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Random record processes and state dependent thinning

Sid Browne[a],*, John Bunge[b]

[a]402 Uris Hall, Graduate School of Business, Columbia University, New York, NY 10027, USA
[b]358 Ives Hall, Department of Economic and Social Statistics, NYSSILR-Cornell, Ithaca, NY 14853-3901 USA

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Abstract

Suppose that a point process $\bar{N}_t = T_1, T_2, \ldots$ on $[0, \infty)$ is thinned by independently retaining T_n with probability p_n. Our main examples are the classical *p-thinning* ($p_n \equiv p$) and the *random record process* ($p_n = 1/n$). When $\bar{N}_t$ is a mixed, nonhomogeneous Poisson process, we find conditions under which the thinned process is Poisson. When $\bar{N}_t$ is a pure birth process (gamma-mixed Poisson with exponential rate), we show that the record process is Markov renewal, with an interesting structure, and we compare this with related asymptotic results. When $\bar{N}_t$ is a Mittag–Leffler renewal process (the homogeneous Poisson is a special case), we give a "Deheuvels-type" representation of the record process (Deheuvels, 1982) and related characterization results.

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1. Introduction

Let $\{T_n, Z_n\}_{n \geq 1}$ denote a marked point process, where $T_1, T_2, \ldots$ are the points of a simple, nonexplosive point process on $[0, \infty)$, and the marks $\{Z_n\}_{n \geq 1}$ form an i.i.d. sequence of continuously distributed random variables (r.v.'s), independent of $\{T_n\}_{n \geq 1}$. The corresponding *random record process* is the process of points at which record marks occur, that is, points T_n at which $Z_n = \max_{1 \leq j \leq n} Z_j$. Random record processes are interesting in their own right, and because of their role in "finding explicit solutions of optimal selection problems based on relative ranks" (Bruss and Rogers, 1991, p. 331), and also because they may possess exact properties that appear as limits in other extremal models (cf. Bunge and Nagaraja, 1992a, Section 7, and Section 3.2, below).

*Corresponding author.

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A theorem of Rényi states that $P\{Z_n \text{ is a record}\} = 1/n$, and that the events $\{Z_n$ is a record$\}$, $n \geq 1$, are independent (see Resnick, 1987, p.169). Thus the random record process can be regarded as a *thinning* of $\{T_n\}_{n\geq 1}$, where the point T_n is independently retained with probability $p_n = 1/n$. The thinning is *state-dependent* because the probability of retention depends on n. In this note we find conditions under which the state-dependent thinning of a mixed, nonhomogeneous Poisson process is Poisson, for general p_n's, and we give exact, explicit representations for the random record process ($p_n = 1/n$) when $\{T_n\}_{n\geq 1}$ is a pure birth process with integer parameter and when $\{T_n\}_{n\geq 1}$ is a "Mittag–Leffler" renewal process.

Specifically, then, let $\{T_n, B_n\}_{n\geq 1}$ denote a bivariate point process on $[0, \infty)$, where $T_1, T_2, \ldots$ are as above and $T_0 := 0$, and $\{B_n\}_{n\geq 1}$ is a sequence of independent Bernoulli r.v.'s, independent of $\{T_n\}_{n\geq 1}$, with success probabilities $0 < p_n := P\{B_n = 1\} \leq 1, n \geq 1$. We assume that $\{T_n, B_n\}_{n\geq 1}$ is defined on a probability space $(\Omega, \mathcal{F}, P)$, and that it is adapted to a history $(\mathcal{F}_t, t \geq 0) \subseteq \mathcal{F}$. Let

$$N_t(i) = \sum_{n\geq 1} 1(T_n \leq t)1(B_n = i), \quad i = 0, 1, \ t \geq 0,$$

where $1(A) :=$ the indicator of the event A, and let $\bar{N}_t = N_t(0) + N_t(1)$ (see Brémaud, 1981, ch. II, regarding such models). Since T_n appears in $N_t(1)$ iff $B_n = 1$, we call $N_t(1)$ a *state-dependent thinning* of $\bar{N}_t$.

The most general formulation of our problem is: give a representation of $N_t(1)$ based on $\{p_n\}_{n\geq 1}$ and the properties of $\bar{N}_t$. Böker and Serfozo (1983) gave weak convergence theory and Arjas et al. (1992) gave filtering formulas for models more general than ours, but it may not be possible to find an exact representation without making some restrictions. However, there are some tractable special cases, such as the random record process ($p_n = 1/n$) and the classical *p-thinning*, where $p_n \equiv p$ for all n (Matthes et al., 1978, p. 91).

