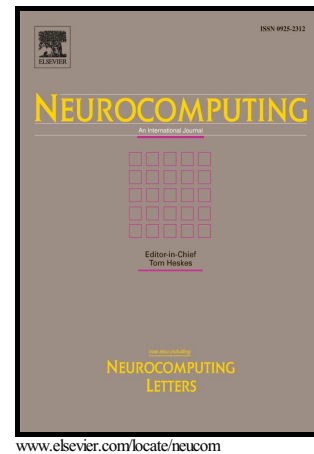


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General decay synchronization stability for a class of delayed chaotic neural networks with discontinuous activations

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Abstract

This paper is concerned with the synchronization problem for a class of delayed chaotic neural networks with discontinuous activations. First a lemma which concerns stability in general decay rate is constructed. Based on this lemma, the general decay synchronization stability criteria of discontinuous neural networks are derived via a designed controller. The general decay synchronization is obtained by introducing a decay function and it contains exponential synchronization and polynomial synchronization as its two special cases. Finally, two examples are given to verify the effectiveness of the obtained results.

Keywords: Delayed neural networks; Discontinuous activations; General decay synchronization; Decay function

1. Introduction

Since Pecora and Carroll firstly introduced chaos synchronization in 1990 [1], chaos synchronization has been extensively studied due to its potential applications such as secure communication, information processing, biological systems [2-7]. It is shown that delayed neural networks can exhibit chaotic behavior provided that the parameters and delays are appropriately chosen [8]. Therefore, synchronization and chaotic control of neural networks has been one of the hot research topics in the past decades. Moreover, lots of synchronization results have been obtained under different control approaches, such as feedback control [9-13], adaptive control [14-17], impulsive control [18], sampled-date control [19], intermittent control [20], finite-time control [21], etc.

It is worth noting that the activations of neural networks model in these papers are assumed to be continuous. A recent paper [22] has pointed out the interest for studying global convergence of general neural networks with discontinuous neuron activations. Discontinuous neuron activations are frequently encountered in the practical applications, and the system of neural networks with discontinuous activations has been proved

really useful as an ideal model for the case where the gain of the neuron amplifiers is very high [23]. So recently, dynamical behaviors including the stability and synchronization of delayed neural networks with discontinuous activations have received a great deal of attention and have been extensively studied in the literature [24-32]. In [25, 26], quasi-synchronization of discontinuous neural networks was investigated, i.e. the synchronization error can only be controlled within a small region around zero. It is also reported in [25] that complete synchronization cannot be achieved between the identical drive and response systems due to the discontinuity of activation functions.

In light of the above analysis, in this paper, we study the synchronization problem for a class of delayed neural networks with discontinuous activations. There are three advantages that make our approach attractive. Firstly, differential inclusion, nonsmooth analysis and control theory are employed to handle system with discontinuous right-hand sides. Secondly, a new crucial lemma which includes and extends the classical exponential stability theorem is constructed. The new lemma provides a new result on the stability in general decay rate by introducing a decay function. Then synchronization in general decay rate for discontinuous neural networks is obtained by using the lemma. Thirdly, the complete synchronization in general decay rate studied in our paper contains exponential synchronization and

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polynomial synchronization as its two special cases.

The rest of this paper is organized as follows. The system and some preliminaries are introduced in Section 2. In Section 3, by constructing a new lemma, the general decay synchronization criteria are established for discontinuous delayed neural networks via a non-linear controller. Then, numerical simulations are given to demonstrate the effectiveness of the obtained results in Section 4. Finally, conclusions are drawn in Section 5.

Notations Through this paper, R_+ denotes the set of all positive real numbers, R^n denotes the n -dimensional Euclidean space and $R^{n \times n}$ denotes the set of all $n \times n$ real matrices. For any vector $x \in R^n$, its Euclidean norm is denoted as $\| \cdot \|$, i.e. $\|x\| = \sqrt{x^T x}$. A^T and A^{-1} stand for the transpose and the inverse of the matrix A , respectively; $A > 0$ ($A \geq 0$) means that the matrix A is symmetric and positive definite (semi-positive definite); $\lambda_{\max}(A)$ denotes the maximum eigenvalue of matrix. $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$, $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$. $\text{diag}(\cdot)$ denotes a block-diagonal matrix. I is the identity matrix with appropriate dimension. $\text{sign}(\cdot)$ denotes the signum function.

2. System description and preliminaries

In this paper, we consider a class of chaotic systems with time-varying delay as follows:

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bg(x(t - \tau(t))) + J, \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in R^n$ is the state vector. $D = \text{diag}(d_1, d_2, \dots, d_n)$ is an $n \times n$ diagonal matrix with $d_i > 0$, $i = 1, 2, \dots, n$. $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n} \in R^{n \times n}$ are the connection weight matrix and delayed connection weight matrix, respectively. $f(x(t)) = (f_1(x_1(t)), \dots, f_n(x_n(t)))^T \in R^n$ and $g(x(t - \tau(t))) = (g_1(x_1(t - \tau(t))), \dots, g_n(x_n(t - \tau(t))))^T \in R^n$ are the neuron activation functions. $\tau(t)$ is the time-varying delay. $J = (J_1, J_2, \dots, J_n)^T$ is the external input vector.

Throughout this paper, the following assumptions are given for system (1).

