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Tails of passage-times and an application to stochastic processes with boundary reflection in wedges

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Abstract

In this paper we obtain lower bounds for the tails of the distributions of the first passage-times for some stochastic processes. We consider first discrete parameter processes with asymptotically small drifts taking values in $\mathbb{R}_+$ and prove for them a general result giving lower bounds for these tails. As an application of the obtained results, we obtain lower bounds for the tails of the distributions of the first passage-times for reflected random walks in a quadrant with zero-drift in the interior. The latter bounds are then used to get explicit conditions for the finiteness or not of the moments of the first passage-time to the origin for a Brownian motion with oblique reflection in a wedge.

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1. Introduction

This paper deals with lower bounds for the tails of the distributions of the first hitting times of compact sets (called simply first passage-times) for one class of one- and two-dimensional stochastic processes with a discrete parameter.

An extensive literature exists on first passage-times for general irreducible countable Markov chains. They are known to be particularly important for the recurrence classification due to the fact that generally under moment conditions on their one-step transition probabilities the tails of the first passage-times have the same asymptotic behavior as the tails of the first return time to a given state. Many of the papers deal with the so-called geometric ergodic Markov chains when the first return and passage times has exponential moments and the rate of convergence to the stationary distribution is exponential (we invite the reader to consult Meyn and Tweedie, 1993; Nummelin, 1984; Fayolle et al., 1995). Another class of well-studied processes is one-dimensional

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Markov processes where one can obtain thorough results on first passage-times, like, for example, in the case of birth and death processes. On the other hand, there are only few results available in the case of multidimensional Markov processes with subgeometric ergodicity when the first return times do not have exponential moments and the rate of convergence is subgeometric. We can mention here the papers by Nummelin and Tuominnen (1983) and Nummelin and Tweedie (1994). Such processes have got recently a rising interest because of their links with various applications, like for instance, statistical physics, the queuing theory, etc. Moreover, the existing results deal only with upper bounds on the tails of the first return times and do not provide one with lower bounds which are especially of interest in the case of Markov processes with subgeometric ergodicity because of their intimate connections with lower bounds on the rate of convergence of the transition probabilities to the invariant distribution.

One of the natural approaches to the first passage-time problem for multidimensional Markov chains consists in reducing it to the one-dimensional one and then to get some information about "transformed" first passage-times. Obviously, by doing so we simplify the state space but bring a new difficulty: the obtained one-dimensional process need not be Markovian. For one-dimensional non-Markov processes with discrete parameter, Lamperti (1963) has obtained general conditions on existence or non-existence of their first passage-time moments of the integer order greater or equal to one. In Aspandiiarov et al. (1994), the criteria of Lamperti were extended to cover the case of the moments of arbitrary order. We also weaken the hypotheses of Lamperti. However, again these results are not entirely satisfactory because they only provide us with upper bounds on the tails of the first passage-times. In the present paper we complete them by proving general conditions formulated in submartingale terms for lower bounds for the tails of the first passage-times for positive stochastic processes that need not be Markov. The main results are given in Theorems 1 and 1′. As a consequence, in Corollary 2 we immediately get sufficient conditions of non-integrability of functions of the first passage-times. It should be mentioned that although in this study we were primarily motivated by discrete-time applications, the recent paper of Menshikov and Williams (1994) on continuous-time analogues of the results in Aspandiiarov et al. (1994) strongly indicates that Theorems 1 and 1′ and Corollary 1 can also be extended to the continuous-time setting.

In Section 3 we illustrate the obtained results on two-dimensional driftless reflected random walks in a quadrant. We get lower bounds for the tails of their first passage-times (Theorem 3) and prove the non-integrability of functions f of the first passage-times of the type $f(x) = x^{\alpha/2} \log^{-1} x \ldots \log_k^{-1} x$ (resp. $f(x) = \log x \log_2^{-1} x \ldots \log_{k+1}^{-1} x$), in the case $\alpha > 0$ (resp. $\alpha = 0$) for any $k \geqslant 1$ (α is one parameter depending on the geometrical data corresponding to the reflected random walk). This gives a negative answer to the question of existence of the moment of the critical order $\alpha/2$ raised in Aspandiiarov et al. (1994) where it has been proved that pth moment of the first passage-times is finite (resp. infinite) if $p < \alpha/2$ (resp. $p > \alpha/2$). Besides, it refutes one conjecture in Fayolle et al. (1995) on ergodicity of the reflected random walks in the case $\alpha = 2$. In a forthcoming paper (Aspandiiarov and Iasnogorodski, 1994) we will show that the estimates of Section 3 give in fact fairly sharp bounds for tails of the distributions of the first passage-times (for instance, in the case $\alpha = 0$, the functions $f(x) = \log x \log_2^{-1} x \dots \log_k^{-1} x \log_{k+1}^{-(1+\chi)} x$ will be integrable for any $\chi > 0$ and $k \geqslant 1$).

The second class of two-dimensional processes treated in the paper is Brownian motion with oblique reflection in a wedge constructed in Varadhan and Williams (1985). We get a sufficient condition for existence and non-existence of the moments of the order $p > 0$ of its first passage-time to the origin (Theorem 5). The criterion is an analogue to the classical one of Spitzer for the first exit times from a wedge by planar Brownian motion and in very particular case $p = 1$ has been proved earlier by Varadhan and Williams (1985). Its proof is based on the lower bounds for the reflected random walks, the results of Aspandiiarov (1995) on the approximation of Brownian motions with oblique reflection and the results on the existence of the first passage-time moments in Aspandiiarov et al. (1994) and suggests another way of using Theorem 1. Let us finally mention that using a different approach based on the continuous-time analogues of the results in Aspandiiarov et al. (1994), Menshikov and Williams (1994) gave a direct proof of Theorem 5 in all except the critical case $p = \alpha/2$. Moreover, they are able to prove finiteness of the pth moments in the case $0 < p < \alpha/2 < 1$ which is not covered by our results.

2. Non-negative stochastic processes with asymptotically small drift

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_n\}_{n\geqslant 0}$. Let x be a positive number and let $\{X_n,\ n\geqslant 0\}$ be a discrete-time $\{\mathscr{F}_n\}$-adapted non-negative stochastic process such that $X_0 = x$. For each $A \geqslant 0$, we will denote by τ_A the following first passage-time into the interval $[0, A]$:

$$\tau_A \equiv \tau^X_{x,A} = \inf\{n \geqslant 0;\ X_n \leqslant A\}.$$

(as usual $\inf \emptyset = +\infty$.) For each $B \geqslant 0$, let

$$\tilde{\tau}_B = \inf\{n \geqslant 0;\ X_n \geqslant B\}.$$

We recall a result proved in Aspandiiarov et al. (1994) (Lemma 2) which plays a key role in the investigation of the non-existence of means of functions of the passage-times τ_A.

Lemma 1. *Suppose there exist positive constants A, C and D such that for any $n \geqslant 0$,*

$$E(X_{n+1}^2 - X_n^2 \mid \mathscr{F}_n) \geqslant -C \quad \text{on } \{\tau_A > n\} \tag{1}$$

and, for some $r > 1$,

$$E(X_{n+1}^{2r} - X_n^{2r} \mid \mathscr{F}_n) \leqslant D X_n^{2r-2} \quad \text{on } \{\tau_A > n\}. \tag{2}$$

Then, for any $v \in (0, 1)$, there exist positive ε and δ that do not depend on A such that for any n:

$$P(\tau_A > n + \varepsilon X^2_{n\wedge\tau_A} \mid \mathscr{F}_n) \geqslant 1 - v \quad \text{on } \{X_{n\wedge\tau_A} > A(1+\delta)\}. \tag{3}$$

Remark 1. Let us now explain why we call the processes $\{X_n,\ n\geqslant 0\}$ the processes with asymptotically small drift. Suppose that in addition to the conditions of Lemma 1 there exists a positive constant c such that for any $n\geqslant 0$

$$E((X_{n+1}-X_n)^2 \mid \mathscr{F}_n)\leqslant c \quad \text{on } \{\tau_A>n\}. \tag{1'}$$

As easy then to see, there exists a positive constant c_1 such that

$$|E(X_{n+1}-X_n \mid \mathscr{F}_n)|\leqslant \frac{c_1}{X_n} \quad \text{on } \{\tau_A>n\}. \tag{4}$$

In fact, it suffices to apply (2) and the inequality $x^{2r}-y^{2r}\geqslant 2ry^{2r-1}(x-y)$, for $x,y>0$ (valid since $f(x)=x^{2r}$ is convex) to get the upper bound in (4), whereas the lower one follows from (1) and (1′).

The main result of this section is the following lower bound for tails of τ_A.

Theorem 1. *Let $\{X_n,\ n\geqslant 0\}$ be an $\{\mathscr{F}_n\}$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$ and satisfying the conditions of Lemma 1 with some positive constants r,A,C and D. Suppose that $X_0=x>A$ and τ_A is finite with probability 1. Suppose also there exist a positive constant $B_0>A$ and a $\{\mathscr{F}_n\}$-adapted process $\{U_n,\ n\geqslant 0\}$ with the following two properties:*

1. *The process $\{U_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ is an uniformly integrable submartingale for all $B\geqslant B_0$;*
2. *There exist functions $G:\mathbb{R}_+\to\mathbb{R}_+$ and $H:\mathbb{R}_+\to\mathbb{R}_+$ such that for all $B\geqslant B_0$:*

$$U_{\tau_A\wedge\tilde{\tau}_B}\leqslant\begin{cases} G(B) & \text{on } \{\tau_A>\tilde{\tau}_B\},\\ H(A) & \text{on } \{\tau_A\leqslant\tilde{\tau}_B\}.\end{cases}$$

Then, for any $v\in(0,1)$ there exist positive constants $c_1=c_1(v,r,C,D)$ and $s_0=s_0(v,r,A,B_0,C,D)$ such that for $s>s_0$:

$$P(\tau_A\geqslant s)\geqslant\frac{(1-v)(U_0-H(A))}{G(c_1\sqrt{s})}. \tag{5}$$

Remark 2. As can be seen from the proof the quantities c_1 and s_0 can be defined by:

$$c_1=1/\sqrt{\varepsilon},\qquad s_0=\varepsilon\max(A^2(1+\delta)^2,B_0^2),$$

with any positive constants $\varepsilon=\varepsilon(v,r,C,D)$ and $\delta=\delta(v,r,C,D)$ satisfying the inequality (3) in Lemma 1.

Remark 3. Even though the process X does not appear clearly in the formulation of Theorem 1, it influences the behavior of U through the random times τ_A, $\tilde{\tau}_B$.

We first give one immediate consequence of Theorem 1 whose intuitive meaning is that if X is a process with asymptotically small drifts and if $\{F(X_n),\ n\geqslant 0\}$ is a submartingale with some positive function F, then lower bounds for the tails of the first passage-time are at least of the asymptotic order $O(1/F(\sqrt{s}))$ as $s \to \infty$. More specifically, we have:

Corollary 1. *Let $\{X_n,\ n \geqslant 0\}$ be a process satisfying the hypotheses of the last theorem. Suppose also that there exists a positive function F increasing on some interval $[\overline{B}, \infty)$ such that the process $\{F(X_{n\wedge\tau_A}),\ n \geqslant 0\}$ is a submartingale. If the process $\{X_n,\ n \geqslant 0\}$ satisfies the condition $|X_{\tilde{\tau}_B} - X_{\tilde{\tau}_B - 1}| \leqslant KB$, for all large enough B and for some positive constant K, then for any $v \in (0,1)$ there exist positive constants $c_1 = c_1(v, r, \overline{B}, C, D, K)$ and $s_0 = s_0(v, r, A, \overline{B}, C, D)$ such that whenever $X_0 = x$ satisfies $x > A \vee \overline{B}$ we have for $s > s_0$:*

$$P(\tau_A \geqslant s) \geqslant \frac{(1-v)(F(x) - F(A))}{F(c_1\sqrt{s})}. \tag{6}$$

Proof. It suffices to set $U_n = F(X_n)$, $G(x) = F(x(1+K))$, $H(x) = F(x)$. $\square$

Let us now discuss the meaning of Theorem 1 and the method of using it for multidimensional Markov processes. Suppose we are given a multidimensional Markov process $\{Z_n,\ n \geqslant 0\}$ and we would like to find lower bounds for the tails of its first passage-times. The algorithm based on Theorem 1 is very natural and is as follows. We first try to find a positive not necessarily one-to-one function f defined on the original state space such that the image-process $\{X_n,\ n \geqslant 0\}$ defined by $X = f(Z)$ satisfies the conditions of Lemma 1. The important restriction on the choice of f is that the knowledge of the asymptotic behavior of tails of the first passage-times for X should provide the adequate information for the original process Z. Once such f is found, the next step consists in constructing a submartingale $\{U_n,\ n \geqslant 0\}$ satisfying the conditions of Theorem 1. As a rule, we look for U among $U = g(Z)$, where g is another positive mapping of the original state space. Such functions g are sometimes called Lyapunov functions. It is important to mention here that generally the processes X and U can be related to each other only implicitly. Finally, applying Theorem 1 we get lower bounds for the tails of the "transformed" positive processes which in turn give us the desired lower bounds for the process $\{Z_n,\ n \geqslant 0\}$.

Let us notice here that one hidden difficulty in applying Theorem 1 for multidimensional Markov chains in the method described above is that generally the jumps of the process $U = g(Z)$ are not bounded at the random time τ_B so that the function G from Theorem 1 may take infinite values which makes the lower bound trivial. The following observation and corresponding modification of Theorem 1 might be of use. Notice first that the condition on G deals only with U_{τ_B} on $\{\tau_A < \tau_B\}$. The modification of Theorem 1 given below (Theorem 1′) shows that if we can "truncate" infinite jumps of the process Z only at τ_B without changing the process $\{g(Z_{n\wedge\tau_A\wedge\tau_B}),\ n \geqslant 0\}$ on $\{n \wedge \tau_A < \tau_B\}$ in such a way that the images $U^B = g(Z^{(B)})$ under g of the "truncated" processes are still submartingales, then we still get lower bounds of the first passage-times for the process $X = f(Z)$ and, hence, for the tails of the first passage-times for the original process $\{Z_n,\ n \geqslant 0\}$. An example of this approach will be given in the next section where we treat two-dimensional reflected random walks in a quadrant.

In a similar way based on truncating, one can extend Corollary 1 to the situations where the process $\{X_n,\ n \geqslant 0\}$ does not satisfy the condition $|X_{\tilde{\tau}_B} - X_{\tilde{\tau}_B - 1}| \leqslant KB$.