We proceed as follows. In Section 2.1 we assume that $\bar{N}_t$ is a mixed, nonhomogeneous Poisson process, and we derive a sufficient condition for $N_t(1)$ to be a Poisson process; the condition holds if and only if the mixing distribution is either degenerate or gamma. In Section 2.2. we relate our results to those of Bruss and Rogers (1991) concerning Pascal processes and k-records. In Section 3.1 we take $\bar{N}_t$ to be a pure birth process with integer parameter (gamma-mixed Poisson with exponential rate), and we show that the record process is a Markov renewal process with an interesting structure; this generalizes results of Bruss and Rogers (1991) and Bunge and Nagaraja (1992a). In Section 3.2 we compare the aforementioned Markov renewal process with the limiting behavior of the record process over an ordinary nonhomogeneous Poisson process. In Section 4 we take $\bar{N}_t$ to be a "Mittag–Leffler" renewal process (the homogeneous Poisson is a special case). In general the Mittag–Leffler process is not mixed Poisson, so the results of Section 2.1 do not apply. However, we give a representation of the record process in the general case, thereby extending a result of Deheuvels (1982) (a different proof of the latter result, in the style used here, was given by Bunge and Nagaraja, 1992a). We also give two related characterizations of the Mittag–Leffler distribution. Finally, Section 5 contains the proofs.

A note on notation: the distribution with density $\lambda^\gamma x^{\gamma-1} e^{-\lambda x}/\Gamma(\gamma)$, $x \geq 0$, with parameters $\lambda > 0$ and $\gamma > 0$, will be denoted by $\Gamma(\lambda, \gamma)$; $\Gamma(\lambda, 1)$ will be called $\exp(\lambda)$.

2. Thinned mixed Poisson processes that are Poisson

2.1. A sufficient condition

Let $\mu(t)$ denote a strictly increasing, positive, differentiable function on $[0, \infty)$ (right differentiable at 0) such that $\mu(0) = 0$ and $\mu(t) \uparrow \infty$. Let Λ denote a positive r.v. on $(\Omega, \mathscr{F}, P)$ with distribution function (d.f.) F and Laplace–Stieltjes transform (LST) $f(t) = E(e^{-\Lambda t})$, $t \geq 0$, and let $f^{(n)}(t)$ denote with nth derivative of f. Let $\mathscr{F}_t^N(i) = \sigma(N_s(i),\ 0 \leq s \leq t)$, $i = 0, 1$, and suppose that $\mathscr{F}_t$ is a history of the form

$$\mathscr{F}_t = \sigma(\Lambda) \vee \left(\bigvee_{i=0}^{1} \mathscr{F}_t^N(i) \right).$$

Finally, suppose that $\bar{N}_t$ is a mixed, nonhomogeneous Poisson process with $\mathscr{F}_t$-intensity $\mu'(t)\Lambda$.

Proposition 1. *If there is a function $\rho: [0, \infty) \to [0, \infty)$ such that*

$$\forall n \geq 1,\ \forall t \geq 0, \qquad -p_n \frac{f^{(n)}(t)}{f^{(n-1)}(t)} = \rho(t), \tag{1}$$

then $N_t(1)$ is a Poisson process, with intensity $\mu'(t)\rho(\mu(t))$.

While condition (1) is appealing, it can be applied in essentially only two cases.

Proposition 2. *Condition* (1) *holds if and only if either*:

(i) *$\Lambda = \lambda$ with probability 1 for some $\lambda > 0$, $p_n \equiv p$ for some $p \in (0, 1]$, and $\rho(t) \equiv p\lambda$, or*
(ii) *$\Lambda \sim \Gamma(\lambda, \gamma)$ for some $\lambda > 0$ and $\gamma > 0$, $p_n = c/(n - 1 + \gamma)$ for some $0 < c \leq \gamma$, and $\rho(t) = c/(\lambda + t)$.*

Thus, $N_t(1)$ is Poisson in these cases, with rate $\mu'(t)p\lambda$ or $c\mu'(t)/(\lambda + \mu(t))$, respectively.

Conversely, suppose that $\bar{N}_t$ is mixed Poisson and $N_t(1)$ is Poisson. Does this imply that (1) holds? In general we do not know, but we can answer affirmatively in two special cases. First, it can be readily shown that if $p_n \equiv p \in (0, 1]$ then $\Lambda \equiv \lambda \in (0, \infty)$. This also means that if Λ is nondegenerate then $\{p_n\}_{n \geq 1}$ must be "nondegenerate" ($p_n \not\equiv p$) in order for $N_t(1)$ to be Poisson. Second, if $p_n = 1/n$, $n \geq 1$, then $\Lambda \sim \exp(\lambda)$ for some $\lambda \in (0, \infty)$. This can be proved using the formula for $P\{N_t(1) = n\}$ given in Theorem 4.4 of Bunge and Nagaraja (1991). But it is also a consequence of the proof of Theorem 2.2 of Bruss and Samuels (1990). In that paper the authors make extensive use of the *order statistic property*, which is characteristic of mixed Poisson processes;

we do not exploit this property directly here, although it underlines the result of Feigin (1979) that we cite below as Fact 2.

2.2. Pascal processes and k-records

Bruss and Rogers (1991) studied the case where $\Lambda \sim \exp(1)$ (i.e., $\lambda = \gamma = 1$ in Proposition 2(ii)); they called $\{T_n\}_{n \geq 1}$ a *Pascal process* in this case. To discuss their results we need the following definitions. Returning to the marked point process $\{T_n, Z_n\}_{n \geq 1}$ of Section 1, define Z_n to be a *k-record* if Z_n is the kth largest among $Z_1, \ldots, Z_n$, $1 \leq k \leq n$. Let $R_t(k)$ denote the k-record process, i.e., the process of points at which k-records occur, and let K be a fixed positive integer. Bruss and Rogers (1991) proved the following theorem.

