdot)$ ($i = 1, \dots, n$) satisfy the following inequality conditions:

$$k_i^- \leq \frac{g_i(u) - g_i(v)}{u - v} \leq k_i^+, \quad u, v \in \mathbb{R}, u \neq v, i = 1, \dots, n. \quad (4)$$

If there is $v = 0$ in (4), then we have

$$k_i^- \leq \frac{g_i(u)}{u} \leq k_i^+, \quad \forall u \neq 0, i = 1, \dots, n. \quad (5)$$

The objective of this paper is to explore the analysis of the asymptotic stability of the considered class of neural networks with time-varying delays via utilizing the representation model (3).

Before deriving the main results, we quote the following lemmas that are used in the subsequent section.

Lemma 1 [17, 32]. For given positive integers n, m , a scalar α in the interval $(0, 1)$, a given $n \times n$ matrix $R > 0$, and two matrices W_1 and W_2 in $\mathbb{R}^{n \times m}$, for all vectors ξ in \mathbb{R}^m let define the function $\Theta(\alpha, R)$ given by

$$\Theta(\alpha, R) = \frac{1}{\alpha} \xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha} \xi^T W_2^T R W_2 \xi.$$

Then, if there exists a matrix X in $\mathbb{R}^{n \times n}$ such that $\begin{bmatrix} R & X \\ * & R \end{bmatrix} > 0$, the following inequality holds:

$$\min_{\alpha \in (0, 1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}.$$

Lemma 2 [32]. For a given matrix $R > 0$, the following inequality holds for all continuously differentiable functions σ in $[a, b] \rightarrow \mathbb{R}^n$:

$$\int_a^b \dot{\sigma}^T(u) R \dot{\sigma}(u) du \geq \frac{1}{b-a} (\sigma(b) - \sigma(a))^T R (\sigma(b) - \sigma(a)) + \frac{3}{b-a} \delta^T R \delta$$

where $\delta = \sigma(b) + \sigma(a) - \frac{2}{b-a} \int_a^b \sigma(u) du$.

Lemma 3 [33]. Let $\xi \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n \times n}$ and $H \in \mathbb{R}^{m \times n}$ such that $\text{rank}(H) < n$. Then, the following statements are equivalent:

- (1) $\xi^T \Phi \xi < 0$, $H \xi = 0$, $\xi \neq 0$,
- (2) $(H^\perp)^T \Phi H^\perp < 0$, where H^\perp is a right orthogonal complement of H .

Lemma 4 [34]. For symmetric matrices of appropriate dimensions $R > 0, \Omega$, and a matrix Γ , the following two statements are equivalent: (1) $\Omega - \Gamma R \Gamma^T < 0$ and (2) there exists a matrix of the appropriate dimension Π such that

$$\begin{bmatrix} \Omega + \Gamma \Pi^T + \Pi \Gamma^T & \Pi \\ \Pi^T & -R \end{bmatrix} < 0. \quad (6)$$

3. Main new results

In this section, by using a novel activation function partitioning method, a new asymptotic stability criterion for system (3) with time-varying delays is proposed. For simplicity of matrix representation, we setup block entry matrices $e_0 = 0_{13n \times n}$, $e_i (i = 1, \dots, 13) \in \mathbb{R}^{13n \times n}$ (for example $e_2^T = [0 \ I \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$) and we define

$$\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-h_M) & \dot{x}^T(t) & \dot{x}^T(t-h(t)) & \dot{x}^T(t-h_M) & g^T(x(t)) & g^T(x(t-h(t))) \end{bmatrix}$$

$$g^T(x(t-h_M)) \quad \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds \quad \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds \quad \int_{t-h(t)}^t g^T(x(s)) ds \quad \int_{t-h_M}^{t-h(t)} g^T(x(s)) ds \quad \Bigg],$$

$$\begin{aligned}
 \omega^T(t) &= \begin{bmatrix} x^T(t) & x^T(t-h_M) & \int_{t-h_M}^t x^T(s)ds & \int_{t-h_M}^t g^T(x(s))ds & x^T(t-h(t)) \end{bmatrix}, \quad H = \begin{bmatrix} -C & 0_{n \times 2n} & -I & 0_{n \times 2n} & A & B & 0_{n \times 5n} \end{bmatrix}, \\
 \alpha^T(t, s) &= \begin{bmatrix} x^T(t) & x^T(s) & \dot{x}^T(s) & g^T(x(s)) & x^T(t-h(t)) \end{bmatrix}, \quad \beta^T(s) = \begin{bmatrix} x^T(s) & \dot{x}^T(s) & \bar{g}^T(x) \end{bmatrix}, \\
 Y_1(\dot{h}) &= \text{diag}\{I, I, I, I, (1-\dot{h}(t))I\}, \quad Y_2(\dot{h}) = \text{diag}\{I, I, I, I, (1-\dot{h}(t))I\}, \quad Y_3(\dot{h}) = \text{diag}\{I, I, I, I, (1-\dot{h}(t))I, (1-\dot{h}(t))I\}, \\
 \Pi_1^0 &= [e_1 \quad e_3 \quad e_0 \quad e_{12}+e_{13} \quad e_2], \quad \Pi_1^1 = [e_0 \quad e_0 \quad e_{10} \quad e_0 \quad e_0], \quad \Pi_1^2 = [e_0 \quad e_0 \quad e_{11} \quad e_0 \quad e_0], \\
 \Pi_2 &= [e_4 \quad e_6 \quad e_1-e_3 \quad e_7-e_9 \quad e_5], \quad \Pi_3 = [e_1 \quad e_4 \quad e_7], \quad \Pi_4 = [e_2 \quad e_5 \quad e_8], \quad \Pi_5^0 = [e_0 \quad e_0 \quad e_1-e_2 \quad e_{12} \quad e_0 \quad e_0], \\
 \Pi_5^1 &= [e_1 \quad e_{10} \quad e_0 \quad e_0 \quad e_2 \quad e_1-e_{10}], \quad \Pi_6 = [e_4 \quad e_0 \quad e_0 \quad e_0 \quad e_5], \quad \Pi_7 = [e_3 \quad e_6 \quad e_9], \\
 \Pi_8^0 &= [e_0 \quad e_0 \quad e_2-e_3 \quad e_{13} \quad e_0 \quad e_0], \quad \Pi_8^2 = [e_1 \quad e_{11} \quad e_0 \quad e_0 \quad e_2 \quad e_2-e_{11}], \quad \Pi_9^0 = [h_M e_1 \quad e_0 \quad e_1-e_3 \quad e_{12}+e_{13} \quad h_M e_2], \\
 \Pi_9^1 &= [e_0 \quad e_{10} \quad e_0 \quad e_0 \quad e_0], \quad \Pi_9^2 = [e_0 \quad e_{11} \quad e_0 \quad e_0 \quad e_0], \quad \Pi_{10}^0 = [e_0 \quad e_1-e_2 \quad e_{12} \quad e_0 \quad e_2-e_3 \quad e_{13}], \\
 \Pi_{10}^1 &= [e_{10} \quad e_0 \quad e_0 \quad e_0 \quad e_0 \quad e_0], \quad \Pi_{10}^2 = [e_0 \quad e_0 \quad e_0 \quad e_0 \quad e_{11} \quad e_0], \\
 K_m &= \text{diag}\{k_1^-, k_2^-, \dots, k_n^-\}, \quad K_M = \text{diag}\{k_1^+, k_2^+, \dots, k_n^+\}, \quad K_\rho = K_m + \rho(K_M - K_m), \quad (0 \leq \rho \leq 1), \\
 \Phi &= \begin{bmatrix} \Xi & S \\ * & \Xi \end{bmatrix}, \quad \Xi = \begin{bmatrix} M & 0_{n \times n} \\ * & 3M \end{bmatrix}, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad X = \begin{bmatrix} Z & W \\ * & Z \end{bmatrix}, \\
 \Theta &= -\text{Sym}\{[e_7 - e_1 K_m] T_1 [e_7 - e_1 K_m]^T + [e_8 - e_2 K_m] T_2 [e_8 - e_2 K_m]^T + [e_9 - e_3 K_m] T_3 [e_9 - e_3 K_m]^T\}, \\
 \Theta_a &= -\text{Sym}\{[e_7 - e_1 K_\rho] T_1 [e_7 - e_1 K_\rho]^T + [e_8 - e_2 K_\rho] T_2 [e_8 - e_2 K_\rho]^T + [e_9 - e_3 K_\rho] T_3 [e_9 - e_3 K_\rho]^T\}, \\
 \Theta_b &= -\text{Sym}\{[e_7 - e_1 K_m] T_4 [e_7 - e_1 K_\rho]^T + [e_8 - e_2 K_m] T_5 [e_8 - e_2 K_\rho]^T + [e_9 - e_3 K_m] T_6 [e_9 - e_3 K_\rho]^T\}, \\
 \Phi_1 &= \text{Sym}\{[e_7 - e_1 K_m] D_1 e_4^T + [e_1 K_m - e_7] D_2 e_4^T + \text{Sym}\{[e_9 - e_3 K_m] D_5 e_6^T + [e_3 K_m - e_9] D_6 e_6^T\}, \\
 \Phi_2(\dot{h}) &= (1-\dot{h}(t)) \text{Sym}\{[e_8 - e_2 K_m] D_3 e_5^T + [e_2 K_m - e_8] D_4 e_5^T\}, \\
 \Phi_3 &= [e_1 \quad \Pi_3 \quad e_2 \quad e_0] Q [e_1 \quad \Pi_3 \quad e_2 \quad e_0]^T, \quad \Phi_4(\dot{h}) = (1-\dot{h}(t)) [e_1 \quad \Pi_4 \quad e_2 \quad e_1-e_2] Q [e_1 \quad \Pi_4 \quad e_2 \quad e_1-e_2]^T, \\
 \Phi_5(\dot{h}) &= (1-\dot{h}(t)) [e_1 \quad \Pi_4 \quad e_2 \quad e_0] R [e_1 \quad \Pi_4 \quad e_2 \quad e_0]^T, \quad \Phi_6 = [e_1 \quad \Pi_7 \quad e_2 \quad e_2-e_3] R [e_1 \quad \Pi_7 \quad e_2 \quad e_2-e_3]^T, \\
 \Phi_7 &= [e_1 \quad \Pi_3 \quad e_2] N [e_1 \quad \Pi_3 \quad e_2]^T, \quad \Phi_8 = [e_1 \quad \Pi_7 \quad e_2] N [e_1 \quad \Pi_7 \quad e_2]^T, \\
 \Omega_a &= -\text{Sym}\{[e_7 - e_8 - (e_1 - e_2) K_\rho] L_1 [e_7 - e_8 - (e_1 - e_2) K_m]^T + [e_8 - e_9 - (e_2 - e_3) K_\rho] L_2 [e_8 - e_9 - (e_2 - e_3) K_m]^T\}, \\
 \Omega_b &= -\text{Sym}\{[e_7 - e_8 - (e_1 - e_2) K_\rho] L_3 [e_7 - e_8 - (e_1 - e_2) K_m]^T + [e_8 - e_9 - (e_2 - e_3) K_\rho] L_4 [e_8 - e_9 - (e_2 - e_3) K_m]^T\}, \\
 \Sigma(\dot{h}) &= \text{Sym}\{\Pi_1^0 P Y_1(\dot{h}) \Pi_2^T\} + \Phi_2(\dot{h}) - \Phi_4(\dot{h}) + \text{Sym}\{\Pi_5^0 Q Y_2(\dot{h}) [\Pi_6 \quad e_4]^T\} \\
 &\quad + \text{Sym}\{\Pi_8^0 R Y_3(\dot{h}) [\Pi_6 \quad e_5]^T\} + \Phi_5(\dot{h}) + \text{Sym}\{\Pi_9^0 N Y_1(\dot{h}) \Pi_6^T\}, \\
 \Sigma_1(\dot{h}) &= \text{Sym}\{\Pi_1^1 P Y_1(\dot{h}) \Pi_2^T + \Pi_5^1 Q Y_2(\dot{h}) [\Pi_6 \quad e_4]^T + \Pi_9^1 N Y_1(\dot{h}) \Pi_6^T\}, \\
 \Sigma_2(\dot{h}) &= \text{Sym}\{\Pi_1^2 P Y_1(\dot{h}) \Pi_2^T + \Pi_8^2 R Y_3(\dot{h}) [\Pi_6 \quad e_5]^T + \Pi_9^2 N Y_1(\dot{h}) \Pi_6^T\}, \\
 Y &= [e_1 - e_2 \quad e_1 + e_2 - 2e_{10} \quad e_2 - e_3 \quad e_2 + e_3 - 2e_{11}]^T, \quad \Psi = \Phi_1 + \Phi_3 - \Phi_6 + \Phi_7 - \Phi_8 + h_M^2 \Pi_3 Z \Pi_3^T + h_M^2 e_4 M e_4^T - Y^T \Phi Y \quad (7)
 \end{aligned}$$

