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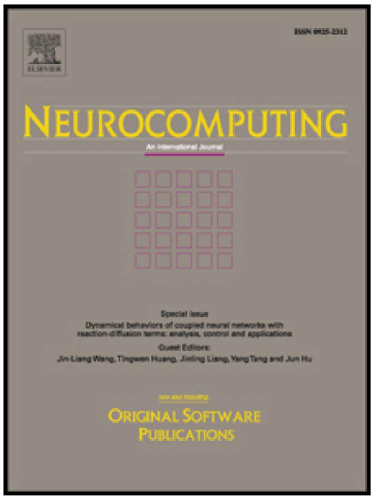

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Auxiliary function-based integral inequality approach to robust passivity analysis of neural networks with interval time-varying delay

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Abstract

In this paper, we study the problem of passivity for uncertain neural networks with interval time-varying delay. Firstly, a suitable augmented Lyapunov-Krasovskii functional (LKF) containing two triple integral terms is constructed and an auxiliary function-based integral inequality (AFBI) is used to manipulate the augmented single integral terms in the derivative of LKF. Secondly, a special form of the AFBI is applied to deal with the delay-product-type term, which was used to be ignored in the time derivative of a triple integral term. As a result, less conservative delay-dependent passivity criteria are derived for normal delayed neural networks (DNNs) in the form of linear matrix inequalities (LMIs). In addition, with the same LKF, delay-dependent passivity criteria are obtained for normal DNNs without the delay-product-type term. Subsequently, these criteria are extended to DNNs with parameter uncertainties. Finally, four numerical examples and simulations are provided to illustrate the effectiveness of the proposed criteria.

Keywords: Neural networks, Parameter uncertainties, Passivity, Time-varying delays, Lyapunov-Krasovskii functional (LKF)

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1. Introduction

In the past few decades, a few scholars have been devoted to the research of neural networks (NNs) due to their potential applications in pattern recognition, associative memories, optimization problem and other scientific fields [1–3]. In the implementations of NNs, time delay is unavoidable due to the finite switching speed of amplifiers and communication time, and its existence may lead to instability, oscillatory or other undesirable system behaviors. Recently, many interesting topics such as filter design [4], synchronization [5, 6], state estimation [7] and dissipativity [8, 9] have been undergoing rapid development. Because such applications greatly rely on whether the equilibrium points of delayed neural networks (DNNs) are stable or not, stability is one of the most important dynamic properties of DNNs [10–13].

On the other hand, in many scientific and engineering problems, stability issues are often related to the theory of dissipative systems. Dissipativity theory was originally introduced by Willems [14] from electrical network theory, which indicates that the quantity of energy dissipated inside dynamical systems is less than the one supplied from outside, and it plays a crucial role in analysis and synthesis of dynamical systems based on the input-output energy-related consideration [15, 16]. Passivity theory, as a part of dissipativity one, is a powerful tool for analyzing the stability of dynamical systems. The reason is that the passive properties of the system can keep the systems internally stable. As was pointed out in [17], the passive system utilizes the product of input and output as energy provision and only burns energy without energy production, and thus passivity embodies the energy attenuation character. In this regard, the passivity theory is a great important subject for analysis and synthesis of DNNs. Meanwhile, the network parameters of a neural system depend on certain resistance and capacitance values, which are subject to uncertainties. The parameter uncertainties are still the potential sources of instability of systems and may result in difficulty or complexity of passivity analysis [18–22].

Generally speaking, delay-dependent passivity criteria [17–33] are less conservative than delay-independent ones [34, 35] especially when the size of time delay is small. Until now, based on the Lyapunov stability theory, there are two effective ways to improve passivity criteria for DNNs. One is to construct a suitable LKF. Some techniques, such as augmenting the terms of a simple type LKF [18, 24], using the idea of the relaxation on the positive-definiteness of every Lyapunov-matrix [19, 25], and introducing triple integral terms [26] or quadruple integral terms [27, 28], were employed to construct LKF. While the other is to develop new inequalities to estimate the derivative of LKF. The free-weighting matrix (FWM) technique [36] was applied to deal with the derivative of LKF in [29]. In [20], Wirtinger-based integral inequality (WBII) [37] was employed to handle the derivative of LKF, and the refined Jensen-based integral inequality (JBII) [38] was used to manipulate the derivative of LKF in [30]. Recently, the free-matrix-based inequality (FMBI) [39], which encompasses the WBII and the Jensen integral inequality (JII), was utilized to derive less conservative passivity criteria [31] than those of [20, 30]. However, the FMBI [39] involves more decision variables than the WBII, and some slack matrices in the FMBI do not seem to be helpful for the reduction of conservativeness. Very recently, the AFBI, which contains several existing integral inequalities as special cases, was applied to derive stability criteria of DNNs in [10].

However, it should be pointed that: (i) the JII was employed to deal with single integral terms with the augmented vectors [10, 30, 31], which may result in conservativeness to some extent. In [30], the derivative of $\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{t+u}^{t}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta duds$ was estimated as $\frac{1}{2}(\tau_2-\tau_1)^2\dot{x}^T(t)R_2\dot{x}(t)-\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds-\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds$, but the term $-(\tau(t)-\tau_1)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)R_2\dot{x}(s)ds$ was ignored, which may lead to conservativeness. (ii) The impact of the delay in system state has been fully considered in [24, 25], but the impact of the time delay in system output [26, 30] was always ignored in the previous literatures [18, 20, 21]. As we all know, the existences of parameter uncertainties in system models [18–22] and time delay may cause performance degradation even instability of NNs. To the best of our knowledge, the problem of passivity analysis of uncertain neural networks with time-varying delay in both the system state and output has not yet been completely investigated in the literature.

Based on the above analysis, this paper focuses on the problem of passivity of uncertain NNs with time-varying delay in both the system state and output, the main contributions of the paper are summarized as follows. Firstly, delay-dependent passivity criteria for normal DNNs will be introduced in Theorem 1 and Corollary 1 by constructing a suitable augmented LKF with two triple integral terms. The AFBI is used to manipulate the augmented single integral terms in the derivative of LKF. Secondly, different from [30], the delay-product-type term $-(\tau(t)-\tau_1)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)R_2\dot{x}(s)ds$ is retained and Lemma 2 is applied to estimate its upper bound. The advantage of such term lies in the fact that the relationship among the time-varying delay, its upper bound and its lower bound is fully taken into account. In this case, a nonlinear function with respective to $\tau(t)$ is induced which cannot be handled directly by Matlab LMI toolbox. Inspired by [40], a new method (Lemma 4) is employed to transform nonlinear matrix inequalities into LMIs. Thirdly, with the same LKF considered in Theorem 1 and Corollary 1, delay-dependent passivity criteria for normal DNNs will be proposed in Theorem 2 and Corollary 2 without the delay-product-type term. In addition, the methods are extended to study the problem of passivity of uncertain DNNs.

Notations: Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space with vector norm $\|\cdot\|$ and the set of $n \times m$ matrices, respectively. $diag\{\cdots\}$ represents the block diagonal matrix. The transposed term in a symmetric matrix is denoted by $*$. For any square matrix A, $Sym\{A\} = A + A^T$. Let $\mathbb{S}_n^+$ denote the set of symmetric positive define matrices in $\mathbb{R}^{n\times n}$. $\mathbb{D}_n^+$ means the set of positive diagonal matrices. I (0) mean identity (zero) matrix with appropriate dimension, respectively.

2. Problem formulation and preliminary

In this section, we will formulate the problem and provide related preliminaries.

2.1. Problem formulation

Consider a class of uncertain NNs with interval time-varying delay

$$\begin{cases} \dot{x}(t) = -A(t)x(t) + W_0(t)f(x(t)) + W_1(t)f(x(t-\tau(t))) + u(t), \\ y(t) = C_1 f(x(t)) + C_2 f(x(t-\tau(t))) + C_3 u(t), \\ x(t) = \phi(t) \quad t \in [-\tau_2, 0], \end{cases} \tag{1}$$

where $x(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T \in \mathbb{R}^n$ denotes the neuron state vector. n is the number of neuron in the network. $y(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^n$ are the output vector and the external input vector, respectively. $\phi(t) \in \mathbb{R}^n$ is the initial condition. $f(\cdot) = [f_1(\cdot)\ f_2(\cdot)\ \cdots\ f_n(\cdot)]^T \in \mathbb{R}^n$ represents the neuron activation function with $f(0) = 0$. $A(t) = A + \triangle A(t)$, $W_0(t) = W_0 + \triangle W_0(t)$, $W_1(t) = W_1 + \triangle W_1(t)$, $A = diag\{a_1, a_2, \cdots, a_n\} \in \mathbb{D}_n^+$ and W_0, W_1 are the interconnection weight matrices. $C_i (i = 1, 2, 3)$ are given real matrices. $\triangle A(t), \triangle W_0(t), \triangle W_1(t)$ are the time-varying structured uncertainties, which are assumed to be of the form

$$[\triangle A(t)\ \triangle W_0(t)\ \triangle W_1(t)] = HF(t)[E_1\ E_2\ E_3], \tag{2}$$

where H and E_i are known real constant matrices with appropriate dimensions. $F(t)$ is the time-varying uncertain matrix and satisfies

$$F^T(t)F(t) \leq I, \forall t \geq 0. \tag{3}$$

$\tau(t)$ is a time-varying delay and satisfies

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \tag{4}$$

where τ_1, τ_2 are known constants and represent the lower bound and the upper bound of delay, respectively.
The neuron activation functions are assumed to satisfy the following assumption.
Assumption 1[12]. For any $i \in \{1, 2 \cdots, n\}$, $f_i(0) = 0$, there exist constants l_i^+, l_i^- such that

$$l_i^- \leq \frac{f_i(\alpha_1) - f_i(\alpha_2)}{\alpha_1 - \alpha_2} \leq l_i^+, \forall \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_1 \neq \alpha_2, \tag{5}$$

where $L_1 = diag\{l_1^-, l_2^-, \cdots, l_n^-\}$, $L_2 = diag\{l_1^+, l_2^+, \cdots, l_n^+\}$ and α_1, α_2 are constants.