Fact 1 (Bruss and Rogers, 1991). *If $\{T_n\}_{n \geq 1}$ is a Pascal process then $(R_t(1), \ldots, R_t(K))$ are i.i.d. nonhomogeneous Poisson processes on (T_{K-1}, ∞).*

Proposition 1, with $\Lambda \sim \exp(1)$ and $p_n = 1/n$, provides a simplified proof of Fact 1 for the case $K = 1$. Furthermore, although our bivariate process $\{T_n, B_n\}_{n \geq 1}$ cannot jointly represent $(R_t(1), \ldots, R_t(K))$ for $K > 1$, it is possible to reformulate our model so that $N_t(1)$ (based on $\{T_n, B_n\}_{n \geq 1}$) can represent $R_t(k)$ *marginally* for any $k \geq 1$, by setting $p_n = 0$, $1 \leq n \leq k - 1$, $p_n = 1/n$, $n \geq k$. (This again is due to Rényi's theorem; see Resnick, 1987, p. 169). By suitably modifying (1) to hold for $n \geq k$, one can then show that *marginally* $R_t(1), \ldots, R_t(K)$ are identically distributed nonhomogeneous Poisson processes on (T_{K-1}, ∞).

3. Record processes over birth processes

3.1. An exact representation

We begin this section with a result of Feigin (1979, p. 303) which connects mixed, nonhomogeneous Poisson processes and pure birth processes. (For further discussion see Resnick, 1992, ch. 5.11.) The setup is the same as in the previous sections; we refer to Anderson (1991, p. 19) in regard to birth processes. Throughout this section m will denote an arbitrary but fixed nonnegative integer.

Fact 2 (Feigin, 1979). *Let $(\nu_t, t \geq 0)$ denote a continuous-time Markov process, defined on $(\Omega, \mathscr{F}, P)$, with state space $\{0, 1, 2, \ldots\}$ and $P\{\nu_0 = 0\} = 1$, and with q-matrix given by*

$$q_{n,n+1} = -q_{n,n} = 1 + m + n, \quad n = 0, 1, \ldots,$$

$q_{ij} = 0$ otherwise. In this case $(\nu_t, t \geq 0)$ is a pure birth process. *Then it is possible to define a mixed, nonhomogeneous Poisson process $(\bar{N}_t, t \geq 0)$ on $(\Omega, \mathscr{F}, P)$ (as in Section 2.1), with $\Lambda \sim \Gamma(1 + m, 1 + m)$ and $\mu(t) = (1 + m)(e^t - 1)$, such that*

$$(\bar{N}_t, t \geq 0) = (\nu_t, t \geq 0) \quad P\text{-almost surely.}$$

In this case $[T_n - T_{n-1}\}_{n \geq 1}$ are independent r.v.'s with $T_n - T_{n-1} \sim \exp(n + m)$, $n \geq 1$.

For this $\bar{N}_t$, (1) holds with $p_n = 1/(n + m)$ (by Proposition 2(ii)), but more can be said about the record process, where $p_n = 1/n$ regardless of m. In fact we have the following representation, special cases of which were proved by Bruss and Rogers (1991) ($m = 0$) and Bunge and Nagaraja (1992a) ($m = 0, 1$). We refer to Cinlar (1975, ch. 10) for a discussion of Markov renewal processes.

Theorem 1. *Suppose that $\bar{N}_t$ is a pure birth process as in Fact 2, and that $p_n = 1/n$, $n \geq 1$. Let $\{\tau_n, \theta_n\}_{n \geq 1}$ denote a* Markov renewal process *with parameter space $\{[0, \infty)\}$ and state space $\{1, 2, \ldots, 1 + m\}$, defined on a probability space $(\Omega^{\#}, \mathcal{F}^{\#}, P^{\#})$, with* semi-Markov kernel

$$\begin{cases} P^{\#}(\tau_1 \leq t\} = 1 - \mathrm{e}^{-(1+m)t} \\ P^{\#}\{\tau_{n+1} - \tau_n \leq t,\ \theta_{n+1} = j \mid \theta_n = i\} = \pi_{ij}(1 - \mathrm{e}^{-jt}),\ n \geq 1, \end{cases} \quad t \geq 0, 1 \leq i \leq j \leq 1 + m,$$

where $\theta_1 = 1 + m$ with probability 1, and

$$[\pi_{ij}] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1+m} & \frac{1}{1+m} & \frac{1}{1+m} & \cdots & \frac{1}{1+m} \end{bmatrix}, \quad 1 \leq i \leq j \leq 1 + m.$$

Let $M_t = \sum_{n \geq 1} 1(\tau_n \leq t)$, $t \geq 0$. Then

$$(N_t(1), t \geq 0) \stackrel{\mathrm{d}}{=} (M_t, t \geq 0),$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution.