(A1) For every $j = 1, 2, \dots, n$, $f_j, g_j : R \rightarrow R$ are continuous except on a countable set of isolate points $\{\rho_k^j\}$, where the finite right and left limits $f_j^+(\rho_k^j)$, $g_j^+(\rho_k^j)$ and $f_j^-(\rho_k^j)$, $g_j^-(\rho_k^j)$ exist, respectively.

(A2) For each $j = 1, 2, \dots, n$, there exist constants $h_j, k_j, r_j, s_j > 0$, such that

$$\begin{aligned} \sup |\xi_j - \zeta_j| &\leq h_j |u - v| + r_j, \\ \sup |\varrho_j - \nu_j| &\leq k_j |u - v| + s_j, \end{aligned} \quad (2)$$

for all $u, v \in R$, where $\xi_j \in K[f_j(u)]$, $\zeta_j \in K[f_j(v)]$, $\varrho_j \in K[g_j(u)]$, $\nu_j \in K[g_j(v)]$, $K[f_j(x)] = [\min\{f_j^-(x), f_j^+(x)\}, \max\{f_j^-(x), f_j^+(x)\}]$, $K[g_j(y)] = [\min\{g_j^-(y), g_j^+(y)\}, \max\{g_j^-(y), g_j^+(y)\}]$ for $x, y \in R$.

(A3) The time-varying delay $\tau(t)$ is bounded and there exist $\tau > 0, \mu > 0$ such that

$$0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq \mu < 1, \quad (3)$$

for all $t \geq 0$.

Remark 1. It is worth noting ξ_j, ϱ_j may not be equal to ζ_j, ν_j even $u = v$ if u is a discontinuous point. So the constants r_j, s_j in assumption (A2) is necessary, which is the essential difference between this paper and the previous literature where the Lipschitz condition was used.

Since system (1) is a discontinuous system, its solution is different from the classic solution and cannot be defined in the conventional sense. So we introduce the Filippov solution [34].

Definition 1 [34]. For a system with discontinuous right-hand sides:

$$\frac{dx}{dt} = F(x), x(0) = x_0, x \in R^n, t \geq 0. \quad (4)$$

A set-valued map is defined as

$$\Phi(x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} K[F(B(x, \delta) \setminus N)],$$

where $K[E]$ is the closure of the convex hull of set E , $E \subset R^n$, $B(x, \delta) = \{y : \|y - x\| < \delta, x, y \in R^n, \delta \in R^+\}$, and $N \subset R^n$, $\mu(N)$ is the Lebesgue measure of set N .

A solution (in Filippov's sense) of system (4) with initial condition $x(0) = x_0 \in R^n$ is an absolutely continuous function $x(t), t \in [0, T], T > 0$, which satisfy $x(0) = x_0$ and differential inclusion:

$$\frac{dx}{dt} \in \Phi(x), \text{ for a.a. } t \in [0, T].$$

Now we extend the concept of the Filippov solution to the discontinuous system (1) as follows:

Definition 2 [23]. A function $x : [-\tau, T] \rightarrow R^n, T \in (0, +\infty]$, is a solution (in Filippov's sense) of the discontinuous system (1) on $[-\tau, T]$, if:

- i) x is continuous on $[-\tau, T]$ and absolutely continuous on $[0, T]$;
- ii) $x(t)$ satisfies

$$\begin{aligned} \dot{x}(t) &\in -Dx(t) + AK[f(x(t))] + BK[g(y(t - \tau(t)))] + J, \\ &\text{for a.a. } t \in [0, T). \end{aligned} \quad (5)$$

Or equivalently,

ii)' there exist two measurable functions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T, \beta = (\beta_1, \beta_2, \dots, \beta_n)^T : [-\tau, T] \rightarrow \mathbb{R}^n$, such that $\alpha(t) \in K[f(x(t))], \beta(t) \in K[g(x(t))]$ for a.a. $t \in [-\tau, T]$ and

$$\dot{x}(t) = -Dx(t) + A\alpha(t) + B\beta(t - \tau(t)) + J, \text{ for a.a. } t \in [0, T]. \quad (6)$$

Definition 3 (IVP) [23]. For any continuous function $v : [-\tau, 0] \rightarrow \mathbb{R}^n$ and any measurable selections $\chi(s) \in K[f(v(s))], \omega(s) \in K[g(v(s))]$ for a.a. $s \in [-\tau, 0]$ by an initial value problem associated to (1) with initial condition (v, χ, ω) , we mean the following problem: find a couple of functions $[x, \alpha, \beta] : [-\tau, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, such that x is a solution of (1) on $[-\tau, T]$ for some $T > 0$, α, β are the outputs associated to x , and

$$\begin{cases} \dot{x}(t) = -Dx(t) + A\alpha(t) + B\beta(t - \tau(t)) + J, \\ \quad \text{for a.a. } t \in [0, T] \\ \alpha(t) \in K[f(x(t))], \beta(t) \in K[g(x(t))], \\ \quad \text{for a.a. } t \in [0, T] \\ x(s) = v(s), \forall s \in [-\tau, 0], \\ \alpha(s) = \chi(s), \beta(s) = \omega(s), \text{ for a.a. } t \in [0, T]. \end{cases} \quad (7)$$

Lemma 1 [25]. Suppose that the Assumptions (A1) and (A2) are satisfied, then there exist at least one solution of system (1) defined on $[0, +\infty)$ in the sense of Eqs. (7).