Theorem 1′. *Let $\{X_n,\ n\geqslant 0\}$ be an $\{\mathscr{F}_n\}$-adapted stochastic process taking values in an unbounded subset of $\mathbb{R}_+$ and satisfying the conditions of Lemma 1 with some positive constants r, A, C and D. Suppose that $X_0 = x > A$ and τ_A is finite with probability 1. Suppose also that for some positive $B_0 > A$ we are given a family of $\{\mathscr{F}_n\}$-adapted processes $\{\{U_n^{(B)},\ n\geqslant 0\},\ B\geqslant B_0\}$ with the following properties:*

1. *The initial values $U_0^{(B)} = U_0$ do not depend on B.*
2. *For all $B\geqslant B_0$, the processes $\{U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ are uniformly integrable submartingales;*
3. *There exist functions $G : \mathbb{R}_+ \to \mathbb{R}_+$ and $H : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $B\geqslant B_0$:*

$$U^{(B)}_{\tau_A\wedge\tilde{\tau}_B} \leqslant \begin{cases} G(B) & \text{on } \{\tau_A > \tilde{\tau}_B\}, \\ H(A) & \text{on } \{\tau_A \leqslant \tilde{\tau}_B\}. \end{cases}$$

Then, for any $v \in (0,1)$, (5) holds with $s > s_0$, where the positive constants $c_1 = c_1(v,r,C,D)$ and $s_0 = s_0(v,r,A,B_0,C,D)$ are given in Remark 2.

Proof of Theorems 1 and 1′. We only prove Theorem 1. The proof of Theorem 1′ can be obtained from it by replacing U by $U^{(B)}$ and is omitted. The idea of the proof is inspired by that of Theorem 3.2 in Lamperti (1963). Let v be any fixed number from $(0,1)$. By Lemma 1 there exist positive $\varepsilon = \varepsilon(v,r,C,D)$ and $\delta = \delta(v,r,C,D)$ such that for any n:

$$P(\tau_A > n + \varepsilon X^2_{n\wedge\tau_A} \mid \mathscr{F}_n) \geqslant 1 - v \quad \text{on } \{X_{n\wedge\tau_A} > A(1+\delta)\}.$$

As is easy to see, this implies that for any stopping time μ we have

$$P(\tau_A > \mu + \varepsilon X^2_{\mu\wedge\tau_A} \mid \mathscr{F}_\mu) \geqslant 1 - v \quad \text{on } \{X_{\mu\wedge\tau_A} > A(1+\delta)\} \cap \{\mu < \infty\}.$$

Let us fix any B such that $B > \max(A(1+\delta), B_0)$. We set $\mu = \tilde{\tau}_B$. Then, the last inequality permits us to deduce that

$$\begin{aligned} P(\tau_A \geqslant \varepsilon B^2) &\geqslant P(\tau_A > \tilde{\tau}_B + \varepsilon X^2_{\tilde{\tau}_B\wedge\tau_A},\ \tilde{\tau}_B < \tau_A) \\ &= E(1_{(\tilde{\tau}_B<\tau_A)} P(\tau_A > \tilde{\tau}_B + \varepsilon X^2_{\tilde{\tau}_B\wedge\tau_A} | \mathscr{F}_{\tilde{\tau}_B})) \\ &\geqslant (1-v) P(\tilde{\tau}_B < \tau_A). \end{aligned} \tag{7}$$

A good control of the last probability in (7) is given by the properties of the process $\{U_n^{(B)},\ n\geqslant 0\}$. Namely, from 1 and $(\tau_A \wedge \tilde{\tau}_B) < \infty$ we have

$$U_0 \leqslant E U^{(B)}_{\tau_A\wedge\tilde{\tau}_B}.$$

Hence, from assumption 2,

$$U_0 \leqslant E(U^{(B)}_{\tau_A} 1_{\tau_A\leqslant\tilde{\tau}_B}) + E(U^{(B)}_{\tilde{\tau}_B} 1_{\tau_A>\tilde{\tau}_B}) \leqslant H(A) + G(B) P(\tau_A > \tilde{\tau}_B).$$

and

$$P(\tau_A > \tilde{\tau}_B) \geqslant \frac{U_0 - H(A)}{G(B)}.$$

Putting this estimate into (7), we get that for $B > \max(A(1+\delta), B_0)$,

$$P(\tau_A \geqslant \varepsilon B^2) \geqslant \frac{(1-v)(U_0 - H(A))}{G(B)},$$

This implies that for all $s > \varepsilon \max(A^2(1+\delta)^2, B_0^2)$,

$$P(\tau_A \geqslant s) \geqslant \frac{(1-v)(U_0 - H(A))}{G(\sqrt{s/\varepsilon})},$$

as was to be shown. □

Theorems 1 and 1′ easily give a sufficient condition of the non-integrability of functions of τ_A. To formulate it we need to introduce the following:

Definition 1. Let f be a positive function defined on $[0, \infty)$. It is said to satisfy the condition (R), if there exist positive $a_f > 1$ and $\widetilde{A}_f$ such that $\limsup_{x\to\infty}[f(a_f x)/f(x)] \leqslant \widetilde{A}_f$.

Corollary 2. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be some positive function such that it increases in some neighborhood of ∞, and $f(s) \to +\infty$ as $s \to +\infty$. In conditions of Theorem 1 (or 1′), suppose that G is a positive continuous function satisfying the condition (R) such that*

$$\int^{+\infty} \frac{f'(s)\,\mathrm{d}s}{G(\sqrt{s})} = \infty. \tag{8}$$

Suppose also that whenever $X_0 = x > A$, we have $U_0 > H(A)$. Then whenever $X_0 = x > A$, $E(f(\tau_A))$ is infinite.

Proof. We see that since G satisfies the condition (R) and G is a positive continuous function, then there exist positive constants $a_G > 1$, $\widetilde{\widetilde{A}}_G$ such that for all $x \geqslant 1$, $G(a_G x)/G(x) \leqslant \widetilde{\widetilde{A}}_G$. By Lemma 1 there exist positive constants ε_0, δ_0 such that (3) holds with $\varepsilon = \varepsilon_0$, $\delta = \delta_0$. Obviously, (3) will also hold with any $\varepsilon \leqslant \varepsilon_0 \wedge 1$, $\delta = \delta_0$. Let n_0 be any positive integer such that $a_G^{-2n_0} \leqslant \varepsilon_0$ and let $\varepsilon_1 = a_G^{-2n_0}$. Then, by Theorem 1′ and Remark 2, (5) holds with

$$c_1 = 1/\sqrt{\varepsilon_1}, \quad s_0 = \varepsilon_1 \max(A^2(1+\delta_0)^2, B_0^2).$$

Using the condition (R), we have that for all $s \geqslant 1$, $G(c_1\sqrt{s})/G(\sqrt{s}) \leqslant \widetilde{\widetilde{A}}_G^{n_0}$. Now the assertion of the corollary follows from the following observations:

$$P(f(\tau_A) \geqslant f(s)) = P(\tau_A \geqslant s) \text{ for all sufficiently large } s \quad \text{and}$$
$$Ef(\tau_A) \text{ is infinite} \quad \text{iff} \int^{\infty} P(f(\tau_A) \geqslant f(s))\, df(s) \quad \text{is infinite.} \qquad \square$$

3. Two-dimensional driftless Markov chains in wedges with boundary reflection

3.1. Notation and statement of results

In the sequel $\widetilde{G}$ is the quadrant given by $\widetilde{G} = \{(x,y) \in \mathbb{R}^2;\ x \geqslant 0,\ y \geqslant 0\}$. The two sides of the quadrant are denoted by $\partial\widetilde{G}_1$ and $\partial\widetilde{G}_2$, where $\partial\widetilde{G}_1 = \{(x,y) \in \widetilde{G};\ x \neq 0, y = 0\}$ and $\partial\widetilde{G}_2 = \{(x,y) \in \widetilde{G};\ y \neq 0,\ x = 0\}$. The interior of $\widetilde{G}$ is referred to as $\widetilde{G}^0$. Next, for any set $F \subseteq \mathbb{R}^d$ and $m \in 0,1,2,\dots$ we denote by $C^m(F)$ the set of real-valued functions that are m times continuously differentiable in some open set containing F. $C_b^m(F)$ denotes the class of functions in $C^m(F)$ which together with their first m derivatives are bounded on F.

Let $(\Omega, \mathscr{F}, P)$ be a probability space equipped with a filtration $\{\mathscr{F}_n\}_{n\geqslant 0}$. We are dealing with a discrete-time homogeneous irreducible aperiodic $\{\mathscr{F}_n\}$-adapted Markov chain $\{\widetilde{Z}_n,\ n \geqslant 0\}$ defined on (Ω, F, P), with values in $\mathbb{Z}_+^2$. Its transition mechanism is given as follows. The Markov chain starting from the point $\widetilde{z} = (x,y)$ of $\mathbb{Z}_+^2$ jumps to the point $(x+i, y+j)$, $i,j \in Z$, $i,j \geqslant -1$ with the probability $p_{i,j}^0$, $i,j \in \mathbb{Z}$, $i,j \geqslant -1$ (respectively $p_{i,j}^1$, $p_{i,j}^2, p_{i,j}^3$) according as $(x,y) \in \widetilde{G}^0$ (respectively $\partial\widetilde{G}^1$, $\partial\widetilde{G}^2$, $\partial\widetilde{G}^3 \equiv (0,0)$). Regarding the transition probabilities we assume the following moment conditions:

1. For any i,j $p_{i,-1}^1 = p_{-1,j}^2 = p_{i,-1}^3 = p_{-1,j}^3 = 0$ (this simply means that our chain cannot jump out of the quadrant) and:

$$\gamma \equiv \sup\left\{\kappa;\ \sum_{i,j}(|i|^\kappa + |j|^\kappa)p_{i,j}^l < \infty, \quad \forall l = 0,1,2\right\} > 2. \tag{9}$$

2. In the interior $\widetilde{G}^0$.

We assume that the Markov chain has one-step zero mean drift, i.e.

$$\sum_{i,j} i p_{i,j}^0 = \sum_{i,j} j p_{i,j}^0 = 0. \tag{10}$$

We also assume that the covariance matrix of the one-step jump distribution is non-degenerate, i.e. $\widetilde{\lambda}_x^0\widetilde{\lambda}_y^0 - (\widetilde{R}^0)^2 > 0$, where $\widetilde{\lambda}_x^0, \widetilde{\lambda}_y^0$ and $\widetilde{R}^0$ by

$$\widetilde{\lambda}_x^0 = \sum_{i,j} i^2 p_{i,j}^0, \qquad \widetilde{R}^0 = \sum_{i,j} ij p_{i,j}^0, \qquad \widetilde{\lambda}_y^0 = \sum_{i,j} j^2 p_{i,j}^0. \tag{11}$$

Geometrically, this condition simply means that the Markov chain cannot jump only along a fixed straight line.

3. On the boundary $\partial\widetilde{G}$.

Let $\widetilde{p}_1 = \sum_{i,j} i p_{i,j}^1$, $\widetilde{p}_2 = \sum_{i,j} j p_{i,j}^1$, $\widetilde{q}_1 = \sum_{i,j} i p_{i,j}^2$, $\widetilde{q}_2 = \sum_{i,j} j p_{i,j}^2$ and let the vectors of boundary reflection be defined by $\boldsymbol{P}' = (\widetilde{p}_1, \widetilde{p}_2)$, $\boldsymbol{Q}' = (\widetilde{q}_1, \widetilde{q}_2)$. We suppose that the boundary reflection field is non-degenerate in the following sense: $\widetilde{p}_2 \neq 0$, $\widetilde{q}_1 \neq 0$.

Definition 2. Let F be any Borel subset of $\mathbb{R}^2$. For any $\tilde{z} \in \mathbb{Z}_+^2$, the first passage-time in F of the Markov chain $\{\tilde{Z}_n,\ n \geqslant 0\}$ with $\tilde{Z}_0 = \tilde{z}$ is defined by

$$\tilde{T}_{\{F\}} \equiv \tilde{T}^{\tilde{Z}}_{z,\{F\}} = \inf\{n \geqslant 0;\ \tilde{Z}_n \in F\}.$$

In particular, if $F_A = \{z;\ |z| \leqslant A\}$, we denote by $\tilde{T}_A$:

$$\tilde{T}_A = \tilde{T}_{\{F_A\}} = \inf\{n \geqslant 0;\ |\tilde{Z}_n| \leqslant A\}.$$

In Aspandiiarov et al. (1994) it has been established that the existence of a parameter $\alpha = \alpha(\tilde{\lambda}_x^0, \tilde{\lambda}_y^0, \tilde{R}^0, \tilde{p}_1/\tilde{p}_2, \tilde{q}_2/\tilde{q}_1)$ with exact value will be given in (18) such that the following result holds:

Theorem 2. (I) *If $\alpha < 0$, then the Markov chain $\{\tilde{Z}_n,\ n \geqslant 0\}$ is transient. If $\alpha > 0$, then the chain is recurrent.*
(II) *Suppose that $0 < \alpha < \gamma$. Then the following statements hold:*
1. *For any $p < \alpha/2$, there exists $\tilde{A}_0 > 0$ such that whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{A}_0$,*

$$E(\tilde{T}^p_{\tilde{A}_0}) \quad \textit{is finite.}$$

Furthermore, if $\alpha > 2$, then for any $p < \alpha/2$ there exist positive $\tilde{A}_0, \tilde{c}_0$ such that for any $\tilde{A} \geqslant \tilde{A}_0$ whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{A}$,

$$E(\tilde{T}^p_{\tilde{A}}) \leqslant \tilde{c}_0 |\tilde{z}|^{2p}. \tag{12}$$

2. *For any $p > \alpha/2$ there exist positive $\tilde{A}_0, \tilde{C}_0$ such that for any $\tilde{A} \geqslant \tilde{A}_0$ whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{C}_0 \tilde{A}$,*

$$E(\tilde{T}^p_{\tilde{A}}) \quad \textit{is infinite.}$$

Let us define the functions $\mathrm{Log}_k\, x$ by

$$\mathrm{Log}_0\, x \equiv 1, \quad \mathrm{Log}_k\, x = \log_k (x + c_k), \ \ k \geqslant 1,$$

where the constants c_k are chosen in such a way that $\mathrm{Log}_k\, 0 = 1$. The principal results of this section are stated as follows.