Intuitively, M_t can be described as follows: $\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots$ are conditionally independent given $\theta_1, \theta_2, \ldots$, with $\tau_n - \tau_{n-1} \sim \exp(\theta_n)$, $n \geq 1$ ($\tau_0 := 0$), where $\theta_1, \theta_2, \ldots$ are the outcomes of a finite Markov chain on $\{1, 2, \ldots, 1 + m\}$ with initial state $1 + m$, transition matrix $[\pi_{ij}]$, and absorbing state 1. In particular, let $A = \min\{n \geq 1: \theta_n = 1\} - 1$; then $\tau_{A+1} - \tau_A, \tau_{A+2} - \tau_{A+1}, \ldots$ are i.i.d. exp(1) r.v.'s; that is, M_t is homogeneous Poisson after τ_A. Note also that $A \equiv 0$ iff $m = 0$ and $P^{\#}\{A = 1\} = P^{\#}\{\theta_2 = 1\} = 1/(1 + m) > 0$ for any $m > 0$.

Now let $T_1(1), T_2(1), \ldots$ denote the points of $N_t(1)$ (with $T_0(1) := 0$), and let $U_n(1) = T_n(1) - T_{n-1}(1), n \geq 1$. Theorem 1 says that when $\bar{N}_t$ is a pure birth process as in Fact 2, $(U_1(1), U_2(1), \ldots) \stackrel{\mathrm{d}}{=} (\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots)$. Heuristically speaking, then, $U_1(1), U_2(1), \ldots$ can be generated as follows in this case. Initially, set $\theta_1 = 1 + m$ and generate $U_1(1) \sim \exp(\theta_1)$. Then for $n \geq 1$, given θ_n, draw θ_{n+1} according to the discrete uniform distribution on $\{1, \ldots, \theta_n\}$, and given θ_{n+1}, generate $U_{n+1}(1) \sim \exp(\theta_{n+1})$. In particular, if A is as above then $U_{A+1}(1), U_{A+2}(1), \ldots$ are i.i.d. exp(1) r.v.'s and $N_t(1)$ is homogeneous Poisson after $T_A(1)$.

3.2. Comparison of exact and limiting behavior

Let $(U_1^*, \ldots, U_n^*)$ denote i.i.d. exp(1) r.v.'s, $n \geq 1$. It was shown in Bunge and Nagaraja (1992b) that if $\Lambda \equiv 1$ (the ordinary nonhomogeneous Poisson case), and $\mu'(t)/\mu(t) \to 1$ as $t \to \infty$, then $(U_{\kappa+1}(1), \ldots, U_{\kappa+n}(1)) \xrightarrow{\mathrm{d}} (U_1^*, \ldots, U_n^*)$ as $\kappa \to \infty$, where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. We can then make the following comparison with Theorem 1.

$$\left.\begin{array}{c} \dfrac{\mu'(t)}{\mu(t)} \to 1 \\ \text{and} \\ \Lambda \equiv 1 \end{array}\right\} \Rightarrow (U_{\kappa+1}(1), \ldots, U_{\kappa+n}(1)) \xrightarrow{\mathrm{d}} (U_1^*, \ldots, U_n^*), \tag{2}$$

$$\left.\begin{array}{c} \mu(t) = (1+m)(\mathrm{e}^t - 1) \\ \text{and} \\ \Lambda \sim \Gamma(1+m, 1+m) \end{array}\right\} \Rightarrow (U_{\Lambda+1}(1), \ldots, U_{\Lambda+n}(1)) \stackrel{\mathrm{d}}{=} (U_1^*, \ldots, U_n^*). \tag{3}$$

Note that $\mu'(t)/\mu(t) \to 1$ as $t \to \infty$ when $\mu(t) = (1+m)(\mathrm{e}^t - 1)$, and that $E(\Lambda) = 1$ when $\Lambda \sim \Gamma(1+m, 1+m)$. In particular, $m = 0$ implies that $A \equiv 0$, and (3) then says that exponential randomization of the rate has the same effect on $N_t(1)$ as does passage to the limit under the nonrandom rate. Furthermore, if $\mu(t) = (1+m)(\mathrm{e}^t - 1)$ and $\Lambda \sim \Gamma(1+m, 1+m)$, then as $m \to \infty$, $\mu'(t)/\mu(t) \to 1$ for all $t \geq 0$, $\Lambda \xrightarrow{\mathrm{p}} 1$, and $A \xrightarrow{\mathrm{p}} \infty$, where $\xrightarrow{\mathrm{p}}$ denotes convergence in probability. Thus as $m \to \infty$, (3) can be regarded as a "random-index" version of (2), with A playing the role of κ.

4. The Mittag–Leffler renewal process

We now consider a different generalization of the Poisson process for $\bar{N}_t$. Let $U_n = T_n - T_{n-1}$, $n \geq 1$, and suppose that

$$\{U_n\}_{n \geq 1} \stackrel{\mathrm{d}}{=} \left\{\frac{1}{\lambda} X_n^{1/\alpha} Y_n\right\}_{n \geq 1},$$

where $\{X_n\}_{n \geq 1}$ is an i.i.d. sequence of exp(1) r.v.'s and $\{Y_n\}_{n \geq 1}$ is an i.i.d. sequence of positive α-stable r.v.'s independent of $\{X_n\}_{n \geq 1}$ (Y_1 has LST e^{-s^α}, $s \geq 0$, where $\alpha \in (0, 1]$). In this case U_1 has LST $1/(1 + (s/\lambda)^\alpha)$, and is said to have a "Mittag–Leffler (α, λ)" distribution (Pillai, 1990); thus $\bar{N}_t$ is a Mittag–Leffler (α, λ) renewal process. If $\alpha = 1$ then $\bar{N}_t$ is a homogeneous Poisson process of rate λ. Now it is readily shown that if $p_n \equiv p$ then