Consider system (1) as the drive system, then the controlled response system is

$$\dot{y}(t) = -Dy(t) + Af(y(t)) + Bg(y(t - \tau(t))) + J + u(t), \quad (8)$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n$ is the state variable of system (8), $u(t)$ is the controller to be designed, the other parameters are the same as in system (1).

In view of Definition 3 and Lemma 1, the IVP of system (8) is

$$\begin{cases} \dot{y}(t) = -Dy(t) + A\eta(t) + B\theta(t - \tau(t)) + J + u(t), \\ \quad \text{for a.a. } t \in [0, T] \\ \eta(t) \in K[f(y(t))], \theta(t) \in K[g(y(t))], \\ \quad \text{for a.a. } t \in [0, T] \\ y(s) = \phi(s), \forall s \in [-\tau, 0], \\ \eta(s) = \vartheta(s), \theta(s) = \varsigma(s), \text{ for a.a. } t \in [0, T]. \end{cases} \quad (9)$$

Define the synchronization error as $e(t) = (e_1(t), e_2(t), \dots, e_n(t))^T = y(t) - x(t)$, then from (7) and

(9), we can obtain the following synchronization error system:

$$\dot{e}(t) = -De(t) + A\pi(t) + B\varpi(t - \tau(t)) + u(t), \quad (10)$$

where $\pi(t) = \eta(t) - \alpha(t)$, $\varpi(t - \tau(t)) = \theta(t - \tau(t)) - \beta(t - \tau(t))$, $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ is the control input that will be designed latter.

Before giving our result, we first introduce the following definitions of ψ -type function (decay function) and ψ -type stability (stability in general decay rate).

Definition 4 [33]. The function $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$ is said to be ψ -type function if this function satisfies the following conditions:

- (i) It is differentiable and nondecreasing;
- (ii) $\psi(0) = 1$ and $\psi(\infty) = \infty$;
- (iii) Let $\psi_1(t) = \psi'(t)/\psi(t)$ be non-increasing and $\phi = \sup_{t \geq 0} \psi_1(t) < \infty$;
- (iv) For any $t, s \geq 0$, $\psi(t + s) \leq \psi(t)\psi(s)$.

It is obvious that functions $\psi(t) = e^{\alpha t}$ and $\psi(t) = (1 + t)^\alpha$ for any $\alpha > 0$ are of ψ -type since they satisfy the above four conditions.

Definition 5 [33]. The error system (10) is said to be ψ -type stable if there exists a constant $\gamma > 0$ such that

$$\limsup_{t \rightarrow \infty} \frac{\log \|e(t)\|}{\log \psi(t)} \leq -\gamma. \quad (11)$$

Remark 2. The function ψ is used as the decay function, so ψ -type stability is also said to be stability with general decay rate. When $\psi(t) = e^{\alpha t}$ and $\psi(t) = (1 + t)^\alpha$ for any $\alpha > 0$, ψ -type stability may be specialized as exponential stability and polynomial stability.

Lemma 2 [24]. If $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is Clarke-regular, and $x(t)$ is absolutely continuous on any compact subinterval of $[0, +\infty)$. Then, $x(t)$ and $V(x(t)) : [0, +\infty) \rightarrow \mathbb{R}$ are differentiable for a.a. $t \in [0, +\infty)$ and

$$\frac{d}{dt} V(x(t)) = \check{\gamma}^T(t) \dot{x}(t), \quad \forall \check{\gamma} \in \partial V(x(t)), \quad (12)$$

where $\partial V(x(t))$ is the Clarke generalized gradient of V at $x(t)$.

3. Main results

Before giving our main result, a new lemma which plays an important role in our proof is constructed. The following assumption (A4) is a preparation for the lemma.

(A4) $\varphi(t) \in C(R, R_+)$ is a continuous function and for some $\lambda > 0$, we have

$$\begin{aligned} \psi(t) - 1 &\leq 0, \quad \sup_{t \in [0, \infty)} \left[\int_0^t \psi^\lambda(s) \varphi(s) ds \right] < \infty, \\ \limsup_{t \rightarrow \infty} \left[\int_0^t \psi^\lambda(s) \varphi(s) ds \right] &< \infty, \end{aligned} \quad (13)$$

where $\psi(t)$ and $\psi_1(t)$ are defined in Definition 4.

Lemma 3. Given a continuous nonlinear system

$$\dot{x}(t) = F(x(t), t), \quad (14)$$

where $x(t)$ is a $n \times 1$ vector, $F(x(t), t) : R^n \times R \rightarrow R^n$ is continuous in x and t . Let $V(x, t)$ be the associated Lyapunov function with following properties:

$$(\lambda_1 \|x\|)^2 \leq V(x, t) \leq (\lambda_2 \|x\|)^2, \quad \forall (x, t) \in R^n \times R_+, \quad (15)$$

$$\dot{V}(x, t) \leq -\lambda_3 V(x, t) + \lambda_4 \varphi(t), \quad \forall (x, t) \in R^n \times R_+, \quad (16)$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive scalar constants, $\varphi(t)$ is defined in (A4). Under the assumption (A4), if the Lyapunov function satisfies (15) and (16), then the state $x(t)$ is ψ -type stable.