Theorem 1. For given positive scalar h_M , any scalars h_D^l and h_D^u with condition C1, diagonal matrices K_m , K_M , network system (3) is asymptotically stable, if there exist positive definite matrices $P \in \mathbb{R}^{5n \times 5n}$, $Q \in \mathbb{R}^{6n \times 6n}$,

$R \in \mathbb{R}^{6n \times 6n}$, $N \in \mathbb{R}^{5n \times 5n}$, $Z \in \mathbb{R}^{3n \times 3n}$, $M \in \mathbb{R}^{n \times n}$, positive diagonal matrices $D_i = \text{diag}\{d_{i1}, d_{i2}, \dots, d_{in}\}$ ($i = 1, \dots, 6$), $T_i = \text{diag}\{t_{i1}, t_{i2}, \dots, t_{in}\}$ ($i = 1, 2, 3$), and any matrix $W \in \mathbb{R}^{3n \times 3n}$, with matrices $S_{ij} \in \mathbb{R}^{n \times n}$ ($i, j = 1, 2$) and matrix Π having appropriate dimensions, such that the following LMIs are feasible for $h = (0, h_M)$ and for $\dot{h} = (h_D^l, h_D^u)$.

$$\begin{bmatrix} (H^\perp)^T \Omega(h, \dot{h})(H^\perp) + \text{Sym}\{(H^\perp)^T \Gamma(h) \Pi^T\} & \Pi \\ * & -X \end{bmatrix} < 0 \quad (8)$$

$$X > 0, \quad \Phi > 0 \quad (9)$$

where $\Omega(h, \dot{h}) = \Sigma(\dot{h}) + h(t)\Sigma_1(\dot{h}) + (h_M - h(t))\Sigma_2(\dot{h}) + \Psi + \Theta$, $\Gamma(h) = \Pi_{10}^0 + h(t)\Pi_{10}^1 + (h_M - h(t))\Pi_{10}^2$, and the other matrices are defined in (7), with H^\perp the right orthogonal complement of H .

Proof. For positive diagonal matrices D_i ($i = 1, \dots, 6$) and positive definite matrices P, Q, R, N, Z, M , we consider the following LKF candidate

$$V = \sum_{i=1}^6 V_i(x_i) \quad (10)$$

where:

$$\begin{aligned} V_1 &= \omega^T(t) P \omega(t) \\ V_2 &= 2 \sum_{i=1}^n (d_{1i} \int_0^{x_i(t)} (g_i(s) - k_i^- s) ds + d_{2i} \int_0^{x_i(t)} (k_i^+ s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{3i} \int_0^{x_i(t-h(t))} (g_i(s) - k_i^- s) ds + d_{4i} \int_0^{x_i(t-h(t))} (k_i^+ s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{5i} \int_0^{x_i(t-h_M)} (g_i(s) - k_i^- s) ds + d_{6i} \int_0^{x_i(t-h_M)} (k_i^+ s - g_i(s)) ds) \\ V_3 &= \int_{t-h(t)}^t \left[\begin{matrix} \alpha(t, s) \\ \int_s^t \dot{x}(u) du \end{matrix} \right]^T Q \left[\begin{matrix} \alpha(t, s) \\ \int_s^t \dot{x}(u) du \end{matrix} \right] ds + \int_{t-h_M}^{t-h(t)} \left[\begin{matrix} \alpha(t, s) \\ \int_s^{t-h(t)} \dot{x}(u) du \end{matrix} \right]^T R \left[\begin{matrix} \alpha(t, s) \\ \int_s^{t-h(t)} \dot{x}(u) du \end{matrix} \right] ds \\ V_4 &= \int_{t-h_M}^t \alpha^T(t, s) N \alpha(t, s) ds \\ V_5 &= h_M \int_{t-h_M}^t \int_s^t \beta^T(u) Z \beta(u) du ds \\ V_6 &= h_M \int_{t-h_M}^t \int_s^t \dot{x}^T(u) M \dot{x}(u) du ds \end{aligned}$$

The time derivative of V_1 can be represented as

$$\dot{V}_1 = \xi^T(t) \{ \text{Sym}((\Pi_1^0 + h(t)\Pi_1^1 + (h_M - h(t))\Pi_1^2) P \Upsilon_1(\dot{h}) \Pi_2^T) \} \xi(t) \quad (11)$$