$$N_{\frac{t}{p^{1/\alpha}}}(1) \stackrel{\mathrm{d}}{=} \bar{N}_t,$$

for all $p \in (0, 1]$. This implies that $\bar{N}_t$ is a Cox process (Matthes et al., 1978, Theorem 7.2.8, p. 295), but it is not mixed Poisson (Grandell, 1976, Theorem 1(i), p. 35), so Section 2 does not apply for $\alpha < 1$. Nevertheless we have the following representation of the record process, for all $\alpha \in (0, 1]$. This generalizes Theorem 2 of Deheuvels (1982) ($\alpha = 1$) and Theorem 7.1 of Bunge and Nagaraja (1992a) ($\alpha = 1$).

Theorem 2. *If $\bar{N}_t$ is a Mittag–Leffler (α, λ) renewal process and $p_n = 1/n$, $n \geq 1$, then*

$$\{U_n(1)\}_{n \geq 1} \stackrel{\mathrm{d}}{=} \left\{\frac{1}{\lambda} U_n \mathrm{e}^{R_{n-1}/\alpha}\right\}_{n \geq 1}, \tag{4}$$

where $R_1, R_2, \ldots$ are the points of a homogeneous unit Poisson process (with $R_0 := 0$), independent of $\{U_n\}_{n \geq 1}$.

The presence of U_n, the renewal time of $\bar{N}_t$, in the right-hand side of (4) suggests that record processes over other renewal processes might admit an analogous representation. However, with a Poisson process "in the exponent," (4) characterizes the Mittag–Leffler process.

Proposition 3. *If $\bar{N}_t$ is an arbitrary renewal process and (4) holds, where $p_n = 1/n$, $\{R_n\}_{n \geq 0}$ is defined in Theorem 2 and $\alpha > 0$, then $\bar{N}_t$ is a Mittag–Leffler (α, λ) renewal process (and $\alpha \in (0, 1]$).*

Finally, it is readily shown that

$$\forall n \geq 1, \quad \sum_{i=1}^{n} X_i^{1/\alpha} Y_i \stackrel{\mathrm{d}}{=} \left(\sum_{i=1}^{n} X_i\right)^{1/\alpha} Y_0, \tag{5}$$

where $\{X_n\}_{n \geq 1}$ and $\{Y_n\}_{n \geq 1}$ are defined above, and $Y_0 \stackrel{\mathrm{d}}{=} Y_1$. That is, the stable r.v. has "factored out" of the sum. The proof of Theorem 2 depends in part upon (5); attempts to extend the theorem to other mixed-exponential renewal processes using an analogous factorization were fruitless, due to the following fact.

Proposition 4. *Let $\{X_n\}_{n \geq 1}$ be an i.i.d. sequence of $\exp(1)$ r.v.'s, let $\{Y_n\}_{n \geq 1}$ be an i.i.d. sequence of nonnegative r.v.'s independent of $\{X_n\}_{n \geq 1}$, let Y_0 be a nonnegative r.v. independent of $\{X_n\}_{n \geq 1}$, and let $\alpha > 0$. Then (5) holds if and only if $Y_0 \stackrel{\mathrm{d}}{=} Y_1$ and Y_1 is a scale multiple of a positive α-stable r.v. (in which case $\alpha \in (0, 1]$).*

In other words, (5) holds with $\exp(1)$ X_i's (for some Y_0) if and only if $X_1^{1/\alpha} Y_1 \sim$ Mittag–Leffler(α, λ).

5. Proofs

Proof of Proposition 1. The $\mathscr{F}_t$-intensity of $N_t(1)$ is $p_{\bar{N}_t+1} \mu'(t) \Lambda$, and by Brémaud (1981), Lemma 5, p. 171) the $\bigvee_{i=0}^{1} \mathscr{F}_t^N(i)$-intensity of $N_t(1)$ is

$$E\left(p_{\bar{N}_t+1} \mu'(t) \Lambda \,\middle|\, \bigvee_{i=0}^{1} \mathscr{F}_t^N(i)\right) = \mu'(t) p_{\bar{N}_t+1} \frac{\int_0^\infty \lambda^{\bar{N}_t+1} \mathrm{e}^{-\lambda\mu(t)} \mathrm{d}F(\lambda)}{\int_0^\infty \lambda^{\bar{N}_t} \mathrm{e}^{-\lambda\mu(t)} \mathrm{d}F(\lambda)}$$

$$= \mu'(t)\left(-p_{\bar{N}_t+1} \frac{f^{(\bar{N}_t+1)}(\mu(t))}{f^{(\bar{N}_t)}(\mu(t))}\right). \tag{6}$$

Under (1), the rightmost term of (6) reduces to $\mu'(t)\rho(\mu(t))$, and Watanabe's theorem (Brémaud, 1981, Theorem 5, p. 25) then implies that $N_t(1)$ is a Poisson process with intensity $\mu'(t)\rho(\mu(t))$. □