Proof. Calculating the derivative of $\psi^{\lambda_3}(t)V(x, t)$ where $\psi(t)$ is defined in Definition 4:

$$\begin{aligned} &\frac{d[\psi^{\lambda_3}(t)V(x, t)]}{dt} \\ &= \lambda_3 \psi^{\lambda_3-1}(t) \dot{\psi}(t) V(x, t) + \psi^{\lambda_3}(t) \dot{V}(x, t) \\ &\leq \lambda_3 \psi^{\lambda_3-1}(t) \dot{\psi}(t) V(x, t) - \lambda_3 \psi^{\lambda_3}(t) V(x, t) \\ &\quad + \lambda_4 \psi^{\lambda_3}(t) \varphi(t) \\ &= [-\lambda_3 + \lambda_3 \psi_1(t)] \psi^{\lambda_3}(t) V(x, t) + \lambda_4 \psi^{\lambda_3}(t) \varphi(t). \end{aligned} \quad (17)$$

Hence

$$\begin{aligned} &\psi^{\lambda_3}(t) V(x, t) \\ &\leq V(x(0), 0) + \int_0^t [-\lambda_3 + \lambda_3 \psi_1(s)] \psi^{\lambda_3}(s) V(x, s) ds \\ &\quad + \lambda_4 \int_0^t \psi^{\lambda_3}(s) \varphi(s) ds. \end{aligned} \quad (18)$$

Combining with definition 4 and assumption (A4), we get

$$\psi^{\lambda_3}(t) V(x, t) \leq V(x(0), 0) + \lambda_4 \int_0^t \psi^{\lambda_3}(s) \varphi(s) ds < \infty, \quad (19)$$

which means that

$$\psi^{\lambda_3}(t) (\lambda_1 \|x\|)^2 < \infty, \quad (20)$$

$$\sup_{t \in [0, \infty)} [\psi^{\lambda_3}(t) (\lambda_1 \|x\|)^2] < \infty. \quad (21)$$

There exists a constant $C \in R_+$ such that

$$\sup_{t \in [0, \infty)} [\psi^{\lambda_3}(t) (\lambda_1 \|x\|)^2] \leq C, \quad (22)$$

and

$$\sup_{t \in [0, \infty)} [\lambda_3 \log \psi + 2 \log \|x\|] \leq \log \frac{C}{\lambda_1^2}. \quad (23)$$

Thus we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log \|x(t)\|}{\log \psi(t)} &\leq \limsup_{t \rightarrow \infty} \frac{\log \frac{C}{\lambda_1^2}}{2 \log \psi(t)} - \frac{\lambda_3}{2} \\ &\leq -\frac{\lambda_3}{2}. \end{aligned} \quad (24)$$

According to the Definition 5, the state $x(t)$ is ψ -type stable. The proof is completed.

Remark 3. When the function $\varphi(t) = 0$, $\psi(t) = e^t$, the condition (13) holds obviously. It follows that

$$\limsup_{t \rightarrow \infty} \frac{\log \|x(t)\|}{t} \leq -\frac{\lambda_3}{2}$$

i.e. the state $x(t)$ of system (14) is exponential stable. Since $\varphi(t) \geq 0$, the right hand of condition (16) can be positive, which means Lemma 3 extends the classical exponential stability theorem.

The controller in the response system (8) is designed as follows.

$$u(t) = -G_1 e(t) - G_2 \text{sign}(e(t)) - \frac{\|A\|_\infty^2 \|e(t)\|^2 e(t)}{2\|A\|_\infty \|e(t)\|^2 + 2\varphi(t)}, \quad (25)$$

where $G_1 = \text{diag}(\epsilon_1, \dots, \epsilon_n)$, $G_2 = \text{diag}(\epsilon_1, \dots, \epsilon_n)$, $\varphi(t) > 0$ is defined in (A3).

Remark 4. when the function $\varphi(t) = 0$, the controller $u(t)$ becomes $-(G_1 + 1/2I)e(t) - G_2 \text{sign}(e(t))$, exponential synchronization of the drive and response systems can be obtained according to Lemma 3.

Theorem 1. Under the assumptions (A1)-(A4), if there exist diagonal matrices $G_1 > 0, G_2 > 0$, such that

$$P - G_2 < 0, \quad (26)$$

$$\begin{aligned} &-D - D^T - 2G_1 + \frac{\|B\|_1 K^2}{1 - \mu} + \tau I + \|A\|_1 H^2 \\ &+ \|B\|_\infty I < 0, \end{aligned} \quad (27)$$

where $P = \text{diag}(\sum_{j=1}^n (r_j |a_{1j}| + s_j |b_{1j}|), \sum_{j=1}^n (r_j |a_{2j}| + s_j |b_{2j}|), \dots, \sum_{j=1}^n (r_j |a_{nj}| + s_j |b_{nj}|))$, $H^2 = \text{diag}(h_1^2, h_2^2, \dots, h_n^2)$,

$K^2 = \text{diag}(k_1^2, k_2^2, \dots, k_n^2)$, $G_1 = \text{diag}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, $G_2 = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$. Then systems (1) and (8) can be synchronized in general decay rate via the controller (25).