Similarly, we get

$$\begin{aligned} \dot{V}_2 &= 2[g(x(t)) - x(t)K_m]^T D_1 \dot{x}(t) + 2[x(t)K_M - g(x(t))]^T D_2 \dot{x}(t) \\ &\quad + (1 - \dot{h}(t)) \{ 2[g(x(t-h(t))) - x(t-h(t))K_m]^T D_3 \dot{x}(t-h(t)) + 2[x(t-h(t))K_M - g(x(t-h(t)))]^T D_4 \dot{x}(t-h(t)) \} \\ &\quad + 2[g(x(t-h_M)) - x(t-h_M)K_m]^T D_5 \dot{x}(t-h_M) + 2[x(t-h_M)K_M - g(x(t-h_M))]^T D_6 \dot{x}(t-h_M) \\ &= \xi^T(t) \{ [\Phi_1 + \Phi_2(\dot{h})] \} \xi(t) \end{aligned} \quad (12)$$

Further, calculation of \dot{V}_3 gives

$$\dot{V}_3 = \begin{bmatrix} \alpha(t, t) \\ 0_{n \times 1} \end{bmatrix}^T Q \begin{bmatrix} \alpha(t, t) \\ 0_{n \times 1} \end{bmatrix} - (1 - \dot{h}(t)) \begin{bmatrix} \alpha(t, t-h(t)) \\ x(t) - x(t-h(t)) \end{bmatrix}^T Q \begin{bmatrix} \alpha(t, t-h(t)) \\ x(t) - x(t-h(t)) \end{bmatrix} + 2 \int_{t-h(t)}^t \begin{bmatrix} \alpha(t, s) \\ \int_s^t \dot{x}(u) du \end{bmatrix}^T Q \begin{bmatrix} \Upsilon_1(\dot{h}) \eta(t) \\ \dot{x}(t) \end{bmatrix} ds$$

$$\begin{aligned}
 & + (1 - \dot{h}(t)) \begin{bmatrix} \alpha(t, t-h(t)) \\ 0_{n \times 1} \end{bmatrix}^T R \begin{bmatrix} \alpha(t, t-h(t)) \\ 0_{n \times 1} \end{bmatrix} - \begin{bmatrix} \alpha(t, t-h_M) \\ x(t-h(t)) - x(t-h_M) \end{bmatrix}^T R \begin{bmatrix} \alpha(t, t-h_M) \\ x(t-h(t)) - x(t-h_M) \end{bmatrix} \\
 & + 2 \int_{t-h_M}^{t-h(t)} \begin{bmatrix} \alpha(t, s) \\ \int_s^{t-h(t)} \dot{x}(u) du \end{bmatrix}^T R \begin{bmatrix} \Upsilon_1(\dot{h})\eta(t) \\ (1-\dot{h}(t))\dot{x}(t-h(t)) \end{bmatrix} ds \\
 & = \xi^T(t) \{ \Phi_3 - \Phi_4(\dot{h}) + \text{Sym}((\Pi_5^0 + h(t)\Pi_5^1)Q\Upsilon_2(\dot{h})[\Pi_6 \quad e_4]^T) + \Phi_5(\dot{h}) \\
 & \quad - \Phi_6 + \text{Sym}((\Pi_8^0 + (h_M - h(t))\Pi_8^2)R\Upsilon_3(\dot{h})[\Pi_6 \quad e_5]^T) \} \xi(t)
 \end{aligned} \tag{13}$$

where $\eta(t) = [\dot{x}(t) \quad 0 \quad 0 \quad \dot{x}(t-h(t))]^T$.

The result of \dot{V}_4 , after an appropriate arrangement, leads to

$$\begin{aligned}
 \dot{V}_4 & = \alpha^T(t, t)N\alpha(t, t) - \alpha^T(t, t-h_M)N\alpha(t, t-h_M) + 2 \int_{t-h_M}^t \alpha^T(t, s)N\Upsilon_1(\dot{h})\eta(t)ds \\
 & = \xi^T(t) \{ \Phi_7 - \Phi_8 + \text{Sym}((\Pi_9^0 + h(t)\Pi_9^1 + (h_M - h(t))\Pi_9^2)N\Upsilon_1(\dot{h})\Pi_6^T) \} \xi(t).
 \end{aligned} \tag{14}$$

Furthermore, by using Jensen's inequality and Lemma 1, \dot{V}_5 is found bounded as

$$\begin{aligned}
 \dot{V}_5 & = h_M^2 \beta^T(t)Z\beta(t) - h_M \int_{t-h(t)}^t \beta^T(s)Z\beta(s)ds - h_M \int_{t-h_M}^{t-h(t)} \beta^T(s)Z\beta(s)ds \\
 & \leq h_M^2 \beta^T(t)Z\beta(t) - \left(\frac{h_M}{h(t)}\right) \left(\int_{t-h(t)}^t \beta(s)ds \right)^T Z \left(\int_{t-h(t)}^t \beta(s)ds \right) - \left(\frac{h_M}{h_M-h(t)}\right) \left(\int_{t-h_M}^{t-h(t)} \beta(s)ds \right)^T Z \left(\int_{t-h_M}^{t-h(t)} \beta(s)ds \right) \\
 & \leq h_M^2 \beta^T(t)Z\beta(t) - \begin{bmatrix} \int_{t-h(t)}^t \beta(s)ds \\ \int_{t-h_M}^{t-h(t)} \beta(s)ds \end{bmatrix}^T \begin{bmatrix} Z & W \\ * & Z \end{bmatrix} \begin{bmatrix} \int_{t-h(t)}^t \beta(s)ds \\ \int_{t-h_M}^{t-h(t)} \beta(s)ds \end{bmatrix} \\
 & = \xi^T(t) \{ h_M^2 \Pi_3 Z \Pi_3^T - \Gamma(h)X\Gamma^T(h) \} \xi(t).
 \end{aligned} \tag{15}$$

Finally, \dot{V}_6 is easily obtained as follows:

$$\dot{V}_6 = h_M^2 \dot{x}^T(t)M\dot{x}(t) - h_M \int_{t-h_M}^t \dot{x}^T(s)M\dot{x}(s)ds \tag{16}$$

By applying Lemma 1 and Lemma 2, we find

$$\begin{aligned}
 & -h_M \int_{t-h_M}^t \dot{x}^T(s)M\dot{x}(s)ds = -h_M \int_{t-h(t)}^t \dot{x}^T(s)M\dot{x}(s)ds - h_M \int_{t-h_M}^{t-h(t)} \dot{x}^T(s)M\dot{x}(s)ds \\
 & \leq -\frac{h_M}{h(t)} [x(t) - x(t-h(t))]^T M [x(t) - x(t-h(t))] - \frac{h_M}{h_M-h(t)} [x(t-h(t)) - x(t-h_M)]^T M [x(t-h(t)) - x(t-h_M)] \\
 & \quad - \frac{3h_M}{h(t)} [x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds]^T M [x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds] \\
 & \quad - \frac{3h_M}{h_M-h(t)} [x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds]^T M [x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds] \\
 & = -\frac{h_M}{h(t)} \begin{bmatrix} x(t) - x(t-h(t)) \\ x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds \end{bmatrix}^T \Xi \begin{bmatrix} x(t) - x(t-h(t)) \\ x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds \end{bmatrix} \\
 & \quad - \frac{h_M}{h_M-h(t)} \begin{bmatrix} x(t-h(t)) - x(t-h_M) \\ x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds \end{bmatrix}^T \Xi \begin{bmatrix} x(t-h(t)) - x(t-h_M) \\ x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds \end{bmatrix} \\
 & \leq - \begin{bmatrix} x(t) - x(t-h(t)) \\ x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds \\ x(t-h(t)) - x(t-h_M) \\ x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds \end{bmatrix}^T \Phi \begin{bmatrix} x(t) - x(t-h(t)) \\ x(t) + x(t-h(t)) - \frac{2}{h(t)} \int_{t-h(t)}^t x(s)ds \\ x(t-h(t)) - x(t-h_M) \\ x(t-h(t)) + x(t-h_M) - \frac{2}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x(s)ds \end{bmatrix}.
 \end{aligned}$$

Hence

$$\dot{V}_6(x_i) \leq \xi^T(t)(h_M^2 e_4 M e_4^T - Y^T \Phi Y) \xi(t) \quad (17)$$

From (5) it follows that for any positive diagonal matrices $T_i = \text{diag}\{t_{i1}, t_{i2}, \dots, t_{in}\}$ ($i=1,2,3$), the following inequality holds:

$$\begin{aligned} 0 \leq & -2 \sum_{i=1}^n t_{1i} \left[g_i(x_i(t)) - k_i x_i(t) \right] \left[g_i(x_i(t)) - k_i x_i(t) \right]^T \\ & -2 \sum_{i=1}^n t_{2i} \left[g_i(x_i(t-h)) - k_i x_i(t-h) \right] \left[g_i(x_i(t-h)) - k_i x_i(t-h) \right]^T \\ & -2 \sum_{i=1}^n t_{3i} \left[g_i(x_i(t-h_M)) - k_i x_i(t-h_M) \right] \left[g_i(x_i(t-h_M)) - k_i x_i(t-h_M) \right]^T \\ & = \xi^T(t) \Theta \xi(t). \end{aligned} \quad (18)$$