Proof of Proposition 2. "If" is readily verified by direct computation. Conversely, taking $n = 1$ and $n = 2$ in (1) and eliminating $\rho(t)$, we obtain the second-order ordinary differential equation

$$p_1 f'^2(t) - p_2 f''(t) f(t) = 0 \tag{7}$$

for all $t \geq 0$. Suppose first that $p_1 = p_2 = p$ for some $p \in (0, 1]$. Then the only LST solutions of (7) are $f(t) = \mathrm{e}^{-\lambda t}$ for arbitrary $\lambda > 0$, in which case assertion (i) of the Proposition follows. On the other hand, if $p_1 \neq p_2$ then it can be shown that the only LST solutions of (7) are $f(t) = (1 + t/\lambda)^{-p_2/(p_1 - p_2)}$ for arbitrary $\lambda > 0$, where necessarily $p_1 > p_2$. That is, $\Lambda \sim \Gamma(\lambda, \gamma)$, where $\gamma = p_2/(p_1 - p_2) > 0$. Then applying (1) again,

$$-p_n \frac{f^{(n)}(t)}{f^{(n-1)}(t)} = p_n(n - 1 + \gamma)\frac{1}{\lambda + t} = \rho(t)$$

for all $n \geq 1$, which implies that $p_n = c/(n - 1 + \gamma)$ for some $0 < c \leq \gamma$, and $\rho(t) = c/(\lambda + t)$. □

Proof of Theorem 1. We begin by computing the joint LST of $(U_1(1), \ldots, U_n(1))$, for arbitrary $n \geq 1$. Let $L_n = \min\{j: \sum_{i=1}^{j} B_i = n\}$, the index of the nth point in the record process, $n \geq 1$ ($L_1 \equiv 1$). By Rényi's theorem,

$$P\{L_2 = a_2, \ldots, L_n = a_n\} = \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n},$$

$1 < a_2 < \cdots < a_n$. Then

$$E\mathrm{e}^{-(s_1 U_1(1) + \cdots + s_n U_n(1))} = E\mathrm{e}^{-s_1 U_1} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \prod_{i=2}^{n} \prod_{j=a_{i-1}+1}^{a_i} E\mathrm{e}^{-s_i U_j}, \tag{8}$$

$s_i \geq 0$, $i = 1, \ldots, n$, $a_1 := 1$. Note that $E\mathrm{e}^{-s_i U_j} = (j + m)/(j + m + s_i) = (j + m)/(j + \xi_i)$, where $\xi_i := m + s_i$, $i = 1, \ldots, n$ (Anderson, 1991, p. 16). Then (8) becomes

$$\begin{aligned} &\frac{1 + m}{1 + \xi_1} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \prod_{i=2}^{n} \prod_{j=a_{i-1}+1}^{a_i} \frac{j + m}{j + \xi_i} \\ &= \frac{1 + m}{1 + \xi_1} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \frac{(a_n + m)!/(1 + m)!}{\prod_{i=2}^{n} \Gamma(a_i + \xi_i + 1)/\Gamma(a_{i-1} + \xi_i + 1)} \\ &= \frac{1}{(1 + \xi_1)m!} \sum_{1 < a_2 < \cdots < a_n} \prod_{i=2}^{n} \frac{\Gamma(a_{i-1} + \xi_i + 1)}{(a_i - 1)\Gamma(a_i + \xi_i + 1)} \Gamma(a_n) \prod_{j=1}^{m} (a_n + j). \end{aligned} \tag{9}$$

Setting $b_i = a_i - 1$ and $\eta_i = \xi_i + 1 = 1 + m + s_i$, we rewrite (9) as

$$\frac{1}{\eta_1 m!} \sum_{1 \le h_2 < \cdots < h_n} \prod_{i=2}^{n} \frac{\Gamma(b_{i-1} + \eta_i + 1)}{b_i \Gamma(b_i + \eta_i + 1)} \Gamma(b_n + 1) \prod_{j=1}^{m} (b_n + 1 + j). \tag{10}$$

It is easy to show by induction on m that

$$\prod_{j=1}^{m} (b_n + 1 + j) = \sum_{j=0}^{m} \frac{m!}{j!} \prod_{r=1}^{j} (b_n + r),$$

$m = 0, 1, \ldots$. Hence (10) is equal to

$$\frac{1}{\eta_1 m!} \sum_{j=0}^{m} \frac{m!}{j!} \sum_{1 \le h_2 < \cdots < h_n} \prod_{i=2}^{n} \frac{\Gamma(b_{i-1} + \eta_i + 1)}{b_i \Gamma(b_i + \eta_i + 1)} \Gamma(b_n + 1) \prod_{r=1}^{j} (b_n + r)$$

$$= \frac{1}{\eta_1} \sum_{k=1}^{1+m} \frac{1}{(k-1)!} \sum_{1 \le h_2 < \cdots < h_n} \prod_{i=2}^{n} \frac{\Gamma(b_{i-1} + \eta_i + 1)}{b_i \Gamma(b_i + \eta_i + 1)} \Gamma(b_n + k). \tag{11}$$