2.2. Problem preliminary

To derive our results, it is necessary to introduce the following definition and lemmas.
Definition 1[18]. The system (1) is said to be passive, if there exists a scalar $\gamma > 0$, such that

$$2\int_0^{t_f} y^T(s)u(s)ds \geq -\gamma \int_0^{t_f} u^T(s)u(s)ds, \tag{6}$$

for all $t_f \geq 0$ and all solution of (1) with $x(0) = 0$.
Lemma 1 [10, 11]. For a matrix $R \in \mathbb{S}_n^+$, scalars $\alpha,\ \beta$ satisfying $\alpha < \beta$, vector

$\omega : [\alpha, \beta] \to \mathbb{R}^n$ and any matrices $N_1, N_2, \delta_1, \delta_2$ such that the integration concerned is well defined, then, the following inequality holds

$$-\int_{\alpha}^{\beta} \omega^T(s) R \omega(s) ds \le \chi^T \Big((\beta - \alpha) \sum_{i=1}^{2} \frac{1}{2i-1} N_i R^{-1} N_i^T + Sym\{ \sum_{i=1}^{2} N_i \delta_i \} \Big) \chi, \tag{7}$$

where χ is any vector and $\delta_1 \chi = \int_{\alpha}^{\beta} \omega^T(s) ds$, $\delta_2 \chi = -\delta_1 \chi + \frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} \int_{s}^{\beta} \omega(u) du ds$. Letting $\omega(s) = \dot{x}(s)$ in (7), we can easily yield the following Lemma 2.

Lemma 2. For a matrix $R \in \mathbb{S}_n^+$, scalars $\alpha,\ \beta$ satisfying $\alpha < \beta$, a differentiable function $x : [\alpha, \beta] \to \mathbb{R}^n$, and any matrices $N_1, N_2, \delta_1, \delta_2$, the following inequality holds

$$-\int_{\alpha}^{\beta} \dot{x}^T(s) R \dot{x}(s) ds \le \chi^T \Big((\beta - \alpha) \sum_{i=1}^{2} \frac{1}{2i-1} N_i R^{-1} N_i^T + Sym\{ \sum_{i=1}^{2} N_i \delta_i \} \Big) \chi, \tag{8}$$

where χ is any vector and $\delta_1 \chi = x(\beta) - x(\alpha)$, $\delta_2 \chi = x(\beta) + x(\alpha) - \frac{2}{\beta - \alpha} \int_{\alpha}^{\beta} x(s) ds$.

Lemma 3 [38, 41]. For matrices $R,\ Z \in \mathbb{S}_n^+$, scalars $\alpha,\ \beta$ satisfying $\alpha < \beta$, a differentiable function $x : [\alpha, \beta] \to \mathbb{R}^n$, the following inequalities hold

$$-\int_{\alpha}^{\beta} \dot{x}^T(s) R \dot{x}(s) ds \le -\frac{1}{\beta - \alpha} \varsigma^T(x, \alpha, \beta) \Gamma_1^T \bar{R} \Gamma_1 \varsigma(x, \alpha, \beta), \tag{9}$$

$$-\int_{\alpha}^{\beta} \int_{s}^{\beta} \dot{x}^T(u) Z \dot{x}(u) du ds \le -\varsigma^T(x, \alpha, \beta) \Gamma_2^T \bar{Z} \Gamma_2 \varsigma(x, \alpha, \beta), \tag{10}$$

where $\bar{R} = diag\{R, 3R, 5R\}$, $\bar{Z} = diag\{2Z, 4Z\}$, and

$$\varsigma(x, \alpha, \beta) = [x^T(\alpha)\ x^T(\beta)\ \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x^T(s) ds\ \frac{2}{(\beta - \alpha)^2} \int_{\alpha}^{\beta} \int_{s}^{\beta} x^T(u) du ds]^T, \tag{11}$$

$$\Gamma_1 = \begin{bmatrix} -I & I & 0 & 0 \\ I & I & -2I & 0 \\ -I & I & 6I & -6I \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & I & -I & 0 \\ 0 & I & 2I & -3I \end{bmatrix}. \tag{12}$$

Remark 1. It is noted that the auxiliary functions $f_i(s), i \in \{0, 1, \cdots, n\}$ satisfy $\int_{\alpha}^{\beta} f_i(s) f_j(s) ds = 0, (0 \le i, j \le n, i \ne j)$, $f_0(s) = 1$ in inequality (10) of Lemma 5 in [10], as a special case, inequality (10) of Lemma 5 in [10] can change into inequalities (7)−(9) of Lemmas 1−3 by appropriately choosing the auxiliary functions, which indicates Lemma 5 in [10] is more general.

Specifically,

• by choosing $f_1(s) = \frac{2s-\beta-\alpha}{\beta-\alpha}$, inequality (10) of Lemma 5 in [10] becomes exactly the same as Lemma 5 in [11], namely Lemma 1 in this paper. Moreover, Lemma 2 can be easily obtained by replacing $\omega(s)$ with $\dot{x}(s)$ in inequality (7) of Lemma 1.

• by choosing $f_1(s) = \frac{2s-\beta-\alpha}{\beta-\alpha}$, $f_2(s) = \frac{6(s-\alpha)^2}{(\beta-\alpha)^2} - \frac{6(s-\alpha)}{\beta-\alpha} + 1$, $\chi = \varsigma(x, \alpha, \beta)$, $N_1 = -\frac{1}{\beta-\alpha}\delta_0^T R$, $N_2 = -\frac{3}{\beta-\alpha}\delta_1^T R$, $N_3 = -\frac{5}{\beta-\alpha}\delta_2^T R$, $\delta_0 = [-I \ I \ 0 \ 0]$, $\delta_2 = [I \ I \ -2I \ 0]$, $\delta_3 = [-I \ I \ 6I \ -6I]$, and $\omega(s) = \dot{x}(s)$, inequality (10) of Lemma 5 in [10] is exactly the same as inequality (12) of Corollary 1 in [38] and inequality (24) of Lemma 5.1 in [41], namely inequality (9) of Lemma 3 in this paper.

Lemma 4 Let us consider a quadratic function $f(x) = k_2x^2 + k_1x + k_0, k_i \in \mathbb{R}, (i = 0, 1, 2)$, if the following inequalities hold

$$(I) \ f(a) < 0, \ (II) \ f(b) < 0, \ (III) \ \ -(b-a)^2k_2 + f(a) < 0, \tag{13}$$

then $f(x) < 0, \forall x \in \big[a, b\big]$.

Remark 2. By employing a similar proof of Lemma 2 in [40], Lemma 4 can be easily obtained. Lemma 4 is a generalized case of Lemma 2 in [40], in which a can be zero, positive or negative.

Lemma 5 [42]. Let H, E and $F(t)$ be real matrices of appropriate dimensions with

$$F^T(t)F(t) \leq I. \tag{14}$$

Then, for any scalar $\varepsilon > 0$,

$$HF(t)E + (HF(t)E)^T \leq \varepsilon^{-1}HH^T + \varepsilon E^T E. \tag{15}$$

3. Main results

3.1. Passivity criteria for normal DNNs

If $\triangle A(t) = \triangle W_0(t) = \triangle W_1(t) = 0$, system (1) is reduced to the following normal system

$$\begin{cases} \dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 f(x(t-\tau(t))) + u(t), \\ y(t) = C_1 f(x(t)) + C_2 f(x(t-\tau(t))) + C_3 u(t), \\ x(t) = \phi(t) \ \ t \in [-\tau_2, 0], \end{cases} \tag{16}$$

In this section, we will develop some delay-dependent criteria which ensure the passivity of the system (16). For simplicity of vector and matrix representations, the following notations are necessary.

$$
\begin{aligned}
e_i &= [0_{n\times(i-1)n}\ I_n\ 0_{n\times(15-i)n}], (i=1,2,\cdots,15), \quad e_0 = [\overbrace{0\ 0\cdots 0}^{15}],\\
x_\tau(t) &= x(t-\tau(t)), \qquad x_{\tau_1}(t) = x(t-\tau_1), \qquad x_{\tau_2}(t) = x(t-\tau_2),\\
f(t) &= f(x(t)), \qquad f_\tau(t) = f(x(t-\tau(t))), \quad f_{\tau_1}(t) = f(x(t-\tau_1)),\\
f_{\tau_2}(t) &= f(x(t-\tau_2)), \qquad \alpha_1(t) = \tau(t)-\tau_1, \qquad \alpha_2(t) = \tau_2-\tau(t),\\
\vartheta_1(t) &= \frac{1}{\tau_1}\int_{t-\tau_1}^{t} x(s)ds, \qquad \vartheta_2(t) = \frac{1}{\alpha_1(t)}\int_{t-\tau(t)}^{t-\tau_1} x(s)ds,\\
\vartheta_3(t) &= \frac{1}{\alpha_2(t)}\int_{t-\tau_2}^{t-\tau(t)} x(s)ds, \qquad \vartheta_4(t) = \frac{2}{\tau_1^2}\int_{t-\tau_1}^{t}\int_{s}^{t} x(u)duds,\\
\vartheta_5(t) &= \frac{2}{\alpha_1^2(t)}\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1} x(u)duds,\ \vartheta_6(t) = \frac{2}{\alpha_2^2(t)}\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)} x(u)duds,\\
\xi_1(t) &= [x^T(t)\ x_{\tau_1}^T(t)\ x_\tau^T(t)\ x_{\tau_2}^T(t)]^T, \qquad \xi_2(t) = [f_\tau^T(t)\ f_\tau^T(t)\ f_{\tau_2}^T(t)\ f_{\tau_1}^T(t)]^T,\\
\xi_3(t) &= [\vartheta_1^T(t)\ \vartheta_2^T(t)\ \vartheta_3^T(t)]^T, \qquad \xi_4(t) = [\vartheta_4^T(t)\ \vartheta_5^T(t)\ \vartheta_6^T(t)]^T,\\
\xi(t) &= [\xi_1^T(t)\ \xi_2^T(t)\ \xi_3^T(t)\ \xi_4^T(t)\ u^T(t)]^T.
\end{aligned}
$$