Now, with (11)-(18) at hand, we get

$$\dot{V} \leq \xi^T(t) \left\{ \Omega(h, \dot{h}) - \Gamma(h) X \Gamma^T(h) \right\} \xi(t). \quad (19)$$

By virtue of Lemma 3, $\xi^T(t) \left\{ \Omega(h, \dot{h}) - \Gamma(h) X \Gamma^T(h) \right\} \xi(t) < 0$ with $0 = H \xi(t)$ is equivalent to

$$(H^\perp)^T [\Omega(h, \dot{h}) - \Gamma(h) X \Gamma^T(h)] (H^\perp) < 0. \quad (20)$$

By virtue of Lemma 4, inequality (20) is equivalent to

$$\begin{bmatrix} (H^\perp)^T \Omega(h, \dot{h}) (H^\perp) + \text{Sym}\{(H^\perp)^T \Gamma(h) \Pi^T\} & \Pi \\ * & -X \end{bmatrix} < 0. \quad (21)$$

where Π is a matrix of appropriate dimensions. The above condition is affine, and consequently convex, with respect to $h(t)$ and $\dot{h}(t)$, and it is necessary and sufficient to ensure that inequality (21) holds at vertices of the intervals $[0, h_M] \times [h_D^l, h_D^u]$ as shown in [32]. Based on this finding, we know that inequality (21) holds if and only if (8)-(9) hold as well, and then network system (3) is asymptotically stable. Therefore then neural network system (1) is stable too. This completes the proof.

Remark 1. Recently, the reciprocally convex optimization technique and Wirtinger integral inequality are proposed in [17] and [32] respectively, and the two methods are utilized in (17) here. We point out that in Lemma 2, the term $\frac{1}{b-a}(\sigma(b) - \sigma(a))^T R(\sigma(b) - \sigma(a))$ is equal to Jensen's inequality, and the new term $\frac{3}{b-a} \delta^T R \delta$ can reduce the enlargement of the estimation of LKF. The usage of reciprocally convex optimization method avoids the enlargement of $h(t)$ and $h_M - h(t)$, and only introduces two matrices W, S . Then, the convex optimization method is used to handle $\dot{V}(x_i)$.

Remark 2. In Theorem 1, first, the terms $\frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds$ and $\frac{1}{h_M - h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds$ are used for the vector $\xi(t)$. Secondly, the states $x(t-h(t))$ and $x(t-h_M)$ as interval of integral terms are taken, as shown in the second and third terms of V_2 . Therefore, more information on the cross terms in $g(x(t-h(t)), x(t-h(t)))$, $\dot{x}(t-h(t))$ and $g(x(t-h_M), x(t-h_M))$, $\dot{x}(t-h_M)$ is being utilized. Thirdly, we introduce new terms $x(t)$, $x(t-h(t))$, $\int_s^{t-h(t)} \dot{x}(u) du$ in V_3 , which is different from the previous works. Thus, the results of time-derivative of the proposed V_3 contain some cross-times such as $2(\frac{1}{h(t)} \int_{t-h(t)}^t x(s) ds)(h(t) Q_{21} \dot{x}(t))$, $2(\frac{1}{h(t)} \int_{t-h(t)}^t x(s) ds)(h(t) Q_{26} \dot{x}(t))$, $2(\frac{1}{h(t)} \int_{t-h(t)}^t x(s) ds)(h(t) Q_{25} (1 - \dot{h}(t)) \dot{x}(t - h(t)))$, which were presented in (13) and does not be used in existing results. These considerations highlight the main differences in the construction of the LKF candidate in this paper.

Remark 3. The maximal order of the LMIs and total number of the scalar variables are usually considered as the index of the calculation complexities, how to get novel stability criteria which spend less time on the calculation and large delay

bound is another key work in this paper. In the stability analysis of neural networks with time delays, a number of works choose delay-partitioning method to investigate. Generally, the delay-partitioning number was taken as the range of two as a tradeoff between the computational burden and the improvement of the feasible region. However, when the condition $0 \leq h(t) \leq h_M$ is divided into $0 \leq h(t) \leq h_M/2$ and $h_M/2 \leq h(t) \leq h_M$, the matrix formulation becomes more complex and the dimension of stability conditions grows larger because it has more augmented vectors. Inspired by the activation functions dividing method for neural networks with time-varying delays in [13], we divide the bounding of activation function $k_i^- \leq f_i(u)/u \leq k_i^+$ of neurons with time-varying delays into $k_i^- \leq f_i(u)/u \leq k_i^\rho$ and $k_i^\rho \leq f_i(u)/u \leq k_i^+$, $k_i^\rho = k_i^- + \rho(k_i^+ - k_i^-)$, $0 \leq \rho \leq 1$. The calculation complexity of this partitioning method is less than delay partitioning method because the stability condition has less augmented vectors. This new activation partitioning technique for neural networks with time delays is more general and less conservative than the one in [13]. The new bounding partitioning approach is utilized instead of using delay-partitioning method, which is used in Theorem 2 further below. Thus, through Theorem 1 and Theorem 2, less conservative stability criteria are derived in this paper.

Now, based on the results of Theorem 1, an improved stability criterion for system (3) is introduced.

Theorem 2. For given positive scalars $\rho \leq 1$ and h_M , any scalars h_D^l and h_D^u with condition C1, diagonal matrices K_m, K_M and K_ρ , network system (3) is asymptotically stable, if there exist positive definite matrices $P \in \mathbb{R}^{5n \times 5n}$, $Q \in \mathbb{R}^{6n \times 6n}$, $R \in \mathbb{R}^{6n \times 6n}$, $N \in \mathbb{R}^{5n \times 5n}$, $Z \in \mathbb{R}^{3n \times 3n}$, $M \in \mathbb{R}^{n \times n}$, positive diagonal matrices $D_i = \text{diag}\{d_{i1}, d_{i2}, \dots, d_{in}\} \geq 0$ ($i=1, \dots, 6$), $T_i = \text{diag}\{t_{i1}, t_{i2}, \dots, t_{in}\} \geq 0$ ($i=1, \dots, 6$), $L_i = \text{diag}\{l_{i1}, l_{i2}, \dots, l_{in}\} \geq 0$ ($i=1, \dots, 4$), and any matrix $W \in \mathbb{R}^{3n \times 3n}$, with matrices $S_{ij} \in \mathbb{R}^{n \times n}$ ($i, j=1, 2$) and matrix Π having appropriate dimensions, such that the following LMIs for $h = (0, h_M)$ and for $\dot{h} = (h_D^l, h_D^u)$

$$\begin{bmatrix} (H^\perp)^T \hat{Q} \hat{h} + \Phi_\Delta + \Omega_\Delta & H^\perp \Gamma h \Pi^T \\ * & -X \end{bmatrix} < 0 \quad \Delta = a, b \quad (22)$$

$$X > 0, \quad \Phi > 0 \quad (23)$$

are satisfied, where $\hat{Q}(h, \dot{h}) = \Sigma(\dot{h}) + h(t)\Sigma_1(\dot{h}) + (h_M - h(t))\Sigma_2(\dot{h}) + \Psi$ and other matrices are defined in (7), and where H^\perp is the right orthogonal complement of H .

Proof. Consider the same LKF proposed in Theorem 1, we derive the actual results from the following two cases.

Case 1:

$$k_i^- \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^\rho, \quad (24)$$

by choosing $v=0$, it is equivalent to

$$[g_i(u) - k_i^- u][g_i(u) - k_i^\rho u] < 0. \quad (25)$$

From (25) it is found that, for any positive diagonal matrices $T_1 = \text{diag}\{t_{11}, \dots, t_{1n}\}$, $T_2 = \text{diag}\{t_{21}, \dots, t_{2n}\}$, and $T_3 = \text{diag}\{t_{31}, \dots, t_{3n}\}$, the following inequality holds:

$$\begin{aligned} 0 &\leq -2 \sum_{i=1}^n t_{1i} [g_i(x_i(t)) - k_i^- x_i(t)] [g_i(x_i(t)) - k_i^\rho x_i(t)] \\ &\quad - 2 \sum_{i=1}^n t_{2i} [g_i(x_i(t-h(t))) - k_i^- x_i(t-h(t))] [g_i(x_i(t-h(t))) - k_i^\rho x_i(t-h(t))] \\ &\quad - 2 \sum_{i=1}^n t_{3i} [g_i(x_i(t-h_M)) - k_i^- x_i(t-h_M)] [g_i(x_i(t-h_M)) - k_i^\rho x_i(t-h_M)] \\ &= \xi^T(t) \Theta_a \xi(t). \end{aligned} \quad (26)$$

From inequality (24), the following conditions hold:

$$k_i^- \leq \frac{g_i(x_i(t)) - g_i(x_i(t-h(t)))}{x_i(t) - x_i(t-h(t))} \leq k_i^+,$$

$$k_i^- \leq \frac{g_i(x_i(t-h(t))) - g_i(x_i(t-h_M))}{x_i(t-h(t)) - x_i(t-h_M)} \leq k_i^+. \quad (27)$$