Theorem 3. *Let $\{\tilde{Z}_n,\ n \geqslant 0\}$ be the Markov chain defined above. Suppose that $0 \leqslant \alpha < \gamma$. Then there exist positive constants $\tilde{A}_1$, $\tilde{C}_1 > 1$, $\tilde{c}_1$ and $\tilde{s}_0$ such that for any $\tilde{A} \geqslant \tilde{A}_1$ whenever $\tilde{Z}_0 = \tilde{z} \in \mathbb{Z}_+^2$ satisfies $|\tilde{z}| > \tilde{C}_1 \tilde{A}$, the following bound holds for all $s > \tilde{s}_0 \tilde{A}^2$:*

$$P(\tilde{T}_{\tilde{A}} \geqslant s) \geqslant \begin{cases} \tilde{c}_1 \tilde{A}^{\alpha} s^{-\frac{\alpha}{2}} & \textit{if } \alpha > 0, \\ \dfrac{\tilde{\tilde{c}}_1}{\log(s)} & \textit{if } \alpha = 0. \end{cases} \tag{13}$$

In particular, if $\alpha > 0$ (resp. $\alpha = 0$), then for any $k \geqslant 1$,

$$E(\tilde{T}^{\frac{\alpha}{2}}_{\tilde{A}}\ \mathrm{Log}_0^{-1}(\tilde{T}_{\tilde{A}}) \ldots \mathrm{Log}_k^{-1}(\tilde{T}_{\tilde{A}})) \quad \textit{is infinite,}$$
$$(\textit{resp.} \quad E(\mathrm{Log}\,(\tilde{T}_{\tilde{A}})\,\mathrm{Log}_2^{-1}(\tilde{T}_{\tilde{A}}) \ldots \mathrm{Log}_{k+1}^{-1}(\tilde{T}_{\tilde{A}})) \quad \textit{is infinite}).$$

Remark 3. This result shows that the moment of the critical order $\alpha/2$ does not exist which answers the open question in [AIM]. Furthermore, it disproves one conjecture in Fayolle et al. (1995) (Section 4.3) stating that in the case $\alpha = 2$ the Markov chain $\{\widetilde{Z}_n,\ n \geqslant 0\}$ can still be ergodic provided that the jumps are uniformly bounded by a constant and some third-order moment conditions of the transition mechanism are imposed.

Remark 4. We will also give a simple proof of the following result proved in Asymont et al. (1994) (Theorem 1): $\{\widetilde{Z}_n,\ n \geqslant 0\}$ is recurrent in the case $\alpha = 0$.

Remark 5. In the forthcoming paper (Aspandiiarov and Iasnogorodski, 1994) we will complete the results of Theorems 2 and 3 by proving the following assertions:

1. Suppose that $0 \leqslant \alpha < \gamma$. Then for any positive δ there exist positive constants $\widetilde{A}_2$, $\widetilde{\widetilde{c}}_2$ and $\widetilde{s}_1$ such that for any $\widetilde{A} \geqslant \widetilde{A}_2$ whenever $\widetilde{Z}_0 = \widetilde{z} \in \mathbb{Z}_+^2$ satisfies $|\widetilde{z}| > \widetilde{A}$, the following bound holds for all $s > \widetilde{s}_1$,

$$P(\widetilde{T}_{\widetilde{A}} \geqslant s) \leqslant \begin{cases} \widetilde{\widetilde{c}}_2 \dfrac{\log^{(1 \vee \alpha/2)+\delta}(s)}{s^{\frac{\alpha}{2}}} & \text{if } \alpha > 0, \\ \dfrac{\widetilde{c}_1 \log_2^{1+\delta}(s)}{\log(s)} & \text{if } \alpha = 0. \end{cases} \tag{14}$$

2. If $2 < \alpha < \gamma$ (resp. $0 \leqslant \alpha \leqslant 2$), then for any $q > \alpha/2$ (resp. $q > 1$) and for any integer $k \geqslant 1$, there exists a positive constant $\widetilde{A}_2$ such that for any $\widetilde{A} \geqslant \widetilde{A}_2$ whenever $\widetilde{Z}_0 = \widetilde{z} \in \mathbb{Z}_+^2$ satisfies $|\widetilde{z}| > \widetilde{A}$,

$$E(\widetilde{T}_{\widetilde{A}}^{\frac{\alpha}{2}} \operatorname{Log}^{-q}(\widetilde{T}_{\widetilde{A}})) \quad \text{is finite in the case } \alpha > 2,$$
$$E(\widetilde{T}_{\widetilde{A}}^{\frac{\alpha}{2}} \operatorname{Log}_1^{-1}(\widetilde{T}_{\widetilde{A}}) \ldots \operatorname{Log}_{k-1}^{-1}(\widetilde{T}_{\widetilde{A}}) \operatorname{Log}_k^{-q}(\widetilde{T}_{\widetilde{A}})) \quad \text{is finite in the case } 0 < \alpha \leqslant 2,$$
$$E(\operatorname{Log}(\widetilde{T}_{\widetilde{A}}) \operatorname{Log}_2^{-1}(\widetilde{T}_{\widetilde{A}}) \ldots \operatorname{Log}_k^{-1}(\widetilde{T}_{\widetilde{A}}) \operatorname{Log}_{k+1}^{-q}(\widetilde{T}_{\widetilde{A}})) \quad \text{is finite if } \alpha = 0.$$

The proof of these results goes along the same lines as those of Theorems 3 and 4 in Aspandiiarov et al. (1994). Namely, we first introduce the "transformed" Markov chain $\{Z_n,\ n \geqslant 0\}$ in a new wedge G as the image of the Markov chain $\{\widetilde{Z}_n,\ n \geqslant 0\}$ under some linear isomorphism Φ of $\mathbb{R}^2$. Then using the results of Section 2 we establish the analogues of our main results for $\{Z_n,\ n \geqslant 0\}$. This will almost immediately give the desired results for the original Markov chain $\{\widetilde{Z}_n,\ n \geqslant 0\}$.

3.2. *"Transformed" setting and some technical results*

As a preliminary step in the proofs of our main results we recall the construction of "transformed" Markov chains given in Aspandiiarov et al. (1994). Let us consider the linear isomorphism Φ of $\mathbb{R}^2$, defined by

$$\begin{aligned} u &= (bx - \bar{r}y)a, \\ v &= \sqrt{1 - \bar{r}^2}\, ya, \end{aligned} \tag{15}$$

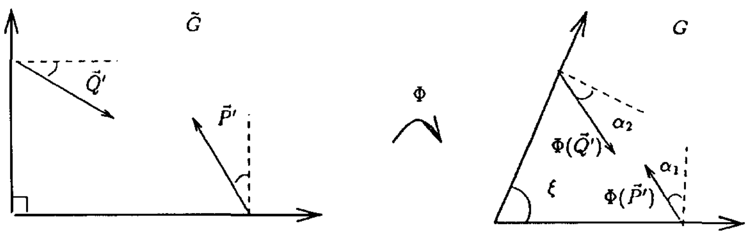

Fig. 1.

where

$$b = \sqrt{\frac{\widetilde{\lambda}_y^0}{\widetilde{\lambda}_x^0}}, \qquad \bar{r} = \frac{\widetilde{R}^0}{\sqrt{\widetilde{\lambda}_x^0 \widetilde{\lambda}_y^0}}, \qquad a = \frac{1}{\sqrt{\widetilde{\lambda}_y^0 (1 - \bar{r}^2)}}. \tag{16}$$

As it is easy to see, this transformation has the following property: the image of the covariance matrix A^0 under Φ is unit. Next, the wedge $\widetilde{G}$ under Φ is transformed into the wedge G of angle ξ, given in the standard polar coordinates by $G = \{(\rho, \theta),\ \rho \geqslant 0,\ \theta \in [0, \xi]\}$, where with ξ is defined by

$$\xi = \arccos(-\bar{r}), \quad \xi \in (0, \pi). \tag{17}$$

Let α_1, α_2 be the angles that makes the vectors $\Phi(\boldsymbol{P}')$ (resp. $\Phi(\boldsymbol{Q}')$) with the inward normals to the corresponding sides of the wedge G positive angles being toward the corner. It follows from the moment condition (3), that $\alpha_1, \alpha_2 \in (-\pi/2, \pi/2)$ (see Fig. 1).

The parameter α appearing in the statement of the last theorems is simply

$$\alpha = \frac{\alpha_1 + \alpha_2}{\xi}. \tag{18}$$

We now introduce the "transformed" Markov chain $\{Z_n,\ n \geqslant 0\}$. For each $n \geqslant 0$, let us set

$$Z_n = \Phi(\widetilde{Z}_n). \tag{19}$$

The Markov chain $\{Z_n,\ n \geqslant 0\}$ takes values in

$$G_4 = \Phi(\mathbb{Z}_+^2) = \{(u, v) \in R^2;\ u = a(bx - \bar{r}y),\ v = ay\sqrt{1 - \bar{r}^2},\ (x, y) \in \mathbb{Z}_+^2\},$$

and is governed by the following transition mechanism. For all integer $i, j \geqslant -1$ the Markov chain jumps from points $(u, v) \in G_4$ to $(u + a(bi - \bar{r}j),\ v + aj\sqrt{1 - \bar{r}^2})$ with probabilities $p_{i,j}^0$, $i, j \in \mathbb{Z}$, $i, j \geqslant -1$ (respectively $p_{i,j}^1$, $p_{i,j}^2, p_{i,j}^3$) according as $(x, y) \in G^0$ (respectively, ∂G^1, ∂G^2, $\partial G^3 \equiv (0, 0)$), where

$$G = \Phi(\widetilde{G}), \qquad G^0 = \Phi(\widetilde{G}^0), \qquad \partial G_1 = \Phi(\partial \widetilde{G}_1), \qquad \partial G_2 = \Phi(\partial \widetilde{G}_2). \tag{20}$$

Definition 3. Let F be any Borel subset of $\mathbb{R}^2$. For any $z \in G_4$, the first passage-time of the Markov chain $\{Z_n, n \geqslant 0\}$ with $Z_0 = z$ in F is defined by

$$T_{\{F\}} \equiv T_{z,\{F\}}^Z = \inf\{n \geqslant 0;\ Z_n \in F\}.$$

In particular, if $F_A = \{z;\ |z| \leqslant A\}$, we denote by T_A:

$$T_A = T_{\{F_A\}} = \inf\{n \geqslant 0;\ |Z_n| \leqslant A\}.$$

In order to prove the main results we need to introduce a family $\{\psi_{\beta_1,\beta_2},\ \beta_1,\beta_2 \in (-\pi/2,\pi/2)\}$ of non-negative functions on G which are defined in polar coordinates (ρ,θ) as follows. For any $\beta_1,\beta_2 \in (-\pi/2,\pi/2)$ we set $\beta = (\beta_1+\beta_2)/\xi$ and

$$\psi_{\beta_1,\beta_2}(\rho,\theta) = \begin{cases} \rho^\beta \cos(\beta\theta - \beta_1), & \beta_1+\beta_2 \neq 0,\ \rho > 0,\ \theta \in [0,\xi], \\ \log\rho + |\tan\beta_1|\theta, & \beta_1+\beta_2 = 0,\ \rho > 0,\ \theta \in [0,\xi], \\ 0, & \rho = 0. \end{cases}$$

Convention.

1. Whenever β_1 and β_2 are fixed the symbol β should be understood as $\beta = (\beta_1+\beta_2)/\xi$.
2. For any positive functions f and h defined on G the equality

$$f(z) = h(\rho,\theta) \text{ should be understood as } f(z) = h(\rho(z),\theta(z)),$$

where $(\rho(z),\theta(z))$ are the standard polar coordinates of z.

At this point we need some further notation.

Notation. For any $n \geqslant 0$ we define $\Delta_n = Z_{n+1} - Z_n$. For any $f \in C^2(G\backslash(0,0)), \Delta = (\Delta_1,\Delta_2) \in \mathbb{R}^2$ we denote by

$$D^2 f(z,\Delta,\Delta) = \left(\Delta_1^2 \frac{\partial^2}{\partial u^2} + 2\Delta_1\Delta_2 \frac{\partial^2}{\partial u \partial v} + \Delta_2^2 \frac{\partial^2}{\partial v^2}\right) f(z).$$

For any $\mathscr{F}_{n+1}$-measurable function F the symbol $E_{Z_n}F$ stands for $E(F \mid Z_n)$. For $\beta_1,\beta_2 \in (-\pi/2,\pi/2)$, $\xi \in (0,\pi)$, and $A > 0$ we introduce:

$$F_{A,\beta_1,\beta_2,\xi} = \begin{cases} \{z \in G;\ {\psi^{1/\beta}}_{\beta_1,\beta_2}(z) \leqslant A\} & \text{if } \beta \neq 0, \\ \{z \in G;\ \exp(\psi_{\beta_1,\beta_2}(z)) \leqslant A\} & \text{if } \beta = 0 \end{cases} \tag{21}$$

and

$$c(\beta) = \begin{cases} \cos^{-1/\beta}(|\beta_1| \vee |\beta_2|) & \text{if } \beta \neq 0, \\ 1 & \text{if } \beta = 0. \end{cases} \tag{22}$$

$$C(\beta) = \begin{cases} 1 & \text{if } \beta \neq 0, \\ \exp(-\xi|\tan\beta_1|) & \text{if } \beta = 0. \end{cases} \tag{23}$$

$$d(\beta) = \begin{cases} \beta & \text{if } \beta \neq 0, \\ \cos^{-1}\beta_1 & \text{if } \beta = 0. \end{cases} \tag{24}$$

Finally, throughout the rest of the paper η is a fixed positive number such that

$$\eta < \min\left(\tfrac{\gamma-2}{2}, \gamma-\alpha, \tfrac{1}{2}\right). \tag{25}$$

Let us fix β_1 and $\beta_2 \in (-\pi/2, \pi/2)$. The following properties of the function ψ_{β_1,β_2} will play a crucial role in the proof of Theorem 3. Their proofs are easy and can be found in Aspandiiarov (1994, Ch. B).

1. ψ_{β_1,β_2} is a harmonic function in $G\backslash(0,0)$.
2. The "monotonicity" property.