Proof. For the error dynamical system (10), consider the following Lyapunov functional

$$V(t) = e^T(t)e(t) + \frac{\|B\|_1}{1-\mu} \int_{t-\tau(t)}^t e^T(s)K^2e(s)ds + \int_{t-\tau}^t \int_{\xi}^t e^T(s)e(s)dsd\xi. \quad (28)$$

By Calculating the derivative of the Lyapunov functional along the trajectory of system (10), we get from Lemma 2 that for a.a. $t \geq 0$

$$\begin{aligned} \dot{V}(t) &= 2e^T(t)\dot{e}(t) + \frac{\|B\|_1}{1-\mu} e(t)K^2e(t) \\ &\quad - \frac{1-\dot{\tau}(t)}{1-\mu} \|B\|_1 e^T(t-\tau(t))K^2e(t-\tau(t)) \\ &\quad + \tau e^T(t)e(t) - \int_{t-\tau}^t e^T(s)e(s)ds. \end{aligned} \quad (29)$$

Noticing that the assumption (A3) $\dot{\tau}(t) \leq \mu < 1$, one has $-\frac{1-\dot{\tau}(t)}{1-\mu} \leq -1$. Hence

$$\begin{aligned} \dot{V}(t) &= 2e^T(t)[-De(t) + A\pi(t) + B\varpi(t-\tau(t)) - G_1e(t) \\ &\quad - G_2\text{sign}(e(t)) - \frac{\|A\|_\infty^2\|e(t)\|^2e(t)}{2\|A\|_\infty\|e(t)\|^2 + 2\varphi(t)}] + \frac{\|B\|_1}{1-\mu} \\ &\quad \times e(t)K^2e(t) - \frac{1-\dot{\tau}(t)}{1-\mu} \|B\|_1 e^T(t-\tau(t))K^2 \\ &\quad \times e(t-\tau(t)) + \tau e^T(t)e(t) - \int_{t-\tau}^t e^T(s)e(s)ds \\ &\leq e^T(t)(-D - D^T - 2G_1 + \frac{\|B\|_1K^2}{1-\mu} + \tau I)e(t) \\ &\quad + 2e^T(t)A\pi(t) + 2e^T(t)B\varpi(t-\tau(t)) \\ &\quad - 2e^T(t)G_2\text{sign}(e(t)) - \frac{\|A\|_\infty^2\|e(t)\|^4}{\|A\|_\infty\|e(t)\|^2 + \varphi(t)} \\ &\quad - \|B\|_1 e^T(t-\tau(t))K^2e(t-\tau(t)) \\ &\quad - \int_{t-\tau}^t e^T(s)e(s)ds. \end{aligned} \quad (30)$$

From assumption (A2), we can get that

$$\begin{aligned} &2e^T(t)A\pi(t) \\ &= 2 \sum_{i=1}^n \sum_{j=1}^n e_i(t)a_{ij}\pi_j(t) \end{aligned}$$

$$\begin{aligned} &\leq 2 \sum_{i=1}^n \sum_{j=1}^n |e_i(t)| |a_{ij}| (|h_j e_j(t)| + r_j) \\ &\leq \|A\|_\infty e^T(t)e(t) + \|A\|_1 e^T(t)H^2e(t) \\ &\quad + 2 \sum_{i=1}^n \sum_{j=1}^n r_j |a_{ij}| |e_i(t)|, \end{aligned} \quad (31)$$

and

$$\begin{aligned} &2e^T(t)B\varpi(t-\tau(t)) \\ &\leq 2 \sum_{i=1}^n \sum_{j=1}^n |e_i(t)| |b_{ij}| (|k_j e_j(t-\tau(t))| + s_j) \\ &\leq \|B\|_\infty e^T(t)e(t) + \|B\|_1 e^T(t-\tau(t))K^2e(t-\tau(t)) \\ &\quad + 2 \sum_{i=1}^n \sum_{j=1}^n s_j |b_{ij}| |e_i(t)|, \end{aligned} \quad (32)$$

where $\|A\|_\infty = \max_i \{\sum_{j=1}^n |a_{ij}|\}$, $\|A\|_1 = \max_j \{\sum_{i=1}^n |a_{ij}|\}$,

$\|B\|_\infty = \max_i \{\sum_{j=1}^n |b_{ij}|\}$, $\|B\|_1 = \max_j \{\sum_{i=1}^n |b_{ij}|\}$.

Substituting (31) and (32) into (30), we derive that

$$\begin{aligned} \dot{V}(t) &\leq e^T(t)[-D - D^T - 2G_1 + \frac{\|B\|_1K^2}{1-\mu} + \tau I \\ &\quad + \|A\|_1H^2 + \|B\|_\infty I]e(t) \\ &\quad + \|A\|_\infty\|e(t)\|^2 - \frac{\|A\|_\infty^2\|e(t)\|^4}{\|A\|_\infty\|e(t)\|^2 + \varphi(t)} \\ &\quad + 2 \sum_{i=1}^n |e_i(t)| (\sum_{j=1}^n r_j |a_{ij}| + \sum_{j=1}^n s_j |b_{ij}| - \epsilon_i) \\ &\quad - \int_{t-\tau}^t e^T(s)e(s)ds \\ &\leq e^T(t)\Delta e(t) + \frac{\|A\|_\infty\|e(t)\|^2 \cdot \varphi(t)}{\|A\|_\infty\|e(t)\|^2 + \varphi(t)} \\ &\quad - \int_{t-\tau}^t \|e(s)\|^2 ds, \end{aligned} \quad (33)$$

where $\Delta = -D - D^T - 2G_1 + \frac{\|B\|_1K^2}{1-\mu} + \tau I + \|A\|_1H^2 + \|B\|_\infty I$.