Theorem 1. For given scalars $0 \le \tau_1 \le \tau_2$, DNNs (16) satisfying (4) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^+, Q_1, S_2 \in \mathbb{S}_{2n}^+, Q_2, Q_3, R_1, R_2, S_1 \in \mathbb{S}_n^+, \Lambda_k, D_j \in \mathbb{D}_n^+ (k=1,2,\cdots,8, j=1,2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}, N_5, N_6 \in \mathbb{R}^{3n\times n}$ and a scalar $\gamma > 0$ such that the following LMIs hold

$$
\Psi_1 = \begin{bmatrix} \Psi(\tau(t))|_{\tau(t)=\tau_2} - \tau_{12}\Xi_1 & \tau_{12}G_7^T N_1 & \tau_{12}G_7^T N_2 \\ * & -\tau_{12}\bar{S}_2 & 0 \\ * & * & -3\tau_{12}\bar{S}_2 \end{bmatrix} < 0, \tag{17}
$$

$$
\Psi_2 = \begin{bmatrix} \Psi(\tau(t))|_{\tau(t)=\tau_1} - \tau_{12}\Xi_2 & \tau_{12}G_7^T N_3 & \tau_{12}G_7^T N_4 \\ * & -\tau_{12}\hat{S}_2 & 0 \\ * & * & -3\tau_{12}\hat{S}_2 \end{bmatrix} < 0, \tag{18}
$$

$$
\Psi_3 = \begin{bmatrix} \Psi_1 & \tau_{12}\aleph_1 & \tau_{12}\aleph_2 \\ * & -R_2 & 0 \\ * & * & -3R_2 \end{bmatrix} < 0, \tag{19}
$$

where

$$\Psi(\tau(t)) = \sum_{i=0}^{3} \Upsilon_i(\tau(t)) + \sum_{i=1}^{10} \Pi_i + \Pi_{14} + Sym\{\Lambda\},\ \tau_{12} = \tau_2 - \tau_1,$$

$$\Upsilon_1(\tau(t)) = Sym\{G^T(\tau(t))PG_0\},\ \Upsilon_2(\tau(t)) = \sum_{i=1}^{2} \alpha_i(t)(\Xi_i + \Pi_{1i}),$$

$$\Upsilon_3(\tau(t)) = \alpha_1(t)\Pi_{13},$$

$$\Pi_1 = e_1^T(Q_2 + Q_3)e_1 - e_2^T Q_2 e_2 - e_4^T Q_3 e_4 + G_1^T Q_1 G_1 - G_2^T Q_1 G_2,$$

$$\Pi_2 = Sym\{[e_5 - L_1 e_1]^T D_1 B + [L_2 e_1 - e_5]^T D_2 B\},$$

$$\Pi_3 = B^T(\tau_1^2 S_1 + \frac{\tau_1^2}{2} R_1 + \frac{\tau_{12}^2}{2} R_2)B + \tau_{12}^2 G_3^T S_2 G_3,$$

$$\Pi_4 = e_2^T P_1 e_2 - e_3^T(P_1 - P_2)e_3 - e_4^T P_2 e_4,$$

$$\Pi_5 = -G_4^T \Gamma_1^T \bar{S}_1 \Gamma_1 G_4, \qquad \Pi_6 = -G_4^T \Gamma_2^T \bar{R}_1 \Gamma_2 G_4,$$

$$\Pi_7 = -G_5^T \Gamma_2^T \bar{R}_2 \Gamma_2 G_5, \qquad \Pi_8 = -G_6^T \Gamma_2^T \bar{R}_2 \Gamma_2 G_6,$$

$$\Pi_9 = Sym\{\sum_{i=1}^{2} G_7^T N_i G_{7+i}\}, \qquad \Pi_{10} = Sym\{\sum_{i=1}^{2} G_7^T N_{i+2} G_{11+i}\},$$

$$\Pi_{11} = Sym\{\sum_{i=1}^{2} G_7^T N_i G_{9+i}\}, \qquad \Pi_{12} = Sym\{\sum_{i=1}^{2} G_7^T N_{i+2} G_{13+i}\},$$

$$\Pi_{13} = Sym\{\sum_{i=1}^{2} G_{16}^T [N_5\ N_6] G_{17}\},$$

$$\Pi_{14} = Sym\{e_{15}^T(C_1 e_5 + C_2 e_6)\} + e_{15}^T(\gamma I_n + C_3^T + C_3)e_{15},$$

$$\Xi_1 = \sum_{i=1}^{2} \frac{1}{2i-1} G_7^T N_i \bar{S}_2^{-1} N_i^T G_7,$$

$$\Xi_2 = \sum_{i=1}^{2} \frac{1}{2i-1} G_7^T N_{i+2} \hat{S}_2^{-1} N_{i+2}^T G_7,$$

$$\Xi_3 = \sum_{i=1}^{2} \frac{1}{2i-1} G_{16}^T N_{i+4} R_2^{-1} N_{i+4}^T G_{16},$$

$$\Lambda = G_{18}^T \Lambda_1 G_{19} + G_{20}^T \Lambda_2 G_{21} + G_{22}^T \Lambda_3 G_{23} + G_{24}^T \Lambda_4 G_{25} + G_{26}^T \Lambda_5 G_{27}$$
$$+ G_{28}^T \Lambda_6 G_{29} + G_{30}^T \Lambda_7 G_{31} + G_{32}^T \Lambda_8 G_{33},$$

$$\bar{S}_1 = diag\{S_1, 3S_1, 5S_1\}, \qquad \bar{R}_i = diag\{2R_i, 4R_i\}\ (i = 1, 2),$$

$$\bar{S}_2 = S_2 + \begin{bmatrix} 0 & P_1 \\ * & 0 \end{bmatrix}, \qquad \hat{S}_2 = S_2 + \begin{bmatrix} 0 & P_2 \\ * & 0 \end{bmatrix},$$
$$B = -Ae_1 + W_0 e_5 + W_1 e_6 + e_{15},$$
$$G(\tau(t)) = [e_1^T \ \tau_1 e_9^T \ \ \alpha_2 e_{11}^T + \alpha_1 e_{10}^T \ \ \frac{1}{2\tau_1^2} e_{12}^T]^T,$$
$$G_0 = [B^T \quad e_1^T - e_2^T \quad e_2^T - e_4^T \quad \tau_1 e_1^T - \tau_1 e_9^T]^T, \qquad G_1 = [e_2^T \ e_8^T]^T,$$
$$G_2 = [e_2^T \ e_7^T]^T, \qquad G_3 = [e_1^T \ B^T]^T, \qquad G_4 = [e_2^T \ e_1^T \ e_9^T \ e_{12}^T]^T,$$
$$G_5 = [e_3^T \ e_2^T \ e_{10}^T \ e_{13}^T]^T, \qquad G_6 = [e_4^T \ e_3^T \ e_{11}^T \ e_{14}^T]^T,$$
$$G_7 = [e_2^T \ e_3^T \ e_4^T \ e_{10}^T \ e_{11}^T \ e_{13}^T \ e_{14}^T]^T, \qquad G_8 = [e_0^T \quad e_2^T - e_3^T]^T,$$
$$G_9 = [e_0^T \quad e_2^T + e_3^T - 2e_{10}^T]^T, \qquad G_{10} = [e_{10}^T \ e_0^T]^T,$$
$$G_{11} = [-e_{10}^T + e_{13}^T \ \ e_0^T]^T, \qquad G_{12} = [e_0^T \quad e_3^T - e_4^T]^T,$$
$$G_{13} = [e_0^T \quad e_3^T + e_4^T - 2e_{11}^T]^T, \qquad G_{14} = [e_{11}^T \ e_0^T]^T,$$
$$G_{15} = [-e_{11}^T + e_{14}^T \quad e_0^T]^T, \qquad G_{16} = [e_3^T \ e_4^T \ e_{11}^T]^T,$$
$$G_{17} = [e_3^T - e_4^T \quad e_3^T + e_4^T - 2e_{11}^T]^T, \qquad G_{18} = [e_5^T - e_1^T L_1]^T,$$
$$G_{19} = [-e_5^T + e_1^T L_2]^T, \qquad G_{20} = [e_6^T - e_3^T L_1]^T,$$
$$G_{21} = [-e_6^T + e_3^T L_2]^T, \qquad G_{22} = [e_5^T - e_6^T - (e_1^T - e_3^T)L_1]^T,$$
$$G_{23} = [-e_5^T + e_6^T + (e_1^T - e_3^T)L_2]^T, \qquad G_{24} = [e_5^T - e_8^T - (e_1^T - e_2^T)L_1]^T,$$
$$G_{25} = [-e_5^T + e_8^T - (e_1^T - e_2^T)L_2]^T, \qquad G_{26} = [e_5^T - e_7^T - (e_1^T - e_4^T)L_1]^T,$$
$$G_{27} = [-e_5^T + e_7^T - (e_1^T - e_4^T)L_2]^T, \qquad G_{28} = [e_7^T - e_8^T - (e_4^T - e_2^T)L_1]^T,$$
$$G_{29} = [-e_7^T + e_8^T - (e_4^T - e_2^T)L_2]^T, \qquad G_{30} = [e_6^T - e_8^T - (e_3^T - e_2^T)L_1]^T,$$
$$G_{31} = [-e_6^T + e_8^T - (e_3^T - e_2^T)L_2]^T, \qquad G_{32} = [e_7^T - e_6^T - (e_4^T - e_3^T)L_1]^T,$$
$$G_{33} = [-e_4^T + e_3^T - (e_7^T - e_6^T)L_2]^T, \qquad \aleph_i = [N_{i+4}^T G_{16} \ 0 \ 0]^T, (i = 1, 2). \tag{20}$$