If $\beta \neq 0$, then for any $z \in G\backslash(0,0)$,

$$|z|^{\beta}\cos(|\beta_1| \vee |\beta_2|) \leqslant \psi_{\beta_1,\beta_2}(z) \leqslant |z|^{\beta}. \tag{26}$$

If $\beta = 0$, then there exist a positive constant c_1 such that for all $|z| > 2$ we have

$$\log|z| \leqslant \psi_{\beta_1,\beta_2}(z) \leqslant c_1 \log|z|. \tag{27}$$

3. If we are given the inward pointing non-degenerate vector field $\{\boldsymbol{v}(z), z \in \partial G_1 \cup \partial G_2\}$ defined by

$$\boldsymbol{v}(z) = \begin{cases} \boldsymbol{v}_1 = (-\sin\alpha_1, \cos\alpha_1) & \text{if } z \in \partial G_1, \\ \boldsymbol{v}_2 = (\sin(\xi - \alpha_2), -\cos(\xi - \alpha_2)) & \text{if } z \in \partial G_2, \end{cases}$$

then for each $i = 1, 2$,

$$(\boldsymbol{v}(z), \nabla\psi_{\beta_1,\beta_2}(z)) = \begin{cases} \beta|z|^{\beta-1}\sin(\beta_i - \alpha_i) & \text{if } \beta \neq 0, z \in \partial G_i, \\ (|z|\cos\beta_1)^{-1}\sin(\beta_i - \alpha_i) & \text{if } \beta = 0, z \in \partial G_i. \end{cases} \tag{28}$$

4. For any integers i, k such that $0 \leqslant i \leqslant k$ and for any $s \neq 0$ there exists a positive constant $c = c(\beta_1, \beta_2, s, k)$ such that for any $z \in G\backslash(0,0)$,

$$\left| \frac{\partial^k (\psi_{\beta_1,\beta_2})^s(z)}{\partial u^i \partial v^{k-i}} \right| \leqslant c|z|^{-k}\psi^s_{\beta_1,\beta_2}(z). \tag{29}$$

In particular, there exist positive constants c_1, c_2 such that for any $z \in G\backslash(0,0), \Delta \in \mathbb{R}^2$,

$$\begin{aligned} &|(\nabla\psi_{\beta_1,\beta_2}(z), \Delta)| \leqslant c_1|z|^{-1}\psi_{\beta_1,\beta_2}(z)|\Delta|, \\ &|D^2\psi_{\beta_1,\beta_2}(z, \Delta, \Delta)| \leqslant c_2|z|^{-2}\psi_{\beta_1,\beta_2}(z)|\Delta|^2. \end{aligned} \tag{30}$$

Furthermore, in the case $\beta = 0$, there exist positive constants c_3, c_4 such that for all $z \in G\backslash(0,0)$,

$$\begin{aligned} &|(\nabla\psi_{\beta_1,\beta_2}(z), \Delta)| \leqslant c_3|z|^{-1}|\Delta|, \\ &|D^2\psi_{\beta_1,\beta_2}(z, \Delta, \Delta)| \leqslant c_4|z|^{-2}|\Delta|^2. \end{aligned} \tag{31}$$

Let $f \in C^3(0, \infty)$. We have the first-order Taylor's expansion:

$$\begin{aligned} &f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n) \\ &\quad = f' \circ \psi_{\beta_1,\beta_2}(Z_n)(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n) + R_n(Z_n, \Delta_n, f, \beta, 1) \end{aligned} \tag{32}$$

and the second-order Taylor's expansion:

$$\begin{aligned} &f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n) \\ &\quad = f' \circ \psi_{\beta_1,\beta_2}(Z_n)(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n) + \tfrac{1}{2}f'' \circ \psi_{\beta_1,\beta_2}(Z_n)(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n)^2 \\ &\qquad + \tfrac{1}{2}f' \circ \psi_{\beta_1,\beta_2}(Z_n)D^2\psi_{\beta_1,\beta_2}(Z_n, \Delta_n, \Delta_n) + R_n(Z_n, \Delta_n, f, \beta, 2), \end{aligned} \tag{33}$$

where the remainders $R_n(Z_n, \Delta_n, f, \beta, k)$ for $k = 1,2$ can be written in the following integral form:

$$R_n(Z_n, \Delta_n, f, \beta, k) = \frac{1}{k!} \int_0^1 \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \{f \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)\}(1-t)^k \,\mathrm{d}t, \quad k = 1,2. \tag{34}$$

The following result shows that under suitable conditions on f and on the means of the increments $Z_{n+1} - Z_n$ the asymptotic behavior of the conditional expectations of $f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)$ can be completely described in terms of the conditional expectations of the principal terms in the Taylor's expansion (33).

Definition 4. Let $\mathscr{G}$ be the following class of non-negative functions defined on $\mathbb{R}_+$:

$$\begin{aligned}\mathscr{G} = \Bigg\{ & f : \mathbb{R}_+ \to \mathbb{R}_+; f \in C^3(0,\infty) \\ & \frac{f'''(x)}{f''(x)} = \mathrm{O}\left(\frac{1}{x}\right) \text{ and } \frac{f''(x)}{f'(x)} = \mathrm{O}\left(\frac{1}{x}\right) \text{ as } x \to \infty, \\ & \forall v > 0, \quad \liminf_{x\to\infty} \left| \frac{f''(x)x^{1+v}}{f'(x)} \right| > 0 \text{ and } \liminf_{x\to\infty} \left| \frac{f'(x)x^{1+v}}{f(x)} \right| > 0, \\ & \text{there exist positive } a_f > 1 \text{ and } \widetilde{A}_f \text{ such that } \limsup_{x\to\infty} \frac{|f''(a_f x)|}{|f''(x)|} \leqslant \widetilde{A}_f \Bigg\}.\end{aligned}$$

Lemma 2. *Let β_1 and $\beta_2 \in (-\pi/2, \pi/2)$ be real numbers such that $\beta_1 + \beta_2 \geqslant 0$. Set $\beta = (\beta_1 + \beta_2)/\xi$. Let f be a function from $\mathscr{G}$ such that in the case $\beta \neq 0$ (resp. $\beta = 0$) $|f''(x^\beta)|x^{2\beta-2}$ (resp. $|f''(\log x)|x^{-2}$) is monotone on some interval $[B,\infty)$. Suppose there exist positive constants $\chi in(0,1)$ and c such that for all $n \geqslant 0$ and for all z, P_z-a. s.*

$$\begin{aligned} &E_{Z_n}(|\Delta_n|^{2+\chi} \max(1, |f''(|\Delta_n|^\beta)||\Delta_n|^{2\beta-2}) \leqslant c \quad \text{if } \beta \neq 0, \\ &E_{Z_n}(|\Delta_n|^{2+\chi} \max(1, |f''(|\log|\Delta_n||)||\Delta_n|^{-2}) \leqslant c \quad \text{if } \beta = 0. \end{aligned} \tag{35}$$

Then there exist positive constants A, b, C such that for any $n \geqslant 0$ and for any $|z| > A$ the following two statements hold P_z-a.s.:

(a) *On $\{Z_n \in G^0\} \cap \{|Z_n| > A\}$,*

$$|E_{Z_n}\left(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)\right)| \leqslant b|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)||Z_n|^{2\theta-2}. \tag{36}$$

Furthermore,

$$\begin{aligned} &\mathrm{sgn}(f'' \circ \psi_{\beta_1,\beta_2}(Z_n)) E_{Z_n}(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)) \\ &\quad \geqslant C|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)||Z_n|^{2\beta-2}. \end{aligned} \tag{37}$$

(b) *For each $i = 1,2$ we have on $\{Z_n \in \partial G_i\} \cap \{|Z_n| > A\}$,*

$$\begin{aligned} &|E_{Z_n}(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n))| \\ &\quad \leqslant \begin{cases} b|f' \circ \psi_{\beta_1,\beta_2}(Z_n)|\psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{-2} & \text{if } \beta_i = \alpha_i, \\ b|f' \circ \psi_{\beta_1,\beta_2}(Z_n)||Z_n|^{\beta-1} & \text{otherwise.} \end{cases} \end{aligned} \tag{38}$$

Furthermore,

$$\operatorname{sgn}(f' \circ \psi_{\beta_1,\beta_2}(Z_n)\sin(\beta_i - \alpha_i))E_{Z_n}(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)) \\ \geqslant C|f' \circ \psi_{\beta_1,\beta_2}(Z_n)\sin(\beta_i - \alpha_i)||Z_n|^{\beta-1}. \tag{39}$$

Proof. Let n be any fixed non-negative integer. We will first estimate the conditional expectation of the remainder $R_n(Z_n, \Delta_n, f, \beta, 2)$. Namely, we will show that there exist positive constants $\widetilde{c} = \widetilde{c}(\chi)$ and $\widetilde{\eta}$ such that for all n, if $|Z_n|$ is large enough, the following estimate holds:

$$E_{Z_n}(|R_n(Z_n, \Delta_n, f, \beta, 2)|) \leqslant \widetilde{c}|Z_n|^{2\beta-2-\widetilde{\eta}}|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|. \tag{40}$$

We will only prove (40) in the case $\beta > 0$. The proof in the other case can be carried out using the same ideas and is left to the reader. Let $\delta \in (0,1)$ be some fixed real number. Then,

$$E_{Z_n}R_n(Z_n, \Delta_n, f, \beta, 2) = E_{Z_n}(R_n(Z_n, \Delta_n, f, \beta, 2)\{1_{(|\Delta_n| \leqslant \delta|Z_n|)} + 1_{(|\Delta_n| > \delta|Z_n|)}\}) \\ = \mathrm{I} + \mathrm{II}. \tag{41}$$

To estimate the first term in (41) we will look at the integral form of R_n. Easy calculations based on properties of functions ψ_{β_1,β_2} show that there exists a positive constant c_1 (here as elsewhere in the proof of the lemma the constants c_i do not depend on n) such that

$$\begin{aligned} |\mathrm{I}| &= |E_{Z_n}(R_n(Z_n, \Delta_n, f, \beta, 2)1_{(|\Delta_n| \leqslant \delta|Z_n|)})| \\ &\leqslant c_1 E_{Z_n}\left\{\int_0^1 (|f' \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)||Z_n + t\Delta_n|^{\beta-3}\right. \\ &\quad + |f'' \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)||Z_n + t\Delta_n|^{2\beta-3} \\ &\quad \left. + |f'''(\psi_{\beta_1,\beta_2})(Z_n + t\Delta_n)||Z_n + t\Delta_n|^{3\beta-3})|\Delta_n|^3(1-t)^2 dt)1_{(|\Delta_n| \leqslant \delta|Z_n|)}\right\}. \end{aligned}$$

Fix any positive $v < \chi/2\beta$. Then, using the assumptions on f we deduce from the last bound that there exists a positive constant c_2 such that for all large enough $|Z_n|$,

$$\begin{aligned} |\mathrm{I}| &\leqslant c_2 E_{Z_n}\left\{\int_0^1 (|f'' \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)|(\psi_{\beta_1,\beta_2}(Z_n + t\Delta_n))^{1+v}|Z_n + t\Delta_n|^{\beta-3}\right. \\ &\quad + |f'' \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)||Z_n + t\Delta_n|^{2\beta-3} + |f'' \circ \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n)| \\ &\quad \left. \times (\psi_{\beta_1,\beta_2}(Z_n + t\Delta_n))^{-1}|Z_n + t\Delta_n|^{3\beta-3})|\Delta_n|^3(1-t)^2 dt)1_{(|\Delta_n| \leqslant \delta|Z_n|)}\right\}. \end{aligned} \tag{42}$$

Next, easy geometrical arguments show that for any $t \in [0,1]$,

$$\begin{aligned} |Z_n| + |\Delta_n| &\geqslant |Z_n + t\Delta_n| \geqslant |Z_n + t(\Delta_n)^+| - |t(\Delta_n)^-| \geqslant \inf_{\Delta \in R_+^2} |Z_n + \Delta| - |(\Delta_n)^-| \\ &\geqslant \sin\left(\frac{\pi}{2} \vee \xi\right)|Z_n| - |(\Delta_n)^-|, \end{aligned} \tag{43}$$

where $(\Delta_n)^+ = ((\Delta_n^x)^+, (\Delta_n^y)^+)$, $(\Delta_n)^- = ((\Delta_n^x)^-, (\Delta_n^y)^-)$. Since the jumps of the Markov chain $\{\widetilde{Z}_n,\ n\geqslant 0\}$ towards the origin are bounded from below, then the last inequalities imply that for any $t\in[0,1]$ and for all large enough $|Z_n|$,

$$|Z_n| + |\Delta_n| \geqslant |Z_n + t\Delta_n| \geqslant \sin\left(\frac{\pi}{2} \vee \xi\right)|Z_n|/2, \tag{44}$$

On the other hand, the "monotonicity" property of ψ_{β_1,β_2} ensures the existence of positive constants c_3, c_4, c_5 and c_6 such that

$$c_3|Z_n + t\Delta_n|^\beta \leqslant \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n) \leqslant c_4|Z_n + t\Delta_n|^\beta \tag{45}$$

and

$$c_5\psi_{\beta_1,\beta_2}(Z_n) \leqslant \psi_{\beta_1,\beta_2}(Z_n + t\Delta_n) \leqslant c_6\psi_{\beta_1,\beta_2}(Z_n) \quad \text{on } \{|\Delta_n| \leqslant \delta|Z_n|\}. \tag{46}$$

Let us put the estimates (45), (46) into (42). Then, using again the assumptions on f and the "monotonicity" property of ψ_{β_1,β_2} we easily see that there exists a positive constant c_7 such that for all large enough $|Z_n|$,

$$|\mathrm{I}| \leqslant c_7|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|\, |Z_n|^{2\beta-3+\nu\beta} E_{Z_n}\{|\Delta_n|^3 1_{(|\Delta_n|\leqslant\delta|Z_n|)}\}.$$

It then follows from (35) that there exist positive constants c_8, c_9 such that whenever $|Z_n|$ is large enough and $Z_n \in \mathscr{A}$,

$$\begin{aligned}|\mathrm{I}| &\leqslant c_8|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|\, |Z_n|^{2\beta-2+\nu\beta-\chi} E_{Z_n}\{|\Delta_n|^{2+\chi} 1_{(|\Delta_n|\leqslant\delta|Z_n|)}\}\\ &\leqslant c_9|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|\, |Z_n|^{2\beta-2+\nu\beta-\chi}.\end{aligned} \tag{47}$$

Let us determine the bounds of the second term in (41) dealing with the big jumps. To this end instead of working with the integral form of $R_n(Z_n, \Delta_n, f, \beta, 2)$ given by (34) we will look directly at its expression that follows from (33). Namely,

$$\begin{aligned}\mathrm{II} = E_{Z_n}\{&1_{(|\Delta_n|>\delta|Z_n|)}(f\circ\psi_{\beta_1,\beta_2}(Z_{n+1}) - f\circ\psi_{\beta_1,\beta_2}(Z_n)\\ &- f'\circ\psi_{\beta_1,\beta_2}(Z_n)(\nabla\psi_{\beta_1,\beta_2}(Z_n),\Delta_n) - \tfrac{1}{2}f''\circ\psi_{\beta_1,\beta_2}(Z_n)(\nabla\psi_{\beta_1,\beta_2}(Z_n),\Delta_n)^2\\ &-\tfrac{1}{2}f'\circ\psi_{\beta_1,\beta_2}(Z_n)D^2\psi_{\beta_1,\beta_2}(Z_n,\Delta_n,\Delta_n))\}.\end{aligned} \tag{48}$$