By using the inequality $0 \leq ab/(a+b) \leq a$, $\forall a, b > 0$, then from (33) we have

$$\dot{V}(t) \leq e^T(t)\Delta e(t) + \varphi(t) - \int_{t-\tau}^t \|e(s)\|^2 ds. \quad (34)$$

Meanwhile, from (28) one has the following two facts: there exists a scalar $\sigma > 1$ such that

$$\|e(t)\|^2 \leq V(t) \leq \sigma \|e(t)\|^2 + \sigma \int_{t-\tau}^t \|e(s)\|^2 ds, \quad (35)$$

and there exists a small enough $0 < \delta < 1$ such that

$$\delta\sigma + \lambda_{\max}(\Delta) < 0, \delta\sigma - 1 < 0.$$

Thus we have

$$\dot{V}(t) < -\delta V(t) + \varphi(t). \quad (36)$$

Then, the error system (10) is ψ -type stable according to Lemma 3. Consequently, systems (1) and (8) can be synchronized in general decay rate via the controller (25). The proof is completed.

If the decay function $\psi(t)$ is chosen to be exponential or polynomial function, then (A4) holds according to the controller. Then based on Theorem 1, the following results can be obtained.

Corollary 1. Under the assumptions (A1)-(A4), let $\varphi(t) = e^{-\rho t}$, $\rho > 0$, and $\psi(t) = e^t$, if there exist diagonal matrices $G_1 > 0, G_2 > 0$, such that (26) and (27) hold. Then systems (1) and (8) are exponentially synchronized via the controller (25).

Corollary 2. Under the assumptions (A1)-(A4), let $\varphi(t) = (1+t)^{-\rho}$, $\rho > 0$, and $\psi(t) = 1+t$, if there exist diagonal matrices $G_1 > 0, G_2 > 0$, such that (26) and (27) hold. Then systems (1) and (8) are polynomially synchronized via the controller (25).

In the case of $\tau(t) = \tau$, $\tau > 0$, $\dot{\tau}(t) = \mu = 0$, we have the following results:

Theorem 2. Under the assumptions (A1)-(A4) and $\tau(t) = \tau > 0, \mu = 0$, if there exist diagonal matrices $G_1 > 0, G_2 > 0$, such that

$$P_1 - G_2 < 0, \quad (37)$$

$$-D - D^T - 2G_1 + \|B\|_1 K^2 + \tau I + \|A\|_1 H^2 + \|B\|_\infty I < 0, \quad (38)$$

where $P_1 = \text{diag}(\sum_{j=1}^n (r_j |a_{1j}|), \sum_{j=1}^n (r_j |a_{2j}|), \dots, \sum_{j=1}^n (r_j |a_{nj}|))$,

$H^2 = \text{diag}(h_1^2, h_2^2, \dots, h_n^2)$, $G_1 = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$, $G_2 = \text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$. Then systems (1) and (8) can be synchronized in general decay rate via the controller (25).

Corollary 3. Under the assumptions (A1)-(A4) and $\tau(t) = \tau > 0, \mu = 0$, let $\varphi(t) = e^{-\rho t}$, $\rho > 0$, and $\psi(t) = e^t$, if there exist diagonal matrices $G_1 > 0, G_2 > 0$, such that (37) and (38) hold. Then systems (1) and (8) are exponentially synchronized via the controller (25).

Corollary 4. Under the assumptions (A1)-(A4) and $\tau(t) = \tau > 0, \mu = 0$, let $\varphi(t) = (1+t)^{-\rho}$, $\rho > 0$, and $\psi(t) = 1+t$, if there exist diagonal matrices $G_1 > 0, G_2 > 0$,

such that (37) and (38) hold. Then systems (1) and (8) are polynomially synchronized via the controller (25).

Remark 5. By constructing a new lemma, general decay synchronization of discontinuous neural networks is obtained. If the ψ -type function is chosen as exponential or polynomial function, then exponential or polynomial synchronization as the special cases of general decay synchronization can be obtained. So our results can be considered as the generalization and extension of previous works on exponential or asymptotical synchronization of delayed neural networks with continuous or discontinuous activations [11-17, 27].

Remark 6. The results in [25] shows that only quasi-synchronization can be achieved when the usual linear state feedback controllers are added to systems with discontinuous right-hand side. In our paper, the drive and response discontinuous neural networks can achieve complete synchronization. So our results are general compared with these in [25].

Remark 7. From the proof of Theorem 1, we can see that the term $-G_2 \text{sign}(e(t))$ in the controller plays a key role in dealing with the uncertain parameters r_j, s_j , $j = 1, 2, \dots, n$. Also, if the function $\varphi(t) = 0$, the drive and response neural networks can achieve exponential synchronization. These mean that the results of this paper are general and can be applied to achieve synchronization of continuous neural networks or other discontinuous delayed nonlinear systems.

4. Numerical examples

In this section, two examples are provided to verify the effectiveness of results obtained in the previous section.

Example 1. Consider a two-dimensional neural networks with time-varying delay

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bg(x(t - \tau(t))) + J, \quad (39)$$

where $x(t) = (x_1(t), x_2(t))^T$, $J = (0, 0)^T$, $D = I$,

$$A = \begin{bmatrix} 2 & -0.1 \\ -5 & 4.5 \end{bmatrix}, B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{bmatrix}, \tau(t) = \frac{e^t}{1 + e^t},$$

$$f(x) = g(x) = \begin{cases} \tanh(x) + 0.03, & x > 0, \\ \tanh(x) - 0.03, & x < 0. \end{cases}$$

The phase plot of system (39) is shown in Fig. 1.

