

# Nonparametric estimation of the local Hurst function of multifractional Gaussian processes

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## Abstract

A new nonparametric estimator of the local Hurst function of a multifractional Gaussian process based on the increment ratio (IR) statistic is defined. In a general frame, the point-wise and uniform weak and strong consistency and a multidimensional central limit theorem for this estimator are established. Similar results are obtained for a refinement of the generalized quadratic variations (QV) estimator. The example of the multifractional Brownian motion is studied in detail. A simulation study is included showing that the IR-estimator is more accurate than the QV-estimator.

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## 1. Introduction

The present paper discusses nonparametric estimation of the local Hurst function  $H(t)$ ,  $t \in (0, 1)$  of a multifractional Gaussian process  $Z = (Z(t))_{t \in (0,1)}$  observed at discrete times  $k/n$ ,  $k = 1, \dots, n-1$ . The process  $Z$  is characterized by the fact that its infinitesimal increments at each point  $t \in (0, 1)$  resemble a multiple of a fractional Brownian motion  $B_{H(t)}$  with Hurst

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parameter  $H(t) \in (0, 1)$ :

$$(\lambda^{-H(t)}(Z(t + \lambda\tau) - Z(t)))_{\tau \geq 0} \xrightarrow[\lambda \rightarrow 0]{\text{f.d.d.}} (c(t)B_{H(t)}(\tau))_{\tau \geq 0}, \quad (1.1)$$

in the sense of the weak convergence of finite-dimensional distributions, where  $c(t) > 0$ ,  $t \in (0, 1)$  is a deterministic function. Recall that a fractional Brownian motion (FBM) with Hurst parameter  $H \in (0, 1)$  is a Gaussian process  $(B_H(\tau))_{\tau \geq 0}$  having stationary increments and satisfying  $EB_H(\tau) = 0$  and  $EB_H^2(\tau) = \tau^{2H}$ ,  $\tau \geq 0$ . Property (1.1) is called asymptotic self-similarity and the limiting process a tangent process at  $t$ . For a Gaussian zero-mean process  $Z$ , relation (1.1) is equivalent to

$$E(Z(t + \tau) - Z(t))^2 \sim c^2(t)\tau^{2H(t)}, \quad \tau \rightarrow 0. \quad (1.2)$$

A typical example of such Gaussian process  $Z$  is a multifractional Brownian motion (MBM) defined through the following harmonizable representation

$$X(t) = B_{H(t)}(t) = K(H(t)) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t)+1/2}} W(dx),$$

$$\text{with } K(H) := (H \Gamma(2H) \sin(\pi H)/\pi)^{1/2}, \quad (1.3)$$

where  $W(dx)$  is a complex-valued Gaussian noise satisfying  $E[\int_{\mathbb{R}} g(x) W(dx) \overline{\int_{\mathbb{R}} h(x) W(dx)}] = \int_{\mathbb{R}} g(x) \overline{h(x)} dx$ ,  $h, g \in L^2(\mathbb{R})$  (see more details in Section 7.2.2 of [21]). A first version of MBM was first introduced by Peltier and Lévy-Véhel [20] using a slightly different time-domain representation, while the MBM in (1.3) was defined, studied and generalized in several papers. In particular, Stoev and Taqqu [22] defined a generalization  $Y_{(a^+, a^-)}$  of (1.3) depending on parameters  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  such that  $Y_{(1,1)}$  agrees with  $X$  of (1.3) up to a multiplicative deterministic function; see Section 4 for details. In the present paper, the functions  $H(\cdot)$  and  $c(\cdot)$  are assumed to belong to a Hölder class  $\mathcal{C}^\eta(0, 1)$ ,  $\eta > 0$ . The smoothness parameter  $\eta$  plays an important role in the asymptotic results discussed below. In particular, for  $0 < \eta < 1$  the local Hurst function of MBM can be very irregular making the problem of its estimation more difficult.

Another important parameter for estimation of  $H(t)$  is the bandwidth parameter  $\alpha \in (0, 1)$ . Following a general approach in non-parametric estimation, discrete observations  $Z(k/n)$ ,  $1 \leq k \leq n-1$  are localized in a neighborhood  $\mathcal{V}_{n,\alpha}(t) = \{k/n : |k/n - t| \leq n^{-\alpha}\}$  of a given point  $t \in (0, 1)$  containing at most  $[2n^{1-\alpha}] + 1$  points, and an estimate of  $H(t)$  from the localized sample  $Z(k/n)$ ,  $k \in \mathcal{V}_{n,\alpha}(t)$  is constructed. Choosing a smaller  $\alpha$  means that the estimate of  $H(t)$  uses more observations of  $Z$  and therefore the variance of it can be expected to be smaller. On the other hand, increasing the “width” of  $\mathcal{V}_{n,\alpha}(t)$  (by taking a smaller  $\alpha$ ) usually increases the bias of the estimator particularly when  $\eta$  is small, since the function  $H(\cdot)$  is more likely to change in a larger interval  $\mathcal{V}_{n,\alpha}(t)$ . Hence the accuracy of estimation of  $H(t)$  depends on  $\alpha$  and  $\eta$  in a crucial way.

The present paper discusses two estimators of  $H(t)$ . The first estimator  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t)$  is a localized version of the increment ratio (IR) estimator introduced in [5]. It is written as a localized sum of ratios of second-order variations  $V_n^{a^*} Z(k/n) = Z((k+2)/n) - 2Z((k+1)/n) + Z(k/n)$ ,  $k \in \mathcal{V}_{n,\alpha}(t)$  and is related to zero intersections of the variations sequence; see (2.7), also [5, Remark 1.1]. We also discuss a new estimator  $\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t)$  which

is a pseudo-generalized least squares (PGLS) version of  $\hat{H}_{n,\alpha}^{(\text{IR})}(t)$  constructed from several dilations of the second-order variations. At the same time, we study the generalized quadratic variations (QV) estimator  $\hat{H}_{n,\alpha}^{(\text{QV})}(t)$  of [15] and its refinement (a PGLS version)  $\hat{H}_{n,\alpha}^{(\text{QV}^2)}(t)$ . The main reasons for including the QV estimators in this paper is the fact that the study and the asymptotic results for IR and QV estimators are very similar (both estimators are expressed in terms of sums of nonlinear functions of Gaussian vectors of the Hermite rank 2), and our desire to compare finite-sample performance of these two classes of estimators.

Let us briefly describe the main results of this paper. Section 2 introduces the two main assumptions  $(A)_\kappa$  and  $(B)_\alpha$  on a Gaussian process  $Z$ . The first assumption specifies the rate of convergence in a similar asymptotic relation to (1.2) and controls the bias of our estimators. The second assumption concerns the decay rate of the covariance of generalized variations of  $Z$  and thus helps to control the variance of the estimators. Under these assumptions, Theorem 1(i)–(iv) obtains the weak and strong consistency of the estimators  $\hat{H}_{n,\alpha}^{(\text{IR})}(t)$  and  $\hat{H}_{n,\alpha}^{(\text{QV})}(t)$  at a given point  $t \in (0, 1)$  and some uniform weak and strong consistency rates of these estimators over arbitrary interval  $t \in (\epsilon, 1 - \epsilon)$ ,  $\epsilon > 0$  under increasingly restrictive conditions on  $\alpha, \eta, \kappa$  and other parameters. A multidimensional central limit theorem (CLT) satisfied by the above estimators is proved in Theorem 2. Corollaries 1 and 2 extend Theorems 1 and 2 for the PGLS versions of the IR and QV estimators which are studied in Section 3. Proposition 3 provides limit theorems to the IR and QV estimators for the MBM (including its generalization in [22]). Section 5 is devoted to simulations while Sections 6 and 7 contain the proofs. Let us note that the simulations in Section 5 performed on the MBM in (1.3) for some typical Hurst functions and three different values of the smoothness parameter  $\eta$  show a very good performance of the estimator  $\hat{H}_{n,\alpha}^{(\text{IR}^2)}(t)$ . The last estimator provides in almost all cases a smaller MISE criterion than  $\hat{H}_{n,\alpha}^{(\text{QV}^2)}(t)$ . We also note that the numerical results for both classes of estimators are clearly (in the case of IR) or slightly (in the case of QV) in favor of their PGLS versions.

Nonparametric estimation of the local Hurst function was the subject of several papers [1,3,10,9,11,12,15,17–19]. However most of these results are less comprehensive and sharp when applied to the MBM while some contain erroneous statements. See Remarks 3, 8 and 9, for more detailed discussion and some controversies. The MBM is probably the most important example of a multifractional Gaussian process at the present time. It is considered as a benchmark for the estimation problem of  $H(t)$  in the present paper and is discussed in detail in Section 4. We believe that the conditions of Proposition 3 for the IR and QV estimators of the Hurst function of the MBM are optimal or close to optimal since they are derived from the asymptotic expansion of the covariance of generalized variations of the MBM in Lemma 4. See also Remark 3.

**Notation.** In what follows,  $C$  stands for a constant whose precise value is unimportant and which may change from line to line. Also, we write  $\xrightarrow[n \rightarrow \infty]{\mathcal{P}}, \xrightarrow[n \rightarrow \infty]{\text{a.s.}}, \xrightarrow[n \rightarrow \infty]{\mathcal{D}}$  for convergence in probability, almost sure (a.s.) convergence and the (weak) convergence of probability distributions, respectively.  $[\eta]$  denotes the integer part of a real number  $\eta$ .  $\mathcal{C}^\eta(a, b)$  is the space of all  $[\eta] \geq 0$  times continuously differentiable functions  $f : (a, b) \mapsto \mathbb{R}$  on interval  $(a, b) \subset \mathbb{R}$  such that  $f^{([\eta])}$ , the  $[\eta]$ th derivative of  $f$ , is an  $(\eta - [\eta])$ -Hölderian function on  $(a, b)$ , i.e. there exists  $K \geq 0$  such that  $|f^{([\eta])}(x) - f^{([\eta])}(y)| \leq K |x - y|^{\eta - [\eta]}$  for all  $x, y \in (a, b)$ .

## 2. General results

### 2.1. Definitions of estimators

Consider a filter  $a := (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$  such that there exists  $m \in \mathbb{N}^*$  satisfying

$$\sum_{\ell=0}^q \ell^p a_\ell = 0 \quad \text{for } p = 0, \dots, (m-1) \quad \text{and} \quad \sum_{\ell=0}^q \ell^m a_\ell \neq 0. \quad (2.1)$$

Denote  $\mathcal{A}(m, q)$  the above class of filters. For  $a \in \mathcal{A}(m, q)$  and  $n \in \mathbb{N}^*$  define the corresponding generalized variations of  $Z$  by

$$V_n^a Z(t) := \sum_{\ell=0}^q a_\ell Z(t + \ell/n), \quad V^a Z(t) \equiv V_1^a Z(t). \quad (2.2)$$

The main examples of generalized variations are the usual simple variations corresponding to  $a = (1, -1) \in \mathcal{A}(1, 1)$ ,  $m = q = 1$ , and the second order variations corresponding to  $a = a^* := (1, -2, 1) \in \mathcal{A}(2, 2)$ ,  $m = q = 2$ . For a filter  $a = (a_0, \dots, a_q) \in \mathcal{A}(m, q)$  and  $j \in \mathbb{N}$  define its  $j$ th dilatation  $a^{(j)} = (a_0^{(j)}, \dots, a_{jq}^{(j)}) \in \mathcal{A}(m, jq)$  by

$$a_{ij}^{(j)} := a_i \quad \text{and} \quad a_k^{(j)} := 0 \quad \text{if } k \notin j\mathbb{N}.$$

Now we are going to define two estimators of the local Hurst function of a process  $(Z(t))_{t \in (0,1)}$  from observed sample  $(Z(1/n), Z(2/n), \dots, Z((n-1)/n))$ . For this, we first define a homogeneous function

$$\psi(x^{(1)}, x^{(2)}) := \frac{|x^{(1)} + x^{(2)}|}{|x^{(1)}| + |x^{(2)}|}, \quad (x^{(1)}, x^{(2)}) \in \mathbb{R}^2, \quad (2.3)$$

taking values in  $[0, 1]$  (with the convention  $0/0 := 1$ ). Note that  $\psi(\lambda x^{(1)}, \lambda x^{(2)}) = \psi(x^{(1)}, x^{(2)})$  for any  $\lambda \in \mathbb{R}$ . In [5], we used the function  $\psi$  for building a statistic, called the Increment Ratio (IR) statistic of order 2, measuring the roughness of a trajectory and defined as

$$\text{IR}_2(Z) := \frac{1}{n-2} \sum_{k=0}^{n-3} \psi \left( V_n^{a^*} Z \left( \frac{k}{n} \right), V_n^{a^*} Z \left( \frac{k+1}{n} \right) \right).$$

For discrete time processes, the IR statistic was first defined in [24]. When  $Z$  is a process having stationary increments and a FBM  $B_H$  as its tangent process, in the sense of (1.1),  $\text{IR}_2(Z) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \Lambda_2(H)$ , where

$$\Lambda_2(H) := \mathbb{E} \psi \left( V_n^{a^*} B_H(0), V_n^{a^*} B_H \left( \frac{1}{n} \right) \right). \quad (2.4)$$

See [5] for details. More explicitly,

$$\Lambda_2(H) = \frac{1}{\pi} \arccos(-\rho_2(H)) + \frac{1}{\pi} \sqrt{\frac{1 + \rho_2(H)}{1 - \rho_2(H)}} \log \left( \frac{2}{1 + \rho_2(H)} \right), \quad (2.5)$$

$$\rho_2(H) := \text{Cor} \left( V_n^{a^*} B_H(0), V_n^{a^*} B_H \left( \frac{1}{n} \right) \right) = \frac{-3^{2H} + 2^{2H+2} - 7}{8 - 2^{2H+1}}. \quad (2.6)$$

The function  $\Lambda_2$  does not depend on  $n$  by self-similarity of FBM and is monotone increasing on  $(0, 1)$  with  $\Lambda_2^{-1}$  as its inverse function; see [5]. Thus,  $\Lambda_2^{-1}(\text{IR}_2(Z))$  provides an estimator of  $H$ . For obtaining a “local” estimator of the value  $H(t)$  at a given point  $t$ , a localized version of  $\Lambda_2^{-1}(\text{IR}_2(Z))$  is defined as follows. For a given bandwidth parameter  $\alpha \in (0, 1)$ , define a neighborhood of  $t \in (0, 1)$  and its cardinal by

$$\mathcal{V}_{n,\alpha}(t) := \left\{ k \in \{1, 2, \dots, n - q - 1\}, \left| \frac{k}{n} - t \right| \leq n^{-\alpha} \right\} \quad \text{and} \quad v_{n,\alpha}(t) := \# \mathcal{V}_{n,\alpha}(t).$$

Note that for any  $t \in (0, 1)$  and  $n$  large enough,  $v_{n,\alpha}(t) = \lfloor 2n^{1-\alpha} \rfloor$  or  $\lfloor 2n^{1-\alpha} \rfloor + 1$  depending on the parity of  $\lfloor 2n^{1-\alpha} \rfloor$  = the integer part of  $2n^{1-\alpha}$ . The corresponding point-wise estimator  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t)$  of  $H(t)$  is defined by

$$\widehat{H}_{n,\alpha}^{(\text{IR})}(t) := \Lambda_2^{-1} \left( \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \psi \left( V_n^{a^*} Z \left( \frac{k}{n} \right), V_n^{a^*} Z \left( \frac{k+1}{n} \right) \right) \right), \quad t \in (0, 1). \quad (2.7)$$

In the literature, non-parametric estimation of the local Hurst function most often uses generalized quadratic variations (QV). In our paper, we consider the more efficient version of the QV estimator defined in [15], which we denote by  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  (see (2.8)). It is based on the fact that when  $Z$  is locally asymptotically self-similar having a FBM  $B_H$  as the tangent process at  $t \in (0, 1)$  (and satisfies some additional conditions), the localized sample mean

$$\text{QV}_{n,\alpha}^{(i)}(t) := \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \left( V_n^{a^{(i)}} Z \left( \frac{k}{n} \right) \right)^2,$$

of its generalized quadratic variations converges to the expectation  $\mathbb{E}(V_n^{a^{(i)}} B_H(t))^2 = -\frac{1}{2} \sum_{\ell=0}^{iq} \sum_{\ell'=0}^{iq} a_\ell^{(i)} a_{\ell'}^{(i)} \left| \frac{\ell - \ell'}{n} \right|^{2H(t)} = C_a \left( \frac{i}{n} \right)^{2H(t)}$  with  $C_a = -\frac{1}{2} \sum_{\ell=0}^q \sum_{\ell'=0}^q a_\ell a_{\ell'} |\ell - \ell'|^{2H(t)}$ . Therefore, the ordinary least squares log-regression of  $(i, \text{QV}_{n,\alpha}^{(i)}(t))_{1 \leq i \leq p}$  provides an estimate of  $2H(t)$  as the slope parameter of the regression line. The corresponding (non-parametric) estimator  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  of  $H(t)$  is defined by

$$\widehat{H}_{n,\alpha}^{(\text{QV})}(t) := \frac{1}{2} \frac{A}{A A^\top} \left( \log \left( \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \left( V_n^{a^{(i)}} Z \left( \frac{k}{n} \right) \right)^2 \right) \right)_{1 \leq i \leq p}^\top, \quad (2.8)$$

where  $A := (\log i - \frac{1}{p} \sum_{j=1}^p \log j)_{1 \leq i \leq p} \in \mathbb{R}^p$  is a row vector,  $A^\top$  its transposed vector (vector-column).