For $i = 1, \dots, n$, the above two conditions are equivalent to

$$\begin{aligned} & \left[g_i(x_i(t)) - g_i(x_i(t-h(t))) - k_i^-(x_i(t) - x_i(t-h(t))) \right] \\ & \times \left[g_i(x_i(t)) - g_i(x_i(t-h(t))) - k_i^+(x_i(t) - x_i(t-h(t))) \right] \leq 0 \end{aligned} \quad (28)$$

$$\begin{aligned} & \left[g_i(x_i(t-h(t))) - g_i(x_i(t-h_M)) - k_i^-(x_i(t-h(t)) - x_i(t-h_M)) \right] \\ & \times \left[g_i(x_i(t-h(t))) - g_i(x_i(t-h_M)) - k_i^+(x_i(t-h(t)) - x_i(t-h_M)) \right] \leq 0. \end{aligned} \quad (29)$$

Therefore, for any positive diagonal matrices $L_1 = \text{diag}\{l_{11}, \dots, l_{1n}\}$, $L_2 = \text{diag}\{l_{21}, \dots, l_{2n}\}$, the following inequality holds:

$$\begin{aligned} 0 & \leq -2 \sum_{i=1}^n \left\{ l_{1i} \left[g_i(x_i(t)) - g_i(x_i(t-h(t))) - k_i^-(x_i(t) - x_i(t-h(t))) \right] \right. \\ & \quad \times \left. \left[g_i(x_i(t)) - g_i(x_i(t-h(t))) - k_i^+(x_i(t) - x_i(t-h(t))) \right] \right\} \\ & \quad - 2 \sum_{i=1}^n \left\{ l_{2i} \left[g_i(x_i(t-h(t))) - g_i(x_i(t-h_M)) - k_i^-(x_i(t-h(t)) - x_i(t-h_M)) \right] \right. \\ & \quad \times \left. \left[g_i(x_i(t-h(t))) - g_i(x_i(t-h_M)) - k_i^+(x_i(t-h(t)) - x_i(t-h_M)) \right] \right\} \\ & = \xi^T(t) \Omega_a \xi(t). \end{aligned} \quad (30)$$

Then, from the proof of Theorem 1, when $k_i^- \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^+$, an upper bound of \dot{V} can be shown

$$\dot{V} \leq \xi^T(t) \{ \hat{\Omega}(h, \dot{h}) + \Theta_a + \Omega_a - \Gamma(h) X \Gamma^T(h) \} \xi(t) \quad (31)$$

Case 2:

$$k_i^+ \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^- \quad (32)$$

For this case, we define positive definite diagonal matrices $T_4 = \text{diag}\{t_{41}, t_{42}, \dots, t_{4n}\}$, $T_5 = \text{diag}\{t_{51}, t_{52}, \dots, t_{5n}\}$, $T_6 = \text{diag}\{t_{61}, t_{62}, \dots, t_{6n}\}$, and $L_3 = \text{diag}\{l_{31}, \dots, l_{3n}\}$, $L_4 = \text{diag}\{l_{41}, \dots, l_{4n}\}$, applying a similar procedure as the one used in Case 1, and therefore ultimately we obtain

$$\dot{V} \leq \xi^T(t) \{ \hat{\Omega}(h, \dot{h}) + \Theta_b + \Omega_b - \Gamma(h) X \Gamma^T(h) \} \xi(t) \quad (33)$$

Finally, we get an upper bound of \dot{V} for $k_i^- \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^+$ as follows:

$$\dot{V} \leq \xi^T(t) \{ \hat{\Omega}(h, \dot{h}) + \Theta_\Delta + \Omega_\Delta - \Gamma(h) X \Gamma^T(h) \} \xi(t) \quad (34)$$

where $\Theta_\Delta, \Omega_\Delta (\Delta = a, b)$ are defined in (7). Similarly to the proof of Theorem 1, if inequalities (22-23) hold, then network system (3) is asymptotically stable for condition C1 fulfilled and $k_i^- \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^+$, and so is the neural network system (1). This completes the proof.

Remark 4. In Theorem 2, a new activation function partitioning method was applied to derive a novel criterion for system (3). Here different novel functional for the two subintervals of the bounding of the activation function was constructed to obtain a new result for system (3). By a procedure similar to the proof of Theorem 2, we also introduce Theorem 3 as follows.

Theorem 3. For given positive scalars $\rho \leq 1$ and h_M , any scalars h_D^l and h_D^u with condition C1, diagonal matrices K_m, K_M and K_ρ , network system (3) is asymptotically stable, if there exist positive definite matrices $P \in \mathbb{R}^{5n \times 5n}$,

$Q \in \mathbb{R}^{6n \times 6n}$, $R \in \mathbb{R}^{6n \times 6n}$, $N \in \mathbb{R}^{5n \times 5n}$, $Z \in \mathbb{R}^{3n \times 3n}$, $M \in \mathbb{R}^{n \times n}$, positive diagonal matrices $D_i = \text{diag}\{d_{i1}, d_{i2}, \dots, d_{in}\}$ ($i = 1, \dots, 12$), $T_i = \text{diag}\{t_{i1}, t_{i2}, \dots, t_{in}\}$ ($i = 1, \dots, 6$), $L_i = \text{diag}\{l_{i1}, l_{i2}, \dots, l_{in}\}$ ($i = 1, \dots, 4$), and any matrix $W \in \mathbb{R}^{3n \times 3n}$, with matrices $S_{ij} \in \mathbb{R}^{n \times n}$ ($i, j = 1, 2$) and matrix Π having appropriate dimensions, such that the following LMIs for $h = (0, h_M)$ and for $\dot{h} = (h_D^l, h_D^u)$

$$\begin{bmatrix} (H^\perp)^T \bar{\Omega}_\Delta(h, \dot{h})(H^\perp) + \text{Sym}\{(H^\perp)^T \Gamma(h) \Pi^T\} & \Pi \\ * & -X \end{bmatrix} < 0 \quad \Delta = a, b \quad (35)$$

$$X > 0, \quad \Phi > 0 \quad (36)$$

are satisfied, where

$$\begin{aligned} \bar{\Omega}_\Delta(h, \dot{h}) &= \bar{\Sigma}_\Delta(\dot{h}) + h(t)\Sigma_1(\dot{h}) + (h_M - h(t))\Sigma_2(\dot{h}) + \bar{\Psi}_\Delta + \Theta_\Delta + \Omega_\Delta, \\ \bar{\Sigma}_\Delta(\dot{h}) &= \text{Sym}(\Pi_1^0 P \Upsilon_1(\dot{h}) \Pi_2^T) + \bar{\Phi}_{2\Delta}(\dot{h}) - \Phi_4(\dot{h}) + \text{Sym}(\Pi_5^0 Q \Upsilon_2(\dot{h}) [\Pi_6 \quad e_4]^T) \\ &\quad + \text{Sym}(\Pi_8^0 R \Upsilon_3(\dot{h}) [\Pi_6 \quad e_5]^T) + \Phi_5(\dot{h}) + \text{Sym}(\Pi_9^0 N \Upsilon_1(\dot{h}) \Pi_6^T), \\ \bar{\Psi}_\Delta &= \bar{\Phi}_{1\Delta} + \Phi_3 - \Phi_6 + \Phi_7 - \Phi_8 + h_M^2 \Pi_3 Z \Pi_3^T + h_M^2 e_4 M e_4^T - Y^T \Phi Y, \\ \bar{\Phi}_{1a} &= \text{Sym}\{[e_7 - e_1 K_m] D_1 e_4^T + [e_1 K_\rho - e_7] D_2 e_4^T + [e_9 - e_3 K_m] D_5 e_6^T + [e_3 K_\rho - e_9] D_6 e_6^T\}, \\ \bar{\Phi}_{1b} &= \text{Sym}\{[e_7 - e_1 K_\rho] D_7 e_4^T + [e_1 K_M - e_7] D_8 e_4^T + [e_9 - e_3 K_\rho] D_{11} e_6^T + [e_3 K_M - e_9] D_{12} e_6^T\}, \\ \bar{\Phi}_{2a}(\dot{h}) &= (1 - \dot{h}(t)) \text{Sym}\{[e_8 - e_2 K_m] D_3 e_5^T + [e_2 K_\rho - e_8] D_4 e_5^T\}, \\ \bar{\Phi}_{2b}(\dot{h}) &= (1 - \dot{h}(t)) \text{Sym}\{[e_8 - e_2 K_\rho] D_9 e_5^T + [e_2 K_M - e_8] D_{10} e_5^T\}, \end{aligned} \quad (37)$$

while other matrices are defined in (7), and H^\perp is the right orthogonal complement of H .