Proof. Construct the following LKF candidate

$$V(x_t) = \eta_1^T(t) P \eta_1(t) + 2\sum_{i=1}^{n} \Big(d_{1i} \int_0^{x_i(t)} \big(f_i(s) - l_i^- s\big) ds + d_{2i} \int_0^{x_i(t)} \big(l_i^+ s$$
$$- f_i(s)\big) ds \Big) + \int_{t-\tau_2}^{t-\tau_1} \eta_2^T(s) Q_1 \eta_2(s) ds + \sum_{i=1}^{2} \int_{t-\tau_i}^{t} x^T(s) Q_{i+1} x(s) ds$$
$$+ \tau_1 \int_{-\tau_1}^{0} \int_{t+s}^{t} \dot{x}^T(u) S_1 \dot{x}(u) du ds + \int_{-\tau_2}^{-\tau_1} \int_{t+s}^{t} \eta_3^T(u) S_2 \eta_3(u) du ds$$

$$+\int_{-\tau_1}^{0}\int_{s}^{0}\int_{t+u}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta duds+\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{t+u}^{t}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta duds, \tag{21}$$

with $\eta_1(t)=[x^T(t)\ \int_{t-\tau_1}^{t}x^T(s)ds\ \int_{t-\tau_2}^{t-\tau_1}x^T(s)ds\ \int_{t-\tau_1}^{t}\int_{s}^{t}x^T(u)duds]^T$,
$\eta_2(t)=[x^T(t)\ f^T(t)]^T$, $\quad \eta_3(t)=[x^T(t)\ \dot{x}^T(t)]^T$.
Taking derivation of $V(x_t)$ in t along the solution of (16), we can obtain

$$\dot{V}(x_t)=\xi^T(t)\Big(Sym\big\{\Upsilon_1(\tau(t))\big\}+\sum_{i=1}^{3}\Pi_i\Big)\xi(t)-\sum_{i=1}^{2}\big(\Im_i(t)+\wp_i(t)\big), \tag{22}$$

where

$$\begin{aligned}\Im_1(t)=&\tau_1\int_{t-\tau_1}^{t}\dot{x}^T(s)S_1\dot{x}(s)ds,\\ \Im_2(t)=&\int_{t-\tau_2}^{t-\tau_1}\eta_3^T(s)S_2\eta_3(s)ds,\\ \wp_1(t)=&\int_{t-\tau_1}^{t}\int_{s}^{t}\dot{x}^T(\theta)R_1\dot{x}(\theta)d\theta ds,\\ \wp_2(t)=&\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds.\end{aligned} \tag{23}$$

Note that $\Im_2(t)=\Im_{21}(t)+\Im_{22}(t)$, $\wp_2(t)=\wp_{21}(t)+\wp_{22}(t)+\wp_{23}(t)$, where

$$\begin{aligned}\Im_{21}(t)=&\int_{t-\tau(t)}^{t-\tau_1}\eta_3^T(s)S_2\eta_3(s)ds,\\ \Im_{22}(t)=&\int_{t-\tau_2}^{t-\tau(t)}\eta_3^T(s)S_2\eta_3(s)ds,\\ \wp_{21}(t)=&\int_{t-\tau(t)}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds,\\ \wp_{22}(t)=&\int_{t-\tau_2}^{t-\tau(t)}\int_{s}^{t-\tau(t)}\dot{x}^T(\theta)R_2\dot{x}(\theta)d\theta ds,\\ \wp_{23}(t)=&\alpha_1(t)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)R_2\dot{x}(s)ds.\end{aligned} \tag{24}$$

Applying Lemma 3 yields

$$-\Im_1(t)\le-\varsigma_1^T(t)\Gamma_1^T\bar{S}_1\Gamma_1\varsigma_2(t),\quad -\wp_1(t)\le-\varsigma_1^T(t)\Gamma_2^T\bar{R}_1\Gamma_1\varsigma_2(t),$$

$$-\wp_{21}(t) \le -\varsigma_2^T(t)\Gamma_2^T \bar{R}_2 \Gamma_2 \varsigma_2(t), \ -\wp_{22}(t) \le -\varsigma_3^T(t)\Gamma_2^T \bar{R}_3 \Gamma_2 \varsigma_3(t), \tag{25}$$

where Γ_1, Γ_2 are defined in (12), and $\varsigma_1(t) = \varsigma(x, t-\tau_1, t), \varsigma_2(t) = \varsigma(x, t-\tau(t), t-\tau_1), \varsigma_3(t) = \varsigma(x, t-\tau_2, t-\tau(t))$ with $\varsigma(\cdot,\cdot,\cdot)$ being defined in (11). For symmetric matrices P_1, P_2, the following zero-value term is obtained

$$\begin{aligned} 0 = & x_{\tau_1}^T(t)P_1 x_{\tau_1}(t) - x_\tau^T(t)P_1 x_\tau(t) - 2\int_{t-\tau(t)}^{t-\tau_1} x^T(s)P_1\dot{x}(s)ds + x_\tau^T(t)P_2 x_\tau(t) \\ & - x_{\tau_2}^T(t)P_2 x_{\tau_2}(t) - 2\int_{t-\tau_2}^{t-\tau(t)} x^T(s)P_2\dot{x}(s)ds. \end{aligned} \tag{26}$$

Substituting (25)−(26) into (22) yields

$$\dot{V}(x_t) = \xi^T(t)\Big(Sym\big\{\Upsilon_1(\tau(t))\big\} + \sum_{i=1}^{8}\Pi_i\Big)\xi(t) - \sum_{i=1}^{2}\bar{\Im}_{2i}(t) - \wp_{23}(t), \tag{27}$$

where $\Upsilon_1(\tau(t)), \Pi_i, (i = 1, 2, \cdots, 8)$ are defined in (20), and $\wp_{23}(t)$ is defined in (24), and

$$\bar{\Im}_{21}(t) = \int_{t-\tau(t)}^{t-\tau_1} \eta_3^T(s)\bar{S}_2\eta_3(s)ds, \ \bar{\Im}_{22}(t) = \int_{t-\tau_2}^{t-\tau(t)} \eta_3^T(s)\hat{S}_2\eta_3(s)ds.$$

For any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}$, letting χ in (7) be $\chi = G_7\xi(t)$, and using Lemma 1 obtains

$$-\bar{\Im}_{21}(t) - \bar{\Im}_{22}(t) \le \xi^T(t)\big(\Upsilon_2(\tau(t)) + \Pi_9 + \Pi_{10}\big)\xi(t), \tag{28}$$

where $\Upsilon_2(\tau(t)), \Pi_i, (i = 9, 10)$ are defined in (20).
Then, for any matrices $N_5, N_6 \in \mathbb{R}^{3n\times n}$, letting χ in (7) be $\chi = G_{16}\xi(t)$, using Lemma 2, we can obtain

$$-\wp_{23}(t) \le \xi^T(t)\big(\alpha_1(t)\alpha_2(t)\Xi_3 + \Upsilon_3(\tau(t))\big)\xi(t), \tag{29}$$

where Ξ_3 and $\Upsilon_3(\tau(t))$ are defined in (20).
Since $\Lambda_i \in \mathbb{D}^+, (i = 1, 2, \cdots, 8)$, it follows from (5) that

$$\varpi_i(s) = 2[L_2x(s) - f(x(s))]^T\Lambda_i[f(x(s)) - L_1x(s)] \ge 0, (i = 1, 2), \tag{30}$$

and

$$\begin{aligned} \varpi_i(s_1, s_2) = & 2[L_2(x(s_1) - x(s_2)) - (f(s_1) - f(s_2))]^T\Lambda_i[(f(s_1) - f(s_2)) \\ & - L_1(x(s_1) - x(s_2))] \ge 0, (i = 3, 4, \cdots, 8), \end{aligned} \tag{31}$$

which imply

$$\begin{aligned}&\varpi_1(t)+\varpi_2(t-\tau(t))+\varpi_3(t,t-\tau(t))+\varpi_4(t,t-\tau_1)+\varpi_5(t,t-\tau_2)\\&+\varpi_6(t-\tau_2,t-\tau_1)+\varpi_7(t-\tau(t),t-\tau_1)+\varpi_8(t-\tau_2,t-\tau(t))\geq 0.\end{aligned} \tag{32}$$

Substituting (28)−(29) and (32) into (27), we can obtain

$$\dot{V}(x_t)-\gamma u^T(t)u(t)-2y^T(t)u(t)\leq \xi^T(t)\big(\Psi(\tau(t))+\alpha_1(t)\alpha_2(t)\Xi_3\big)\xi(t). \tag{33}$$

On the other hand, we should note that $\Psi(\tau(t))+\alpha_1(t)\alpha_2(t)\Xi_3$ is a quadratic function with respective to $\tau(t)$ and can not directly be solved by Matlab LMI toolbox. Combining Lemma 4 with the convex combination method [43], if (17)−(19) satisfy, then $\Psi(\tau(t))+\alpha_1(t)\alpha_2(t)\Xi_3<0$.
Hence,

$$\dot{V}(x_t)-\gamma u^T(t)u(t)-2y^T(t)u(t)<0. \tag{34}$$

By integrating (34) over the time period from 0 to t_p, we have

$$\begin{aligned}2\int_0^{t_p}y^T(s)u(s)ds&\geq V(x(t_p),t)-V(x(0),0)-\gamma\int_0^{t_p}u^T(s)u(s)ds\\&\geq-\gamma\int_0^{t_p}u^T(s)u(s)ds,\end{aligned} \tag{35}$$

which shows that DNNs (16) is passive in the sense of Definition 1. The proof is completed.