## 2.2. Assumptions

First, recall (see e.g. [24]) that for a filter  $a = (a_0, \dots, a_q) \in \mathcal{A}(m, q)$  with  $m \geq 1$  and for any fixed  $H \in (0, 1)$  and  $n \in \mathbb{N}$ ,  $(V_n^a B_H(j/n))_{j \in \mathbb{Z}}$  is a stationary Gaussian process,  $(V_n^a B_H(\frac{j}{n}))_{j \in \mathbb{Z}} \stackrel{\text{f.d.d.}}{=} n^{-H} (V^a B_H(j))_{j \in \mathbb{Z}}$  by self-similarity, furthermore,

$$\begin{aligned} \text{Var}(V_n^{a^{(i)}} B_H(j)) &= \frac{i^{2H}}{n^{2H}} C(H, a), \quad C(H, a) := \text{Var}(V^a B_H(0)) \\ &= -\frac{1}{2} \sum_{k,k'=0}^q a_k a_{k'} |k - k'|^{2H}, \quad i \in \mathbb{N}^*, \end{aligned}$$

$$\begin{aligned} \text{Cov}\left(V_n^a B_H\left(\frac{k}{n}\right), V_n^a B_H\left(\frac{k'}{n}\right)\right) &= -\frac{1}{2n^{2H}} \sum_{i=0}^q \sum_{j=0}^q a_i a_j |k' - k + j - i|^{2H} \\ &\sim \left( (-1)^{m+1} \frac{\left(\sum_{i=0}^q a_i i^m\right)^2}{2(m!)^2} \prod_{\ell=0}^{2m-1} (2H - \ell) \right) \\ &\quad \times \frac{|k' - k|^{2H-2m}}{n^{2H}}, \quad |k' - k| \rightarrow \infty. \end{aligned} \quad (2.9)$$

Note  $C(H, a) > 0$  for any  $H \in (0, 1)$ ,  $a \in \mathcal{A}(m, q)$  which fact follows from the harmonizable representation of FBM analogous to (1.3) yielding  $C(H, a) = K^2(H) \int_{\mathbb{R}} |\sum_{k=0}^q a_k (e^{ikx} - 1)|^2 |x|^{-2H-1} dx > 0$  since the integrand does not identically vanish on  $\mathbb{R}$ . In the particular case  $a = a^* = (1, -2, 1)$  ( $m = 2$ ), then

$$\begin{aligned} \text{Var}(V^{a^*} B_H(j)) &= 4 - 4^H, \\ \text{Cor}(V^{a^*} B_H(j), V^{a^*} B_H(j+1)) &= \rho_2(H), \end{aligned}$$

where  $\rho_2(H)$  is defined in (2.6).

For a given  $m \geq 1$ , a general Gaussian process  $Z = (Z(t))_{t \in (0,1)}$  and real numbers  $\kappa \geq 0$  and  $\alpha \in (0, 1)$ , we introduce the following assumptions.

(A) $_{\kappa}$  There exist continuous functions  $0 < H(t) < 1$  and  $c(t) > 0$  for  $t \in (0, 1)$  satisfying  $H(\cdot) \in \mathcal{C}^{\eta}(0, 1)$  and  $c(\cdot) \in \mathcal{C}^{\eta}(0, 1)$ ,  $\eta > 0$  such that for any  $0 < \varepsilon < 1/2$ ,  $j \geq 1$  and any filter  $a \in \mathcal{A}(m, q)$ ,  $q \geq 1$ ,

$$\max_{[n\varepsilon] \leq k \leq [(1-\varepsilon)n]} n^{\kappa} \left| \frac{\text{Var}(V_n^a Z(k/n))}{\text{Var}(V_n^a B_H(k/n)(0))} - c\left(\frac{k}{n}\right) \right| \xrightarrow{n \rightarrow \infty} 0. \quad (2.10)$$

(B) $_{\alpha}$  There exist  $C > 0$ ,  $\gamma > 1/2$  and  $0 \leq \theta < \gamma/2$  such that for any  $n \in \mathbb{N}^*$ ,  $0 < \varepsilon < 1/2$ ,  $[n\varepsilon] \leq k < k' \leq [(1-\varepsilon)n]$  and any filter  $a \in \mathcal{A}(m, q)$ ,  $q \geq 1$ ,

$$\left| \text{Cor}\left(V_n^a Z\left(\frac{k}{n}\right), V_n^a Z\left(\frac{k'}{n}\right)\right) \right| \leq C n^{(1-\alpha)\theta} (|k' - k| \wedge n^{1-\alpha})^{-\gamma}.$$

**Remark 1.** If condition (A) $_0$  is satisfied for  $m = 1$  then  $Z$  admits a FBM as a tangent process at each point  $t \in (0, 1)$ , in the sense that for any  $k \in \mathbb{Z}$ ,

$$n^{H(t)} \left( Z\left(\frac{[nt] + k + j}{n}\right) - Z\left(\frac{[nt] + k}{n}\right) \right)_{j \in \mathbb{Z}} \xrightarrow[n \rightarrow \infty]{\text{f.d.d.}} \left( \sqrt{c(t)} B_{H(t)}(j) \right)_{j \in \mathbb{Z}}. \quad (2.11)$$

Assumption (A) $_{\kappa}$  with  $\kappa > 0$  allows to control the uniform convergence rate of generalized variations of  $Z$  to the corresponding variations of the tangent process. Condition (B) $_{\alpha}$  provides a bound for the correlation of the process  $(V_n^a Z(k/n))_{[n\varepsilon] \leq k, k' \leq [(1-\varepsilon)n]}$  (the choice of this bound can be deduced from the expansion (2.9)). Under such conditions, the strong consistency and a CLT for the estimators  $\widehat{H}_{n,\alpha}^{(\text{QV})}$ ,  $\widehat{H}_{n,\alpha}^{(\text{IR})}$  respectively defined in (2.8), (2.7) can be established (see Section 2.3).

The following statement deduced from assumption  $(A)_\kappa$  will be often used in the remaining sections. Indeed the local asymptotic behavior of variances of  $V_n^a Z$  for any  $a \in \mathcal{A}(m, q)$  induced by  $(A)_\kappa$  implies an exhaustive local behavior of the processes  $V_n^a Z$  (given by the covariances since  $Z$  is supposed to be Gaussian). Hence,  $(A)_\kappa$  totally specifies the local behavior of  $V_n^a Z$  while  $(B)_\alpha$  deals about its “global” behavior.

**Property 1.** *Assumption  $(A)_\kappa$  implies that for a given  $m \geq 1$  and any  $a \in \mathcal{A}(m, q)$ ,  $q \geq 1$  and any  $\varepsilon > 0$ ,  $j \in \mathbb{N}$*

$$\begin{aligned} \max_{[n\varepsilon] \leq k \leq [(1-\varepsilon)n]} n^\kappa \left| \text{Cor} \left( V_n^a Z \left( \frac{k}{n} \right), V_n^a Z \left( \frac{k+j}{n} \right) \right) \right. \\ \left. - \text{Cor} \left( V^a B_{H\left(\frac{k}{n}\right)}(0), V^a B_{H\left(\frac{k}{n}\right)}(j) \right) \right| \xrightarrow{n \rightarrow \infty} 0. \end{aligned} \quad (2.12)$$

**Proof.** We have

$$\begin{aligned} \mathbb{E} \left[ V_n^a Z \left( \frac{k}{n} \right) V_n^a Z \left( \frac{k+j}{n} \right) \right] &= \frac{1}{2} \left\{ \text{Var} \left( V_n^a Z \left( \frac{k}{n} \right) + V_n^a Z \left( \frac{k+j}{n} \right) \right) \right. \\ &\quad \left. - \text{Var} \left( V_n^a Z \left( \frac{k}{n} \right) - V_n^a Z \left( \frac{k+j}{n} \right) \right) \right\} \\ &= \frac{1}{2} \left\{ \text{Var} \left( V_n^{a^+} Z \left( \frac{k}{n} \right) \right) - \text{Var} \left( V_n^{a^-} Z \left( \frac{k}{n} \right) \right) \right\}, \end{aligned}$$

where generalized variations  $V_n^{a^\pm} Z \left( \frac{k}{n} \right) = \sum_{i=0}^{j+q} a_i^\pm Z \left( \frac{k+i}{n} \right)$  correspond to filters  $a^\pm = (a_0^\pm, \dots, a_{q+j}^\pm) \in \mathbb{R}^{q+j+1}$  with

$$\begin{aligned} a_i^+ &:= a_i \mathbf{1}(0 \leq i \leq q) + a_{i-j} \mathbf{1}(j \leq i \leq j+q), \\ a_i^- &:= a_i \mathbf{1}(0 \leq i \leq q) - a_{i-j} \mathbf{1}(j \leq i \leq j+q). \end{aligned} \quad (2.13)$$

Note that  $a \in \mathcal{A}(m, q)$  implies  $a^\pm \in \mathcal{A}(m, j+q)$  for any  $j = 0, 1, \dots$ . From  $(A)_\kappa$  with  $a = a^\pm$  we have that uniformly in  $[n\varepsilon] \leq k \leq [(1-\varepsilon)n]$ ,  $c \left( \frac{k}{n} \right) \text{Var} V_n^{a^\pm} B_{H(k/n)}(0) (1 + o(n^{-\kappa}))$ , implying

$$\begin{aligned} \text{Cov} \left( V_n^a Z \left( \frac{k}{n} \right), V_n^a Z \left( \frac{k+j}{n} \right) \right) &= \frac{1}{2} c \left( \frac{k}{n} \right) \left\{ \text{Var} (V_n^{a^+} B_{H(k/n)}(0)) (1 + o(n^{-\kappa})) \right. \\ &\quad \left. - \text{Var} (V_n^{a^-} B_{H(k/n)}(0)) (1 + o(n^{-\kappa})) \right\} \\ &= \frac{1}{2} c \left( \frac{k}{n} \right) \left\{ \text{Var} (V_n^{a^+} B_{H(k/n)}(0)) \right. \\ &\quad \left. - \text{Var} (V_n^{a^-} B_{H(k/n)}(0)) \right\} + o(n^{-2H(k/n)-\kappa}) \\ &= c \left( \frac{k}{n} \right) \text{Cov} \left( V_n^a B_{H(k/n)}(0), V_n^a B_{H(k/n)} \left( \frac{j}{n} \right) \right) \\ &\quad + o(n^{-2H(k/n)-\kappa}). \end{aligned} \quad (2.14)$$

We also have

$$\begin{aligned} V_n^a Z\left(\frac{k}{n}\right) &= c\left(\frac{k}{n}\right) \text{Var}\left(V_n^a B_{H(k/n)}(0)\right)(1 + o(n^{-\kappa})), \\ V_n^a Z\left(\frac{k+j}{n}\right) &= c\left(\frac{k}{n}\right) \text{Var}\left(V_n^a B_{H(k/n)}(0)\right)(1 + o(n^{-\kappa})), \end{aligned} \quad (2.15)$$

where the last relation follows by writing  $\text{Var}\left(V_n^a Z\left(\frac{k+j}{n}\right)\right) = \text{Var}\left(V_n^{a+} Z\left(\frac{k}{n}\right)\right) - \text{Var}\left(V_n^a Z\left(\frac{k}{n}\right)\right) - 2\text{Cov}\left(V_n^a Z\left(\frac{k}{n}\right), V_n^a Z\left(\frac{k+j}{n}\right)\right)$  and using the facts established above. Then, (2.12) is deduced from (2.14) and (2.15).  $\square$

### 2.3. Asymptotic results

Denote

$$\mu := \frac{2(1-\alpha)\left(1 - \frac{\theta}{\gamma}\right) - \alpha}{4}, \quad \mu_1 := \frac{2(1-\alpha)\left(1 - \frac{\theta}{\gamma}\right) - \alpha - 1}{4}. \quad (2.16)$$

Note  $\mu > 0$  (respectively,  $\mu_1 > 0$ ) is equivalent to  $\alpha < \frac{2\gamma-2\theta}{3\gamma-2\theta}$  (respectively,  $\alpha < \frac{\gamma-2\theta}{3\gamma-2\theta}$ ).

**Theorem 1.** Let  $Z = (Z(t))_{t \in (0,1)}$  be a zero-mean Gaussian process.

(i) If  $Z$  satisfies condition  $(A)_0$  and  $(B)_\alpha$  with  $m = 2$ ,  $\alpha \in (0, 1)$  then for any  $t \in (0, 1)$

$$\widehat{H}_{n,\alpha}^{(\text{IR})}(t) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t).$$

(ii) If  $Z$  satisfies conditions  $(A)_0$  and  $(B)_\alpha$  with  $m = 2$ ,  $0 < \alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$  then for any  $t \in (0, 1)$

$$\widehat{H}_{n,\alpha}^{(\text{IR})}(t) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} H(t).$$

(iii) If  $Z$  satisfies conditions  $(A)_\kappa$  and  $(B)_\alpha$  with  $m = 2$ ,  $\kappa \geq \mu$  and  $\frac{2\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta \wedge 2)} \leq \alpha < \frac{2\gamma-2\theta}{3\gamma-2\theta}$  then for any  $\epsilon > 0$

$$\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(\text{IR})}(t) - H(t)| = O_p(n^{-\mu}).$$

(iv) If  $Z$  satisfies conditions  $(A)_\kappa$  and  $(B)_\alpha$  with  $m = 2$ ,  $\kappa \geq \mu_1$  and  $\frac{\gamma-2\theta}{3\gamma-2\theta+4\gamma(\eta \wedge 2)} \leq \alpha < \frac{\gamma-2\theta}{3\gamma-2\theta}$  then for any  $\epsilon > 0$ ,  $\delta > 0$

$$\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(\text{IR})}(t) - H(t)| = O(n^{-(\mu_1-\delta)}) \quad \text{a.s.}$$

**Theorem 2.** Let  $Z = (Z(t))_{t \in (0,1)}$  be a zero-mean Gaussian process satisfying  $(A)_\kappa$  and  $(B)_\alpha$ , with  $m = 2$ ,  $\alpha > \frac{1}{1+2(\eta \wedge 2)}$ ,  $\kappa \geq \frac{1-\alpha}{2}$  and  $\theta = 0$ . Then for any  $u \in \mathbb{N}^*$  and any  $t_1, \dots, t_u \in (0, 1)$ ,  $t_i \neq t_j$  ( $i \neq j$ )

$$\sqrt{2n^{1-\alpha}} \left( \widehat{H}_{n,\alpha}^{(\text{IR})}(t_i) - H(t_i) \right)_{1 \leq i \leq u} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \left( W^{(\text{IR})}(t_i) \right)_{1 \leq i \leq u}, \quad (2.17)$$



where  $W^{(\text{IR})}(t_i)$ ,  $i = 1, \dots, u$  are independent centered Gaussian r.v.'s such as, with  $\Lambda_2(\cdot)$  defined in (2.5),

$$\mathbb{E}[W^{(\text{IR})}(t_i)]^2 := \left[ \frac{\partial}{\partial x} (\Lambda_2)^{-1} (\Lambda_2(H(t_i))) \right]^2 \sigma^2(H(t_i)) \quad (2.18)$$

$$\text{where } \sigma^2(H) := \sum_{k \in \mathbb{Z}} \text{Cov} \left( \psi(V^{a^*} B_H(0), V^{a^*} B_H(1)), \right. \\ \left. \times \psi(V^{a^*} B_H(k), V^{a^*} B_H(k+1)) \right) \quad \text{for } H \in (0, 1). \quad (2.19)$$

The following Theorem 3 extends Theorems 1 and 2 to the estimator  $\hat{H}_{n,\alpha}^{(\text{QV})}$ . As noted in the Introduction, both estimators  $\hat{H}_{n,\alpha}^{(\text{IR})}(t)$  and  $\hat{H}_{n,\alpha}^{(\text{QV})}$  are written in terms of sums of nonlinear functions of Gaussian vectors of the Hermite rank 2 and their asymptotic analysis is very similar. The proof of Theorem 3 can be deduced *mutatis mutandis* from the proofs of Theorems 1 and 2 and is omitted.