We can get $\tau = 1, \dot{\tau}(t) \leq \mu = \frac{1}{4} < 1, h_j = k_j = 1, r_j = s_j = 0.06, j = 1, 2$. And $H^2 = K^2 = I, \|A\|_\infty = 9.5, \|A\|_1 = 7, \|B\|_\infty = 4.2, \|B\|_1 = 4.1$.

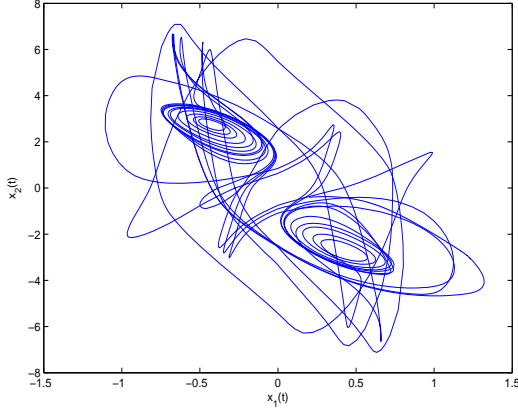


Figure 1: The phase plot of system (39) with the initial condition $x_1(s) = 0.2, x_2(s) = -0.4, \forall s \in [-1, 0]$.

Consider system (39) as the drive system. Then the corresponding response system is as follows

$$\dot{y}(t) = -Dy(t) + Af(y(t)) + Bg(y(t - \tau(t))) + J + u(t), \quad (40)$$

where $y(t) = (y_1(t), y_2(t))^T$, the other parameters are the same as in system (39).

Choose $\varphi(t) = \exp(-t)$, $G_1 = \text{diag}(8, 8)$, $G_2 = \text{diag}(0.5, 1)$, then the controller $u(t) = (u_1(t), u_2(t))^T$ is

$$u(t) = - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} e(t) - \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \text{sign}(e(t)) - \frac{9.5^2 \|e(t)\|^2 e(t)}{19 \|e(t)\|^2 + 2 \exp(-t)}. \quad (41)$$

It is easy to check that

$$\begin{aligned} P - G_2 &= \text{diag}(-0.2780, -0.1780) < 0, \\ -D - D^T - 2G_1 + \frac{\|B\|_1 K^2}{1 - \mu} + \tau I + \|A\|_1 H^2 + \|B\|_\infty I \\ &= \text{diag}(-0.3333, -0.3333) < 0. \end{aligned}$$

It follows from Corollary 1 that systems (39) and (40) are exponentially synchronized via the nonlinear controller (41). Fig.2. depicts the trajectories of the error states between systems (39) and (40) via the controller (41). We can see all errors converge to zero fast as time goes by. The state trajectories of variables $x_i(t)$ and $y_i(t)$, $i = 1, 2$ between systems (39) and (40) are shown in Fig.3.

Example 2. Consider the following two neural networks with time-varying delay

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bg(x(t - \tau(t))) + J, \quad (42)$$

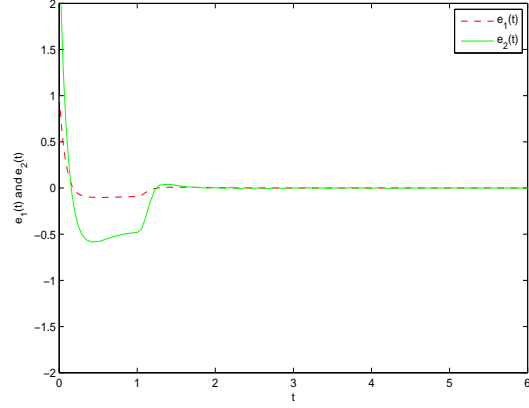


Figure 2: The synchronization errors $e_1(t)$ and $e_2(t)$ between systems (39) and (40) via the controller (41).

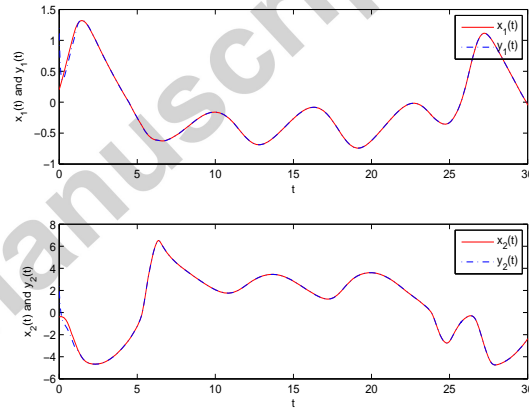


Figure 3: The state trajectories of variables $x_i(t)$ and $y_i(t)$, $i = 1, 2$ between systems (39) and (40).

$$\dot{y}(t) = -Dy(t) + Af(y(t)) + Bg(y(t - \tau(t))) + J + u(t), \quad (43)$$

where $x(t) = (x_1(t), x_2(t))^T$, $y(t) = (y_1(t), y_2(t))^T$, $J = (0, 0)^T$, $D = I$, $\tau(t) = 1$,

$$\begin{aligned} A &= \begin{bmatrix} 3 & 5 \\ 0.1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2.5 & 0.2 \\ 0.1 & -1.5 \end{bmatrix}, \\ f(x) &= \begin{cases} \tanh(x) + 0.03, & x > 0, \\ \tanh(x) - 0.03, & x < 0, \end{cases} \\ g(x) &= \begin{cases} \tanh(x) + 0.02x + 0.028, & x > 0, \\ \tanh(x) + 0.02x - 0.028, & x < 0. \end{cases} \end{aligned}$$

The phase plot of system (42) is shown in Fig.4.