Remark 3. In the proof of Theorem 1, slack matrices P_1, P_2 are introduced in the zero-value FWM equation (26). In addition, different from [10, 30, 31], the AFBI is used to deal with $-\int_{t-\tau(t)}^{t-\tau_1}\eta_3^T(s)\bar{S}_2\eta_3(s)ds$ and $-\int_{t-\tau_2}^{t-\tau(t)}\eta_3^T(s)\hat{S}_2\eta_3(s)ds$. The term $(\tau(t)-\tau_1)\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)R_2\dot{x}(s)ds$ is retained and is handled by the AFBI. After the treatment with the AFBI, more matrix variables are introduced in LMIs, accordingly, Theorem 1 is less conservativeness.

As considered in many existing works, when the rate of change of delay $\tau(t)$ satisfies

$$\dot{\tau}(t)\leq\mu, \tag{36}$$

where μ is a known constant, we slightly modify the LKF as follows

$$\bar{V}(x_t)=V(x_t)+\int_{t-\tau(t)}^{t}\eta_2^T(s)Q_4\eta_2(s)ds,\quad Q_4\in\mathbb{S}_{2n}^+. \tag{37}$$

Similar to the proof procedure of Theorem 1, we can obtain the following Corollary 1.

Corollary 1. For given scalars $0 \leq \tau_1 \leq \tau_2, \mu$, DNNs (16) satisfying (4) and (36) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^{+}, Q_1, Q_4, S_2 \in \mathbb{S}_{2n}^{+}, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}_{n}^{+}, \Lambda_k, D_j \in \mathbb{D}_{n}^{+}(k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}, N_5, N_6 \in \mathbb{R}^{3n\times n}$ and a scalar $\gamma > 0$ such that the following LMIs hold

$$\Psi_i + diag\{\bar{Q}_4, 0, 0\} < 0, \ (i = 1, 2), \tag{38}$$

$$\Psi_3 + diag\{\bar{Q}_4, 0, 0, 0, 0\} < 0, \tag{39}$$

where

$$\bar{Q}_4 = G_{34}^T Q_4 G_{34} + (1-\mu) G_{35}^T Q_4 G_{35}, \ G_{34} = [e_1^T \ e_5^T]^T, \ G_{35} = [e_3^T \ e_6^T]^T, \tag{40}$$

and other notations are the same as given in Theorem 1.

Remark 4. Different from [30], it can be seen that the delay-product-type term $-\alpha_1(t)\int_{t-\tau_2}^{t-\tau(t)} \dot{x}^T(s) R_2 \dot{x}(s) ds$ in $\dot{V}(x_t)$ is retained in Theorem 1 and Corollary 1. In order to clearly check its contribution to conservativeness, the following Theorem 2 and Corollary 2, which just need to ignore the delay-product-type term from Theorem 1 and Corollary 1, are given as follows.

Theorem 2. For given scalars $0 \leq \tau_1 \leq \tau_2$, DNNs (16) satisfying (4) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^{+}, Q_1, S_2 \in \mathbb{S}_{2n}^{+}, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}_{n}^{+}, \Lambda_k, D_j \in \mathbb{D}_{n}^{+}(k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}$ and a scalar $\gamma > 0$ such that the following LMIs hold

$$\Psi_i + diag\{(i-2)^i \tau_{12} \Pi_{13}, 0, 0\} < 0, \ (i = 1, 2). \tag{41}$$

When $\tau(t)$ satisfies (36), we also obtain the corresponding result.

Corollary 2. For given scalars $0 \leq \tau_1 \leq \tau_2, \mu$, DNNs (16) satisfying (4) and (36) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^{+}, Q_1, Q_4, S_2 \in \mathbb{S}_{2n}^{+}, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}_{n}^{+}, \Lambda_k, D_j \in \mathbb{D}_{n}^{+}(k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}$ and a scalar $\gamma > 0$ such that the following LMIs hold

$$\Psi_i + diag\{\bar{Q}_4 + (i-2)^i \tau_{12} \Pi_{13}, 0, 0\} < 0, \ (i = 1, 2). \tag{42}$$

3.2. Passivity criteria for uncertain system

Now we extend Theorems 1−2 and Corollaries 1−2 to uncertain DNNs (1), then the following theorems and corollaries can be derived.

Theorem 3. For given scalars $0 \leq \tau_1 \leq \tau_2$, DNNs (1) satisfying (2)−(4) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^+, Q_1, S_2 \in \mathbb{S}_{2n}^+, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}_n^+, \Lambda_k, D_j \in \mathbb{D}_n^+ (k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}, N_5, N_6 \in \mathbb{R}^{3n\times n}$ and scalars $\gamma > 0$, $\varepsilon_i > 0 (i = 1, 2, 3)$ such that the following LMIs hold

$$\Omega_i = \begin{bmatrix} \Phi_i & \tau_1\aleph_3 & \tau_1\aleph_4 & \tau_{12}\aleph_5 & \mho(\tau(t))|_{\tau(t)=\tau_{\frac{3+(-1)^{i-1}}{2}}} & \varepsilon_i\aleph_6 \\ * & -S_1 & 0 & 0 & \tau_1 S_1^T H & 0 \\ * & * & -2R_1 & 0 & \tau_1 R_1^T H & 0 \\ * & * & * & -2R_2 & \tau_{12} R_2^T H & 0 \\ * & * & * & * & -\varepsilon_i I & 0 \\ * & * & * & * & * & -\varepsilon_i I \end{bmatrix} < 0, \ (i = 1, 2), \tag{43}$$

$$\Omega_3 = \begin{bmatrix} \Phi_1 & \tau_{12}\aleph_1 & \tau_{12}\aleph_2 & \tau_1\aleph_3 & \tau_1\aleph_4 & \tau_{12}\aleph_5 & \mho(\tau(t))|_{\tau(t)=\tau_2} & \varepsilon_3\aleph_6 \\ * & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -3R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -S_1 & 0 & 0 & \tau_1 S_1^T H & 0 \\ * & * & * & * & -2R_1 & 0 & \tau_1 R_1^T H & 0 \\ * & * & * & * & * & -2R_2 & \tau_{12} R_2^T H & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_3 I \end{bmatrix} < 0, \tag{44}$$

where

$$\begin{aligned} &\Phi_i = \Psi_i + diag\{-\Pi_3 - \tau_{12}^2 G_3^T S_2 G_3, 0, 0\}, (i = 1, 2), \\ &\aleph_3 = [S_1^T B \ 0 \ 0]^T, \quad \aleph_4 = [R_1^T B \ 0 \ 0]^T, \quad \aleph_5 = [R_2^T B \ 0 \ 0]^T, \\ &\aleph_6 = [C \ 0 \ 0]^T, \qquad C = -E_1 e_1 + E_2 e_5 + E_3 e_6, \\ &M(\tau(t)) = G^T(\tau(t)) P [I \ 0 \ 0 \ 0]^T + e_1^T(L_2 D_2 - L_1 D_1) - e_5^T(D_2 - D_1), \\ &\mho(\tau(t)) = [H^T M^T(\tau(t)) \ 0 \ 0]^T, \end{aligned} \tag{45}$$

and other notations are the same as Theorem 1.