Proof. By considering the same LKF proposed in Theorem 1 except for the term V_2 , in Case 1 and Case 2, we choose different V_2 for the two subintervals of the bounding of the activation function as follows:

Case 1: when $k_i^- \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^\rho$, we choose V_2 as

$$\begin{aligned} V_{2a} &= 2 \sum_{i=1}^n (d_{1i} \int_0^{x_i(t)} (g_i(s) - k_i^- s) ds + d_{2i} \int_0^{x_i(t)} (k_i^\rho s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{3i} \int_0^{x_i(t-h(t))} (g_i(s) - k_i^- s) ds + d_{4i} \int_0^{x_i(t-h(t))} (k_i^\rho s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{5i} \int_0^{x_i(t-h_M)} (g_i(s) - k_i^- s) ds + d_{6i} \int_0^{x_i(t-h_M)} (k_i^\rho s - g_i(s)) ds). \end{aligned} \quad (38)$$

Case 2: when $k_i^\rho \leq (g_i(u) - g_i(v)) / (u - v) \leq k_i^+$, we choose V_2 as

$$\begin{aligned} V_{2b} &= 2 \sum_{i=1}^n (d_{7i} \int_0^{x_i(t)} (g_i(s) - k_i^\rho s) ds + d_{8i} \int_0^{x_i(t)} (k_i^+ s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{9i} \int_0^{x_i(t-h(t))} (g_i(s) - k_i^\rho s) ds + d_{10i} \int_0^{x_i(t-h(t))} (k_i^+ s - g_i(s)) ds) \\ &\quad + 2 \sum_{i=1}^n (d_{11i} \int_0^{x_i(t-h_M)} (g_i(s) - k_i^\rho s) ds + d_{12i} \int_0^{x_i(t-h_M)} (k_i^+ s - g_i(s)) ds). \end{aligned} \quad (39)$$

The calculation of $\dot{V}_{2a}, \dot{V}_{2b}$ is similar to that of \dot{V}_2 in Theorem 1. Thus, if inequalities (35-36) hold, then network system (3) is asymptotically stable, and so is neural network (1). This completes the proof.

Remark 5. In Theorem 3, we consider that $h(t)$ satisfies condition C1, but there are many systems satisfying condition C2. Therefore we introduced Corollary 1 in order to analyze the stability of neural networks satisfying the condition C2 by setting $D_3, D_4, D_9, D_{10} = 0$, $R = 0$ and changing V_1, V_2, V_3, V_4 .

In Corollary 1, block entry matrices $\tilde{e}_0 = 0_{12n \times n}$, $\tilde{e}_i \in \mathbb{R}^{12n \times n}$ ($i = 1, \dots, 12$) are used and the following notations are defined for the sake of the simplicity of the matrix notation:

$$\begin{aligned} \xi^T(t) &= \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-h_M) & \dot{x}^T(t) & \dot{x}^T(t-h_M) & g^T(x(t)) & g^T(x(t-h(t))) & g^T(x(t-h_M)) \\ \frac{1}{h(t)} \int_{t-h(t)}^t x^T(s) ds & \frac{1}{h_M-h(t)} \int_{t-h_M}^{t-h(t)} x^T(s) ds & \int_{t-h(t)}^t g^T(x(s)) ds & \int_{t-h_M}^{t-h(t)} g^T(x(s)) ds \\ \tilde{\omega}^T(t) &= \begin{bmatrix} x^T(t) & x^T(t-h_M) & \int_{t-h_M}^t x^T(s) ds & \int_{t-h_M}^t g^T(x(s)) ds \end{bmatrix}, \\ \tilde{H} &= \begin{bmatrix} -C & 0_{n \times 2n} & -I & 0_{n \times n} & A & B & 0_{n \times 5n} \end{bmatrix}, \quad \tilde{Y} = [\tilde{e}_1 - \tilde{e}_2 \quad \tilde{e}_1 + \tilde{e}_2 - 2\tilde{e}_9 \quad \tilde{e}_2 - \tilde{e}_3 \quad \tilde{e}_2 + \tilde{e}_3 - 2\tilde{e}_{10}]^T, \\ \tilde{\Pi}_1^0 &= [\tilde{e}_1 \quad \tilde{e}_3 \quad \tilde{e}_0 \quad \tilde{e}_{11} + \tilde{e}_{12}], \quad \tilde{\Pi}_1^1 = [\tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_9 \quad \tilde{e}_0], \quad \tilde{\Pi}_1^2 = [\tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_{10} \quad \tilde{e}_0], \\ \tilde{\Pi}_2 &= [\tilde{e}_4 \quad \tilde{e}_5 \quad \tilde{e}_1 - \tilde{e}_3 \quad \tilde{e}_6 - \tilde{e}_8], \quad \tilde{\Pi}_3 = [\tilde{e}_1 \quad \tilde{e}_6 \quad \tilde{e}_0], \quad \tilde{\Pi}_4 = [\tilde{e}_2 \quad \tilde{e}_7 \quad \tilde{e}_1 - \tilde{e}_2], \quad \tilde{\Pi}_5^0 = [\tilde{e}_0 \quad \tilde{e}_{11} \quad \tilde{e}_0], \quad \tilde{\Pi}_5^1 = [\tilde{e}_9 \quad \tilde{e}_0 \quad \tilde{e}_1 - \tilde{e}_9], \\ \tilde{\Pi}_6 &= [\tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_4], \quad \tilde{\Pi}_7 = [\tilde{e}_1 \quad \tilde{e}_4 \quad \tilde{e}_6], \quad \tilde{\Pi}_8 = [\tilde{e}_3 \quad \tilde{e}_5 \quad \tilde{e}_8], \quad \tilde{\Pi}_9^0 = [\tilde{e}_0 \quad \tilde{e}_1 - \tilde{e}_2 \quad \tilde{e}_{11} \quad \tilde{e}_0 \quad \tilde{e}_2 - \tilde{e}_3 \quad \tilde{e}_{12}], \\ \tilde{\Pi}_9^1 &= [\tilde{e}_9 \quad \tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_0], \quad \tilde{\Pi}_9^2 = [\tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_0 \quad \tilde{e}_{10} \quad \tilde{e}_0 \quad \tilde{e}_0], \\ \tilde{\Theta}_a &= -\text{Sym}([\tilde{e}_6 - \tilde{e}_1 K_\rho] T_1 [\tilde{e}_6 - \tilde{e}_1 K_m]^T + [\tilde{e}_7 - \tilde{e}_2 K_\rho] T_2 [\tilde{e}_7 - \tilde{e}_2 K_m]^T + [\tilde{e}_8 - \tilde{e}_3 K_\rho] T_3 [\tilde{e}_8 - \tilde{e}_3 K_m]^T), \\ \tilde{\Theta}_b &= -\text{Sym}([\tilde{e}_6 - \tilde{e}_1 K_m] T_4 [\tilde{e}_6 - \tilde{e}_1 K_\rho]^T + [\tilde{e}_7 - \tilde{e}_2 K_m] T_5 [\tilde{e}_7 - \tilde{e}_2 K_\rho]^T + [\tilde{e}_8 - \tilde{e}_3 K_m] T_6 [\tilde{e}_8 - \tilde{e}_3 K_\rho]^T), \\ \tilde{\Omega}_a &= -\text{Sym}([\tilde{e}_6 - \tilde{e}_7 - (\tilde{e}_1 - \tilde{e}_2) K_\rho] L_1 [\tilde{e}_6 - \tilde{e}_7 - (\tilde{e}_1 - \tilde{e}_2) K_m]^T + [\tilde{e}_7 - \tilde{e}_8 - (\tilde{e}_2 - \tilde{e}_3) K_\rho] L_2 [\tilde{e}_7 - \tilde{e}_8 - (\tilde{e}_2 - \tilde{e}_3) K_m]^T), \\ \tilde{\Omega}_b &= -\text{Sym}([\tilde{e}_6 - \tilde{e}_7 - (\tilde{e}_1 - \tilde{e}_2) K_\rho] L_3 [\tilde{e}_6 - \tilde{e}_7 - (\tilde{e}_1 - \tilde{e}_2) K_m]^T + [\tilde{e}_7 - \tilde{e}_8 - (\tilde{e}_2 - \tilde{e}_3) K_\rho] L_4 [\tilde{e}_7 - \tilde{e}_8 - (\tilde{e}_2 - \tilde{e}_3) K_m]^T), \\ \tilde{\Sigma} &= \text{Sym}(\tilde{\Pi}_1^0 \tilde{P} \tilde{\Pi}_1^T), \quad \tilde{\Sigma}_1 = \text{Sym}(\tilde{\Pi}_1^1 \tilde{P} \tilde{\Pi}_1^T) + \text{Sym}(\tilde{\Pi}_1^2 \tilde{Q} \tilde{\Pi}_1^T), \quad \tilde{\Sigma}_2 = \text{Sym}(\tilde{\Pi}_1^2 \tilde{P} \tilde{\Pi}_1^T), \\ \tilde{\Phi}_{1a} &= \text{Sym}([\tilde{e}_6 - \tilde{e}_1 K_m] D_1 \tilde{e}_4^T + [\tilde{e}_1 K_\rho - \tilde{e}_6] D_2 \tilde{e}_4^T + [\tilde{e}_8 - \tilde{e}_3 K_m] D_5 \tilde{e}_3^T + [\tilde{e}_3 K_\rho - \tilde{e}_8] D_6 \tilde{e}_3^T), \\ \tilde{\Phi}_{1b} &= \text{Sym}([\tilde{e}_6 - \tilde{e}_1 K_\rho] D_7 \tilde{e}_4^T + [\tilde{e}_1 K_m - \tilde{e}_6] D_8 \tilde{e}_4^T + [\tilde{e}_8 - \tilde{e}_3 K_\rho] D_{11} \tilde{e}_5^T + [\tilde{e}_3 K_m - \tilde{e}_8] D_{12} \tilde{e}_5^T), \\ \tilde{\Psi}_\Delta &= \tilde{\Sigma} + \tilde{\Phi}_{1\Delta} + \tilde{\Pi}_3 \tilde{Q} \tilde{\Pi}_3^T - (1 - h_D^u) \tilde{\Pi}_4 \tilde{Q} \tilde{\Pi}_4^T + \text{Sym}(\tilde{\Pi}_5^0 \tilde{Q} \tilde{\Pi}_5^T) + \tilde{\Pi}_7 \tilde{N} \tilde{\Pi}_7^T - \tilde{\Pi}_8 \tilde{N} \tilde{\Pi}_8^T + h_M^2 \tilde{\Pi}_7 Z \tilde{\Pi}_7^T + h_M^2 \tilde{e}_4 M \tilde{e}_4^T - \tilde{Y}^T \Phi \tilde{Y} \quad (40) \end{aligned}$$