We can get $\tau = 1$, $\dot{\tau}(t) = 0$, $h_j = 1$, $k_j = 1.02$, $r_j = 0.06$, $s_j = 0.056$, $j = 1, 2$. And $H^2 = \text{diag}(1, 1)$, $K^2 =$

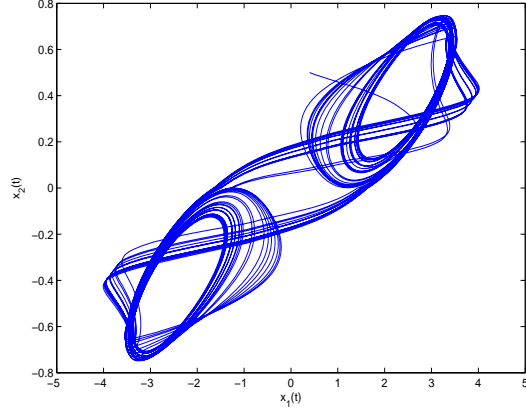


Figure 4: The phase plot of system (42) with the initial condition $x_1(s) = 0.4, x_2(s) = 0.5, \forall s \in [-1, 0)$.

$\text{diag}(1.0404, 1.0404), \|A\|_\infty = 8, \|A\|_1 = 7, \|B\|_\infty = 2.7, \|B\|_1 = 2.6$.

Choose $\varphi(t) = (1+t)^{-2}, G_1 = \text{diag}(6, 6), G_2 = \text{diag}(1, 0.5)$, then the controller $u(t) = (u_1(t), u_2(t))^T$ is

$$u(t) = - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} e(t) - \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \text{sign}(e(t)) - \frac{8^2 \|e(t)\|^2 e(t)}{16 \|e(t)\|^2 + 2(1+t)^{-1}}. \quad (44)$$

It is easy to check that

$$\begin{aligned} P - G_2 &= \text{diag}(-0.3688, -0.2844) < 0, \\ -D - D^T - 2G_1 + \|B\|_1 K^2 + \tau I + \|A\|_1 H^2 + \|B\|_\infty I \\ &= \text{diag}(-0.5950, -0.5950) < 0. \end{aligned}$$

It follows from Corollary 4 that systems (42) and (43) are polynomially synchronized via the controller (44). Fig.5. depicts the trajectories of the error states between systems (42) and (43) via the controller (44). We can see all errors converge to zero fast as time goes by. The state trajectories of variables $x_i(t)$ and $y_i(t), i = 1, 2$ between systems (42) and (43) are shown in Fig.6.

5. Conclusions

In this paper, we have studied the synchronization problem for a class of delayed chaotic neural networks with discontinuous activations. First a new crucial lemma which concerns stability in general decay rate has been constructed. What's more, this lemma includes and extends the classical exponential stability theorem. Then a new nonlinear controller has been designed to

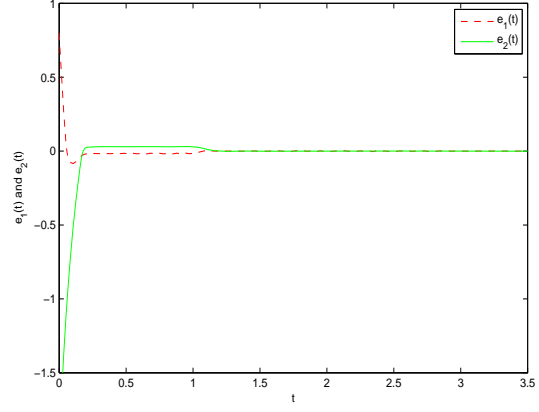


Figure 5: The synchronization errors $e_1(t)$ and $e_2(t)$ between systems (42) and (43) via the controller (44).

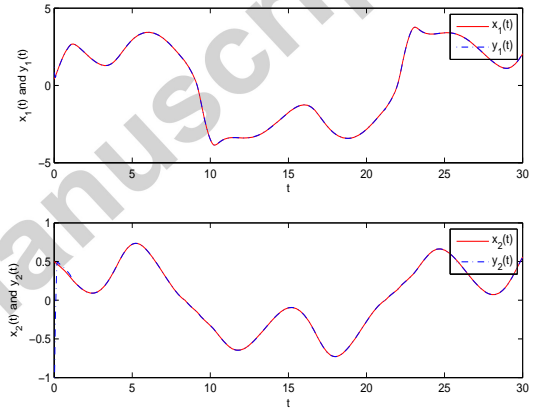


Figure 6: The state trajectories of variables $x_i(t)$ and $y_i(t), i = 1, 2$ between systems (42) and (43).

realize the general decay synchronization of discontinuous neural networks. It is worth noting that complete synchronization of the discontinuous drive and response systems can be achieved with the controller and the general decay synchronization contains exponential synchronization and polynomial synchronization as its two special cases. The results of this paper are general and can be applied to continuous neural networks or other discontinuous delayed nonlinear systems. Finally, two examples have been provided to illustrate the effectiveness and validity of the results.

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