Proof. Using Schur complement, it shows that (17)−(19) are equivalent to

(46)−(47),

$$\bar{\Psi}_i = \begin{bmatrix} \Phi_i & \tau_1\aleph_3 & \tau_1\aleph_4 & \tau_{12}\aleph_5 \\ * & -S_1 & 0 & 0 \\ & * & -2R_1 & 0 \\ & * & * & -2R_2 \end{bmatrix} < 0,\ (i=1,2), \tag{46}$$

$$\bar{\Psi}_3 = \begin{bmatrix} \Phi_1 & \tau_{12}\aleph_1 & \tau_{12}\aleph_2 & \tau_1\aleph_3 & \tau_1\aleph_4 & \tau_{12}\aleph_5 \\ * & -R_2 & 0 & 0 & 0 & 0 \\ * & * & -3R_2 & 0 & 0 & 0 \\ * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & -2R_1 & 0 \\ * & * & * & * & * & -2R_2 \end{bmatrix} < 0. \tag{47}$$

Replacing A, W_0, W_1 in (46)−(47) with $A + HF(t)E_1, W_0 + HF(t)E_2, W_1 + HF(t)E_3$, we can obtain

$$\bar{\Psi}_i + \Theta_{di}F(t)\Theta_{ei} + \Theta_{ei}^T F^T(t)\Theta_{di}^T < 0,\ (i=1,2), \tag{48}$$

and

$$\bar{\Psi}_3 + \Theta_{d3}F(t)\Theta_{e3} + \Theta_{e3}^T F^T(t)\Theta_{d3}^T < 0, \tag{49}$$

with
$\Theta_{di} = [\mho^T(\tau(t))|_{\tau(t)=\tau_{\frac{3+(-1)^{i-1}}{2}}}\ \ \tau_1 H^T S_1\ \tau_1 H^T R_1\ \tau_{12} H^T R_2]^T, (i=1,2)$,
$\Theta_{d3} = [\mho^T(\tau(t))|_{\tau(t)=\tau_2}\ \ 0\ \ 0\ \ \tau_1 H^T S_1\ \ \tau_1 H^T R_1\ \ \tau_{12} H^T R_2]^T$,
$\Theta_{ei} = [\aleph_6^T\ 0\ 0\ 0\ 0], (i=1,2), \quad \Theta_{e3} = [\aleph_6^T\ 0\ 0\ 0\ 0\ 0\ 0]$.
Using Lemma 5, for any scalars $\varepsilon_i > 0$ $(i=1,2,3)$, one can obtain

$$\bar{\Psi}_i + \varepsilon_i^{-1}\Theta_{di}\Theta_{di}^T + \varepsilon_i\Theta_{ei}^T\Theta_{ei} < 0,\ (i=1,2), \tag{50}$$

and

$$\bar{\Psi}_3 + \varepsilon_3^{-1}\Theta_{d3}\Theta_{d3}^T + \varepsilon_3\Theta_{e3}^T\Theta_{e3} < 0. \tag{51}$$

Based on Schur complement, (50)−(51) are equivalent to (43)−(44). The proof is completed.
Similar to the proof of Theorem 3, we can obtain the following theorem 4 and Corollaries 3−4. Their detailed proofs are omitted here.
Corollary 3. For given scalars $0 \le \tau_1 \le \tau_2, \mu$, DNNs (1) satisfying (2)−(4) and (36) is passive, if there exist matrices $P \in \mathbb{S}_{4n}^+, Q_1, Q_4, S_2 \in \mathbb{S}_{2n}^+, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}_n^+, \Lambda_k, D_j \in \mathbb{D}_n^+ (k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices

$P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}, N_5, N_6 \in \mathbb{R}^{3n\times n}$ and scalars $\gamma > 0$, $\varepsilon_i > 0 (i = 1, 2, 3)$ such that the following LMIs hold

$$\Omega_i + diag\{\bar{Q}_4, 0, 0, 0, 0, 0, 0, 0\} < 0, (i = 1, 2), \tag{52}$$

$$\Omega_3 + diag\{\bar{Q}_4, 0, 0, 0, 0, 0, 0, 0, 0, 0\} < 0, \tag{53}$$

where $\bar{Q}_4$ is defined in (40) and other notations are the same as given in Theorem 3.

Theorem 4. For given scalars $0 \le \tau_1 \le \tau_2$, DNNs (1) satisfying (2)−(4) is passive,if there exist matrices $P \in \mathbb{S}^+_{4n}, Q_1, S_2 \in \mathbb{S}^+_{2n}, Q_2, Q_3, S_1, R_1, R_2 \in \mathbb{S}^+_n, \Lambda_k, D_j \in \mathbb{D}^+_n (k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}$ and scalars $\gamma > 0$, $\varepsilon_i > 0 (i = 1, 2, 3)$ such that the following LMIs hold

$$\Omega_i + diag\{(i-2)^i \tau_{12}\Pi_{13}, 0, 0, 0, 0, 0, 0, 0\} < 0, (i = 1, 2). \tag{54}$$

Corollary 4. For given scalars $0 \le \tau_1 \le \tau_2, \mu$, DNNs (1) satisfying (2)−(4) and (36) is passive, if there exist matrices $P \in \mathbb{S}^+_{4n}, Q_1, Q_4, S_2 \in \mathbb{S}^+_{2n}, Q_2, Q_3, S_1$, $R_1, R_2 \in \mathbb{S}^+_n, \Lambda_k, D_j \in \mathbb{D}^+_n (k = 1, 2, \cdots, 8, j = 1, 2)$, symmetric matrices $P_1, P_2 \in \mathbb{R}^{n\times n}$, any matrices $N_1, N_2, N_3, N_4 \in \mathbb{R}^{7n\times 2n}$ and scalars $\gamma > 0$, $\varepsilon_i > 0 (i = 1, 2, 3)$ such that the following LMIs hold

$$\Omega_i + diag\{\bar{Q}_4 + (i-2)^i \tau_{12}\Pi_{13}, 0, 0, 0, 0, 0, 0, 0\} < 0, (i = 1, 2). \tag{55}$$

Remark 5. In this paper, Theorems 1−4 and Corollaries 1−4 are presented to guarantee passive stability for NNs with time-varying delay in the form of LMIs, which can be easily solved by Matlab LMI toolbox [44]. The allowable maximum upper bounds (AMUBs) τ_2 can be determined by solving the following constraint optimization problem (see Theorem 3)

$$\begin{array}{ll} & \max\limits_{\substack{P, S_i, R_i, P_i, D_i, Q_j, \varepsilon_j, N_l, \Lambda_k, \gamma \\ (i=1,2,\, j=1,2,3,\, l=1,2,\cdots,6,\, k=1,2,\cdots,8)}} \quad \tau_2 \\ s.t. & (43) - (44) \text{ hold} \end{array} \tag{56}$$

4. Numerical examples

In the section, four numerical examples are given to demonstrate the effectiveness of the obtained results in this paper.

Table 1: The AMUBs τ_2 with different τ_1 (Example 1)

τ_1	0.5	1.5	2.0
[30]	2.3918	3.2386	3.7023
Theorem 1	3.9120	4.5899	5.0897
Theorem 2	3.2090	3.6200	4.1162

Example 1. Consider DNNs (16) with the following parameters [30]

$$A = diag\{2.5, 2\},\ W_0 = \begin{bmatrix} 1.3 & 1 \\ -0.5 & 0.5 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.9 & 0.5 \\ -0.3 & 0.4 \end{bmatrix},$$
$$u(t) = [5cos(2t)\ \ -3sin(0.5t)]^T, C_i = I\ (i = 1, 2, 3).$$

The activation functions are given by $f_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|), (i = 1, 2)$, which satisfies (5) with $l_i^- = 0,\ l_i^+ = 1, (i = 1, 2)$. The delay $\tau(t) = \tau_1 + (\tau_2 - \tau_1)|sin(\omega t)|$, where $0 \le \tau_1 \le \tau_2$, and ω is a positive scalar. Fig. 1 gives the curve of $\tau(t)$ with $\omega = 1, t \in [0\ 4\pi], \tau_1 = 0.5$, and $\tau_2 = 3.2090$, which indicates that $\tau(t)$ is continuous but non-differentiable at $t_k = k\pi, (k = 1, 2, 3)$. Thus, the results derived in [19, 20] can not directly ensure passivity of DNNs (16) satisfying (4). Based on Theorems 1−2, we calculate the allowable maximum upper bounds (AMUBs) τ_2 for different τ_1, such that DNNs (16) satisfying (4) is passive, which are given in Table 1. It shows that the obtained results are clearly better than those in [30], and Theorem 1 indeed can produce much better results than Theorem 2.

When $\tau(t) = 0.5 + 2.7090|sint|$, Fig. 2 shows the state trajectories of DNNs (16) with zero input $u(t)$ and 10 random initial states, which indicates DNNs (16) is stable.

Example 2. Consider DNNs (16) with the following parameters

$$A = diag\{2.2, 1.8\},\ W_0 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}, C_1 = I,$$

$C_i = 0\ (i = 1, 2)$.The activation functions are given by $f_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|), (i = 1, 2)$.

This example has been extensively discussed in [30, 31, 33]. The AMUBs τ_2 with various μ are listed in Table 2 by using Corollaries 1−2. Table 2 also gives the improvements between Corollary 1 and the recent work [31].

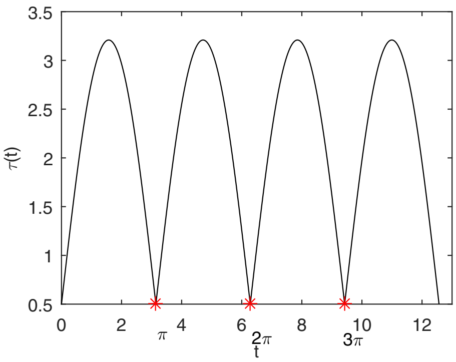

Figure 1: The curve of time delay $\tau(t)$ (Example 1)

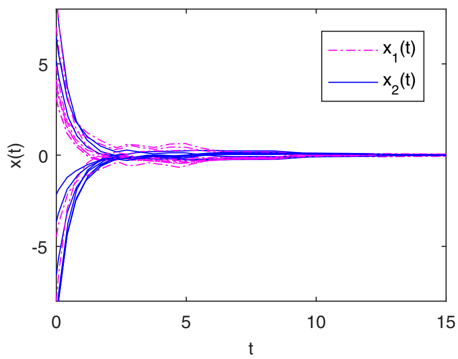

Figure 2: The state trajectories with $u(t) = [0\ 0]^T$ (Example 1)

Table 2: The AMUBs τ_2 with different μ and $\tau_1 = 0$ (Example 2)

μ	0.5	0.9	1
[21, 23, 29]	<1.8500	<1.7700	<1.7500
[20]	3.0430	2.8428	2.8036
[30]	3.1127	2.9415	2.9059
[33]	3.2019	3.0620	3.0612
[31]	3.2659	3.1300	3.0918
Corollary 2	3.2548	3.1277	3.1162
Corollary 1	4.7413	3.7219	3.7133
Improvements over [31]	45.18%	18.19%	20.10%

Table 3: The AMUBs τ_2 with different μ and $\tau_1 = 0$ (Example 3)

μ	0	0.5	0.7
[25]	0.7340	0.6834	0.6335
[19]	0.9722	0.9368	0.9109
[8]	∞	1.0760	1.0704
Corollary 2	0.9645	0.9274	0.9263
Corollary 1	1.4945	1.4089	1.3855

Corollary 1 is less conservative than those of [30, 31, 33].