**Theorem 3.** Let  $Z$  be a zero-mean Gaussian process and  $a \in \mathcal{A}(m, q)$ ,  $m \geq 1$  a filter:

- (a) The statements of Theorem 1 (i)–(iv) hold for  $\hat{H}_{n,\alpha}^{(\text{QV})}(t)$  instead of  $\hat{H}_{n,\alpha}^{(\text{IR})}(t)$  under respective assumptions (i)–(iv) with  $m = 2$  replaced by  $m \geq 1$ .
- (b) Let  $Z$  satisfy the assumptions of Theorem 2 with  $m = 2$  replaced by  $m \geq 1$ . Then for any  $u \in \mathbb{N}^*$  and any  $t_1, \dots, t_u \in (0, 1)$ ,  $t_i \neq t_j$  ( $i \neq j$ )

$$\sqrt{2n^{1-\alpha}} \left( \hat{H}_{n,\alpha}^{(\text{QV})}(t_i) - H(t_i) \right)_{1 \leq i \leq u} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \left( W^{(\text{QV})}(t_i) \right)_{1 \leq i \leq u}, \quad (2.20)$$

where  $W^{(\text{QV})}(t_i)$ ,  $i = 1, \dots, u$  are independent centered Gaussian r.v.'s such as, with  $A$  defined in (2.8),

$$\mathbb{E}[W^{(\text{QV})}(t_i)]^2 := \frac{A^\top \Gamma(H(t_i)) A}{4(A^\top A)^2} \quad \text{and} \\ \Gamma(H) := \left( \frac{2}{i_1^{2H} i_2^{2H}} \sum_{j \in \mathbb{Z}} \left[ \frac{\sum_{k_1, k_2=0}^q a_{k_1}^{(1)} a_{k_2}^{(1)} |i_1 k_1 - i_2 k_2 + j|^{2H}}{\sum_{k_1, k_2=0}^q a_{k_1}^{(1)} a_{k_2}^{(1)} |k_1 - k_2|^{2H}} \right]^2 \right)_{1 \leq i_1, i_2 \leq p}. \quad (2.21)$$

**Remark 2.** Let us note that in the case of stationary increment process  $Z$ , the functions  $H(t)$  and  $c(t)$  in assumption (A) $_\kappa$  do not depend on  $t$ . In this case, it is natural to consider “global” estimators  $\hat{H}_{n,0}^{(\text{IR})}$ ,  $\hat{H}_{n,0}^{(\text{QV})}$  defined by replacing  $\mathcal{V}_{n,\alpha}(t)$  in (2.7), (2.8) by the full grid  $\mathcal{V}_{n,0} = \{1, 2, \dots, n - q - 1\}$ . Such estimators of constant Hurst parameter  $H \in (0, 1)$  were discussed in papers [5,4], whose results can be compared with Theorems 1–3. In particular, the “pointwise results” in Theorem 1(i), (ii) for  $H(t) = \text{const}$  essentially follow from [5, Proposition 2.2 and Theorem 4.1]. On the other hand, the “uniform results” in parts (iii), (iv) of Theorems 1 and 3 are new also in the case of  $H(t) = \text{const}$  and a stationary increment process  $Z$ , including the case of FBM  $Z = B_H$  (see below).

**Example 2.1.** For the (classical) FBM  $Z = B_H$ , assumption  $(A)_\kappa$  (with  $H(t) \equiv H \in (0, 1)$ ,  $c(t) \equiv 1$ ) is satisfied for all  $\kappa > 0$ . Using a Taylor expansion, it easily follows that there exists  $C > 0$  such that  $|\text{Cor}(V_n^a B_H(\frac{k}{n}), V_n^a B_H(\frac{k'}{n}))| \leq C(1 + |k - k'|)^{2H-2m}$ . Thus,  $Z = B_H$  satisfies assumption  $(B)_\alpha$  for any  $\alpha \in (0, 1)$  with  $\theta = 0$  and  $\gamma = 2m - 2H$ . Hence, for  $Z = B_H$ , statements (iii) of Theorems 1 and 3 hold with  $\alpha = 2/11$ ,  $\mu = 4/11$ , while statements (iv) of the same theorems are valid for  $\alpha = 1/11$ ,  $\mu_1 = 2/11$ .

### 3. Pseudo-generalized least squares versions of the IR and QV estimators

Asymptotic and finite-sample performance of the estimators  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t)$  and  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  can be improved by using their pseudo-generalized least squares (PGLS) versions  $\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t)$  and  $\widehat{H}_{n,\alpha}^{(\text{QV}2)}(t)$  as defined below.

*Definition of  $\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t)$ .* Let  $a^{(i*)}$  denote the  $i$ th dilatation of the filter  $a^* = (1, -2, 1)$ ,  $i \in \mathbb{N}^*$ , so that  $V_n^{a^{(i*)}} Z(\ell/n) = Z((\ell + 2i)/n) - 2Z((\ell + i)/n) + Z(\ell/n)$ . The corresponding IR estimator of  $H(t)$  is defined as

$$\widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) := [\Lambda_2^{(i)}]^{-1} \left( \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \psi \left( V_n^{a^{(i*)}} Z \left( \frac{k}{n} \right), V_n^{a^{(i*)}} Z \left( \frac{k+1}{n} \right) \right) \right),$$

where the function  $\Lambda_2^{(i)} : (0, 1) \rightarrow (0, 1)$  is defined as the expectation

$$\begin{aligned} \Lambda_2^{(i)}(H) &:= E \psi \left( V_n^{a^{(i*)}} B_H(0), V_n^{a^{(i*)}} B_H \left( \frac{1}{n} \right) \right) \\ &= \frac{1}{\pi} \arccos(-\rho_2^{(i)}(H)) + \frac{1}{\pi} \sqrt{\frac{1 + \rho_2^{(i)}(H)}{1 - \rho_2^{(i)}(H)}} \log \left( \frac{2}{1 + \rho_2^{(i)}(H)} \right), \end{aligned}$$

$$\begin{aligned} \text{and } \rho_2^{(i)}(H) &:= \text{Cor} \left( V_n^{a^{(i*)}} B_H(0), V_n^{a^{(i*)}} B_H \left( \frac{1}{n} \right) \right) \\ &= \frac{-|2i + 1|^{2H} - |2i - 1|^{2H} + 4|i + 1|^{2H} + 4|i - 1|^{2H} - 6}{8 - 2^{2H+1}}. \end{aligned}$$

Under the assumptions of Theorem 2, these estimators satisfy the  $p$ -dimensional CLT:

$$n^{(1-\alpha)/2} \left( \widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) - H(t) \right)_{1 \leq i \leq p} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}_p \left( 0, \Sigma^{(p)}(H(t)) \right) \quad (3.1)$$

with the limit covariance matrix  $\Sigma^{(p)}(H(t))$ , where for  $H \in (0, 1)$

$$\Sigma^{(p)}(H) := \left( \left[ \frac{\partial}{\partial x} (\Lambda_2^{(i)})^{-1} (\Lambda_2^{(i)}(H)) \right] \sigma_{ij}(H) \left[ \frac{\partial}{\partial x} (\Lambda_2^{(j)})^{-1} (\Lambda_2^{(j)}(H)) \right] \right)_{1 \leq i, j \leq p}, \quad (3.2)$$

$$\begin{aligned} \sigma_{ij}(H) &:= \sum_{k \in \mathbb{Z}} \text{Cov} \left( \frac{|V_n^{a^{(i*)}} B_H(0) + V_n^{a^{(i*)}} B_H(1)|}{|V_n^{a^{(i*)}} B_H(0)| + |V_n^{a^{(i*)}} B_H(1)|}, \right. \\ &\quad \left. \times \frac{|V_n^{a^{(j*)}} B_H(k) + V_n^{a^{(j*)}} B_H(k+1)|}{|V_n^{a^{(j*)}} B_H(k)| + |V_n^{a^{(j*)}} B_H(k+1)|} \right) \end{aligned} \quad (3.3)$$

(note that  $V^{a^{(j*)}} B_H(k) = B_H(2j+k) - 2B_H(j+k) + B_H(k)$  for  $j = 1, \dots, p$  and  $k \in \mathbb{Z}$ ). Now define

$$\widehat{\Sigma}^{(p)}(t) := \Sigma^{(p)}(\widehat{H}_{n,\alpha}^{(\text{IR})}(t)), \quad (3.4)$$

which is a consistent estimator of  $\Sigma^{(p)}(H(t))$  since  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t)$  is a consistent estimator of  $H(t)$ . Then, with the vector-column  $\mathbf{1}_p = (1, 1, \dots, 1)^\top \in \mathbb{R}^p$ , the PGLSE  $\widehat{H}_{n,\alpha}^{(\text{IR2})}(t)$  of  $H(t)$  is defined by

$$\widehat{H}_{n,\alpha}^{(\text{IR2})}(t) := (\mathbf{1}_p^\top (\widehat{\Sigma}^{(p)}(t))^{-1} \mathbf{1}_p)^{-1} \mathbf{1}_p^\top (\widehat{\Sigma}^{(p)}(t))^{-1} \left( \widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) \right)_{1 \leq i \leq p}. \quad (3.5)$$

*Definition of  $\widehat{H}_{n,\alpha}^{(\text{QV2})}(t)$ .* Analogously to (3.4), define

$$\widehat{\Gamma}(t) := \Gamma(\widehat{H}_{n,\alpha}^{(\text{QV})}(t)), \quad (3.6)$$

which is a consistent estimator of  $\Gamma(H(t))$  since  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  is a consistent estimator of  $H(t)$ . The one-dimensional CLT satisfied by  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  at a given point  $t \in (0, 1)$  (see (2.20)) is a consequence of the following  $p$ -dimensional CLT:

$$\sqrt{2n^{1-\alpha}} \left( \widehat{T}_{n,\alpha}^{(p)}(t) - 2H(t) (\log(i/n))_{1 \leq i \leq p} - C(t) \mathbf{1}_p \right) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}_p(0, \Gamma(H(t)))$$

with  $C(t) := \log\left(-\frac{c(t)}{2} \sum_{k,k'=0}^q a_k^{(1)} a_{k'}^{(1)} |k' - k|^{2H(t)}\right)$  and

$$\widehat{T}_{n,\alpha}^{(p)}(t) := \left( \log \left( \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} |V_n^{a^{(i)}} Z(k/n)|^2 \right) \right)_{1 \leq i \leq p};$$

see [6] for details. Then  $2\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  is the slope of the ordinary least squares regression of the vector  $\widehat{T}_{n,\alpha}^{(p)}(t)$  onto the vector  $(\log(i/n))_{1 \leq i \leq p}$ . The PGLS estimators  $\widehat{H}_{n,\alpha}^{(\text{QV2})}(t)$  and  $\widehat{C}_{n,\alpha}^{(\text{QV2})}(t)$  of  $H(t)$  and  $C(t)$  are obtained from the corresponding generalized least squares regression with the covariance matrix  $\Gamma(H(t))$  replaced by its estimate  $\widehat{\Gamma}(t)$  (3.6):

$$\begin{pmatrix} 2\widehat{H}_{n,\alpha}^{(\text{QV2})}(t) \\ \widehat{C}_{n,\alpha}^{(\text{QV2})}(t) \end{pmatrix} := \left( Z_n^{(p)} \widehat{\Gamma}(t)^{-1} (Z_n^{(p)})^\top \right)^{-1} Z_n^{(p)} \widehat{\Gamma}(t)^{-1} (\widehat{T}_{n,\alpha}^{(p)}(t))^\top, \quad (3.7)$$

$$\text{with } Z_n^{(p)} := \begin{pmatrix} \log(1/n) & \log(2/n) & \cdots & \log(p/n) \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \quad (3.8)$$

**Corollary 1.** (i) Assume that for each  $0 < H < 1$  the matrix  $\Sigma^{(p)}(H)$  of (3.2) is non-degenerate and continuously depends on  $H$ . Then all statements of Theorem 1 (i)–(iv) hold for the estimator  $\widehat{H}_{n,\alpha}^{(\text{IR2})}$  instead of  $\widehat{H}_{n,\alpha}^{(\text{IR})}$ .

(ii) Assume that for each  $0 < H < 1$  the matrix  $\Gamma(H)$  in (2.21) is non-degenerate and continuously depends on  $H$ . Then the statement of Theorem 3(a) holds for the estimator  $\widehat{H}_{n,\alpha}^{(\text{QV2})}$  instead of  $\widehat{H}_{n,\alpha}^{(\text{QV})}$ .

**Corollary 2.** Let  $Z$  satisfy the assumptions of Theorem 2 and the matrices  $\Sigma^{(p)}(H)$ ,  $\Gamma(H)$ ,  $0 < H < 1$  satisfy the conditions of Corollary 1 (i)–(ii). Then for any  $u \in \mathbb{N}^*$  and any  $t_1, \dots, t_u \in$

$(0, 1)$ ,  $t_i \neq t_j$  ( $i \neq j$ ),

$$\sqrt{2n^{1-\alpha}} (\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t_i) - H(t_i))_{1 \leq i \leq u} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (W^{(\text{IR}2)}(t_i))_{1 \leq i \leq u} \quad (3.9)$$

and

$$\sqrt{2n^{1-\alpha}} \begin{pmatrix} 2(\widehat{H}_{n,\alpha}^{(\text{QV}2)}(t_i) - H(t_i)) \\ \widehat{C}_{n,\alpha}^{(\text{QV}2)}(t_i) - C(t_i) \end{pmatrix}_{1 \leq i \leq u} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (W^{(\text{QV}2)}(t_i))_{1 \leq i \leq u}. \quad (3.10)$$

In (3.9),  $W^{(\text{QV}2)}(t_i)$ ,  $1 \leq i \leq u$  are independent centered Gaussian r.v.'s with respective variances  $E[W^{(\text{IR}2)}(t_i)]^2 := (\mathbf{1}_p^\top (\Sigma^{(p)}(H(t_i)))^{-1} \mathbf{1}_p)^{-1}$ . In (3.10),  $W^{(\text{QV}2)}(t_i)$ ,  $1 \leq i \leq u$  are independent centered Gaussian random vectors with respective  $2 \times 2$  covariance matrices  $E[W^{(\text{QV}2)}(t_i) W^{(\text{QV}2)}(t_i)^\top] := (Z_1^{(p)} \Gamma(H(t_i))^{-1} (Z_1^{(p)})^\top)^{-1}$ .