As is easy to see, the properties of ψ_{β_1,β_2} ensure that there exist positive constants $c_{10},\ldots,c_{16}$ such that

$$\psi_{\beta_1,\beta_2}(Z_n + \Delta_n) \leqslant c_{10}|\Delta_n|^\beta \quad \text{on } \{|\Delta_n| > \delta|Z_n|\}. \tag{49}$$

and

$$\begin{aligned}&|(\nabla\psi_{\beta_1,\beta_2}(Z_n),\Delta_n)| \leqslant c_{11}|Z_n|^{-1}\psi_{\beta_1,\beta_2}(Z_n)|\Delta_n| \leqslant c_{12}|Z_n|^{\beta-1}|\Delta_n|,\\ &|D^2\psi_{\beta_1,\beta_2}(Z_n,\Delta_n,\Delta_n)| \leqslant c_{13}|Z_n|^{-2}\psi_{\beta_1,\beta_2}(Z_n)|\Delta_n|^2 \leqslant c_{14}|Z_n|^{\beta-2}|\Delta_n|^2.\end{aligned} \tag{50}$$

Then, the assumptions on f and (48)–(50) yield that there exist positive constants c_{15}, c_{16} such that

$$\begin{aligned}|\mathrm{II}| &\leqslant c_{15} E_{Z_n}\{|f''(|\Delta_n|^\beta)|\,|\Delta_n|^{2\beta+2\beta v} 1_{(|\Delta_n|>\delta|Z_n|)}\} + c_{15}|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|\,|Z_n|^{2\beta-2}\\ &\quad\times E_{Z_n}\{(|Z_n|^{2\beta v+2}|Z_n|^{\beta v+1}|\Delta_n| + |\Delta_n|^2 + |Z_n|^{\beta v}|\Delta_n|^2) 1_{(|\Delta_n|>\delta|Z_n|)}\}\\ &\leqslant c_{15} E_{Z_n}\{|f''(|\Delta_n|^\beta)|\,|\Delta_n|^{2\beta+2\beta v} 1_{(|\Delta_n|>\delta|Z_n|)}\}\\ &\quad + c_{16}|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|\,|Z_n|^{2\beta-2+2\beta v-\chi}. \end{aligned} \tag{51}$$

To estimate the term including $f''(|\Delta_n|^\beta)$ we recall that $|f''(x^\beta)|x^{2\beta-2}$ is a monotone function on some interval $[B,\infty)$. If it is non-increasing, then by "monotonicity" property of ψ_{β_1,β_2} there exist positive constants c_{17}, c_{18}, c_{19} such that for all large enough $|Z_n|$,

$$\begin{aligned}&\frac{E_{Z_n}\{|f''(|\Delta_n|^\beta)|\,|\Delta_n|^{2\beta+2\beta v} 1_{(|\Delta_n|>\delta|Z_n|)}\}}{|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|\,|Z_n|^{2\beta-2}}\\ &\leqslant c_{17}\frac{|f''(|Z_n|^\beta)|}{|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|} E_{Z_n}\{|\Delta_n|^{2+2\beta v} 1_{(|\Delta_n|>\delta|Z_n|)}\}\\ &\leqslant c_{18}|Z_n|^{2\beta v-\chi} E_{Z_n}\{|\Delta_n|^{2+\chi}\} \leqslant c_{19}|Z_n|^{2\beta v-\chi}. \end{aligned} \tag{52}$$

In the other case, when the function $|f''(x^\beta)|x^{2\beta-2}$ is non-decreasing, one can see from the assumptions on f that there exist positive constants c_{20}, c_{21} such that for all large enough $|Z_n|$, $|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|\,|Z_n|^{2\beta-2} \geqslant c_{20}$ and

$$\begin{aligned}\frac{E_{Z_n}\{|f''(|\Delta_n|^\beta)|\,|\Delta_n|^{2\beta+2\beta v} 1_{(|\Delta_n|>\delta|Z_n|)}\}}{|f''\circ\psi_{\beta_1,\beta_2}(Z_n)|\,|Z_n|^{2\beta-2}} &\leqslant c_{20}^{-1}|Z_n|^{2\beta v-\chi} E_{Z_n}\{|f''(|\Delta_n|^\beta)|\,|\Delta_n|^{2\beta+\chi}\}\\ &\leqslant c_{21}|Z_n|^{2\beta v-\chi}. \end{aligned}$$

This result and Eqs. (51) and (52) imply immediately that there exists a positive constant c_{22} such that for all large enough $|Z_n|$,

$$|\mathrm{II}| \leqslant c_{22}|f''(Z_n^\beta)|\,|Z_n|^{2\beta-2+2\beta v-\chi}. \tag{53}$$

This and (47) give the desired conclusion with $\tilde{\eta} = \chi - 2\beta v$.

Next, we investigate the principal terms in the second-order Taylor's expansion (33). We separate two cases.

(a) Suppose $Z_n \in G^0$. Then it can be seen from the moment conditions on the transition mechanism of the Markov chain $\{\tilde{Z}_n,\ n \geqslant 0\}$, the form of transformation Φ and properties of ψ_{β_1,β_2} that for any $n \geqslant 0$,

$$E_{Z_n}(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n) = 0,$$

$$E_{Z_n}(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n)^2 = |\nabla\psi_{\beta_1,\beta_2}(Z_n)|^2 = d^2(\beta)|Z_n|^{2\beta-2}$$

and

$$E_{Z_n}(D^2\psi_{\beta_1,\beta_2}(Z_n, \Delta_n, \Delta_n)) = \triangle\psi_{\beta_1,\beta_2}(Z_n) = 0, \tag{54}$$

where $d(\beta)$ was introduced in (24). Therefore, we have from (33),

$$E_{Z_n}(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)) = \tfrac{1}{2} d^2(\beta) f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{2\beta-2} + E_{Z_n} R_n(Z_n, \Delta_n, f, \beta, 2). \tag{55}$$

But (40) shows that for all large enough $|Z_n|$:

$$|E_{Z_n} R_n(Z_n, \Delta_n, f, \beta, 2)| \leqslant |f'' \circ \psi_{\beta_1,\beta_2}(Z_n)||Z_n|^{2\beta-2-\tilde{\eta}}. \tag{56}$$

with some positive constant $\tilde{\eta}$. The bounds (36) and (37) follow readily from (55) and (56).

(b) Let $Z_n \in \partial G_i$ for some $i \in \{1,2\}$. In this case we have from the moment conditions on the transition mechanism of the Markov chain $\{\tilde{Z}_n,\ n \geqslant 0\}$, the form of transformation Φ and (28) that for any $n \geqslant 0$,

$$E_{Z_n}(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n) = \tilde{d}_i d(\beta)|Z_n|^{\beta-1}\sin(\beta_i - \alpha_i), \tag{57}$$

where $\tilde{d}_1 = |\Phi(\boldsymbol{P}')|, \tilde{d}_2 = |\Phi(\boldsymbol{Q}')|$. Hence, on $\{Z_n \in \partial G_i\}$, $i = 1,2$,

$$\begin{aligned} &E_{Z_n}(f \circ \psi_{\beta_1,\beta_2}(Z_{n+1}) - f \circ \psi_{\beta_1,\beta_2}(Z_n)) \\ &\quad = \tilde{d}_i d(\beta)\sin(\beta_i - \alpha_i)|Z_n|^{\beta-1} f' \circ \psi_{\beta_1,\beta_2}(Z_n) \\ &\qquad + \tfrac{1}{2} f'' \circ \psi_{\beta_1,\beta_2}(Z_n) E_{Z_n}\{(\nabla\psi_{\beta_1,\beta_2}(Z_n), \Delta_n)^2\} \\ &\qquad + \tfrac{1}{2} f' \circ \psi_{\beta_1,\beta_2}(Z_n) E_{Z_n}\{D^2\psi_{\beta_1,\beta_2}(Z_n, \Delta_n, \Delta_n)\} + E_{Z_n}\{R_n(Z_n, \Delta_n, f, \beta, 2)\} \\ &\quad = \tilde{d}_i d(\beta)\sin(\beta_i - \alpha_i)|Z_n|^{\beta-1} f' \circ \psi_{\beta_1,\beta_2}(Z_n) + \mathrm{I}. \end{aligned} \tag{58}$$

Next, it can be easily deduced from the bounds (30), (40), the "monotonicity" property of ψ_{β_1,β_2} and the assumptions on f, that there exist positive constants c_{23}, c_{24} such that for all large enough $|Z_n|$,

$$\begin{aligned} |\mathrm{I}| \leqslant c_{23}\{&|f'' \circ \psi_{\beta_1,\beta_2}(Z_n)|\psi^2_{\beta_1,\beta_2}(Z_n)|Z_n|^{-2} + |f' \circ \psi_{\beta_1,\beta_2}(Z_n)|\psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{-2} \\ &+ |f'' \circ \psi_{\beta_1,\beta_2}(Z_n)||Z_n|^{2\beta-2-\tilde{\eta}}\} \leqslant c_{24}|f' \circ \psi_{\beta_1,\beta_2}(Z_n)|\psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{-2}. \end{aligned} \tag{59}$$

Now the inequalities (38) and (39) are immediate consequences of (58) and (59) and (26) written in the following form:

$$\psi_{\beta_1,\beta_2}(z)|z|^{-2} = \mathrm{o}(|z|^{\beta-1}) \quad \text{as } |z| \to \infty. \qquad \square$$

In the sequel we will denote by $\{X_n,\ n \geqslant 0\}$ the process defined by

$$X_n = \begin{cases} \psi^{\frac{1}{\alpha}}_{\alpha_1,\alpha_2}(Z_n) & \text{if } \alpha \neq 0, \\ \exp(\psi_{\alpha_1,\alpha_2}(Z_n)) & \text{if } \alpha = 0. \end{cases} \tag{60}$$

Lemma 3. *The process $\{X_n,\ n \geqslant 0\}$ satisfies the conditions of Lemma 1 with some $r, A = \bar{A}, C, D$.*

Proof. The proof in the case $\alpha > 0$ is a direct consequence of the previous lemma. As far as the other case is concerned, the proof is very similar to that of Lemma 2, and we omit it here. $\square$

3.3. Proof of Theorem 3

The essential part of the proof consists in proving the following:

Theorem 3′. *Let $\{Z_n,\ n\geqslant 0\}$ be the Markov chain defined above. Suppose that $0\leqslant\alpha<\gamma$. Then there exist positive constants A_1, $C_1>1$, $\bar{c}_1$ and s_0 such that for any $A\geqslant A_1$ whenever $Z_0=z\in\mathbb{Z}_+^2$ satisfies $|z|>C_1A$, the following bound holds for all $s>s_0A^2$:*

$$P(T_A\geqslant s)\geqslant\begin{cases}\bar{c}_1A^{\alpha}s^{-\frac{\alpha}{2}} & \text{if } \alpha>0,\\ \dfrac{\bar{c}_1}{\log(s)} & \text{if } \alpha=0.\end{cases}\tag{61}$$

Let $\{X_n,\ n\geqslant 0\}$ be the process defined in (60). Using the "monotonicity" property of ψ_{α_1,α_2} it is easy to see that in order to prove the last theorem it suffices to demonstrate the corresponding statements for τ_A (recall that $\tau_A=\inf\{n\geqslant 0;\ X_n\leqslant A\}$). Suppose for a while that we have proved Remark 4. From Theorem 2 and Remark 4 it follows that the Markov chain $\{\widetilde{Z}_n,\ n\geqslant 0\}$ is recurrent (recall that $\alpha\geqslant 0$). This and the "monotonicity" property of ψ_{α_1,α_2} imply in turn that with probability 1, $\tau_A<\infty$ for any positive A.

Our plan of the proof is as follows. We first prove the following weaker result on non-integrability of functions of the first passage-times and during the proof we obtain as well the desired estimates of Theorem 3′.

Proposition 1. *Let $\{Z_n,\ n\geqslant 0\}$ be the Markov chain defined above. Suppose that $0\leqslant\alpha<\gamma$. Then there exist positive constants C_1, A_1 such that for any $A\geqslant A_1$,*

1. *In the case $\alpha>0$, whenever $Z_0=z\in G_4$ satisfies $|z|>C_1A$,*

$$E(T_A^{\alpha/2}\log^{-1}(T_A))\ \textit{is infinite};\tag{62}$$

2. *In the case $\alpha=0$, whenever $Z_0=z\in G_4$ satisfies $|z|>C_1A$,*

$$E(\log(T_A)\ \log_2^{-1}(T_A))\ \textit{is infinite.}\tag{63}$$

To prove this proposition we would like to construct a family of the processes $U^{(B)}$, functions G,H satisfying the conditions of Theorem 1′ such that in the case $\alpha>0$ (resp. $\alpha=0$),

$$\begin{gathered}\int^{+\infty}\frac{s^{\alpha/2-1}\,\mathrm{d}s}{G(\sqrt{s})\log s}\ \text{ is infinite;}\\ \left(\text{resp. }\int^{+\infty}\frac{\mathrm{d}s}{sG(\sqrt{s})\log_2 s}\ \text{ is infinite}\right)\end{gathered}\tag{64}$$

and to use Corollary 2.