To further comparison, when μ is unknown, we calculate the AMUBs τ_2, such that DNNs (16) is passive. Applying Theorems 1−2, the AMUBs τ_2 are 3.7113 and 3.1163, respectively, which are larger than 2.9068 by [30].

Example 3. Consider DNNs (16) with the following parameters

$$A = diag\{1.4, 1.5\},\ W_0 = \begin{bmatrix} 1.2 & 1.0 \\ -1.2 & 1.3 \end{bmatrix},\ W_1 = \begin{bmatrix} -0.2 & 0.5 \\ 0.3 & -0.8 \end{bmatrix},$$

$C_i (i = 1, 2, 3)$ and the activation functions are the same as described in Example 2. For various μ, we calculate the MAUBs τ_2 by using Corollaries 1−2. The obtained results in this paper and those in [8, 19, 25] are listed in Table 3. It is clear that Corollary 1 is less conservative than others especially when $\mu = 0.5$ and $\mu = 0.7$.

Example 4. Consider DNNs (1) with the following parameters

$$A = diag\{2.2, 1.5\},\ W_0 = \begin{bmatrix} 1 & 0.6 \\ 0.1 & 0.3 \end{bmatrix},\ W_1 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, C_i = 0 (i = 2, 3)$$
$$F(t) = diag\{sint, cost\}, C_1 = I, H = E_1 = 0.1I, E_2 = 0.2I, E_3 = 0.3I.$$

The activation functions are assumed to be $f_i(x_i) = tanh(x_i), (i = 1, 2)$.

By applying Corollaries 3−4, we can obtain the AMUBs τ_2 with various μ, which are listed in Table 4, in which Th. indicates Theorem. Table 4 also gives the improvements between Corollary 3 and the recent work [20]. The improvements over the existing best results in [20] are 77.49% for $\mu = 0.3$, 56.04% for $\mu = 0.5$, and 49.33% for $\mu = 0.7$, which implies that Corollary 3 has significant improvements over [18, 20, 22].

In addition, we give a comparative result on the total number of the scalar decision variables (NDVs) to obtain AMUBs τ_2 in Table 5. From Table 5, one can see that the NDVs of Corollaries $3-4$ are bigger than the ones provided by [18, 20, 22].

To confirm one of the obtained results in Table 4 ($\tau_1 = 0, \mu = 0.5, \tau_2 = 2.1287$), the state trajectories and output trajectories of DNNs (1) with and without the input $u(t)$ are depicted in Figs. 3−6. From Figs. 3−4, we can see that DNNs (1) with input $u = [1 + cos(2t)\ 1 - sin(2t)]^T$ is passive in the sense of Definition 1 and can keep internally stable. By adopting the product of input and output as the energy provision, it embodies energy attenuation character. That is, passive system (1) will not produce energy by itself. From Figs. 5−6, it can be verified that DNNs (1) is stable.

Remark 6. As we can see in (7) and (8), the AFBI needs more matrix variables. This means that more matrix variables are contained in LMIs. In addition, Tables $4-5$ show that Corollaries $3-4$ can significantly reduce conservativeness but need more decision variables, which increases the calculation complexity to some extent.

5. Conclusion

The problem of robust passivity for uncertain NNs with time-varying delay has been investigated in this paper. An AFBI is employed to manipulate the augmented single integral terms and its special form is used to deal with the delay-product-type term. Delay-dependent passivity criteria have been derived for normal DNNs. Meanwhile, with the same LKF, delay-dependent

Table 4: The AMUBs τ_2 with different μ and $\tau_1 = 0$ (Example 4)

μ	0.3	0.5	0.7	0.9	1
[18] (Th.3)	0.8171	0.7581	0.7029	0.6380	0.6059
[22] (Th.2, m=3)	1.1921	1.1590	1.1297	1.1081	1.1008
[20] (Th.2)	1.9091	1.9005	1.8956	1.8911	1.8873
Corollary 4	2.1415	2.1287	2.1270	2.1265	2.1265
Corollary 3	3.3884	2.9655	2.8307	2.7882	2.7851
Improvements over [20]	77.49%	56.04%	49.33%	47.44%	47.57%

m is delay-partitioning number.

Table 5: The total number of the scalar decision variables (NDVs) (Example 4)

Methods	NDVs
[18] (Th.3)	$12.5n^2$+10.5n+2
[22] (Th.2, m=3)	$31.5n^2$+21.5n+2
[20] (Th.2)	$17n^2$+9n+2
Corollary 4	$73.5n^2$+18.5n+4
Corollary 3	$79.5n^2$+18.5n+4

m is delay-partitioning numbers.

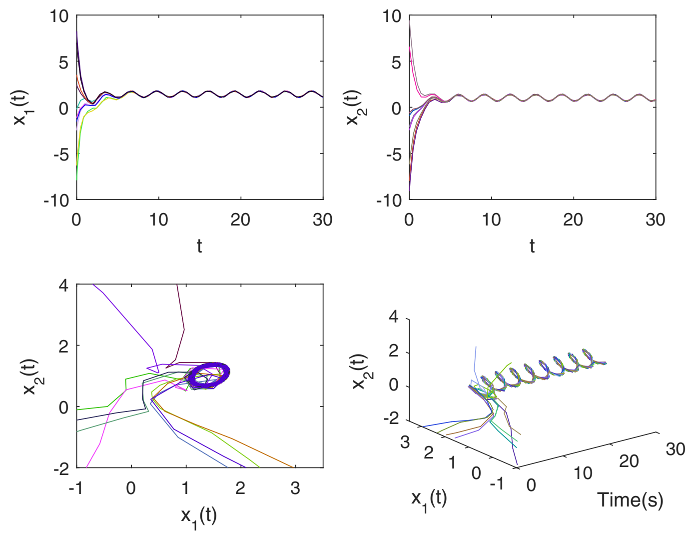

Figure 3: The state trajectories with $u = [1 + cos(2t)\ 1 - sin(2t)]^T$ (Example 4)

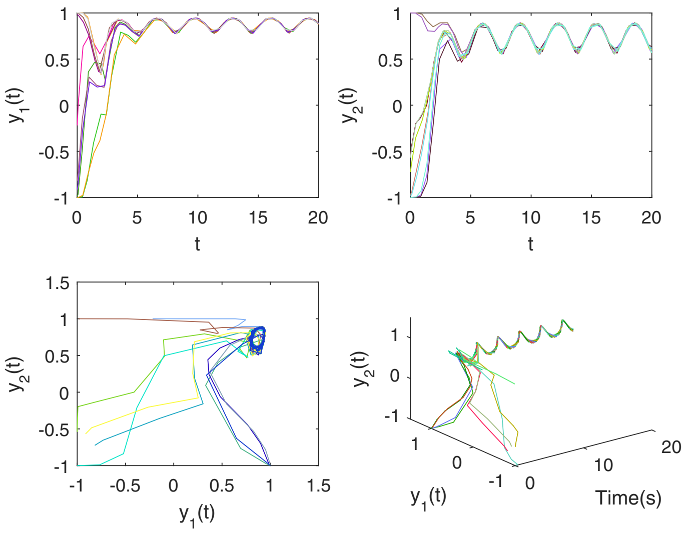

Figure 4: The output trajectories with $u = [1 + cos(2t)\ 1 - sin(2t)]^T$ (Example 4)

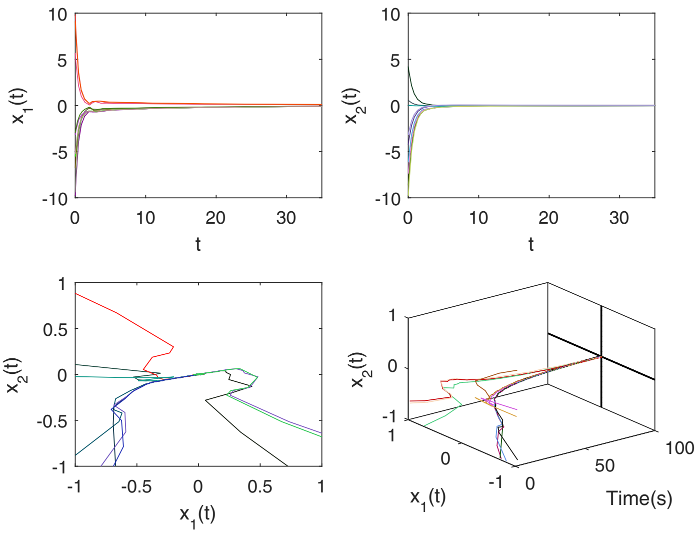

Figure 5: The state trajectories with $u(t) = [0\ 0]^T$ (Example 4)

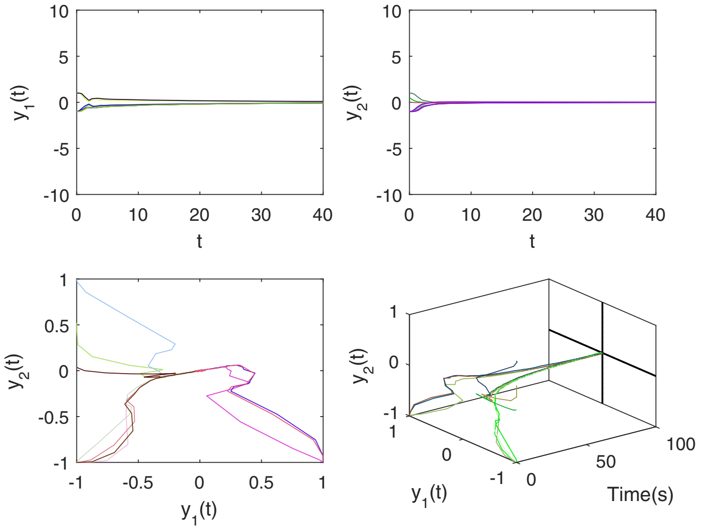

Figure 6: The output trajectories with $u(t) = [0\ 0]^T$ (Example 4)

passivity criteria have been obtained without considering the delay-product-type term. Moreover, the methods are extended to deal with the problem of passivity analysis of uncertain DNNs. Finally, the effectiveness of the proposed criteria has been illustrated by four numerical examples.