#### 4. The case of multifractional Brownian motion

This section details assumptions  $(A)_\kappa$  and  $(B)_\alpha$  for the MBM (1.3) and its generalization due to Stoev and Taqqu [22]. As a consequence, the asymptotic behaviors of  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t)$ ,  $\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t)$ ,  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  and  $\widehat{H}_{n,\alpha}^{(\text{QV}2)}(t)$  for the above process are established under explicit conditions on the parameters  $\alpha, \eta$  and the function  $H(\cdot)$ . These limit theorems generalize those already obtained for the MBM (1.3), notably in [12,15].

The above mentioned generalization of the MBM in [22] is defined as

$$Y_{(a^+, a^-)}(t) := Y_{(a^+, a^-)}(H(t), t), \quad t \in \mathbb{R}, \quad (4.1)$$

where  $\{Y_{(a^+, a^-)}(H, t), H \in (0, 1), t \in \mathbb{R}\}$  is a Gaussian random field given by stochastic integral representation

$$Y_{(a^+, a^-)}(H, t) := \frac{\Gamma\left(H + \frac{1}{2}\right)}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H+\frac{1}{2}}} U_{(a^+, a^-)}(H, x) W(dx), \quad (4.2)$$

where  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  are parameters,  $W(dx)$  is the same white noise as in (1.3), and

$$U_{(a^+, a^-)}(H, x) := a^+ e^{-i \operatorname{sign}(x) \left(H + \frac{1}{2}\right) \frac{\pi}{2}} + a^- e^{i \operatorname{sign}(x) \left(H + \frac{1}{2}\right) \frac{\pi}{2}}, \quad H \in (0, 1), x \in \mathbb{R}. \quad (4.3)$$

The above definition implies  $EY_{(a^+, a^-)}^2(t) = EY_{(a^+, a^-)}^2(H(t), 1)|t|^{2H(t)}$  and a covariance structure of  $Y_{(a^+, a^-)}$  similar to that of the MBM in (1.3) (see [22, Theorem 4.1]). In particular,  $Y_{(1,0)}$  gives a harmonizable representation of the time-domain MBM of [20]. The so-called well-balanced case  $a^+ = a^- = a \neq 0$ , or

$$Y_{(a,a)}(t) = \frac{2a}{\sqrt{2\pi}} \cos\left(\left(H(t) + \frac{1}{2}\right) \frac{\pi}{2}\right) \Gamma\left(H(t) + \frac{1}{2}\right) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t)+\frac{1}{2}}} W(dx)$$

is more delicate, see [22, Section 5], since the function  $H \mapsto \cos\left(\left(H + \frac{1}{2}\right) \frac{\pi}{2}\right)$  vanishes and changes its sign at  $H = 1/2$ . In particular,  $Y_{(a,a)}(t)/(EY_{(a,a)}^2(H(t), 1))^{1/2} = -\operatorname{sign}(H(t) - \frac{1}{2})X(t)$ , where  $X$  is defined at (1.3). Since  $X$  has a.s. continuous trajectories under the condition

(4.6) on  $H(\cdot)$ , see [1], this implies that the normalized process  $Y_{(a,a)}(t)/(EY_{(a,a)}^2(H(t), 1))^{1/2}$  is generally a.s. discontinuous at each  $t \neq 0$  with  $H(t) = 1/2$ .

In order to avoid the above complication, we introduce the following generalization  $X_{(a^+, a^-)}$  of the MBM in (1.3):

$$X_{(a^+, a^-)}(t) := \begin{cases} X(t), & a^+ = a^- \neq 0, \\ Y_{(a^+, a^-)}(t)/\text{Var}^{1/2}(Y_{(a^+, a^-)}(H(t), 1)), & a^+ \neq a^-, \end{cases}$$

where  $X$  and  $Y_{(a^+, a^-)}$  are defined at (1.3) and (4.1)–(4.2), respectively. Accordingly, the process  $X_{(a^+, a^-)}$  is defined by the following harmonizable representation:

$$X_{(a^+, a^-)}(t) = K(H(t)) \int_{\mathbb{R}} \frac{e^{itx} - 1}{|x|^{H(t)+\frac{1}{2}}} \mathcal{U}_{(a^+, a^-)}(H(t), x) W(dx), \quad t \in (0, 1), \quad (4.4)$$

where  $K(H)$  is defined in (1.3) and

$$\mathcal{U}_{(a^+, a^-)}(H, x) := \begin{cases} 1, & a^+ = a^- \neq 0, \\ \frac{a^+ e^{-i \text{sign}(x)(H+\frac{1}{2})\frac{\pi}{2}} + a^- e^{i \text{sign}(x)(H+\frac{1}{2})\frac{\pi}{2}}}{((a^+)^2 + (a^-)^2 - 2a^+a^- \sin(\pi H))^{1/2}}, & a^+ \neq a^-. \end{cases} \quad (4.5)$$

Note that the function  $H \mapsto \mathcal{U}_{(a^+, a^-)}(H, x)$  is analytic on  $(0, 1)$  for all  $x \in \mathbb{R}$  and  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . This definition of  $X_{(a^+, a^-)}$  induces that for all  $t \in (0, 1)$  and  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$

$$EX_{(a^+, a^-)}(t) = 0 \quad \text{and} \quad EX_{(a^+, a^-)}^2(t) = t^{2H(t)}.$$

The following assumption  $(C)_\eta$  is crucial for regularity properties of the MBM; see [1,3].

$(C)_\eta$  There exists  $\eta > 0$  such that  $H(\cdot) \in \mathcal{C}^\eta(0, 1)$  and

$$0 < \inf_{t \in (0, 1)} H(t) \leq \sup_{t \in (0, 1)} H(t) < \min(1, \eta).$$

It is known that assumption  $(C)_\eta$  guarantees that the MBM  $X = X_{(1,1)}$  of (1.3) is locally asymptotically self-similar at each point  $t \in (0, 1)$  having a FBM  $B_{H(t)}$  as its tangent process at  $t$  [13] and its pointwise Hölder exponent coincides with  $H(t)$  [3]. A particular case ( $a = (1, 0, \dots, 0, -1)$  and  $m = 1$ ) of the following proposition shows that a FBM  $B_{H(t)}$  is also a tangent process at  $t$  for  $X_{(a^+, a^-)}$  defined in (4.4); see Remark 1 and Property 1.

**Proposition 1.** *Let  $H(\cdot)$  satisfy assumption  $(C)_\eta$ . Then, for any  $a \in \mathcal{A}(m, q)$ , with  $m \geq 1$ ,  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  and  $0 < \varepsilon < 1/2$ , there exists a constant  $C_\ell(\varepsilon) > 0$  such that for  $n \geq 1$ ,*

$$\begin{aligned} & \max_{[n\varepsilon] \leq k \leq [(1-\varepsilon)n]} \left| \frac{\text{Var} \left( V_n^a X_{(a^+, a^-)} \left( \frac{k}{n} \right) \right)}{\text{Var} \left( V_n^a B_{H(\frac{k}{n})} \left( \frac{k}{n} \right) \right)} - 1 \right| \\ & \leq C_\ell(\varepsilon) \left( \frac{\log n}{n^{(\eta \wedge 1)}} + \frac{1}{n^{2((\eta \wedge m) - \sup_{t \in (0, 1)} H(t))}} \right). \end{aligned} \quad (4.6)$$

The next proposition provides an expansion of the covariance of the process  $V_n^{a*} X_{(a^+, a^-)}$ . For the ease of writing, we consider here the case  $a = a^*$ ,  $m = 2$  only. The case of general filter  $a \in \mathcal{A}(m, q)$  is discussed in Lemma 4 (see Section 7).

**Proposition 2.** Let  $H(\cdot)$  satisfy assumption (C) $_{\eta}$  and  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ . Then for any  $0 < \varepsilon < 1/2$ ,  $n \in \mathbb{N}^*$ ,  $[n\varepsilon] \leq k < k' \leq [(1 - \varepsilon)n]$ ,  $k' - k > 2q$ ,

$$\begin{aligned} \text{Cov} \left( V_n^{a*} X_{(a^+, a^-)} \left( \frac{k}{n} \right), V_n^{a*} X_{(a^+, a^-)} \left( \frac{k'}{n} \right) \right) &= A \left( \frac{k}{n}, \frac{k'}{n} \right) V_n^{a*} H \left( \frac{k}{n} \right) V_n^{a*} H \left( \frac{k'}{n} \right) \\ &+ B_2 \left( \frac{k}{n}, \frac{k'}{n} \right) \frac{|k - k'|^{H(\frac{k}{n}) + H(\frac{k'}{n}) - 4}}{n^{H(\frac{k}{n}) + H(\frac{k'}{n})}} + \lambda_n(k, k'), \end{aligned} \quad (4.7)$$

where (recall)  $V_n^{a*} H \left( \frac{k}{n} \right) = H \left( \frac{k+2i}{n} \right) - 2H \left( \frac{k+i}{n} \right) + H \left( \frac{k}{n} \right)$  are the second-order variations of  $H(\cdot)$  and  $A(t, t')$ ,  $B_2(t, t')$  are defined in (7.11)–(7.12). The remainder term  $\lambda_n(k, k')$  in (4.7) satisfies the following bound: for any  $\delta > 0$  there exist  $n_0 \in \mathbb{N}^*$ ,  $k_0 \in \mathbb{N}^*$  such that for any  $n > n_0$ ,  $|k - k'| > k_0$ ,  $[n\varepsilon] \leq k, k' \leq [(1 - \varepsilon)n]$

$$|\lambda_n(k, k')| \leq \delta \left( \frac{1}{n^{2(\eta \wedge 2)}} + \frac{|k' - k|^{H(\frac{k}{n}) + H(\frac{k'}{n}) - 4}}{n^{H(\frac{k}{n}) + H(\frac{k'}{n})}} \right). \quad (4.8)$$

**Remark 3.** From definitions (7.11)–(7.12) it immediately follows that  $A(t, t')$  and  $B_2(t, t')$  are bounded on  $(0, 1) \times (0, 1)$  and have finite and generally non-vanishing limits

$$\lim_{t' \rightarrow t} A(t, t') = t^{2H(t)} \log^2 t \quad \text{and} \quad \lim_{t' \rightarrow t} B_2(t, t') = -\frac{1}{2} \prod_{\ell=0}^3 (2H(t) - \ell).$$

Thus, Proposition 2 and Lemma 4 allow to obtain lower bounds in (4.7) and a decorrelation rate of generalized variations of the MBM, see Corollary 3, (4.9), which is close to optimal. In turn, the decorrelation rate of generalized variations determines the decay rate of the variance and the 4th moment of the estimators written in terms of sums of nonlinear functions of Hermite rank 2 of normalized generalized variations (see the proof of Theorem 2, also (6.10), (6.11)). Proposition 2 and Lemma 4 also permit to construct explicit counter-examples to some earlier results pertaining to this issue. In particular, we can show that the expansions obtained in [15, Lemma 1] and [12, Lemma 2] are erroneous. A detailed discussion of the above counter-examples is given in the extended version of this paper [6]. Let us note also that the Erratum in [16] does not concern [15, Lemma 1] but another error in the last paper.

**Remark 4.** Let us note that in [19, 8], some correct bounds of the decorrelation rate of  $V_n^{a*} X \left( \frac{k}{n} \right)$  are obtained, but their bounds are less sharp than the bound in (4.9). Another correct bound was also obtained in the preprint [2, Lemma 2.4].

Propositions 1 and 2 help to verify assumptions (A) $_{\kappa}$  and (B) $_{\alpha}$  for the MBM in (4.4).

**Corollary 3.** Let the conditions of Proposition 2 hold. Then for any  $0 < \varepsilon < 1/2$ , there exist  $C(\varepsilon) > 0$  and  $n_0(\varepsilon) \in \mathbb{N}^*$  such that for any  $n \geq n_0(\varepsilon)$  and any  $[n\varepsilon] \leq k < k' \leq [(1 - \varepsilon)n]$ ,

$$\left| \text{Cor} \left( V_n^{a^*} X_{(a^+, a^-)} \left( \frac{k}{n} \right), V_n^{a^*} X_{(a^+, a^-)} \left( \frac{k'}{n} \right) \right) \right| \leq C(\varepsilon) \left( \frac{1}{|k' - k|^{4-H(\frac{k}{n})-H(\frac{k'}{n})}} + \frac{1}{n^{2(\eta \wedge 2)-H(\frac{k}{n})-H(\frac{k'}{n})}} \right). \quad (4.9)$$

Moreover,

- (i)  $X_{(a^+, a^-)}$  satisfies assumption  $(A)_\kappa$  with  $m = 2$  and any  $0 \leq \kappa < \min\{(\eta \wedge 1), 2(\eta - \sup_{t \in (0,1)} H(t))\}$ ;
- (ii)  $X_{(a^+, a^-)}$  satisfies assumption  $(B)_\alpha$  with  $m = 2$  and any  $0 \leq \theta < \gamma/2$ ,  $2 \geq \gamma > 1/2$  satisfying the following condition:

$$\gamma - \theta \leq \min \left( \frac{2(\eta \wedge 2) - 2H(t)}{1 - \alpha}, 4 - 2H(t) \right), \quad \text{for any } t \in (0, 1). \quad (4.10)$$

**Remark 5.** The presence of the second terms on the r.h.s. of (4.6) and (4.9) indicates that dependence properties of increments of the MBM are sensitive to the smoothness parameter  $\eta$  of the Hurst function. These terms have a negative effect on the estimation of  $H(\cdot)$  and the convergence rates, by imposing restrictions on the bandwidth  $\alpha$ ; see Proposition 3. [23] argued that the dependence properties of the MBM are rather peculiar and proposed a different class of multifractional Gaussian processes defined via nonhomogeneous fractional integration of white noise. These processes are locally self-similar in the sense of (1.1) under less restrictive conditions on  $H(\cdot)$  and have better decorrelation properties than the MBM [23]. On the other hand, the covariance function of the above-mentioned time-varying fractionally integrated processes is not a local function of  $H(t)$  and  $H(t')$  as in the case of the MBM (see (7.1)) and its study is more difficult. Extending Propositions 1 and 2 to time-varying fractionally integrated processes of [23] is an interesting open problem.

**Proposition 3.** Let  $X_{(a^+, a^-)}$  be the MBM of (4.4) with  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  and  $H(\cdot)$  satisfying assumption  $(C)_\eta$ ,  $\eta > 0$ . Then, with  $a = a^*$  and  $\mathcal{E} = \text{IR}, \text{IR2}, \text{QV}$  or  $\text{QV2}$ , we have the following.

- (i) For any  $t \in (0, 1)$  and any  $\alpha \in (0, 1)$ ,  $\widehat{H}_{n,\alpha}^{(\mathcal{E})}(t) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} H(t)$ .
- (ii) For any  $t \in (0, 1)$ ,  $\widehat{H}_{n,\alpha}^{(\mathcal{E})}(t) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} H(t)$ , provided  $H(t) < \eta - \frac{1}{8}$  and  $\max(0, 1 - 4((\eta \wedge 2) - H(t))) < \alpha < 1/2$  hold.
- (iii) For any  $\epsilon > 0$ ,  $\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(\mathcal{E})}(t) - H(t)| = O_p(n^{-(2-3\alpha)/4})$ , provided  $\alpha$  and  $\eta$  satisfy

$$\alpha > \sup_{t \in (0,1)} (1 - 4((\eta \wedge 2) - H(t))), \quad (4.11)$$

$$\sup_{t \in (0,1)} H(t) < \eta - \frac{1}{12} \quad \text{and} \quad \frac{2}{3 + 4(\eta \wedge 2)} \leq \alpha < \frac{2}{3}. \quad (4.12)$$

(iv) For any  $\epsilon > 0$  and  $\delta > 0$ ,  $\sup_{\epsilon < t < 1-\epsilon} |\widehat{H}_{n,\alpha}^{(\mathcal{E})}(t) - H(t)| = O(n^{-(1-3\alpha-\delta)/4})$  a.s., provided  $\alpha$  and  $\eta$  satisfy (4.11),

$$\sup_{t \in (0,1)} H(t) < \eta - \frac{1}{6} \quad \text{and} \quad \frac{1}{3+4(\eta \wedge 2)} \leq \alpha < \frac{1}{3}. \quad (4.13)$$

(v) For any  $u \in \mathbb{N}^*$  and  $t_1, \dots, t_u \in (0, 1)$ ,  $t_i \neq t_j$  ( $i \neq j$ ) the multidimensional CLTs (2.17), (2.20), (3.9) and (3.10) hold, provided the matrix-valued functions  $\Sigma^{(p)}$  and  $\Gamma$  satisfy the condition of Corollary 1 and the parameters  $\alpha$  and  $\eta$  satisfy (4.11) and

$$\alpha > \max \left\{ \frac{1}{1+2(\eta \wedge 2)}, 1 - 4((\eta \wedge 2) - \sup_{t \in (0,1)} H(t)) \right\}. \quad (4.14)$$

**Remark 6.** Roughly speaking, conditions in (ii)–(v) require that  $\eta$  and  $H(t)$  are sufficiently separated, or the difference  $\eta - H(t)$  is large enough. Else, it may happen that  $\alpha$  satisfying these conditions does not exist. However, if  $\eta \geq (3 + \sqrt{41})/8 \simeq 1.175$  then  $\alpha$  satisfying (ii)–(iv) exists for any  $H(\cdot) \in \mathcal{C}^\eta(0, 1)$  with  $\sup_{t \in (0,1)} H(t) < 1$ , while condition (4.14) in (v) reduces to  $\alpha > \frac{1}{1+2(\eta \wedge 2)}$ .