Before going into details, let us briefly describe the ideas of the proof and make some preliminary observations. Suppose for a while the original process $\{Z_n,\ n\geqslant 0\}$ has the bounded jumps. Then the process $\{X_n,\ n\geqslant 0\}$ also has bounded jumps and, in particular, $|X_{\tilde{\tau}_B}-X_{\tilde{\tau}_B-1}|\leqslant B$ for all large enough B. In this case, if we find a non-negative increasing function F such that the processes $\{F(X_{n\wedge\tau_A\wedge\tilde{\tau}_B}),\ n\geqslant 0\}$ are submartingales for all large enough A, B such that $A<B$ and in the case $\alpha>0$ (resp. $\alpha=0$),

$$\int^{+\infty} \frac{s^{\alpha-1}\,\mathrm{d}s}{F(s)\log s} \text{ is infinite} \qquad \left(\text{resp. } \int^{+\infty} \frac{\mathrm{d}s}{sF(s)\log_2 s} \text{ is infinite}\right), \tag{65}$$

then $U^{(B)}$, G, H can be simply defined by $U_n^{(B)} = F(X_n)$, $G(x) = F(2x)$, $H(x) = F(x)$. Let us try to find such F. We will search for F in the class of functions satisfying the conditions of Lemma 2. One restriction on F follows immediately from (65): this condition shows that in the case $\alpha>0$ (resp. $\alpha = 0$) the order of the asymptotic growth of $F(x)$ at infinity should not exceed $\mathrm{O}(x^\alpha \log^\varepsilon(x))$ (resp. $\mathrm{O}(\log x \log_2^\varepsilon(x))$) for any $\varepsilon>0$. Let us find out what other properties of F can be deduced from the fact that for all large enough $A<B$, the processes $\{F(X_{n\wedge\tau_A\wedge\tilde\tau_B}),\ n\geqslant 0\}$ are submartingales. Let us define the function $\widetilde{F}$ by

$$\widetilde{F}(x) = \begin{cases} F(x^{\frac{1}{\alpha}}) & \text{if } \alpha>0, \\ F(\exp(x)) & \text{if } \alpha = 0. \end{cases} \tag{66}$$

Then $\widetilde{F}$ also satisfies the conditions of Lemma 2 and we know that there exist positive constants c, A_0 such that on $\{Z_n \in G^0\} \cup \{|Z_n|>A_0\}$,

$$\begin{aligned} &\operatorname{sgn}\{\widetilde{F}'' \circ \psi_{\alpha_1,\alpha_2}(Z_n)\} E(F(X_{n+1}) - F(X_n)|\mathscr{F}_n) \\ &\quad = \operatorname{sgn}\{\widetilde{F}'' \circ \psi_{\alpha_1,\alpha_2}(Z_n)\} E_{Z_n}(\widetilde{F} \circ \psi_{\alpha_1,\alpha_2}(Z_{n+1}) - \widetilde{F} \circ \psi_{\alpha_1,\alpha_2}(Z_n)) \\ &\quad \geqslant c|\widetilde{F}'' \circ \psi_{\alpha_1,\alpha_2}(Z_n)||Z_n|^{2\alpha-2}. \end{aligned} \tag{67}$$

These inequalities show that if the processes $\{F(X_{n\wedge\tau_A\wedge\tilde\tau_B}),\ n\geqslant 0\}$ are submartingales for all large enough A,B such that $B>A>A_0$, then $\limsup_{x\to\infty} \widetilde{F}''(x)\geqslant 0$. Easy calculations show that $\widetilde{F}''(x)\geqslant 0$ for all large enough x, if and only if the function F satisfies for all large enough x,

$$xF''(x) + (1-\alpha)F'(x)\geqslant 0, \tag{68}$$

(notice that this condition readily implies that $\liminf_{x\to\infty} F(x)x^{-\alpha}>0$). Suppose now F satisfies (68). Then (67) and the "monotonicity" property of ψ_{α_1,α_2} imply that there exists a positive constant $A_1>A_0$ such that for all A,B such that $B>A>A_1$,

$$E(F(X_{(n+1)\wedge\tau_A\wedge\tilde\tau_B}) - F(X_{n\wedge\tau_A\wedge\tilde\tau_B})|\mathscr{F}_n)\geqslant 0 \quad \text{on } \{Z_n \in G^0\}.$$

On the other hand, even if the function F satisfies the condition (68) and $A>A_1$, on $\{Z_n \in \partial G_i\} \cap \{|Z_n|>A\}$, $i=1,2$ we only have

$$\begin{aligned} |E(F(X_{n+1}) - F(X_n)|\mathscr{F}_n)| &= |E_{Z_n}(\widetilde{F} \circ \psi_{\alpha_1,\alpha_2}(Z_{n+1}) - \widetilde{F} \circ \psi_{\alpha_1,\alpha_2}(Z_n))| \\ &\leqslant \tilde{c}|\widetilde{F}' \circ \psi_{\alpha_1,\alpha_2}(Z_n)|\psi_{\alpha_1,\alpha_2}(Z_n)|Z_n|^{-2}, \end{aligned} \tag{69}$$

with some positive constant $\tilde{c}$. The last bound shows that in general the processes $\{F(X_{n\wedge\tau_A\wedge\tilde\tau_B}),\ n\geqslant 0\}$ are not submartingales and we cannot simply set $U_n = F(X_n)$ in our construction. The idea that will permit us to overcome this problem is very simple and is as follows. We perturb the process $\{F(X_{n\wedge\tau_A\wedge\tilde{\tau}_B}),\ n\geqslant 0\}$ by adding a process $\{e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_{n\wedge\tau_A\wedge\tilde{\tau}_B}),\ n\geqslant 0\}$, with some $\beta_1,\beta_2\in(-\pi/2,\pi/2)$ such that $\beta_1>\alpha_1$, $\beta_2>\alpha_2$ and some positive function e satisfying the following conditions:

1. e satisfies the conditions of Lemma 2 with just defined β_i;
2. $e'(x)>0$ on some interval (B,∞) and $e(x)=\mathrm{o}(F(x))$ as $x\to\infty$;
3. As $|z|\to\infty$,

$$\psi_{\alpha_1,\alpha_2}(z)|\widetilde{F}'\circ\psi_{\alpha_1,\alpha_2}(z)|\,|z|^{-2}=\mathrm{o}(\widetilde{e}'\circ\psi_{\beta_1,\beta_2}(z)|z|^{\beta-1})$$

and

$$\widetilde{e}''\circ\psi_{\beta_1,\beta_2}(z)|z|^{2\beta-2}=\mathrm{o}(\widetilde{F}''\circ\psi_{\alpha_1,\alpha_2}(z)|z|^{2\alpha-2}). \tag{70}$$

where the function $\widetilde{e}(x)$ is defined by $\widetilde{e}(x)=e(x^{1/\beta})$.

For example, we can take $F(x)=cx^{\alpha}(1-x^{-\chi\alpha})$, $e(x)=x^{2p_0}$ with positive constants c, χ and p_0 such that $(\alpha-1)_+<2p_0<\alpha-\chi\alpha$, $\chi\in(0,1\wedge\alpha^{-1})$ in the case $\alpha>0$ and $F(x)=\log(x)(1-\log_2^{-\chi}(x))$, $e(x)=(1-\log_2^{-\chi\zeta}(x))$ with $\chi\in(0,1)$, $\zeta>1$ in the case $\alpha=0$.

Next, applying Lemma 2 we see that there exist positive constants $\widetilde{\widetilde{c}}$ and b such that for all large enough $|Z_n|$:

1. On $\{Z_n\in G^0\}$,

$$|E(e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_{n+1})-e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_n)|\mathscr{F}_n)|\leqslant b\widetilde{e}''\circ\psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{2\beta-2}; \tag{71}$$

2. On $\{Z_n\in\partial G_1\}\cup\{Z_n\in\partial G_2\}$,

$$\begin{aligned}E(e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_{n+1})-e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_n)|\mathscr{F}_n)&=E(\widetilde{e}(Z_{n+1})-\widetilde{e}(Z_n)|\mathscr{F}_n)\\&\geqslant\widetilde{\widetilde{c}}\widetilde{e}'\circ\psi_{\beta_1,\beta_2}(Z_n)|Z_n|^{\beta-1}.\end{aligned} \tag{72}$$

Taking into account (70) and comparing (67), (69) with (71)–(72) we see that for all large enough A and B such that $B>A$, the processes $\{U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ defined by

$$U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B}=F(X_{n\wedge\tau_A\wedge\tilde{\tau}_B})+e\circ\psi^{1/\beta}_{\beta_1,\beta_2}(Z_{n\wedge\tau_A\wedge\tilde{\tau}_B}),\quad\forall n\geqslant 0.$$

have the desired submartingale property. Moreover, using the assumption $e(x)=\mathrm{o}(F(x))$ as $x\to\infty$, it can be seen that the function G corresponding to the "perturbed" processes $\{U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ still verifies the condition (64). This will then finish the proof of Proposition 1. The same arguments can be applied in the general case, when the jumps of the process $\{Z_n,\ n\geqslant 0\}$ are not necessarily bounded. In this case we will first "truncate" the process and then apply the last arguments to the obtained "truncated" process.

Let us now proceed. For each positive B, we define the following "truncated" process $Z^{(B)}$:

$$Z^{(B)}_0=Z_0,\quad Z^{(B)}_{n+1}=Z^{(B)}_n+\Delta^{(B)}_n,$$

where

$$\Delta^{(B)}_n=\begin{cases}\Delta_n & \text{if } |\Delta_n|<2B,\\ 0 & \text{if } |\Delta_n|\geqslant 2B.\end{cases}$$

Let $\beta_1 = \beta_1(\alpha_1, \alpha_2, \xi)$ and $\beta_2 = \beta_2(\alpha_1, \alpha_2, \xi)$ be any fixed real numbers such that $\beta_i \in (-\pi/2, \pi/2)$ and $\alpha_i < \beta_i$, for $i = 1, 2$. Set $\beta = (\beta_1 + \beta_2)/\xi$. Let $\chi \in (0, 1 \wedge \eta\alpha^{-1})$, $\zeta > 1$ and let p_0 be any fixed number such that $\alpha - (\eta \wedge \alpha) < 2p_0 < \alpha - \chi\alpha$. Let us define the function g by

$$g(z) = \begin{cases} \psi_{\alpha_1,\alpha_2}(z)(1 - (1 + \psi_{\alpha_1,\alpha_2}(z))^{-\chi}) + \psi_{\beta_1,\beta_2}^{2p_0/\beta}(z) & \text{if } \alpha > 0,\ |z| > 0, \\ \psi_{\alpha_1,\alpha_2}(z)(1 - \mathrm{Log}^{-\chi}(\psi_{\alpha_1,\alpha_2}(z))) & \\ \quad +(1 - \mathrm{Log}_2^{-\chi\zeta}(\psi_{\beta_1,\beta_2}^{\frac{1}{\beta}}(z))), & \text{if } \alpha = 0,\ |z| > 0, \\ 0 & \text{if } |z| = 0. \end{cases}$$

For each $n \geqslant 0$, we finally set

$$U_n^{(B)} = g(Z_n^{(B)}).$$

Our previous discussion in the case when the process $\{Z_n,\ n \geqslant 0\}$ has bounded jumps makes credible the following claim.

Lemma 4. *For all initial values $Z_0^{(B)} = Z_0$ and for all large enough A and B such that $A < B$ the processes $\{U_{n\wedge\tau_A\wedge\tilde{\tau}_B}^{(B)},\ n \geqslant 0\}$ are submartingales.*

Suppose for a while that this lemma has been proved.

Let us check the conditions of Theorem 1′. As is easy to see, the "monotonicity" property of $\psi_{\alpha_1,\alpha_2}, \psi_{\beta_1,\beta_2}$ and the choice of p_0, β_1, β_2 imply that there exists a positive constant $\approx{A}$ such that for all $|z| \geqslant \approx{A}$,

$$g(z) \leqslant \begin{cases} 2\psi_{\alpha_1,\alpha_2}(z) & \text{if } \alpha > 0, \\ \psi_{\alpha_1,\alpha_2}(z)(1 - \mathrm{Log}^{-\chi}(\psi_{\alpha_1,\alpha_2}(z))) + 1 & \text{if } \alpha = 0. \end{cases}$$

Therefore, for all $A > \approx{A}$, $B > A$ and for all n we have on $\{n < (\tau_A \wedge \tilde{\tau}_B)\}$:

$$U_n^{(B)} \leqslant \begin{cases} 2X_n^\alpha \leqslant 2B^\alpha & \text{if } \alpha > 0, \\ \log(X_n)(1 - \mathrm{Log}^{-\chi}(\log X_n)) & \\ \quad + 1 \leqslant \log(B)(1 - \mathrm{Log}^{-\chi}(\log B)) + 1 & \text{if } \alpha = 0. \end{cases}$$

Similarly, for any $A > \approx{A}$ and any $B > A$ we have on $\{\tau_A \leqslant \tilde{\tau}_B\}$:

$$U_{\tau_A}^{(B)} \leqslant \begin{cases} 2A^\alpha & \text{if } \alpha > 0, \\ \log(A)(1 - \mathrm{Log}^{-\chi}(\log A)) + 1 & \text{if } \alpha = 0. \end{cases}$$

By our construction of the process $\{Z_n^{(B)},\ n \geqslant 0\}$ and the "monotonicity" property of ψ_{α_1,α_2} we have that there exist positive constants c_0, c_1, c_2 such that for all large enough $B > A$:

$$|X_{\tilde{\tau}_B}^{(B)}| \leqslant c_0|Z_{\tilde{\tau}_B}^{(B)}| \leqslant c_0(|Z_{\tilde{\tau}_B - 1}^{(B)}| + |\Delta_{\tilde{\tau}_B - 1}^{(B)}|) \leqslant c_0(c_1|X_{\tilde{\tau}_B - 1}^{(B)}| + 2B) \leqslant c_2 B \quad \text{on } \{\tau_A > \tilde{\tau}_B\}.$$

Therefore, there exists a positive constant c_3 such that for all large enough $B > A$ we have on $\{\tau_A > \tilde{\tau}_B\}$

$$U^{(B)}_{\tilde{\tau}_B} \leqslant \begin{cases} c_3 B^\alpha & \text{if } \alpha > 0, \\ 2\log(B) & \text{if } \alpha = 0. \end{cases}$$

This demonstrates that for all large enough A and B the family $\{U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ is uniformly integrable and satisfies condition (3) of Theorem 1′ is satisfied with the following functions G,H:

$$H(s) = \begin{cases} 2s^\alpha & \text{if } \alpha > 0, \\ \log(s)(1 - \mathrm{Log}^{-\chi}(\log s)) + 1 & \text{if } \alpha = 0. \end{cases} \tag{73}$$

and

$$G(s) = \begin{cases} c_3 s^\alpha & \text{if } \alpha > 0, \\ 2\log(s) & \text{if } \alpha = 0. \end{cases} \tag{74}$$

Notice that the function G satisfies condition (R). We are now ready to conclude the proof of the theorem.

Let us fix any $\overline{A}_1$ such that

(1) $\overline{A}_1 > (\overline{A} \vee \tilde{\tilde{A}} \vee 1)$ ($\overline{A}$ as in Lemma 3).

(2) For any B,A such that $B > A > \overline{A}_1$ the process $\{U^{(B)}_{n\wedge\tau_A\wedge\tilde{\tau}_B},\ n\geqslant 0\}$ is an uniformly integrable submartingale.