Corollary 1. For given positive scalars $\rho \leq 1$ and h_M , any scalar h_D^u with condition C2, diagonal matrices K_m , K_M and K_ρ , network system (3) is asymptotically stable, if there exist positive definite matrices $\tilde{P} \in \mathbb{R}^{4n \times 4n}$, $\tilde{N} \in \mathbb{R}^{3n \times 3n}$, $\tilde{Q} \in \mathbb{R}^{3n \times 3n}$, $Z \in \mathbb{R}^{3n \times 3n}$, $M \in \mathbb{R}^{n \times n}$, positive diagonal matrices $D_i = \text{diag}\{d_{i1}, d_{i2}, \dots, d_{in}\}$ ($i = 1, 2, 5, 6, 7, 8, 11, 12$), $T_i = \text{diag}\{t_{i1}, t_{i2}, \dots, t_{in}\}$ ($i = 1, \dots, 6$), $L_i = \text{diag}\{l_{i1}, l_{i2}, \dots, l_{in}\}$ ($i = 1, \dots, 4$), and any matrix $W \in \mathbb{R}^{3n \times 3n}$ with matrices $S_{ij} \in \mathbb{R}^{n \times n}$ ($i, j = 1, 2$), and matrix $\tilde{\Pi}$ having appropriate dimensions, such that the following LMIs for $h = (0, h_M)$

$$\begin{bmatrix} (\tilde{H}^\perp)^T \tilde{\Omega}_\Delta(h) (\tilde{H}^\perp) + \text{Sym}\{(\tilde{H}^\perp)^T \tilde{\Gamma}(h) \tilde{\Pi}^T\} & \tilde{\Pi} \\ * & -X \end{bmatrix} < 0 \quad \Delta = a, b \quad (41)$$

$$X > 0, \quad \Phi > 0 \quad (42)$$

are satisfied, where $\tilde{\Omega}_\Delta(h) = h(t) \tilde{\Sigma}_1 + (h_M - h(t)) \tilde{\Sigma}_2 + \tilde{\Psi}_\Delta + \tilde{\Theta}_\Delta + \tilde{\Omega}_\Delta$, $\Delta = a, b$, $\tilde{\Gamma}(h) = \tilde{\Pi}_9^0 + h(t) \tilde{\Pi}_9^1 + (h_M - h(t)) \tilde{\Pi}_9^2$, and other matrices are defined in (7) or (40), while \tilde{H}^\perp is the right orthogonal complement of \tilde{H} .

Proof. Consider the following LKF candidate

$$\tilde{V}(x_t) = \sum_{i=1}^6 \tilde{V}_i(x_t)$$

where

$$\tilde{V}_1 = \tilde{\omega}^T(t) \tilde{P} \tilde{\omega}(t),$$

$$\begin{aligned}
\tilde{V}_{2a} &= 2 \sum_{i=1}^n (d_{1i} \int_0^{x_i(t)} (g_i(s) - k_i^- s) ds + d_{2i} \int_0^{x_i(t)} (k_i^\rho s - g_i(s)) ds) \\
&\quad + 2 \sum_{i=1}^n (d_{5i} \int_0^{x_i(t-h_M)} (g_i(s) - k_i^- s) ds + d_{6i} \int_0^{x_i(t-h_M)} (k_i^\rho s - g_i(s)) ds), \\
\tilde{V}_{2b} &= 2 \sum_{i=1}^n (d_{7i} \int_0^{x_i(t)} (g_i(s) - k_i^\rho s) ds + d_{8i} \int_0^{x_i(t)} (k_i^+ s - g_i(s)) ds) \\
&\quad + 2 \sum_{i=1}^n (d_{11i} \int_0^{x_i(t-h_M)} (g_i(s) - k_i^\rho s) ds + d_{12i} \int_0^{x_i(t-h_M)} (k_i^+ s - g_i(s)) ds), \\
\tilde{V}_3 &= \int_{t-h(t)}^t \begin{bmatrix} x(s) \\ g(x(s)) \\ \int_s^t \dot{x}(u) du \end{bmatrix}^T \tilde{Q} \begin{bmatrix} x(s) \\ g(x(s)) \\ \int_s^t \dot{x}(u) du \end{bmatrix} ds, \\
\tilde{V}_4 &= \int_{t-h_M}^t \beta^T(s) \tilde{N} \beta(s) ds, \quad \tilde{V}_5 = V_5, \tilde{V}_6 = V_6.
\end{aligned}$$

The proof is very similar to the one of Theorem 3 so that we can see inequalities (41-42) to guarantee the asymptotic stability of network system (3) hence of the neural network (1).

In some circumstances, the information on the derivative of the delay may not be always available after all. The criterion for such a case can be readily derived from Corollary 1 by setting $\tilde{Q} = 0$.

4. Numerical examples

In this section, two illustrative examples are presented and the respective results are summarized in comparison with relevant ones in the literature. These clearly the effectiveness of the here proposed method and demonstrate the achieved improvements.

Example 1. Consider the neural network system (3) with the following parameters

$$\begin{aligned}
C &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \\
K_m &= \text{diag}\{0,0\}, \quad K_M = \text{diag}\{0.4,0.8\}.
\end{aligned} \tag{43}$$

With the condition $-h_D \leq \dot{h}(t) \leq h_D$, our results obtained by Theorems 1-3, are shown in Table I. Also, when $\dot{h}(t) \leq h_D$ and unknown too, the corresponding results obtained by Corollary 1 are included in Table I as well. It can be seen that our results based on Theorems 1-3, compared to the results of [11, 13, 28] indeed improve the feasible region of stability criteria. The upper bounds obtained by Theorem 1 are larger than the results of Theorem 1 in [13], which shows that our new LKF, Wirtinger integral inequality and combined convex method can and do reduce the conservatism. The results based on Corollary 1 improve the feasible region of stability criteria compared to those of [11] and [28], however, fall short compared to the results of [13] when h_D is unknown. It is worth pointing out that the results of Theorem 2 and Theorem 3 clearly provide larger delay bounds than those of Theorem 1 when $\rho = 0.66$, which shows the effectiveness of technique of partitioning the bounding of the activation function conditions.