**Remark 7.** In the case of estimators  $\mathcal{E} = \text{QV}$  or  $\text{QV2}$ , Proposition 3 can be easily extended to a general filter  $a \in \mathcal{A}(m, q)$ ,  $m \geq 1$ . For  $m = 1$ , the corresponding results in (ii)–(v) can be proved under weaker conditions on  $\alpha$ ,  $\eta$  but the convergence rates are worse. On the other hand, if  $a \in \mathcal{A}(m, q)$ ,  $m \geq 3$ , the corresponding conditions on  $\alpha$ ,  $\eta$  in (ii)–(iv) are the same as in the case  $a = a^*$  except that  $\eta \wedge 2$  in (4.11) and (4.13) can be replaced by  $\eta \wedge m$ . In (v), condition (4.14) holds. Note also that this proposition also holds for a process  $(\sigma(t) X_{(a^+, a^-)}(t))_{t \in (0,1)}$  when  $\sigma \in \mathcal{C}^\eta(0, 1)$ .

**Remark 8.** For a class of harmonizable Lévy processes, Lacaux [19, Theorem 4.1] obtained the weak consistency rate of the QV estimator  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t) - H(t) = O_p(n^{-\rho} \log n)$  with  $\rho \leq \eta - H(t)$ ,  $\rho < \frac{\eta}{1+2\eta}$  in the case when  $H(\cdot) \in \mathcal{C}^\eta(0, 1)$ ,  $0 < \eta \leq 1$ . For the MBM the above result follows from the CLT in Proposition 3(v) with  $\rho = \frac{1-\alpha}{2}$  satisfying  $\rho < 2(\eta - H(t))$  and  $\rho < \frac{\eta}{1+2\eta}$ . This means that concerning the weak consistency of the point-wise QV estimator of  $H(t)$ , our results for the MBM are more accurate than those in [19]. Moreover, the last paper does not discuss strong and uniform consistency rates and the CLT as in Proposition 3.

**Remark 9.** Begyn [10] obtained strong consistency and asymptotic normality of sums of quadratic variations for general non-stationary Gaussian processes under different assumptions on the covariance function which exclude the case of MBM. In particular, if  $\eta < 2$  then the covariance  $R(t, t') = \mathbb{E}X(t)X(t')$  in (7.1) is not twice differentiable in  $t$  or  $t'$  and the derivative  $\partial^4 R(t, t') / \partial^2 t \partial t'^2$  for  $t \neq t'$  does not exist, contrary to what is assumed in [10, Theorems 1 and 2]. In the case of quadratic variations localized in a neighborhood  $\mathcal{V}_{n,\alpha}(t)$  of given point  $t \in (0, 1)$  as in the estimator  $\widehat{H}_{n,\alpha}^{(\text{QV})}(t)$  (2.8), the imposed conditions in [10, Eqs. (5), (21), (23)] exclude the appearance of the first term of the order  $O(n^{-2(\eta \wedge 2)})$  on the r.h.s. of (4.7) which cannot be ignored and which plays a crucial role in our asymptotic results.

**Remark 10.** Since  $\alpha$  depends on  $\eta$  and  $H(t)$  that are not available, it is a problem to apply the CLTs of Proposition 3 in concrete situations. However, when  $\eta \geq 2$  it is possible to select



any  $\alpha > 1/5$  and even to use  $n^{1/5} \log^2 n$  instead of  $n^\alpha$ ; then the convergence rate of the CLT is  $n^{2/5}$  up to a logarithm term. Note that [15] proposed a procedure by minimization of the MISE to select an optimal  $\hat{\alpha}$  and then an adaptive estimator of  $H(t)$  is  $\hat{H}_{n,\hat{\alpha}}^{(QV)}(t)$ .

## 5. Simulations

All the softwares used in this section are available with a free access on <http://samm.univ-paris1.fr/Jean-Marc-Bardet> (in Matlab language).

We use the original version of the MBM  $X$  in (1.3). Since trajectories of the MBM in our simulation study are generated (for a given Hurst function  $H(\cdot)$ ) using the Cholesky decomposition of the covariance matrix, the number of observation points is limited to  $n = 6000$ . Although this data length may appear rather small in the present context, some interesting features can nevertheless be noted. Three cases are considered: Case 1:  $H(\cdot)$  is a smooth function; Case 2:  $H(\cdot)$  is a trajectory of an integrated FBM with Hurst parameter  $0 < h < 1$  and independent of  $X$ ; therefore  $H(\cdot) \in \mathcal{C}^{\eta-}$  with  $\eta = 1 + h \in (1, 2)$ ; and Case 3:  $H(\cdot)$  is a trajectory of a FBM with Hurst parameter  $0 < \eta < 1$ , independent of  $X$ .

For each local Hurst function  $H(\cdot)$ , Monte-Carlo experiments are realized from 100 independent replications of observed paths  $(X(1/n), X(2/n), \dots, X(1))$  of the MBM (1.3) for the following choices of parameters:

- $\alpha = 0.2, 0.3, 0.4$  and  $0.5$ ;
- $p = 5$  (= the number of dilatations of  $\hat{H}_{n,\alpha}^{(QV)}$  and  $\hat{H}_{n,\alpha}^{(QV2)}$ , = the regression length providing  $\hat{H}_{n,\alpha}^{(IR2)}$ ) in all cases;
- $a = a^* = (1, -2, 1)$  and therefore  $m = 2$  in all cases.

Each estimator of  $H(\cdot)$  is computed for  $t = \{n^{-\alpha}, n^{-\alpha} + 0.01, \dots, \min(1 - n^{-\alpha}, n^{-\alpha} + 0.99)\}$  and therefore an approximation of  $\sqrt{\text{MISE}} = \left(\int_0^1 \mathbb{E}(\hat{H}_n(t) - H(t))^2 dt\right)^{1/2}$  can be computed. Here there are the results of simulations.

*Case 1:  $H(\cdot) \in \mathcal{C}^\infty$*

We have chosen 4 different functions  $t \mapsto H(t)$ . These functions are the following.

- $H_1(t) = 0.6$  for any  $t \in (0, 1)$ .
- $H_2(t) = 0.1 + 0.8t$  for any  $t \in (0, 1)$ .
- $H_3(t) = 0.5 + 0.4 \sin(5t)$  for any  $t \in (0, 1)$ .
- $H_4(t) = 0.1 + 0.8(1 - t) \sin^2(10t)$  for any  $t \in (0, 1)$ .

From  $H_1(\cdot)$  to  $H_4(\cdot)$ , the functions  $H(\cdot)$  display more and more ample fluctuations and less regularity, even if these functions are all  $\mathcal{C}^\infty(0, 1)$  functions. Fig. 1 provides two kinds of graphs (the mean trajectory and single trajectory) of the two estimators  $\hat{H}_{n,\alpha}^{(QV2)}$  and  $\hat{H}_{n,\alpha}^{(IR2)}$  of the function  $H_4$  for  $n = 6000$  and two values of  $\alpha$  ( $\alpha = 0.3$  and  $0.4$ ). The corresponding MISE are given in Table 1.

*Case 2:  $H(\cdot) \in \mathcal{C}^{\eta-}$ ,  $1 < \eta < 2$*

We have chosen  $\eta = 1.5$  but other simulations with different values of  $\eta$  lead to similar conclusions. To ensure impartial results, we chose to compute the estimators for 50 different functions  $H(\cdot)$  generated from 50 trajectories of integrated FBM. Then 100 independent

Table 1

Values of the (empirical) MISE for estimators  $\hat{H}_{n,\alpha}^{(QV)}$ ,  $\hat{H}_{n,\alpha}^{(QV2)}$ ,  $\hat{H}_{n,\alpha}^{(IR)}$  and  $\hat{H}_{n,\alpha}^{(IR2)}$  of  $\mathcal{C}^\infty$ -Hurst functions for  $n \in \{2000, 6000\}$ ,  $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$  and  $a = (1, -2, 1)$ .

$H_1(t) = 0.6$	$\alpha$	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.044	0.055	0.073	0.104
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.041	0.051	0.069	0.096
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.111	0.137	0.186	0.260
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.061	0.077	0.106	0.145
$n = 6000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.026	0.035	0.053	0.079
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.025	0.033	0.050	0.074
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.065	0.091	0.128	0.202
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.037	0.049	0.076	0.115
$H_2(t) = 0.1 + 0.8t$					
$n = 2000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.170	0.076	0.075	0.101
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.170	0.073	0.072	0.096
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.115	0.143	0.184	0.247
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.059	0.071	0.098	0.135
$n = 6000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.115	0.045	0.051	0.074
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.114	0.044	0.048	0.070
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.070	0.094	0.134	0.195
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.036	0.046	0.069	0.103
$H_3(t) = 0.5 + 0.4 \sin(5t)$					
$n = 2000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.362	0.125	0.084	0.102
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.362	0.123	0.080	0.096
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.129	0.133	0.171	0.229
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.093	0.071	0.091	0.124
$n = 6000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.260	0.078	0.056	0.077
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.260	0.077	0.052	0.072
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.078	0.089	0.125	0.180
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.057	0.047	0.065	0.097
$H_4(t) = 0.1 + 0.8(1-t) \sin^2(10t)$					
$n = 2000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.321	0.165	0.121	0.120
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.320	0.164	0.117	0.112
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.178	0.138	0.160	0.210
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.165	0.098	0.091	0.112
$n = 6000$	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.251	0.136	0.074	0.084
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.251	0.135	0.071	0.078
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.158	0.088	0.115	0.164
	$\sqrt{\widehat{\text{MISE}}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.148	0.062	0.067	0.091

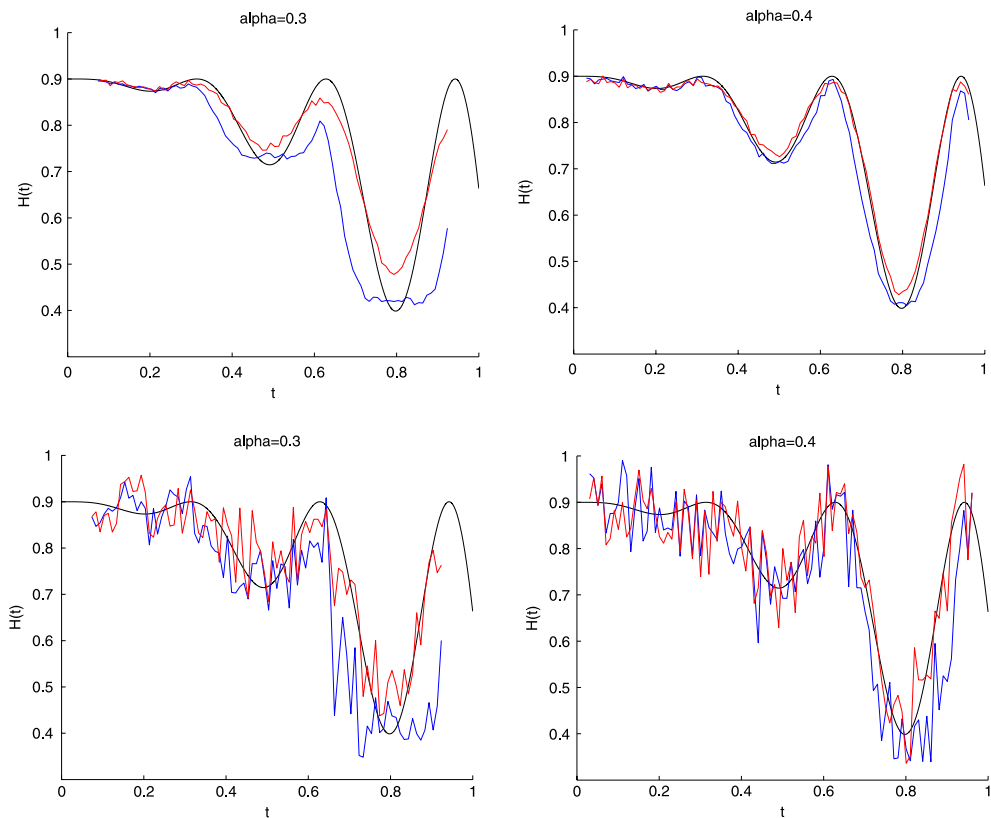


Fig. 1. Estimates of the function  $H_4(t) = 0.1 + 0.8(1 - t) \sin^2(10t)$  with  $t \in (0, 1)$  for  $n = 6000$  and  $\alpha = 0.3$  and  $0.4$  (from left to right). The top row represents the mean trajectories of  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  obtained from 100 independent replications of MBM with the above function  $H(\cdot)$ . The bottom row represents a trajectory of  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  obtained from a trajectory of a MBM with the same function  $H(\cdot)$ . The graphs of  $H(t)$ ,  $\hat{H}_{n,\alpha}^{(QV2)}(t)$ , and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  are in black, blue and red, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

replications of processes generated with each function  $H(\cdot)$ . Table 2 contains the empirical MISE computed from these  $50 \times 100$  processes. Fig. 2 provides the trajectories of  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  for one of the 50 differentiable Hurst functions  $H(\cdot) \in \mathcal{C}^{1.5-}$  for  $n = 6000$  and  $\alpha = 0.3, 0.4$ .

*Case 3:*  $H(\cdot) \in \mathcal{C}^{\eta-}$ ,  $0 < \eta < 1$

We have chosen here  $\eta = 0.6$ . As in the case  $\eta = 1.5^-$ , we generated 50 different trajectories  $H(\cdot)$  obtained from trajectories of FBM with parameter  $H = 0.6$ . Then for each function  $H(\cdot)$ , 100 independent replications of MBM with Hurst function  $H(\cdot)$  are generated. Table 2 contains the empirical MISE computed from these  $50 \times 100$  processes. An example of a graph of a function  $H(\cdot) \in \mathcal{C}^{0.6-}$  and the mean trajectories (obtained from the 100 replications) of both the estimators are drawn in Fig. 3.

Table 2

Values of the (empirical) MISE for estimators  $\hat{H}_{n,\alpha}^{(QV)}$ ,  $\hat{H}_{n,\alpha}^{(QV2)}$ ,  $\hat{H}_{n,\alpha}^{(IR)}$  and  $\hat{H}_{n,\alpha}^{(IR2)}$  of  $\mathcal{C}^{\eta-}$ -Hurst functions for  $\eta = 1.5$  (up) and  $\eta = 0.6$  (down),  $n \in \{2000, 6000\}$ ,  $\alpha \in \{0.2, 0.3, 0.4, 0.5\}$  and  $a = (1, -2, 1)$ .