Let us take any fixed A greater than $\overline{A}_1$. Then, a routine checking shows that there exists a positive constant $\overline{C}_1 > 1$ that does not depend on A such that whenever $Z_0 = z \in G_4$ satisfies $|z| > \overline{C}_1 A$:

$$U_0 = U_0^{(B)} \geqslant \begin{cases} \psi_{\alpha_1,\alpha_2}(z)/2 \geqslant 2H(A) & \text{if } \alpha > 0, \\ \psi_{\alpha_1,\alpha_2}(z)(1 - \mathrm{Log}^{-\chi}(\psi_{\alpha_1,\alpha_2}(z))) \geqslant H(A) + 1 & \text{if } \alpha = 0. \end{cases}$$

Next, using the choice $A > \tilde{A}$ and Lemma 3, we see that the process $\{X_n,\ n\geqslant 0\}$ satisfies the conditions of Lemma 1 and there exist constants ε, δ independent of A such that inequality (3) holds with $v = \frac{1}{2}$. Obviously, we can suppose that $\varepsilon < 1$. Let us fix such ε, δ. Finally, let B_0 be any fixed positive number such that $B_0 > A(1+\delta)$. Let us take, for instance, $B_0 = 2A(1+\delta)$.

Summing up the last arguments, we see that the family of the stochastic processes $\{\{U_n^{(B)},\ n\geqslant 0\},\ B\geqslant B_0\}$ with $U_0^{(B)} = g(z)$ satisfies conditions of Theorem 1′ with the functions G,H defined by (73). It then follows from the assertions of Theorem 1′ and Remark 5 that whenever $Z_0 = z \in G_4$ satisfies $|z| > \overline{C}_1 A$ the following bound holds for all $s > 4\varepsilon(1+\delta)^2 A^2$,

$$P(\tau_A > s) \geqslant \frac{g(z) - H(A)}{2G(\sqrt{s/\varepsilon})} \geqslant \begin{cases} \dfrac{H(A)}{2G(\sqrt{s/\varepsilon})} & \text{if } \alpha > 0, \\[2ex] \dfrac{1}{2G(\sqrt{s/\varepsilon})} & \text{if } \alpha = 0. \end{cases}$$

But from the "monotonicity" property of ψ_{α_1,α_2} we know that for any $A>0$ and any $s>0$, we have the following inclusion: $\{\tau_A > s\} \subset \{T_{AC(\alpha)} > s\}$ (recall that $C(\alpha)$ was defined in (23)). This observation easily implies that (61) holds with $A_1 = \overline{A}_1 C(\alpha)$, $C_1 = \overline{C}_1 / C(\alpha)$, $s_0 (= 4(1+\delta)^2/(C(\alpha))^2$ and some positive $\overline{c}_1 < 1$. The statements (62) and (63) are now immediate consequences of (61).

To complete the proof of Theorem 3′ we only need to prove Lemma 4. Notice that to this end it suffices to prove

Lemma 5. *For each A and B, let us set $S_B = \inf\{n \geqslant 0;\ |Z_n| \geqslant B\}$. Then, for all initial values Z_0 the processes $\{U^{(B)}_{n\wedge T_A \wedge S_B},\ n\geqslant 0\}$ are submartingales for all large enough A, B such that $B > A$.*

In fact, suppose that we have proved Lemma 5. We have to show that for all large enough A and B such that $A < B$:

$$E_{Z_n^{(B)}}(g(Z^{(B)}_{n+1}) - g(Z^{(B)}_n)) \geqslant 0 \quad \text{on } \{n < (\tau_A \wedge \tilde{\tau}_B)\}.$$

But, the statement of Lemma 5 ensures us that for all large enough $\overline{\overline{A}}$ and $\overline{\overline{B}}$ such that $\overline{\overline{A}} < \overline{\overline{B}}$, the LHS in the last formula is non-negative on $\{n < (T^{(\overline{\overline{B}})}_{\overline{\overline{A}}} \wedge S_{\overline{\overline{B}}})\}$. It only remains to set $\overline{\overline{B}} = c(\alpha)B, \overline{\overline{A}} = C(\alpha)A$ and to notice that by our construction of the process $\{Z_n^{(\overline{\overline{B}})},\ n \geqslant 0\}$ and the "monotonicity" property of the function ψ_{α_1,α_2} we have

$$\{n < (\tau_A \wedge \tilde{\tau}_B)\} = \{\forall k \leqslant n,\ A < X_k < B\} \subset \{\forall k \leqslant n,\ \overline{\overline{A}} < |Z_k| < \overline{\overline{B}}\}. \qquad \square$$

Proof of Lemma 5. To prove the lemma we have to show that there exists $\tilde{\tilde{A}} > 0$ such that for all $A > \tilde{\tilde{A}}$ and $B > A$:

$$E_{Z_n^{(B)}}(g(Z^{(B)}_{n+1}) - g(Z^{(B)}_n)) \geqslant 0 \quad \text{on } \{n < (T_A \wedge S_B)\}.$$

We notice that on $\{n < (T_A \wedge S_B)\}$, $Z_n^{(B)} = Z_n$. Then, on $\{n < (T_A \wedge S_B)\}$,

$$\begin{aligned}
&E_{Z_n^{(B)}}(g(Z^{(B)}_{n+1}) - g(Z^{(B)}_n)) \\
&\quad = E_{Z_n}(g(Z_{n+1}) - g(Z_n)) + E_{Z_n}(g(Z_n + \Delta_n 1_{(|\Delta_n| < 2B)}) - g(Z_{n+1})) \\
&\quad = E_{Z_n}(g(Z_{n+1}) - g(Z_n)) + E_{Z_n}\{(g(Z_n) - g(Z_n + \Delta_n))1_{(|\Delta_n| \geqslant 2B)})\} \\
&\quad \geqslant E_{Z_n}(g(Z_{n+1}) - g(Z_n)) - E_{Z_n}(g(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2B)}).
\end{aligned} \tag{75}$$

Let us investigate the asymptotic behavior of the second term in the last expression. We easily see that there exists a positive constant c_0 such that for all large enough A and $B > A$ on $\{n < (T_A \wedge S_B)\}$:

$$E_{Z_n}(g(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2B)}) \leqslant c_0 E_{Z_n}((\psi_{\alpha_1,\alpha_2}(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2B)})).$$

Recall that on $\{n < (T_A \wedge S_B)\}$, $A < |Z_n| < B$. Hence, there exists a positive constant $\bar{c}_0$ such that for all large enough A and $B > A$ on $\{n < (T_A \wedge S_B)\}$,

$$E_{Z_n}(g(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2B)}) \leqslant c_0 E_{Z_n}(E_{Z_n}((\psi_{\alpha_1,\alpha_2}(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2|Z_n|)}))$$
$$\leqslant \bar{c}_0 E_{Z_n}(|Z_n + \Delta_n|^{\alpha+\eta} 1_{(|\Delta_n| \geqslant 2|Z_n|)}) \leqslant \left(\frac{3}{2}\right)^{\alpha} \bar{c}_0 E_{Z_n}(|\Delta_n|^{\alpha+\eta} 1_{(|\Delta_n| \geqslant 2|Z_n|)}).$$

On the other hand, using the choice of η: $\eta < ((\gamma - 2)/2 \wedge \gamma - \alpha)$ we get that there exist positive constants c_1, c_2 such that

$$E_{Z_n}(|\Delta_n|^{\alpha+\eta} 1_{(|\Delta_n| \geqslant 2|Z_n|)})$$
$$\leqslant \begin{cases} c_1 \leqslant c_1 |Z_n|^{\alpha-2-\eta}, & \text{if } \alpha > 2 + \eta \\ E_{Z_n}(|\Delta_n|^{\alpha-2-\eta} |\Delta_n|^{2+2\eta} 1_{(|\Delta_n| \geqslant 2|Z_n|)}) \leqslant c_2 |Z_n|^{\alpha-2-\eta} & \text{if } \alpha \leqslant 2 + \eta. \end{cases}$$

Finally, we obtain that there exists a positive constant c_3 such that for large enough A and $B > A$ we have on $\{n < (T_A \wedge S_B)\}$:

$$E_{Z_n}(g(Z_n + \Delta_n)1_{(|\Delta_n| \geqslant 2B)}) \leqslant c_3 |Z_n|^{\alpha-2-\eta}. \tag{76}$$

We turn now to the estimate of the first term on the RHS of (75). Applying twice Lemma 2 and the "monotonicity" property of ψ_{α_1,α_2} it can be easily seen that there exist positive constants $c_4 \ldots c_{19}$ such that for all large enough $|Z_n|$:

1. If $\alpha > 0$, then on $\{Z_n \in G^0\}$,

$$E_{Z_n}(g(Z_{n+1}) - g(Z_n)) \geqslant c_4 \psi_{\alpha_1,\alpha_2}^{-\chi-1}(Z_n)|Z_n|^{2\alpha-2} - c_5 \psi_{\beta_1,\beta_2}^{2p_0\beta^{-1}-2}(Z_n)|Z_n|^{2\beta-2}$$
$$\geqslant c_6 |Z_n|^{\alpha-\chi\alpha-2} - c_7 |Z_n|^{2p_0-2}. \tag{77}$$

and on $\{Z_n \in \partial G_i\}$, $i = 1,2$,

$$E_{Z_n}(g(Z_{n+1}) - g(Z_n)) \geqslant c_8 \sin(\beta_i - \alpha_i) \psi_{\beta_1,\beta_2}^{2p_0\beta^{-1}-1}(Z_n)|Z_n|^{\beta-1}$$
$$- c_9 c_4 \psi_{\alpha_1,\alpha_2}^{-\chi-1} |Z_n|^{2\alpha-2} \geqslant c_{10} |Z_n|^{2p_0-1} - c_{11} |Z_n|^{\alpha-\chi\alpha-2}. \tag{78}$$

2. If $\alpha = 0$, then on $\{Z_n \in G^0\}$,

$$E_{Z_n}(g(Z_{n+1}) - g(Z_n)) \geqslant c_{12} \psi_{\alpha_1,\alpha_2}^{-1}(Z_n) \log^{-1-\chi} \psi_{\alpha_1,\alpha_2}(Z_n)|Z_n|^{-2}$$
$$- c_{13}(\psi_{\beta_1,\beta_2}^{-2}(Z_n) \log^{-1}(\psi_{\beta_1,\beta_2}(Z_n)) \log_2^{-\chi\zeta-1}(\psi_{\beta_1,\beta_2}(Z_n))|Z_n|^{2\beta-2}$$
$$\geqslant |Z_n|^{-2} \log^{-1}(|Z_n|)(c_{14} \log_2^{-\chi-1}(|Z_n|) - c_{15} \log_2^{-\chi\zeta-1}(|Z_n|)) \tag{79}$$

and on $\{Z_n \in \partial G_i\}$, $i = 1,2$,

$$E_{Z_n}(g(Z_{n+1}) - g(Z_n)) \geqslant -c_{16} \psi_{\alpha_1,\alpha_2}(Z_n)|Z_n|^{-2}$$
$$+ c_{17} \sin(\beta_i - \alpha_i) \{(\psi_{\beta_1,\beta_2}(Z_n) \log(\psi_{\beta_1,\beta_2}(Z_n))\}^{-1} \log_2^{-\chi\zeta-1}(\psi_{\beta_1,\beta_2}(Z_n))|Z_n|^{\beta-1}$$
$$\geqslant -c_{18} |Z_n|^{-2} \log(|Z_n|) + c_{19} |Z_n|^{-1} \log^{-1}(|Z_n|) \log_2^{-\chi\zeta-1}(|Z_n|). \tag{80}$$

Therefore, using the choice of χ, ζ, p_0 ($\chi \in (0, 1 \wedge \eta\alpha^{-1})$, $\zeta > 1$ and $\alpha - (\eta \wedge \alpha) < 2p_0 < \alpha - \chi\alpha$), we deduce from (77)–(80) that there exists a positive constant c_{20} such that for all large enough A and for all $B > A$, on $\{n < (T_A \wedge S_B)\}$:

$$E_{Z_n}(g(Z_{n+1}) - g(Z_n)) \geqslant \begin{cases} c_{20}|Z_n|^{\alpha-\chi\alpha-2} & \text{if } \alpha > 0, \\ c_{20}|Z_n|^{-2}\log^{-1}(|Z_n|)\log_2^{-\zeta-1}(|Z_n|) & \text{if } \alpha = 0. \end{cases}$$

Joining together this and (76) we obtain that for all large enough A and for all $B > A$ the RHS in (75) is non-negative on $\{n < (T_A \wedge S_B)\}$. □

The rest of the proof of Theorem 3 is very easy. Let, for instance, $0 < \alpha < \gamma$. Theorem 3′ yields the existence of positive constants $C_1, A_1, \bar{c}_1, s_0$, such that for any $A \geqslant A_1$ whenever $Z_0 = z \in G_4$ satisfies $|z| > C_1 A$, we have for all $s > s_0 A^2$,

$$P(T_A > s) \geqslant \bar{c}_1 A^\alpha s^{-\alpha/2}. \tag{81}$$

Recall that the Markov chain $\{Z_n,\ n \geqslant 0\}$ was introduced as the image of the Markov chain $\{\widetilde{Z}_n,\ n \geqslant 0\}$ under Φ defined by (15) and (16). It is easy to see that there exist positive constants $\widetilde{c}(\Phi)$, $\widetilde{C}(\Phi)$ such that for any positive A, $\{\widetilde{z} \in \mathbb{R}_+^2;\ |\widetilde{z}| \leqslant A\widetilde{c}(\Phi)\} \subset \{\widetilde{z} \in \mathbb{R}_+^2;\ |\Phi(\widetilde{z})| \leqslant A\} \subset \{\widetilde{z} \in \mathbb{R}_+^2 : |\widetilde{z}| \leqslant A\widetilde{C}(\Phi)\}$. These inclusions and (81) show that for any $A > A_1$ whenever $\widetilde{Z}_0 = \widetilde{z} \in Z_+^2$ satisfies $|\widetilde{z}| > C_1 A\widetilde{C}(\Phi)$ we have for all $s > s_0 A^2$,

$$P(\widetilde{T}_{\widetilde{c}(\Phi)A} > s) \geqslant \bar{c}_1 A^\alpha s^{-\alpha/2}.$$

Letting

$$\widetilde{A}_1 = A_1 \widetilde{c}(\Phi),\ \widetilde{C}_1 = \frac{C_1 \widetilde{C}(\Phi)}{\widetilde{c}(\Phi)},\ \widetilde{s}_0 = \frac{s_0}{(\widetilde{c}(\Phi))^2} \quad \text{and} \quad \widetilde{\bar{c}}_1 = \frac{\bar{c}_1}{(\widetilde{c}(\Phi))^\alpha}$$

terminates the proof of (13). □

Proof of Remark 4. It suffices to show that the Markov chain $\{Z_n,\ n \geqslant 0\}$ is recurrent. To this end it will now be checked that the conditions of the following well-known result (see, for instance, Asmussen (1987), Proposition 5.3) are satisfied.