It is readily shown by induction on k that

$$\sum_{1 \le h_2 < \cdots < h_n} \prod_{i=2}^{n} \frac{\Gamma(b_{i-1} + \eta_i + 1)}{b_i \Gamma(b_i + \eta_i + 1)} \Gamma(b_n + k)$$

$$= (k-1)! \sum_{\substack{i_1 + \cdots + i_k = n-1 \\ i_1, \ldots, i_k \ge 0}} \prod_{j_1=2}^{i_1+1} \eta_{j_1}^{-1} \prod_{j_2=i_1+2}^{i_1+i_2+1} (\eta_{j_2} - 1)^{-1} \cdots$$

$$\prod_{j_k=i_1+\cdots+i_{k-1}+2}^{n} (\eta_{j_k} - (k-1))^{-1},$$

$k = 1, 2, \ldots$ (for $k = 1$ see Bunge and Nagaraja, 1992a, Lemma 2.2(i), p. 24). Thus, resubstituting for the η_i's, (11) is finally

$$\left(\left(1 + \frac{s_1}{1+m}\right)(1+m)\right)^{-1} \sum_{k=1}^{1+m} \sum_{\substack{i_1 + \cdots + i_k = n-1 \\ i_1, \ldots, i_k \ge 0}} (1+m)^{-i_1} \prod_{j_1=2}^{i_1+1} \left(1 + \frac{s_{j_1}}{1+m}\right)^{-1}$$

$$\times m^{-i_2} \prod_{j_2=i_1+2}^{i_1+i_2+1} \left(1 + \frac{s_{j_2}}{(1+m)-1}\right)^{-1} \cdots$$

$$\times ((1+m) - (k-1))^{-i_k} \prod_{j_k=i_1+\cdots+i_{k-1}+2}^{n} \left(1 + \frac{s_{j_k}}{(1+m)-(k-1)}\right)^{-1}, \tag{12}$$

$$s_i \ge 0,\ i = 1, \ldots, n.$$

On the other hand, it can be shown that (12) is also the joint LST of $(\tau_1 - \tau_0, \ldots, \tau_n - \tau_{n-1})$ (the renewal times of M_t), by conditioning on $(\theta_1, \ldots, \theta_n)$ and using $[\pi_{ij}]$ and the law of total probability. Therefore $(T_1(1) - T_0(1), \ldots, T_n(1) - T_{n-1}(1)) = (U_1(1), \ldots, U_n(1)) \stackrel{d}{=} (\tau_1 - \tau_0, \ldots, \tau_n - \tau_{n-1})$ for all $n \ge 1$, and hence $(N_t(1), t \ge 0) \stackrel{d}{=} (M_t, t \ge 0)$. $\square$

Proof of Theorem 2. We compute the joint characteristic function (ch.f.) of $(\log U_1(1), \ldots, \log U_n(1))$, first for $\lambda = 1$. Setting $z_j = \imath t_j$, where $\imath := \sqrt{-1}$ and $t_j \in \mathscr{R}$, $j = 1, \ldots, n$, this is

$$E\mathrm{e}^{z_1 \log U_1(1) + \cdots + z_n \log U_n(1)}$$

$$= E\mathrm{e}^{z_1 \log U_1} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \prod_{i=2}^{n} E \exp\left\{ z_i \log\left(\sum_{j=a_{i-1}+1}^{a_i} U_j \right)\right\}$$

$$= E U_1^{z_1} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \prod_{i=2}^{n} E\left(\sum_{j=a_{i-1}+1}^{a_i} U_j \right)^{z_i}. \tag{13}$$

Now it is easy to show that $\sum_{j=a_{i-1}+1}^{a_i} U_i \overset{\mathrm{d}}{=} W_i^{1/\alpha} Y_i$, where $W_i \sim \Gamma(a_i - a_{i-1}, 1)$, $Y_i \sim$ positive α-stable, and W_i and Y_i are independent. Hence

$$E\left(\sum_{j=a_{i-1}+1}^{a_i} U_j \right)^{z_i} = E(W_i^{1/\alpha} Y_i)^{z_i} = E W_i^{z_i/\alpha} E Y_i^{z_i}.$$

So (13) becomes

$$E U_1^{z_1} \prod_{i=2}^{n} E Y_i^{z_i} \sum_{1 < a_2 < \cdots < a_n} \frac{1}{(a_2 - 1)\cdots(a_n - 1)a_n} \prod_{i=2}^{n} E W_i^{z_i/\alpha}. \tag{14}$$

But

$$E W_i^{z_i/\alpha} = \int_0^\infty w^{z_i/\alpha} \frac{w^{a_i - a_{i-1} - 1} \mathrm{e}^{-w}}{\Gamma(a_i - a_{i-1})} \mathrm{d}w = \frac{\Gamma(a_i - a_{i-1} + z_i/\alpha)}{\Gamma(a_i - a_{i-1})},$$

so setting $b_i = a_i - 1$, (14) is

$$E U_1^{z_1} \prod_{i=2}^{n} E Y_i^{z_i} \sum_{1 \le b_2 < \cdots < b_n} \frac{1}{b_2 \cdots b_n (b_n + 1)} \prod_{i=2}^{n} \frac{\Gamma(b_i - b_{i-1} + z_i/\alpha)}{\Gamma(b_i - b_{i-1})}$$

$$= E U_1^{z_1} \prod_{i=2}^{n} E Y_i^{z_i} \Gamma(z_i/\alpha + 1) \left(1 - \sum_{j=i}^{n} z_j/\alpha \right)^{-1}, \tag{15}$$

where the last equality is a consequence of the proof of Theorem 7.1, Bunge and Nagaraja (1992a, p. 37). It is readily shown that (15) is the joint ch.f. of $\{\log(U_i \mathrm{e}^{R_{i-1/\alpha}})\}_{i=1}^n$, so $\{U_i(1)\}_{i=1}^n \overset{\mathrm{d}}{=} \{U_i \mathrm{e}^{R_{i-1/\alpha}}\}_{i=1}^n$, and multiplication by $1/\lambda$ completes the proof. □