$H \in \mathcal{C}^{1.5}(0, 1)$	$\alpha$	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.261	0.113	0.088	0.103
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.261	0.112	0.085	0.098
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.139	0.141	0.175	0.233
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.098	0.077	0.093	0.128
$n = 6000$	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.164	0.067	0.055	0.074
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.164	0.066	0.053	0.070
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.084	0.094	0.131	0.186
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.054	0.047	0.066	0.098
$H \in \mathcal{C}^{0.6}(0, 1)$	$\alpha$	0.2	0.3	0.4	0.5
$n = 2000$	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.140	0.087	0.083	0.096
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.140	0.086	0.081	0.094
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.148	0.156	0.192	0.249
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.088	0.078	0.096	0.135
$n = 6000$	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV)}$	0.129	0.067	0.057	0.074
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(QV2)}$	0.130	0.067	0.056	0.071
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR)}$	0.096	0.106	0.143	0.201
	$\sqrt{\text{MISE}}$ for $\hat{H}_{n,\alpha}^{(IR2)}$	0.066	0.052	0.067	0.103

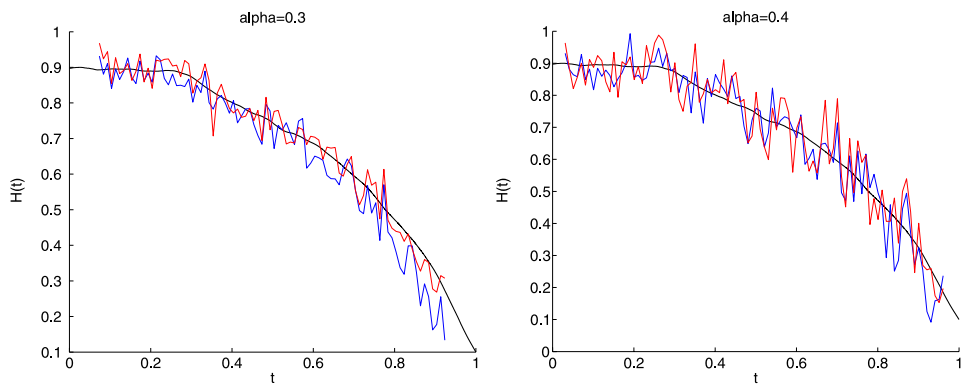


Fig. 2. Trajectories of  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  for one of the 50 differentiable Hurst functions  $H(\cdot) \in \mathcal{C}^{1.5-}$  for  $n = 6000$  and  $\alpha = 0.3, 0.4$  (from left to right). The graphs of  $H(t)$ ,  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  are respectively in black, blue and red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### Conclusions of simulations

1. In any cases of function  $H(\cdot)$ , for  $n = 2000$  or  $n = 6000$  and  $\alpha = 0.2, 0.3, 0.4$  or  $0.5$ , the empirical MISE of  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  are smaller than the ones of  $\hat{H}_{n,\alpha}^{(QV)}(t)$  and

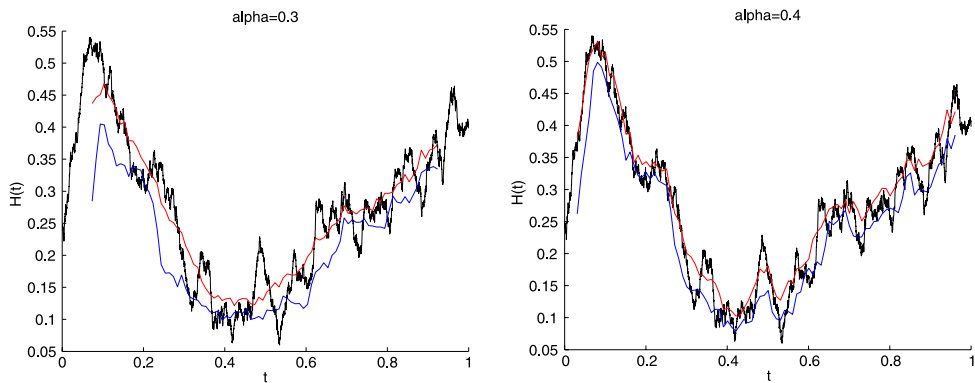


Fig. 3. Mean trajectory (from 100 independent replications) of  $\hat{H}_{n,\alpha}^{(QV)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR)}(t)$  for one of the 50 differentiable Hurst functions  $H(\cdot) \in \mathcal{C}^{0.6-}$  for  $n = 6000$  and  $\alpha = 0.3, 0.4$  (from left to right). The graphs of  $H(t)$ ,  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  and  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  are respectively in black, blue and red. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$\hat{H}_{n,\alpha}^{(IR)}(t)$ , respectively. This theoretically corresponds to the Gauss–Markov Theorem which is also empirically satisfied. Note that PGLSE  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  allows for certain cases to divide by 2 the empirical  $\sqrt{\text{MISE}}$  of  $\hat{H}_{n,\alpha}^{(IR)}(t)$ , while the gain obtained with  $\hat{H}_{n,\alpha}^{(QV2)}(t)$  instead of  $\hat{H}_{n,\alpha}^{(QV)}(t)$  is more limited (less than 10%).

2. In agreement with the theory (see the remark at the end of Section 4) in the case  $H(\cdot) \in \mathcal{C}^\eta$  with  $\eta = 1.5^-$  or  $\eta = \infty$ , our simulation suggests to select  $\alpha$  greater than  $(1 + 2(\eta \wedge 2))^{-1}$  for “optimal” results. More precisely, it seems that a good choice is  $\alpha \simeq 0.3$  for  $\hat{H}_{n,\alpha}^{(IR)}$  and  $\hat{H}_{n,\alpha}^{(IR2)}$ , and  $\alpha \simeq 0.4$  for  $\hat{H}_{n,\alpha}^{(QV)}$  and  $\hat{H}_{n,\alpha}^{(QV2)}$ . This empirical rule seems also valid for  $H(\cdot) \in \mathcal{C}^{0.6-}$  even if the theoretical choice should be  $\alpha \geq \max\{5/11, 2(2 \sup_{t \in (0,1)} H(t) - 0.7)\}$ . However a bias clearly appears for  $\alpha = 0.3$  when  $H(t)$  is close to  $\eta$ , while this bias is reduced when  $\alpha = 0.4$  (see Fig. 3). The tables also confirm that, with the above choice of the bandwidth  $\alpha$ , the accuracy of both estimators increases according to the theoretical convergence rate  $n^{(1-\alpha)/2}$  ( $\simeq n^{1/3}$ ) as  $n$  increases from  $n = 2000$  to  $n = 6000$  (for instance for  $\alpha = 0.3$  the empirical  $\sqrt{\text{MISE}}$  of  $\hat{H}_{n,\alpha}^{(IR2)}(\cdot)$  seems to be divided by  $1.5 \simeq 3^{0.35}$ ).
3. *Mutatis mutandis*, the values of the empirical  $\sqrt{\text{MISE}}$  are almost the same for any functions  $H(\cdot) \in \mathcal{C}^\eta$  with  $\eta \geq 1$ . However, even if this observation seems to be able to be extended for  $\eta = 0.6$ , this does not hold since in this case the function  $H(\cdot)$  is required to vary from 0 to 0.6 while for  $\eta \geq 1$  it varies from 0 to 1.
4. Finally, in all cases, the best estimator is clearly  $\hat{H}_{n,\alpha}^{(IR2)}$  with the choice  $\alpha = 0.3$  (even when  $H$  is a constant, the empirical  $\sqrt{\text{MISE}}$  of  $\hat{H}_{n,\alpha}^{(IR2)}$  for  $\alpha = 0.3$  is smaller than the one of  $\hat{H}_{n,\alpha}^{(QV2)}$  for  $\alpha = 0.4$ ). However the asymptotic variance of  $\hat{H}_{n,\alpha}^{(IR2)}(t)$  is a little larger than the one of  $\hat{H}_{n,\alpha}^{(QV2)}$ . But, as it can be seen in Figs. 1 and 3,  $\hat{H}_{n,\alpha}^{(IR2)}$  is nearly unbiased when  $\alpha = 0.3$  (and sometimes when  $\alpha = 0.2$ ) while the one of  $\hat{H}_{n,\alpha}^{(QV2)}$  is still biased for  $\alpha = 0.3$ . Then  $\hat{H}_{n,\alpha}^{(IR2)}$  can be used when  $\alpha = 0.3$  with an asymptotic variance varying with  $n^{-(1-\alpha)/2} = n^{-0.35}$  while  $\hat{H}_{n,\alpha}^{(QV2)}$  requires  $\alpha = 0.4$  and an asymptotic variance varying with  $n^{-0.3}$ .

## 6. Proofs of Theorems 1 and 2 and Corollaries 1 and 2

The proof of [Theorem 1](#) uses the moment inequality in [Lemma 1](#), which is a particular case of a more general moment bound in [\[7\]](#).

Let  $(Y_1, \dots, Y_N)$  be a collection of Gaussian vectors  $Y_t = (Y_t^{(1)}, \dots, Y_t^{(v)}) \in \mathbb{R}^v$  with zero mean  $EY_t = 0$  and non-degenerated covariance matrices  $\Sigma_t = (\text{Cov}(Y_t^{(u)}, Y_t^{(v)}))_{1 \leq u, v \leq v}$ , having a joint Gaussian distribution in  $\mathbb{R}^{vN}$ . Let  $\varepsilon \in [0, 1]$  be a fixed number. Call  $(Y_1, \dots, Y_N)$   $\varepsilon$ -correlated if  $|\text{Cor}(Y_t^{(u)}, Y_s^{(v)})| \leq \varepsilon$  for any  $t \neq s$ ,  $1 \leq t, s \leq N$  and any  $1 \leq u, v \leq v$ . Finally,  $\sum'$  denotes the sum over all distinct integers  $1 \leq t_1, \dots, t_p \leq N$ ,  $t_i \neq t_j$  ( $i \neq j$ ).

Let  $|x| = (\sum_{i=1}^v (x^{(i)})^2)^{1/2}$  denote the Euclidean norm in  $\mathbb{R}^v$ ,  $A = (a_{ij})$  a real  $v \times v$  matrix,  $A^\top$  the transposed matrix,  $I$  the unit matrix, and  $\|A\| := \sup_{|x|=1} |Ax|$  the matrix spectral norm, respectively. [Lemma 1](#) is a particular case of the moment bound established in [\[7, Corollary 1\]](#).

**Lemma 1.** Let  $(Y_1, \dots, Y_N) \in \mathbb{R}^{vN}$  be an  $\varepsilon$ -correlated Gaussian vector such that

$$\max_{1 \leq t \leq N} \|\Sigma_t^{-1}\| \leq c_{\max} \quad (6.1)$$

for some constant  $c_{\max} > 0$ . Let  $G_{j,t} : \mathbb{R}^v \rightarrow \mathbb{R}$ ,  $1 \leq j \leq p$  ( $p \geq 2$ ),  $1 \leq t \leq N$  be measurable functions such that  $\|G_{j,t}\|^2 := E|G_{j,t}(Y_t)|^2 < \infty$ . Assume that for some integer  $0 \leq \alpha \leq p$ ,

$$E[G_{j,t}(Y_t)] = 0, \quad E[Y_t G_{j,t}(Y_t)] = 0, \quad \text{for all } 1 \leq j \leq \alpha, 1 \leq t \leq N \quad (6.2)$$

and that  $\varepsilon v^2 c_{\max} < \frac{1}{vp-1}$ . Then

$$\sum' E[G_{1,t_1}(Y_{t_1}) \cdots G_{p,t_p}(Y_{t_p})] \leq C(\varepsilon, p, \alpha, c_{\max}) K N^{p-\frac{\alpha}{2}} Q_N^{\frac{\alpha}{2}}, \quad (6.3)$$

where

$$Q_N := \max_{1 \leq t \leq N} \sum_{1 \leq s \leq N, s \neq t} \max_{1 \leq u, v \leq v} |EY_t^{(u)} Y_s^{(v)}|^2, \quad (6.4)$$

$K := \prod_{j=1}^p \max_{1 \leq t \leq N} \|G_{j,t}\|$ , and the constant  $C(\varepsilon, p, \alpha, c_{\max})$  depends on  $\varepsilon, p, \alpha, c_{\max}$  only.

In what follows, we use [Lemma 1](#) with  $2 \leq p \leq 4$ ,  $v = 2$  only. Let

$$S_n(t) := \frac{1}{v_{n,\alpha}(t)} \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \eta_n(k), \quad \eta_n(k) := \frac{|V_n^{a*} Z(k/n) + V_n^{a*} Z((k+1)/n)|}{|V_n^{a*} Z(k/n)| + |V_n^{a*} Z((k+1)/n)|}. \quad (6.5)$$

Thus,  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t) = \Lambda_2^{-1}(S_n(t))$ .

**Lemma 2.** Let  $Z = (Z(t))_{t \in (0,1)}$  be a zero-mean Gaussian process satisfying  $(A)_\kappa$  with  $m = 2$ . Then for any  $0 < \varepsilon < 1/2$ ,

$$\sup_{\varepsilon < t < 1-\varepsilon} |ES_n(t) - \Lambda_2(H(t))| = o(n^{-\kappa}) + O(n^{-1} + n^{-\alpha(\eta \wedge 2)}). \quad (6.6)$$

**Proof of Lemma 2.** The first relation in (6.6) follows from

$$\max_{[n\varepsilon/2] \leq k \leq [(1-\varepsilon/2)n]} |\mathbb{E}\eta_n(k) - \Lambda_2(H(k/n))| = o(n^{-\kappa}), \quad (6.7)$$

$$\sup_{\varepsilon < t < 1-\varepsilon} v_{n,\alpha}^{-1}(t) \left| \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \{ \Lambda_2(H(k/n)) - \Lambda_2(H(t)) \} \right| \leq C n^{-\alpha(\eta \wedge 2)}. \quad (6.8)$$

To show (6.7), write  $\mathbb{E}\eta_n(k) = \mathbb{E}[\psi(V_n^{a*} Z(k/n), V_n^{a*} Z((k+1)/n))] = \Lambda(\text{Cor } V_n^{a*} Z(k/n), V_n^{a*} Z((k+1)/n))$  where  $\Lambda(\rho) := \frac{1}{\pi} \arccos(-\rho) + \frac{1}{\pi} \sqrt{\frac{1+\rho}{1-\rho}} \log(\frac{2}{1+\rho})$ ; see [5, p. 777], also (2.5).

From Property 1,

$$\max_{[n\varepsilon/2] \leq k \leq [(1-\varepsilon/2)n]} n^\kappa |\text{Cor}(V_n^{a*} Z(k/n), V_n^{a*} Z((k+1)/n)) - \rho_2(H(k/n))| \xrightarrow{n \rightarrow \infty} 0.$$

Since  $\Lambda_2(H) = \Lambda(\rho_2(H))$  and the function  $\rho \in (-1, 1) \mapsto \Lambda(\rho)$  is analytic we deduce (6.7).