TABLE I. DELAY BOUNDS h_M WITH DIFFERENT h_D

Methods	Condition of $\dot{h}(t)$	$h_D = 0.8$	$h_D = 0.9$	Unknown or ≥ 1
Theorem 1 [28]	$0 \leq \dot{h}(t) \leq h_D$	3.0604	1.9956	-
Corollary 1 [28]	-	-	-	1.7860
Theorem 1 [11]	$\dot{h}(t) \leq h_D$	3.0640	2.0797	-
Corollary 1[11]	-	-	-	1.9207
Theorem 1 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	5.4741	3.7440	-

Theorem 2 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	6.5848	4.1767	-
Theorem 3 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	7.5173	5.3993	-
Corollary 1 [13]	$\dot{h}(t) \leq h_D$	3.7236	2.9229	-
Corollary 2 [13]	-	-	-	2.9208
Theorem 1	$-h_D \leq \dot{h}(t) \leq h_D$	7.1744	4.0356	-
Theorem 2 ($\rho = 0.66$)	$-h_D \leq \dot{h}(t) \leq h_D$	12.1028	6.3209	-
Theorem 3 ($\rho = 0.66$)	$-h_D \leq \dot{h}(t) \leq h_D$	16.8744	9.3778	-
Corollary 1 ($\rho = 0.66$)	$\dot{h}(t) \leq h_D$	4.6785	3.1153	-
Corollary 1 ($\tilde{Q} = 0, \rho = 0.66$)	-	-	-	2.3631

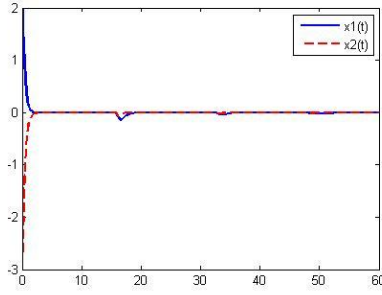


Fig. 1. State trajectories of the system of Example 1 with time-varying delay $h(t) = 16.0744 + 0.8\sin(t)$.

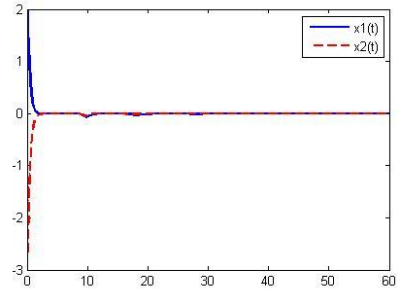


Fig. 2. State trajectories of the system of Example 1 with time-varying delay $h(t) = 8.4778 + 0.9\sin(t)$.

The time evolution responses of the neural network with a time-varying delay in Example 1 when $h_M = 16.8744$ for $h_D = 0.8$, $h(t) = 16.0744 + 0.8\sin(t) \leq 16.8744$ and $h_M = 9.3778$ for $h_D = 0.9$, $h(t) = 8.4778 + 0.9\sin(t) \leq 9.3778$ are shown in Figure 1 and 2. These were obtained by setting $x(0) = [2, -3]^T$, $g(x(t)) = [0.4\tanh(x_1(t)), 0.8\tanh(x_2(t))]^T$. The resulting responses clearly indicate fast asymptotic stability of the simulated neural networks with time-varying delay.

Example 2. Next, let consider the neural networks (3) with the following parameters:

$$C = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6231 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix}, \quad A = \begin{bmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$

$$K_m = \text{diag}\{0, 0, 0, 0\}, \quad K_M = \text{diag}\{0.1137, 0.1279, 0.7994, 0.2368\}. \quad (44)$$

TABLE II. DELAY BOUNDS h_M WITH DIFFERENT h_D

Methods	Condition of $\dot{h}(t)$	$h_D = 0.1$	$h_D = 0.5$	$h_D = 0.9$	Unknown or ≥ 1
Theorem 1 [26] ($m = 2$)	$\dot{h}(t) \leq h_D$	3.7525	2.7353	2.2760	-
Theorem 1 without v_2 [26]	-	-	-	-	2.1326
Theorem 2 [31]	$\dot{h}(t) \leq h_D$	3.7857	3.0546	2.6703	-
Theorem 1 without v_3 [31]	-	-	-	-	2.6575

Theorem 1 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	3.9269	3.4072	2.8337	-
Theorem 2 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	3.9332	3.5277	3.2025	-
Theorem 3 [13]	$-h_D \leq \dot{h}(t) \leq h_D$	3.9337	3.5307	3.2627	-
Corollary 1 [13]	$\dot{h}(t) \leq h_D$	3.8102	3.1518	2.8402	-
Corollary 2 [13]	-	-	-	-	2.8379
Theorem 1	$-h_D \leq \dot{h}(t) \leq h_D$	4.5086	3.8091	3.2895	-
Theorem 2 ($\rho = 0.40$)	$-h_D \leq \dot{h}(t) \leq h_D$	4.5045	3.8334	3.4840	-
Theorem 3 ($\rho = 0.40$)	$-h_D \leq \dot{h}(t) \leq h_D$	4.5106	3.8716	3.5482	-
Corollary 1 ($\rho = 0.40$)	$\dot{h}(t) \leq h_D$	4.4338	3.5344	3.1410	-
Corollary 1 ($\tilde{Q} = 0, \rho = 0.40$)	-	-	-	-	3.1122

For condition C1 satisfied, the acceptable maximal upper bounds of the time delay based on Theorems 1-3 are shown in Table II. Also, when $\dot{h}(t) \leq h_D$ and it is unknown, the corresponding results obtained by means of Corollary 1 are also included in Table II. It can be seen that our results based on Theorems 1-3 and Corollary 1 compare favorably to the results of [13, 26, 31] hence improve the feasible region of stability criteria.

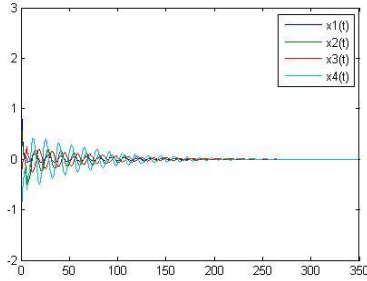


Fig.3. State trajectories of the system of Example 2 with time-varying delay $h(t) = 4.4106 + 0.1\sin(t)$.

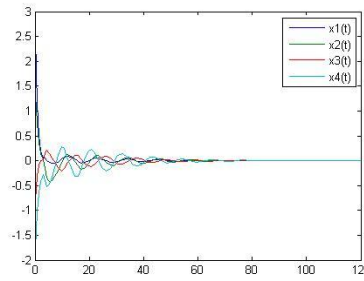


Fig. 4. State trajectories of the system of Example 2 with time-varying delay $h(t) = 3.3716 + 0.5\sin(t)$

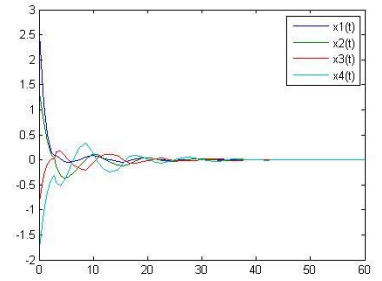


Fig. 5. State trajectories of the system of Example 2 with time-varying delay $h(t) = 2.6482 + 0.9\sin(t)$

Figure 3, Figure 4, Figure 5 depict simulation results for the state curves of the neural network in Example 2 with the following settings: $h(t) = 4.4106 + 0.1\sin(t) \leq 4.5106$ and $h_D = 0.1$; $h(t) = 3.3716 + 0.5\sin(t) \leq 3.8716$ and $h_D = 0.5$; $h(t) = 2.6482 + 0.9\sin(t) \leq 3.5482$ and $h_D = 0.9$. The chosen initial condition is given as $x(0) = [3, 1.5, -1, -2]^T$. The activation functions are $g(x(t)) = [0.1137 \tanh(x_1(t)), 0.1279 \tanh(x_2(t)), 0.7994 \tanh(x_3(t)), 0.2368 \tanh(x_4(t))]^T$. Although these state time responses have longer transient oscillations, nonetheless, very rapidly fall within boundaries smaller than an amplitude of 0.5. These responses again demonstrate fast asymptotic stability of neural networks with time-varying delays (1) is guaranteed.

5. Conclusions

The problem of delay-dependent stability for neural networks with time-varying delays is investigated in this paper and new less conservative asymptotic stability criteria derived. By constructing a suitable LKF and using a novel partitioning method for the bounding of the activations functions and by employing Wirtinger integral inequality to deal with the derivative of LKF, less conservative delay-dependent stability criteria expressed in terms of LMIs are presented. The results for two illustrative examples from the literature are given in comparison with the previous ones that clearly to demonstrate the improvements achieved.

This paper only consider the stability problems of CNNs, other problems such as robust stability, exponential stability, synchronization, and so on, can be investigated using the new methodological approach. Also, it is worth noting, constructing a more suitable LKF and reducing the calculation enlargement in estimating the derivative also needs further investigation. Systematic stability analysis and controller design for Takagi–Sugeno (T–S) fuzzy systems have witnessed growing interests[42–46]. Generally, most of the approaches involve the employment of a simple LKF[47], and the application of some more or less tight techniques[48], such as Moon's inequality, free-weighting matrix, or Jensen's inequality, to derive the stability criteria for the time-delayed T–S fuzzy system. How to extend the

new constructed LKF and Wirtinger inequality to investigate the stability and controller design problems for time-delayed T–S fuzzy system is our another future work direction.

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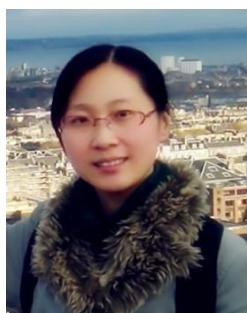
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