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References

[1] S. S. Young, P. D. Scott, N. M. Nasrabadi, Object recognition using multilayer Hopfield neural network, IEEE Trans. Image Process. 6 (1997) 357-372.

[2] L. O. Chua, L. Yang, Cellular neural networks: applications, IEEE Trans. circuits syst. 35 (1988) 1273-1290.

[3] Y. H. Chen, S. C. Fang, Neurocomputing with time delay analysis for solving convex quadratic programming problems, IEEE Trans. Neural Net. 11 (2000) 230-240.

[4] P. Shi, Y. Zhang, M. Chadli, P. K. Agarwal, Mixed H_∞ and passive filtering to discrete fuzzy neural networks with stochastic jumps and time delays, IEEE Trans. Neural Netw. Learn. Syst. 27 (2016) 903-909.

[5] H. G. Zhang, J. Y. Wang, Z. S. Wang, H. J. Liang, Sampled-data synchronization analysis of markovian neural networks with generally incomplete transition rates, IEEE Trans. Neural Netw. Learn. Syst. 28 (2017) 740-752.

[6] J. Y. Xiao, S. M. Zhong, Y. T. Li, F. Xu, Finite-time mittag-leffler synchronization of fractional-order memristive BAM neural networks with time delays. Neurocomputing 219 (2017) 431-439.

[7] R. Saravanakumar, M. S. Ali, J. D. Cao, H. Huang, H_∞ state estimation of generalised neural networks with interval time-varying delays, Int. J. Syst. Sci. 47 (2016) 3888-3899.

[8] J. Y. Xiao, S. M. Zhong, Y. T. Li, Relaxed dissipativity criteria for memristive neural networks with leakage and time-varying delays, Neurocomputing 171 (2016) 708-718.

[9] R. Saravanakumar, G. Rajchakit, M. S. Ali, Y. H. Joo, Extended dissipativity of generalised neural networks including time delays, Int. J. Syst. Sci. 48 (2017) 2311-2320.

[10] B. Yang, J. Wang, J. Wang, Stability analysis of delayed neural networks via a new integral inequality, Neural Netw. 88 (2017) 49-57.

[11] C. K. Zhang, Y. He, L. Jiang, W. J. Lin, M. Wu, Delay-dependent stability analysis of neural networks with time-varying delay: a generalized free-weighting-matrix approach, Appl. Math. Comput. 294 (2017) 102-120.

[12] Y. Liu, Z. Wang, X. Liu, Global exponential stability of generalized recurrent neural networks with discrete and distributed delays, Neural Netw. 19 (2006) 667-675.

[13] G. Rajchakit, R. Saravanakumar, C. K. Ahn, H. R. Karimi, Improved exponential convergence result for generalized neural networks including interval time-varying delayed signals, Neural Netw. 86 (2017) 10-17.

[14] J. C. Willems, Dissipative dynamical systems part I: general theory, Arch. Rational Mech. Anal. 45 (1972) 321-351.

[15] K. Yasue, Quantization of dissipative dynamical systems, Phys. Lett. B. 64 (1976) 239-241.

[16] R. Lozano, B. Brogliato, O. Egeland, B. Maschke, Dissipative systems analysis and control: theory and applications, Springer-Verlag, London, 2013.

[17] A. Wu , Z. Zeng, Exponential passivity of memristive neural networks with time delays, Neural Netw. 49(2014) 11-18.

[18] O. M. Kwon, S. M. Lee, J. H. Park, On improved passivity criteria of uncertain neural networks with time-varying delays, Nonlinear Dyn. 67 (2012) 1261-1271.

[19] Y. Li, S. Zhong, J. Cheng, K. Shi, J. Ren, New passivity criteria for uncertain neural networks with time-varying delay, Neurocomputing 171 (2016) 1003-1012.

[20] H. B. Zeng, H. P. Ju, H. Shen, Robust passivity analysis of neural networks with discrete and distributed delays, Neurocomputing 149 (2015) 1092-1097.

[21] Z. G. Wu, H. P. Ju, H. Su, J. Chu, Robust dissipativity analysis of neural networks with time-varying delay and randomly occurring uncertainties, Nonlinear Dyn. 69 (2012) 1323-1332.

[22] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, E. J. Cha, Passivity analysis of uncertain neural networks with mixed time-varying delays, Nonlinear Dyn. 73 (2013) 2175-2189.

[23] S. Xu, W. X. Zheng, Y. Zou, Passivity Analysis of Neural Networks With Time-Varying Delays, IEEE Trans. Circuits Syst. II 56 (2009) 325-329.

[24] J. Y. Xiao, S. M. Zhong, Extended dissipative conditions for memristive neural networks with multiple time delays, Applied Mathematics and Computation 323 (2018) 145-163.

[25] B. Zhang, S. Xu, J. Lam, Relaxed passivity conditions for neural networks with time-varying delays, Neurocomputing 142 (2014) 299-306.

[26] J. Y. Xiao, Y. T. Li, S. M. Zhong, F. Xu, Extended dissipative state estimation for memristive neural networks with time-varying delay, ISA Trans. 64 (2016) 113-128.

[27] G. Nagamani, T. Radhika, Q. Zhu, An improved result on dissipativity and passivity analysis of Markovian jump stochastic neural networks with two delay components, IEEE Trans. Neural Netw. Learn. Syst. 28 (2017) 3018-3031.

[28] M. S. Ali, R. Saravanakumar, J. D. Cao, New passivity criteria for memristor-based neutral-type stochastic BAM neural networks with mixed time-varying delays, Neurocomputing 171 (2016) 1533-1547.

[29] H. B. Zeng, Y. He, M. Wu, S. P. Xiao, Passivity analysis for neural networks with a time-varying delay, Neurocomputing 74 (2011) 730-734.

[30] M. V. Thuan, H. Trinh, L. V. Hien, New inequality-based approach to passivity analysis of neural networks with interval time-varying delay, Neurocomputing 194 (2016) 301-307.

[31] H. B. Zeng, Y. He, P. Shi, M. Wu, S. P. Xiao, Dissipativity analysis of neural networks with time-varying delays, Neurocomputing 168 (2015) 741-746.

[32] S. Rajavel, R. Samidurai, J. Cao, A. Alsaedi, B. Ahmad, Finite-time non-fragile passivity control for neural networks with time-varying delay, Appl. Math. Comput. 297 (2017) 145-158.

[33] Y. J. Shu, X. G. Liu, S. B. Qiu, F. Wang, Dissipativity analysis for generalized neural networks with markovian jump parameters and time-varying delay. Nonlinear Dyn. 89 (2017) 2125-2140.

[34] Y. Wu, Passivity analysis for dynamic multilayer neuro identifier, IEEE Trans. Circuits Syst. I 50 (2003) 173-178.

[35] C. Li, X. Liao, Passivity analysis of neural networks with time delay, IEEE Trans. Circuits Syst. II 52 (2005) 471-475.

[36] M. Wu, Y. He, J. H. She, G. P. Liu, Delay-dependent criteria for robust stability of time-varying delay systems, Automatica 40 (2004) 1435-1439.

[37] A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: application to time-delay systems, Automatica 49 (2013) 2860-2866.

[38] V. H. Le, H. Trinh, Refined Jensen-based inequality approach to stability analysis of time-delay systems, IET Control Theory Appl. 9 (2015) 2188-2194.

[39] Z. H. Beng, Y. He, M. Wu, J. She, Free-matrix-based integral inequality for stability analysis of systems with time-varying delay, IEEE Trans. Autom. Control 60 (2015) 2768-2772.

[40] J. H. Kim, Further improvement of Jensen inequality and application to stability of time-delayed systems, Automatica 64 (2016) 121-125.

[41] P. G. Park, W. I. Lee, S. Y. Lee, Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems, J. Frankl. Inst. 352 (2015) 1378-1396.

[42] L. H. Xie, Output feedback hcontrol of systems with parameter uncertainty, Int. J. Control 63 (1996) 741-750.

[43] P. G. Park, J. W. Ko, Stability and robust stability for systems with a time-varying delay, Automatica 43 (2007) 1855-1858.

[44] S. Boyd, L. E. Ghaoui, E. Feron, V. Balakrishnan, Linear matrix inequalities in system and control theory, SIAM, Philadelphia, 1994.

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