Proof of Proposition 3. It suffices to show that

$$U_1 + \cdots + U_N \overset{\mathrm{d}}{=} U \mathrm{e}^{R/\alpha},$$

if and only if $U \sim$ Mittag–Leffler(α, λ), where $\{U_j\}_{j \ge 1}$ are i.i.d. nonnegative r.v.'s, N is independent of $\{U_j\}_{j \ge 1}$ with $P\{N = n\} = 1/(n(n+1))$, $U \overset{\mathrm{d}}{=} U_1$, and $R \sim \exp(1)$ independent of U. The proof of this is a simple extension of the proof of Theorem 7.3 in Bunge and Nagaraja (1992a) ($1/\alpha$ plays the role of a scale factor). □

Proof of Proposition 4. "If" follows by a direct computation. So assume that (5) holds. First, taking $n = 1$ in (5), we have $X_1^{1/\alpha} Y_1 \stackrel{d}{=} X_1^{1/\alpha} Y_0$, or

$$\frac{1}{\alpha}\log X_1 + \log Y_1 \stackrel{d}{=} \frac{1}{\alpha}\log X_1 + \log Y_0.$$

Then

$$\forall t \in \mathscr{R}, \quad \phi_{\log X_1/\alpha}(t)\,\phi_{\log Y_1}(t) = \phi_{\log X_1/\alpha}(t)\,\phi_{\log Y_0}(t),$$

where ϕ_{rv} denotes the ch.f. of rv. But $\log X_1$ is infinitely divisible (Steutel, 1973, p.131), so $\phi_{\log X_1/\alpha}(t) \neq 0$ $\forall t \in \mathscr{R}$ and hence $\phi_{\log Y_1}(t) = \phi_{\log Y_0}(t)$ $\forall t \in \mathscr{R}$, i.e., $Y_1 \stackrel{d}{=} Y_0$.

Next, it is required to show that the LST of Y_1 is $\mathrm{e}^{-(\lambda s)^\alpha}$ for some $\lambda > 0$. Taking the LST of both sides of (5), with $f(s) := E\mathrm{e}^{-sY_1}$, we have

$$\left(\int_0^\infty f(sr^{1/\alpha})\mathrm{e}^{-r}\,\mathrm{d}r\right)^n = \int_0^\infty f(sw^{1/\alpha})\frac{w^{n-1}\mathrm{e}^{-w}}{\Gamma(n)}\,\mathrm{d}w. \tag{16}$$

Let $c(s) = \int_0^\infty f(sr^{1/\alpha})\mathrm{e}^{-r}\,\mathrm{d}r$. Then (16) implies that

$$\Gamma(n) = \int_0^\infty w^{n-1} f(sw^{1/\alpha})\frac{\mathrm{e}^{-w}}{c^n(s)}\,\mathrm{d}w,$$

or

$$(n-1)! = \int_0^\infty w^{n-1} f(s(c(s)w)^{1/\alpha})\mathrm{e}^{-c(s)w}\,\mathrm{d}w = \int_0^\infty w^{n-1} p(w; s)\,\mathrm{d}w,$$

where $p(w; s)$ is a probability density function in w with parameter $s \geq 0$. But the exponential distribution is uniquely determined by the sequence of its moments, so

$$p(w; s) = f(s(c(s)w)^{1/\alpha})\mathrm{e}^{-c(s)w} = \mathrm{e}^{-w}$$

for almost every w (with respect to Lebesgue measure). Hence by the continuity of both sides

$$f(s(c(s)w)^{1/\alpha})\mathrm{e}^{-c(s)w} = \mathrm{e}^{-w}$$

for every $w \geq 0$. Since f is an LST it has a (continuous) inverse $f^{\leftarrow}$, and we have

$$s(c(s))^{1/\alpha} = w^{-1/\alpha} f^{\leftarrow}(\mathrm{e}^{(c(s)-1)w})$$

for all $w > 0$ and $s \geq 0$. Therefore

$$\lim_{s\to\infty} s(c(s))^{1/\alpha} = \lim_{s\to\infty} w^{-1/\alpha} f^{\leftarrow}(\mathrm{e}^{(c(s)-1)w}) = w^{-1/\alpha} f^{\leftarrow}(\mathrm{e}^{-w})$$

for all $w > 0$, and hence $w^{-1/\alpha} f^{\leftarrow}(\mathrm{e}^{-w}) = \lambda^*$ for some $\lambda^* > 0$, that is, $\mathrm{e}^{-w} = f(\lambda^* w^{1/\alpha})$ or $f(s) = \mathrm{e}^{-(\lambda s)^\alpha}$, $s \geq 0$. □

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