Lemma 6. *Let $\{U_n,\ n \geqslant 0\}$ be a discrete-time irreducible aperiodic Markov chain with some countable state space $\mathscr{U}$. Then the Markov chain is recurrent, iff there exist a function f defined on $\mathscr{U}$, and a finite set F, such that for any m:*

$$E\{f(U_{m+1}) - f(U_m) \mid U_m = a\} \leqslant 0, \quad \forall a \notin F, \tag{82}$$

and the set $\{a \in \mathscr{U};\ f(a) < K\}$ is finite for each K.

In fact, let us fix any real numbers $\beta_1 = \beta_1(\alpha_1, \xi)$ and $\beta_2 = \beta_2(\alpha_1, \xi)$ such that: $\beta_i \in (-\pi/2, \pi/2)$ and $\alpha_i < \beta_i$, for $i = 1, 2$ and any positive number p_0 such that $2p_0 < \eta$. We define the function f by

$$f(z) = \psi_{\alpha_1,\alpha_2}^s(z) + \psi_{\beta_1,\beta_2}^{-2p_0/\beta}(z), \quad z \in G, \tag{83}$$

where s is any fixed real number such that $s \in (0, 1)$. Applying twice Lemma 2, one can see using the choice of p_0 (namely, $2p_0 < \eta$), s and β_1, β_2 that for all large enough A the processes $\{Z_{n\wedge\tau_A},\ n \geqslant 0\}$ are supermartingales. On the other hand, it is immediate that the set $\{z \in G_4;\ f(z) < K\}$ is finite for each K. This finishes the proofs of Remark 4 and Theorem 3. $\square$

4. Reflected Brownian motion in a wedge

In this section we will obtain partial analogues of the results of Section 3 for reflected Brownian motions in a wedge. To state these results we need to introduce some additional notation. Let F be any subset of $\mathbb{R}^2$. We denote by $D_F = D([0,\infty),F)$ the space of right continuous functions $\omega : [0,\infty) \to F$ with left-hand limits endowed with the Skorohod topology. The subset of D_F of continuous functions is denoted by C_F. The symbol $\mathscr{F}_t$ will stand for the σ-algebra of D_F generated by coordinate functionals π_u, $0 \leqslant u \leqslant t$. Let Y_u denote the canonical process on D_F. Let G be the wedge with the corner at the origin 0, and with an arbitrary angle $\xi < \pi$, given in polar coordinates by $G = \{(\rho,\theta), 0 \leqslant \theta \leqslant \xi,\ \rho > 0\}$. Let α_1, α_2 be any fixed numbers such that $\alpha_1 + \alpha_2 > 0$ and $|\alpha_j| < \pi/2$, $j = 1,2$. As usual, let $\alpha = (\alpha_1 + \alpha_2)/\xi$. Let us fix any non-zero $z_0 \in G$.

Suppose $\alpha_1 + \alpha_2 < 2\xi$ (resp. $\alpha_1 + \alpha_2 \geqslant 2\xi$). By definition, Brownian motion (resp. stopped Brownian motion) with oblique reflection $\boldsymbol{v}_1, \boldsymbol{v}_2$ in the wedge G starting from z_0 is a G-valued process that solves the submartingale problem (resp. stopped submartingale problem) associated with $(\frac{1}{2}\Delta, \boldsymbol{v}_1, \boldsymbol{v}_2)$ starting from z_0 (Δ is the standard laplacian operator in $\mathbb{R}^2$). Intuitively, Brownian motion (resp. stopped Brownian motion) with oblique reflection is a process that behaves in the interior of the wedge like a planar Brownian motion, reflects instantaneously at the boundary of the quadrant, the directions of reflection being given by the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$) and spends zero time at the origin (resp. reach almost surely the origin and remains there). For precise definitions we refer to Varadhan and Williams (1985) or Aspandiiarov (1995). As it follows from the results of Varadhan and Williams (see Theorems 3.4, 3.10, 3.11), if $\alpha_1 + \alpha_2 < 2\xi$ (resp. $\alpha_1 + \alpha_2 \geqslant 2\xi$), then Brownian motion (resp. stopped Brownian motion) with oblique reflection exists and is unique in law. In both cases, it will be denoted by $\{W_t,\ t \geqslant 0\}$.

For each $\chi \geqslant 0$, we introduce the first passage-time S_χ by

$$S_\chi = \inf\{t \geqslant 0;\ |W_t| \leqslant \chi\}.$$

Then Theorem 2.2 in Varadhan and Williams (1985) shows that $P(S_0 < \infty) = 1$, if and only if $\alpha > 0$.

The main result of this section is the following condition for the existence of pth moments of S_0.

Theorem 4. 1. *If $\alpha > 2$, then for all $p < \alpha/2$, ES_0^p is finite.*

2. *If $\alpha > 0$, then there exist positive T and c such that for all $t > T$,*

$$P(S_0 > t) \geqslant \frac{c}{t^{\alpha/2}}.$$

In particular, for all $p \geqslant \alpha/2$, ES_0^p is infinite.

Remark 6. As a consequence of the last theorem we immediately obtain the following criterion of existence of ES_0 derived in Varadhan and Williams (1985) (see Corollary 2.3):

$$ES_0 \quad \text{is finite iff} \quad \alpha > 2.$$

Remark 7. Using a direct approach based on the analogues of the results in Aspandiiarov, et al. (1994), Menshikov and Williams (1994) have recently showed that the condition $\alpha > 2$ in the first part of Theorem 4 can be removed.

Proof. The idea of the proof is to use the results on approximation of the process $\{W_t,\ t \geqslant 0\}$ by Markov chains in G with boundary reflection and the results on the first passage-times for these Markov chains of Section 3 (more precisely, Theorem 4 and (61)).

Step 1: We need to recall some preliminary results.

(a) As was shown in Aspandiiarov (1995), there exists a Markov chain $\{Z_k,\ k \geqslant 0\}$ with bounded jumps governed by the transition mechanism described in Section 3 such that its corresponding α_1, α_2 defined as in Section 3.2 coincide with the angles of boundary reflection α_1, α_2 for the process $\{W_t,\ t \geqslant 0\}$.

Let us state the result on existence of the approximating family of Markov chains for the process $\{W_t,\ t \geqslant 0\}$ established in Aspandiiarov (1995) (see Theorem 2 and Remark 3 therein). For each $N \geqslant 1$, we set

$$G_4^N = \left\{ \left(\frac{i}{\sqrt{N}}, \frac{j}{\sqrt{N}} \right);\ (i,j) \in Z_+^2 \right\}.$$

Proposition 2. *Let $(z_0^N)_{N \geqslant 1}$ be any sequence such that $\forall N \geqslant 1$, $z_0^N \in G_4^N$ and $\lim z_0^N = z_0$ as $N \to \infty$. Then there exists a family of G_4-valued discrete parameter Markov chains $\{\{Z_k^N,\ k \geqslant 0\},\ N \geqslant 1\}$ with the same transition probabilities as $\{Z_k,\ k \geqslant 0\}$ such that $Z_0^N = z_0^N \sqrt{N}$ and the "rescaled" Markov chains $\{W_t^N,\ t \geqslant 0\}$ defined by*

$$W_t^N = \frac{Z^N[Nt]}{\sqrt{N}}, \quad t \geqslant 0$$

converge weakly in D_G to the process $\{W_t,\ t \geqslant 0\}$.

(b) Let us now fix any sequence $(z_0^N)_{N \geqslant 1}$ such that $\forall N \geqslant 1$, $z_0^N \in G_4^N$ and $\lim z_0^N = z_0$ as $N \to \infty$. We also fix the Markov chains $\{\{Z_k^N,\ k \geqslant 0\},\ N \geqslant 1\}$ and $\{W_t^N,\ t \geqslant 0\}$ with the properties described in the last proposition. Let $P_{z_0^N}^N$ (resp. P_{z_0}) be the distribution of the process $\{W_t^N,\ t \geqslant 0\}$ (resp. $\{W_t,\ t \geqslant 0\}$) on D_G under P.

(c) The next ingredient of the proof is the following:

Proposition 3. *For any fixed $\chi > 0$, there exists a set $H_\chi \subset D_G$ such that*

1. $P_{z_0}(H_\chi) = 1$.
2. *The function $\mathscr{S}_\chi : D_G \to [0, \infty)$ defined by*

$$\mathscr{S}_\chi(\omega) = \inf\{t \geqslant 0,\ |\omega_t| \leqslant \chi\}, \quad \omega \in D_G,$$

is continuous in the Skorohod topology of D_G on H_χ.

Proof. The main role is played by the continuity of the coordinate functionals π_t : $D_G \to G(\pi_t(\omega) = \omega(t))$ on some subset of D_G of P_{z_0} – probability 1 and the following easy consequence of the strong Markov property of planar Brownian motion and reflected Brownian motion in a half-plane. For any fixed $\chi > 0$, $\Delta > 0$, there exists a set $F_{\chi,\Delta} \subset D_G$ such that $P_{z_0}(F_{\chi,\Delta}) = 1$ and, for any $\omega \in F_{\chi,\Delta}$, there exists $t \in (\mathscr{S}_\chi, \mathscr{S}_\chi + \Delta)$ such that $|\omega(t)| < \chi$. The rest of the proof is carried over using standard arguments and we omit it here. □

(d) Let us set, for each $\chi > 0$ and $N \geqslant 1$,

$$T_\chi^N = \inf\{k \geqslant 0;\ |Z_k^N| \leqslant \chi\},$$

$$S_\chi^N = \inf\{t \geqslant 0;\ |W_t^N| \leqslant \chi\}.$$

It follows from the last proposition and the continuous mapping theorem that for each $\chi > 0$, S_χ^N converges weakly to S_χ as $N \to \infty$. Notice also that for any $\chi > 0, N \geqslant 1$,

$$S_\chi^N \overset{(\mathrm{d})}{=} T_{\chi\sqrt{N}}^N / N. \tag{84}$$

Step 2: Here we relate the results on the first passage-times S_χ^N for the Markov chains $\{W_t^N,\ t \geqslant 0\}$ to their counterparts for the process $\{W_t,\ t \geqslant 0\}$. This will give the desired result.

1. Let $\alpha > 2$ and let $p < \alpha/2$. We fix any positive χ such that $\chi < |z_0|/2$. It then follows that for all large enough N we have, $\chi < |z_0^N| < 2|z_0|$. We recall that the Markov chain Z has bounded jumps and the parameter γ corresponding to it is infinite. Therefore, we can apply Theorem 4 to obtain that for any $p < \frac{\alpha}{2}$, there exist positive $\widetilde{c}(p)$ and $A_0(p)$ such that for all $A \geqslant A_0(p)$ whenever $Z_0 = z \in G_4$ satisfies $|z| > A$,

$$ET_A^p \leqslant \widetilde{c}(p)|z|^{2p}. \tag{85}$$

Let p_0 be any number such that $p < p_0 < \alpha/2$. It easily follows from (84)–(85) that there exists $N_0 = N_0(\chi, p, p_0)$ such that for all $N \geqslant N_0$,

$$E(S_\chi^N)^p = E(T_{\chi\sqrt{N}}^N)^p / N^p \leqslant \widetilde{c}(p)|z_0^N|^{2p},$$

$$E(S_\chi^N)^{p_0} = E(T_{\chi\sqrt{N}}^N)^{p_0} / N^{p_0} \leqslant \widetilde{c}(p_0)|z_0^N|^{2p_0}.$$

This shows that the family $\{(S_\chi^N)^p,\ N \geqslant N_0\}$ is uniformly integrable. Therefore, by weak convergence of S_χ^N to $\widetilde{S}_\chi$ (see, for instance, Theorem 5.4 in Billingsley (1968)),

$$E(S_\chi^N)^p \to E(\widetilde{S}_\chi)^p \quad \text{and} \quad E(S_\chi)^p = E(\widetilde{S}_\chi)^p \leqslant 2^{2p}\widetilde{c}(p)|z_0|^{2p}.$$

Notice that the last inequality holds for any $\chi < |z_0|/2$ and $\widetilde{c}(p)$ is independent of χ. Therefore, letting in the last inequality $\chi \to 0$ and using the monotone convergence theorem we finally arrive at

$$E(S_0)^p \leqslant 2^{2p}\widetilde{c}(p)|z_0|^{2p}. \tag{86}$$

2. Let now $p \geqslant \alpha/2$. Recall that by (61) there exist fixed positive constants $C_1 > 1$, A_1 and $\bar{c}_1, s_0$ such that for any $A \geqslant A_1$, whenever $Z_0 = z \in G_4$ satisfies $|z| > C_1 A$, the following estimate holds for all $s > s_0 A^2$:

$$P(T_A > s) \geqslant \bar{c}_1 A^\alpha s^{-\alpha/2}. \tag{87}$$

Let us fix any positive χ such that $\chi < |z_0|/2C_1$. This choice of χ ensures that for all large enough N, $C_1 \chi < |z_0^N|$. Using this, (84) and (87) we get the existence of N_1 such that for any $N \geqslant N_1$ and for all $t > s_0 \chi^2$:

$$P(S_\chi^N > t) = P(T_{\chi\sqrt{N}}^N > Nt) \geqslant \bar{c}_1 \chi^\alpha t^{-\alpha/2}.$$

Passing to the limit as $N \to \infty$ and using the weak convergence of S_χ^N to S_χ, we obtain that for all except countably many $t > s_0 \chi^2$,

$$P(S_\chi > t) = \lim_{N \to \infty} P(S_\chi^N > t) \geqslant \bar{c}_1 \chi^\alpha t^{-\alpha/2}.$$

This obviously implies that for all except countably many $t > s_0 \chi^2$,

$$P(S_0 > t) \geqslant \bar{c}_1 \frac{\chi^\alpha}{t^{\alpha/2}}. \qquad \square \tag{88}$$

Remark 8. It can be obtained from the estimates (86) and (88) and the explicit construction of constants $\tilde{c}(p), \bar{c}_1, s_0, C_1$ and χ that if $\alpha > 2$, then for any $p < \alpha/2$ there exist constants b_1, b_2 depending on $p, \alpha_1, \alpha_2, \xi$ such that $0 < b_1 \leqslant b_2 < \infty$ and

$$b_1 |z_0|^{2p} \leqslant E(S_0)^p \leqslant b_2 |z_0|^{2p}.$$

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