To show (6.8), write

$$\begin{aligned} \Lambda_2(H(k/n)) - \Lambda_2(H(t)) &= \Lambda'_2(H(t))(H(k/n) - H(t)) + O((H(k/n) - H(t))^2) \\ &= \Lambda'_2(H(t))(H(k/n) - H(t)) + O\left(\left(\frac{k}{n} - t\right)^{2(\eta \wedge 1)}\right) \\ &= \Lambda'_2(H(t))(H(k/n) - H(t)) + O(n^{-2\alpha(\eta \wedge 1)}) \end{aligned}$$

for  $k \in \mathcal{V}_{n,\alpha}(t)$ . Hence

$$\begin{aligned} v_{n,\alpha}^{-1}(t) \left| \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \{ \Lambda_2(H(k/n)) - \Lambda_2(H(t)) \} \right| &\leq C v_{n,\alpha}^{-1}(t) \left| \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \{ H(k/n) - H(t) \} \right| \\ &\quad + C n^{-2\alpha(\eta \wedge 1)} \\ &\leq C(n^{-\alpha(\eta \wedge 2)} + n^{-1} + n^{-2\alpha(\eta \wedge 1)}), \end{aligned}$$

where for  $\eta > 1$  we used  $H(k/n) - H(t) = H'(t)((k/n) - t) + O(n^{-\alpha(\eta \wedge 2)})$  and  $v_{n,\alpha}^{-1}(t) \sum_{k \in \mathcal{V}_{n,\alpha}(t)} \{(k/n) - t\} = O(n^{-1})$ . This proves (6.8) and Lemma 2, too.  $\square$

**Proof of Theorem 1.** (i) Recall  $\widehat{H}_{n,\alpha}^{(\text{IR})}(t) = \Lambda_2^{-1}(S_n(t))$ . Note that the map  $H \mapsto \Lambda_2(H)$  is continuous and strictly increasing on  $[0, 1]$  and the inverse map  $s \mapsto \Lambda_2^{-1}(s)$  is continuously differentiable on  $s \in [\Lambda_2(0), \Lambda_2(1)]$ . Therefore, since  $S_n(t) \in [0, 1]$  a.s., part (i) follows from  $S_n(t) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} \Lambda_2(H(t))$  which is a consequence of

$$\mathbb{E}S_n(t) \xrightarrow[n \rightarrow \infty]{} \Lambda_2(H(t)) \quad \text{and} \quad S_n(t) - \mathbb{E}S_n(t) \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (6.9)$$

The first term of (6.9) is obtained from Lemma 2. The second member of Lemma 2 can be deduced from the following bound:

$$\mathbb{E}(S_n(t) - \mathbb{E}S_n(t))^4 < C n^{-\rho}, \quad \text{for some } \rho := 2(1 - \alpha) \left(1 - \frac{\theta}{\gamma}\right) > 0, \quad (6.10)$$

from assumption  $(B)_\alpha$  with  $\alpha \in (0, 1)$ .

We shall prove a slightly stronger bound: there exists a constant  $C < \infty$  such that for any integers  $0 \leq s < t < n$ ,  $|t - s| \leq 2n^{1-\alpha}$

$$\mathbb{E} \left( \sum_{k=s+1}^t \tilde{\eta}_n(k) \right)^4 < C(t-s)^2 n^{\frac{2\theta(1-\alpha)}{\gamma}}, \quad \text{where } \tilde{\eta}_n(k) := \eta_n(k) - \mathbb{E}\eta_n(k). \quad (6.11)$$

Relation (6.11) implies (6.10) with  $\rho := 4(1-\alpha) - 2(1-\alpha) - \frac{2\theta(1-\alpha)}{\gamma} = 2(1-\alpha)(1 - \frac{\theta}{\gamma})$ .

In order to apply Lemma 1 to prove (6.11), we rewrite the ratios  $\eta_n(k) = \psi(Y_n(k))$  as functions of Gaussian vectors  $Y_n(k) = (Y_n^{(1)}(k), Y_n^{(2)}(k)) \in \mathbb{R}^2$ , where

$$Y_n^{(1)}(k) := \frac{V_n^{a*} Z(k/n)}{\sigma_n(k)}, \quad Y_n^{(2)}(k) := \frac{V_n^{a*} Z((k+1)/n)}{\sigma_n(k+1)}. \quad (6.12)$$

with

$$\sigma_n^2(k) := \text{Var}(V_n^{a*} Z(k/n)). \quad (6.13)$$

Also define

$$W_n^{(1)}(k) := \frac{V^{a*} B_{H(k/n)}(0)}{\sqrt{\text{Var}(V^{a*} B_{H(k/n)}(0))}}, \quad W_n^{(2)}(k) := \frac{V^{a*} B_{H(k/n)}(1)}{\sqrt{\text{Var}(V^{a*} B_{H(k/n)}(1))}}. \quad (6.14)$$

Note  $\psi(-y^{(1)}, -y^{(2)}) = \psi(y^{(1)}, y^{(2)})$ , implying  $\mathbb{E}[Y_n(k)\psi(Y_n(k))] = 0$ ; c.f. condition (6.2) of Lemma 1. Property 1 implies that the covariance matrices  $\Sigma_n(k)$  of the  $Y_n(k)$ 's approach as  $n \rightarrow \infty$  uniformly in  $[n\varepsilon] \leq k \leq [(1-\varepsilon)n]$  the corresponding non-degenerate covariance matrices  $\tilde{\Sigma}_n(k) := \begin{pmatrix} 1 & \rho_2(H(k/n)) \\ \rho_2(H(k/n)) & 1 \end{pmatrix}$  of Gaussian vectors  $W_n(k) = (W_n^{(1)}(k), W_n^{(2)}(k))$  and therefore these matrices satisfy the bound in (6.1), viz.,

$$\max_{1 \leq k \leq n-2} n^k \|\Sigma_n(k) - \tilde{\Sigma}_n(k)\| \rightarrow 0, \quad \max_{1 \leq k \leq n-2} \|\Sigma_n(k)^{-1}\| \leq c_{\max} \quad (6.15)$$

with some  $c_{\max} < \infty$  independent of  $n$ .

Next, we “decimate” the sum on the l.h.s. of (6.11) so that the “remaining” vectors  $Y_n(k)$ 's are  $\varepsilon'$ -correlated, as follows. Let  $\ell = \ell_n$  be the sequence of integers increasing to  $\infty$  at a rate  $o(n^{1-\alpha})$  which is specified in (6.21). Write

$$R_n(s, t) := \sum_{k=s+1}^t \tilde{\eta}_n(k) = \sum_{j=0}^{\ell-1} R_{n,\ell}(j), \quad R_{n,\ell}(j) := \sum_{s < k \leq t: k=j \pmod{\ell}} \tilde{\eta}_n(k). \quad (6.16)$$

Then

$$\begin{aligned} \mathbb{E}(R_n(s, t))^4 &\leq \ell^4 \max_{0 \leq j < \ell} \mathbb{E}(R_{n,\ell}(j))^4 \quad \text{and} \\ \mathbb{E}(R_{n,\ell}(j))^4 &\leq C \left( \sum_4 + \sum_3 + \sum_{2,1} + \sum_{2,2} \right), \end{aligned} \quad (6.17)$$

where

$$\begin{aligned} \sum_4 &:= \sum_4' |\mathbb{E}[\tilde{\eta}_n(k_1) \cdots \tilde{\eta}_n(k_4)]|, & \sum_3 &:= \sum_3' |\mathbb{E}[\tilde{\eta}_n(k_1) \tilde{\eta}_n(k_2) \tilde{\eta}_n^2(k_3)]|, \\ \sum_{2,1} &:= \sum_{2,1}' |\mathbb{E}[\tilde{\eta}_n^3(k_1) \tilde{\eta}_n(k_2)]|, & \sum_{2,2} &:= \sum_{2,2}' \mathbb{E}[\tilde{\eta}_n^2(k_1) \tilde{\eta}_n^2(k_2)], \end{aligned}$$



where  $\sum'$  (respectively,  $\sum$ ) stands for the sum over all *different* integers  $s < k_p \leq t, k_p = u \pmod{\ell}, k_p \neq k_q (p \neq q)$  (respectively, over *all*  $s < k_1, k_2 \leq t, k_1 = u \pmod{\ell}, k_2 = u \pmod{\ell}$ ). But, since  $|\tilde{\eta}_n(k)| \leq 1$  for all  $k, n$ ,

$$\sum_{2,1} + \sum_{2,2} \leq C (|t - s|/\ell)^2. \quad (6.18)$$

Let us estimate  $\sum_3$  and  $\sum_4$ . Let

$$Q_{n,\ell}(j) := \max_{s < k_1 \leq t, k_1 = j \pmod{\ell}} \sum_{s < k_2 \leq t, k_2 = j \pmod{\ell}, k_2 \neq k_1} \bar{\rho}_n^2(k_1, k_2), \quad (6.19)$$

where

$$\bar{\rho}_n(k_1, k_2) := \max_{u,v=1,2} |E[Y_n^{(u)}(k_1)Y_n^{(v)}(k_2)]|$$

and  $Y_n^{(1)}(k), Y_n^{(2)}(k)$  are defined in (6.12). Assumptions  $(B)_\alpha$  and  $(A)_0$  imply the following bound. There exists a constant  $C_1 < \infty$  such that for any  $n \geq 1$  and any  $[n\varepsilon] \leq k_1 < k_2 \leq [(1 - \varepsilon)n]$

$$\bar{\rho}_n(k_1, k_2) \leq C_1 n^{\theta(1-\alpha)} (|k_1 - k_2| \wedge n^{1-\alpha})^{-\gamma}; \quad (6.20)$$

see also [7]. Now, given  $C_1 > 0$ , define

$$C_2 := C_1/\varepsilon' \quad \text{and} \quad \ell := [C_2 n^{(\theta/\gamma)(1-\alpha)}]. \quad (6.21)$$

Relations (6.20) and (6.21) imply that for any  $0 \leq j < \ell$ , the Gaussian vector  $(Y_n(k), k = j \pmod{\ell}, k = 1, \dots, n)$  is  $\varepsilon'$ -correlated. We choose  $\varepsilon' < 1/(28c_{\max})$ , thus guaranteeing condition  $\varepsilon' < 1/(c_{\max} v^2 (vp - 1))$  of Lemma 1 for  $v = 2$  and  $p = 3, 4$ .

Using (6.20), (6.21) and  $\gamma > 1/2$ ,  $|t - s| \leq 2n^{1-\alpha}$  we obtain

$$\begin{aligned} \max_{0 \leq j < \ell} Q_{n,\ell}(j) &\leq C \left( \frac{n^{\theta(1-\alpha)}}{\ell^\gamma} \right)^2 \max_{1 \leq k_1 \leq |t-s|/\ell} \sum_{k_2=1, k_1 \neq k_2}^{|t-s|/\ell} |k_1 - k_2|^{-2\gamma} \\ &\leq 2C \left( \frac{n^{\theta(1-\alpha)}}{\ell^\gamma} \right)^2 \sum_{k'=1}^{\infty} |k'|^{-2\gamma} \leq C. \end{aligned} \quad (6.22)$$

Thus, the sums  $\sum_4$  and  $\sum_3$  can be bounded using Lemma 1 with  $N = [|t - s|/\ell]$ . Accordingly, for  $\sum_4$  with  $p = \alpha = 4$  we have

$$\sum_4 \leq C (|t - s|/\ell)^{4-\frac{4}{2}} \left( \max_{0 \leq j < \ell} Q_{n,\ell}(j) \right)^{\frac{4}{2}} \leq C (|t - s|/\ell)^2. \quad (6.23)$$

Similarly, for  $\sum_3$ , with  $p = 3$  and  $\alpha = 2$  we get

$$\sum_3 \leq C (|t - s|/\ell)^{3-\frac{2}{2}} \left( \max_{0 \leq j < \ell} Q_{n,\ell}(j) \right)^{\frac{2}{2}} \leq C (|t - s|/\ell)^2. \quad (6.24)$$

Relation (6.11) follows by combining (6.17), (6.18), (6.23) and (6.24).

(ii) Similarly as in the proof (i), it suffices to show

$$S_n(t) - ES_n(t) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (6.25)$$

In turn, (6.25) is a consequence of (6.10) with  $\rho > 1$  (since  $\alpha < \frac{\gamma-2\theta}{2(\gamma-\theta)}$ ) and Chebyshev's inequality imply that for any  $0 < \delta < (\rho - 1)/4$

$$\sum_{n=1}^{\infty} \mathbb{P}(|S_n(t) - \mathbb{E}S_n(t)| > n^{-\delta}) < \sum_{n=1}^{\infty} n^{4\delta} \mathbb{E}(S_n(t) - \mathbb{E}S_n(t))^4 \leq C \sum_{n=1}^{\infty} n^{4\delta-\rho} < \infty.$$

Hence, (6.25) follows by the Borel–Cantelli lemma.

(iii) We first estimate  $\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \Lambda_2(H(t))|$ , where  $S_n(t)$  is defined in (6.5). Split  $S_n(t) - \Lambda_2(H(t)) = (S_n(t) - \mathbb{E}S_n(t)) + (\mathbb{E}S_n(t) - \Lambda_2(H(t)))$ . Since  $v_n \equiv v_{n,\alpha}(t) \sim 2n^{1-\alpha}$  do not depend on  $t \in (\epsilon, 1-\epsilon)$  for  $n$  large enough, so

$$\begin{aligned} \sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \mathbb{E}S_n(t)| &\leq v_n^{-1} \max_{0 \leq s < n-v_n} |R_n(s, s+v_n)| \\ &\leq v_n^{-1} \sum_{1 \leq i < n/v_n} \max_{(i-1)v_n \leq s < iv_n} |R_n(s, s+v_n)|, \end{aligned} \quad (6.26)$$

with  $R_n(s, t)$  as defined in (6.16). Therefore for any  $a > 0$

$$\begin{aligned} &\mathbb{P}\left(\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \mathbb{E}S_n(t)| > a\right) \\ &\leq \sum_{1 \leq i < n/v_n} \mathbb{P}\left(\max_{(i-1)v_n \leq s < iv_n} |R_n(s, s+v_n)| > av_n\right). \end{aligned} \quad (6.27)$$

Since  $|R_n(s, s+v_n)| = |R_n((i-1)v_n, s+v_n) - R_n((i-1)v_n, s)| \leq |R_n((i-1)v_n, s+v_n)| + |R_n((i-1)v_n, s)|$ , we have

$$\begin{aligned} &\mathbb{P}\left(\max_{(i-1)v_n \leq s < iv_n} |R_n(s, s+v_n)| > av_n\right) \\ &\leq 2\mathbb{P}\left(\max_{(i-1)v_n \leq s < (i+1)v_n} |R_n((i-1)v_n, s)| > av_n/2\right). \end{aligned}$$

We will use the following fact from Billingsley [14, Theorem 12.1, see also (12.5), (12.10)]. Let  $\xi_j$ ,  $1 \leq j \leq m$  be arbitrary r.v.'s. Suppose there exist nonnegative numbers  $u_j \geq 0$ ,  $1 \leq j \leq m$  such that

$$\mathbb{E}\left(\sum_{j=t+1}^s \xi_j\right)^4 \leq \left(\sum_{j=t+1}^s u_j\right)^2, \quad 0 \leq t \leq s \leq m. \quad (6.28)$$

Then for any  $\epsilon > 0$

$$\mathbb{P}\left(\max_{1 \leq s \leq m} \left|\sum_{j=1}^s \xi_j\right| \geq \epsilon\right) \leq K\epsilon^{-4} \left(\sum_{j=1}^m u_j\right)^2 + \mathbb{P}\left(\left|\sum_{j=1}^m \xi_j\right| \geq \epsilon/2\right), \quad (6.29)$$

where  $K > 0$  is an absolute constant.

We shall apply (6.29) with

$$\xi_j := \tilde{\eta}_n((i-1)v_n + j), \quad u_j := C^{1/2} n^{\frac{\theta(1-\alpha)}{\gamma}}, \quad m := v_n \sim 2n^{1-\alpha},$$

where  $C > 0$  is the same as in (6.11). The validity of condition (6.28) follows from the bound (6.11). Hence from (6.29), (6.11) and Chebyshev's inequality we obtain

$$\begin{aligned} & \mathbb{P}\left(\max_{(i-1)v_n < s \leq (i+1)v_n} |R_n((i-1)v_n, s)| > av_n/2\right) \\ & \leq C(av_n/4)^{-4} v_n^2 n^{\frac{2\theta(1-\alpha)}{\gamma}} + \mathbb{P}(|R_n((i-1)v_n, iv_n)| > av_n/4) \\ & \leq Ca^{-4} v_n^{-2} n^{\frac{2\theta(1-\alpha)}{\gamma}} + C(av_n)^{-4} \mathbb{E}|R_n((i-1)v_n, iv_n)|^4 \\ & \leq Ca^{-4} v_n^{-2} n^{\frac{2\theta(1-\alpha)}{\gamma}} \\ & \leq Ca^{-4} n^{-2(1-\alpha)(1-\frac{\theta}{\gamma})}. \end{aligned}$$

Hence and from (6.27) we obtain

$$\mathbb{P}\left(\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \mathbb{E}S_n(t)| > a\right) \leq Ca^{-4} n^{\alpha-2(1-\alpha)(1-\frac{\theta}{\gamma})}, \quad (6.30)$$

where  $C > 0$  does not depend on  $a$  and  $n$ . Taking  $a = \delta n^{-\mu}$  with  $\mu$  given in (2.16) results in  $\mathbb{P}(\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \mathbb{E}S_n(t)| > \delta n^{-\mu}) \leq C\delta^{-4}$ ,  $\forall \delta > 0$ , or

$$\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \mathbb{E}S_n(t)| = O_p(n^{-\mu}). \quad (6.31)$$

Next, from Lemma 2 we have

$$\sup_{\epsilon < t < 1-\epsilon} |\mathbb{E}S_n(t) - \Lambda_2(H(t))| = o(n^{-\kappa}) + O(n^{-\alpha(\eta \wedge 2)}) = O(n^{-\mu}), \quad (6.32)$$

since  $2\alpha < 1$ . Finally, (6.32) and (6.31) imply

$$\sup_{\epsilon < t < 1-\epsilon} |S_n(t) - \Lambda_2(H(t))| = O_p(n^{-\mu}). \quad (6.33)$$

The result in (iii) follows from (6.32) and the properties of the inverse map  $s \mapsto \Lambda_2^{-1}(s)$  mentioned in the proof of (i).

(iv) follows from the Borel–Cantelli lemma, (6.30) with  $a = n^{-(\mu_1 - \delta)}$  and (6.6) with  $\mu$  replaced by  $\mu_1$  (which in turn follows from (6.7) and (6.8)). Here, we use the facts that  $2(1-\alpha)(1-(\theta/\gamma)) - \alpha - 4(\mu_1 - \delta) = 1 + 4\delta > 1$  and  $\alpha(\eta \wedge 2) \geq \mu_1$ . Theorem 1 is proved.  $\square$

The proof of Theorem 2, uses the following lemma which is a particular case of a more general result in [7] (see also [5, Theorem A.1]).

**Lemma 3.** Let  $\mathbf{Y}_{k,N} = (Y_{k,N}^{(1)}, \dots, Y_{k,N}^{(v)}) \in \mathbb{R}^v$ ,  $1 \leq k \leq N$ ,  $N \in \mathbb{N}$  be a triangular array of jointly Gaussian vectors, with zero mean  $\mathbb{E}\mathbf{Y}_{k,N} = 0$  and non-generated covariance matrices  $\Sigma_{k,N} = (\text{Cov}(Y_{k,N}^{(u)}, Y_{k,N}^{(v)}))_{1 \leq u, v \leq v}$ ,  $1 \leq k \leq N$ ,  $N \in \mathbb{N}$ . Assume that there exists a function  $\rho : \mathbb{Z} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} & \forall (j, k) \in \{1, \dots, N\}^2, \quad \max_{1 \leq u, v \leq v} |\mathbb{E}Y_{j,N}^{(u)} Y_{k,N}^{(v)}| \leq |\rho(j-k)| \\ & \text{with } \sum_{j \in \mathbb{Z}} |\rho(j)|^2 < \infty. \end{aligned} \quad (6.34)$$

Moreover, assume that for any  $\tau \in (0, 1)$  any  $J \in \mathbb{N}^*$ ,

$$(\mathbf{Y}_{j+[N\tau], N})_{-J \leq j \leq J} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} (\mathbf{W}_j)_{-J \leq j \leq J}, \quad (6.35)$$

where  $\mathbf{W}_j = (W_j^{(1)}, \dots, W_j^{(v)})$ ,  $j \in \mathbb{Z}$  is a stationary Gaussian process taking values in  $\mathbb{R}^v$  such that the covariance matrix  $\Sigma := (\text{Cov}(W_0^{(u)}, W_0^{(v)}))_{1 \leq u, v \leq v}$  is non-degenerated:  $\det \Sigma \neq 0$ . In addition, assume that

$$\max_{1 \leq k \leq N} \|\Sigma_{k, N} - \Sigma\| \rightarrow 0 \quad (N \rightarrow \infty). \quad (6.36)$$

Let  $G : \mathbb{R}^v \rightarrow \mathbb{R}$  be a bounded measurable function such that

$$\mathbb{E}[Y_{k, N} G(\mathbf{Y}_{k, N})] = 0, \quad \forall 1 \leq k \leq N, \quad N \in \mathbb{N}. \quad (6.37)$$

Then, with  $\sigma^2 := \sum_{j \in \mathbb{Z}} \text{Cov}(G(\mathbf{W}_0), G(\mathbf{W}_j)) < \infty$ ,

$$N^{-1/2} \sum_{k=1}^N \{G(\mathbf{Y}_{k, N}) - \mathbb{E}G(\mathbf{Y}_{k, N})\} \xrightarrow[N \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2). \quad (6.38)$$

**Proof of Theorem 2.** For notational simplicity, we restrict the proof of the  $p$ -dimensional CLT in (2.17) to the case  $p = 2$ . Let  $0 < t_1 < t_2 < 1$  be arbitrary time moments and  $S_n(t)$  be defined as in (6.5). First, we prove the following bivariate CLT for  $(S_n(t_1), S_n(t_2))$ :

$$\begin{aligned} & \sqrt{2n^{1-\alpha}} (S_n(t_1) - \Lambda_2(H(t_1)), S_n(t_2) - \Lambda_2(H(t_2))) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \Gamma), \\ & \Gamma := \begin{pmatrix} \sigma^2(H(t_1)) & 0 \\ 0 & \sigma^2(H(t_2)) \end{pmatrix}, \end{aligned} \quad (6.39)$$

where  $2n^{1-\alpha} \sim v_n = \#\mathcal{V}_{n, \alpha}(t_i)$ ,  $i = 1, 2$ , for all  $n$  large enough. Relation (6.39) follows from  $\mathbb{E}S_n(t_i) - \Lambda_2(H(t_i)) = o(n^{(1-\alpha)/2})$  (which is a consequence of (6.6) and the inequalities for  $\kappa$  and  $\alpha$  in the statement of Theorem 2) and

$$(2n^{1-\alpha})^{1/2} (S_n - \mathbb{E}S_n) \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2), \quad (6.40)$$

$$S_n := c_1 S(t_1) + c_2 S(t_2), \quad \sigma^2 := c_1^2 \sigma^2(H(t_1)) + c_2^2 \sigma^2(H(t_2)), \quad (6.41)$$

where  $c_1, c_2$  are arbitrary numbers.

To show (6.40) we use Lemma 3, with  $N := v_n$ ,  $v = 4$ . Let  $\psi(y^{(1)}, y^{(2)}) = \frac{|y^{(1)} + y^{(2)}|}{|y^{(1)}| + |y^{(2)}|}$  as above and rewrite

$$S_n = N^{-1} \sum_{k=1}^N G(\mathbf{Y}_{k, N}), \quad G(\mathbf{y}) := c_1 \psi(y^{(1)}, y^{(2)}) + c_2 \psi(y^{(3)}, y^{(4)}),$$

$$\mathbf{y} = (y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}) \in \mathbb{R}^4,$$

$$\mathbf{Y}_{k, N} := (Y_{k, N}^{(1)}, \dots, Y_{k, N}^{(4)}), \quad j_{ni} := [nt_i - n^{1-\alpha}], \quad i = 1, 2,$$

$$Y_{k,N}^{(1)} := \frac{V_n^{a*} Z\left(\frac{k+j_{n1}}{n}\right)}{\sigma_n(k+j_{n1})}, \quad Y_{k,N}^{(2)} := \frac{V_n^{a*} Z\left(\frac{k+j_{n1}+1}{n}\right)}{\sigma_n(k+j_{n1})},$$

$$Y_{k,N}^{(3)} := \frac{V_n^{a*} Z\left(\frac{k+j_{n2}}{n}\right)}{\sigma_n(k+j_{n2})}, \quad Y_{k,N}^{(4)} := \frac{V_n^{a*} Z\left(\frac{k+j_{n2}+1}{n}\right)}{\sigma_n(k+j_{n2})}.$$

Similarly to (6.14) define

$$W_k := \left(W_k^{(1)}, \dots, W_k^{(4)}\right), \quad k \in \mathbb{Z}, \quad \sigma^{*2}(H) := \text{Var}(V^{a*} B_H(0)) = 4 - 4^H,$$

$$W_k^{(1)} := \frac{V^{a*} B_{1,H(t_1)}(k)}{\sigma^*(H(t_1))}, \quad W_k^{(2)} := \frac{V^{a*} B_{1,H(t_1)}(k+1)}{\sigma^*(H(t_1))},$$

$$W_k^{(3)} := \frac{V^{a*} B_{2,H(t_2)}(k)}{\sigma^*(H(t_2))}, \quad W_k^{(4)} := \frac{V^{a*} B_{2,H(t_2)}(k)}{\sigma^*(H(t_2))},$$

where  $(B_{i,H(t_i)}(t)), i = 1, 2$  are independent FBM's with respective parameters  $H(t_i)$ . By definition,  $(W_k)_{k \in \mathbb{Z}}$  is a stationary Gaussian process with  $W_0$  having a non-degenerated covariance matrix  $\Sigma = \left(EW_0^{(u)} W_0^{(v)}\right)_{1 \leq u, v \leq 4}$ ,  $E(W_0^{(u)})^2 = 1$ ,  $u = 1, \dots, 4$ ,  $E[W_0^{(1)} W_0^{(2)}] = \rho_2(H(t_1))$ ,  $E[W_0^{(3)} W_0^{(4)}] = \rho_2(H(t_2))$ ,  $E[W_0^{(u)} W_0^{(v)}] = 0$ ,  $u \in \{1, 2\}$ ,  $v \in \{3, 4\}$ .

By definition,  $E(Y_{k,N}^{(i)})^2 = 1$ ,  $i = 1, 3$ . From Property 1, it follows that  $\max_{1 \leq k \leq N} |E(Y_{k,N}^{(i)})^2 - 1| \rightarrow 0$ ,  $i = 2, 4$  and

$$\max_{1 \leq k, j \leq N} \left| \text{Cov}(Y_{k,N}^{(u)}, Y_{j,N}^{(v)}) - \text{Cov}(W_k^{(u)}, W_j^{(v)}) \right| \rightarrow 0 \quad (6.42)$$

for  $u, v \in \{1, 2\}$  and  $u, v \in \{3, 4\}$ . For  $u \in \{1, 2\}$  and  $v \in \{3, 4\}$  relation (6.42) follows from (B) and (2.10), implying for the above  $u, v$  that  $\max_{1 \leq k, j \leq N} |\text{Cov}(Y_{k,N}^{(u)}, Y_{j,N}^{(v)})| \leq C(n|t_2 - t_1|)^{-\gamma} \rightarrow 0$ . Relation (6.42) guarantees the finite-dimensional convergence in (6.35) for the Gaussian vectors  $(Y_{k,N})$  and  $(W_k)$  defined above, including the convergence (6.36). The covariance bound in (6.34) also follows from (B) since  $\gamma > 1/2$ . Condition (6.37) follows from symmetries  $Y_{k,N} \stackrel{\text{law}}{=} -Y_{k,N}$  and  $G(y) = G(-y)$ . Finally, the expression for  $\sigma^2$  in (6.41) follows by independence of  $B_{1,H(t_1)}$  and  $B_{2,H(t_2)}$  and the definitions of  $G(y)$  and  $\sigma^2$  in (6.38). This proves (6.40) and (6.39) as well. Then, the clt of Theorem 2 (2.17) for  $p = 2$ , together with the expression (2.18) for the limit variance, follows from (6.39) by an application of the Delta-method. Theorem 2 is proved.  $\square$

**Proof of Corollary 1.** By definition,

$$\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t) = \sum_{i=1}^p b_i(\widehat{H}_{n,\alpha}^{(\text{IR})}(t)) \widehat{H}_{n,\alpha}^{(\text{IR}),i}(t), \quad \text{where } b_i(H) := \frac{\mathbf{1}_p^\top (\Sigma^{(p)}(H))^{-1} \mathbf{e}_i}{\mathbf{1}_p^\top (\Sigma^{(p)}(H))^{-1} \mathbf{1}_p},$$

where  $\mathbf{e}_i := \underbrace{(0, \dots, 0}_{i-1}, 1, 0, \dots, 0)^\top \in \mathbb{R}^p$  is a unit vector. Using the fact that  $\sum_{i=1}^p b_i(H) = 1$ ,

we can decompose

$$\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t) - H = W_n(t) + U_n(t), \quad (6.43)$$

where

$$W_n(t) = \sum_{i=1}^p b_i(H) (\widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) - H),$$

$$U_n(t) := \sum_{i=1}^p (b_i(\widehat{H}_{n,\alpha}^{(\text{IR})}(t)) - b_i(H)) (\widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) - H).$$

If view of the assumptions on  $\Sigma^{(p)}(H)$ , the functions  $H \mapsto b_i(H)$ ,  $1 \leq i \leq p$  are continuous on  $(0, 1)$  implying that  $|U_n(t)| \leq (\max_{1 \leq i \leq p} |\widehat{H}_{n,\alpha}^{(\text{IR}),i}(t) - H|) \delta(|\widehat{H}_{n,\alpha}^{(\text{IR})}(t) - b_i(H)|)$ , where  $\delta(x) = o(x)$ ,  $x \rightarrow 0$ . As a consequence, in all cases (i)–(iv) the term  $U_n(t)$  is negligible w.r.t. the term  $W_n(t)$ . From the proof of [Theorem 1](#) it is evident that all statements of [Theorem 1](#) hold with  $\widehat{H}_{n,\alpha}^{(\text{IR})}$  replaced by any estimator  $\widehat{H}_{n,\alpha}^{(\text{IR}),i}$ ,  $1 \leq i \leq p$ . This proves the corollary.  $\square$

**Proof of Corollary 2.** From the decomposition (6.43),

$$\sqrt{2n^{1-\alpha}} (\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t) - H) = \sqrt{2n^{1-\alpha}} (W_n(t) + o_p(1)). \quad (6.44)$$

Clearly, it suffices to prove the corollary with  $\widehat{H}_{n,\alpha}^{(\text{IR}2)}(t) - H$  replaced by  $W_n(t)$ . In turn, this follows from the  $(p \times m)$ -dimensional CLT

$$\sqrt{2n^{1-\alpha}} \left( (\widehat{H}_{n,\alpha}^{(\text{IR}),i}(t_j) - H(t_j))_{1 \leq i \leq p} \right)_{1 \leq j \leq m} \xrightarrow[n \rightarrow \infty]{\mathcal{D}} \left( (W^{(\text{IR}),i}(t_j))_{1 \leq i \leq p} \right)_{1 \leq j \leq m}, \quad (6.45)$$

where  $(W^{(\text{IR}),i}(t_j))_{1 \leq i \leq p}$ ,  $j = 1, \dots, m$  are independent Gaussian vectors with zero mean and respective covariance matrices  $\Sigma^{(p)}(H(t_j))$ . The proof of (6.45) mimics that of [Theorem 2](#) and we omit the details.  $\square$

## 7. Proofs of Section 4

This section contains the proofs of [Propositions 1](#) and [2](#) concerning the asymptotic behavior of (co)variances of generalized variations of the general MBM  $X_{(a^+, a^-)}$  for  $(a^+, a^-) \in \mathbb{R}^2 \setminus \{(0, 0)\}$  with local Hurst function  $H(\cdot)$  satisfying assumption  $(C)_\eta$ . We discuss the general case of filters  $a := (a_0, \dots, a_q) \in \mathbb{R}^{q+1}$  satisfying condition (2.1) for some  $m \in \mathbb{N}^*$ .

First introduce the following notations. Using [Theorem 4.1](#) of [\[22\]](#), for  $t, t' \in (0, 1)$

$$EX_{(a^+, a^-)}(t)X_{(a^+, a^-)}(t') = Q(H(t), H(t'), t, t'), \quad (7.1)$$

where  $Q$  is a function of 4 variables  $t, t' \in (0, 1)$ ,  $H, H' \in (0, 1)$  satisfying  $Q(H, H', t, t') = Q(H', H, t', t)$  and defined for  $t \leq t'$  by

$$Q(H, H', t, t') := \begin{cases} L(H, H') \left( \frac{1}{\pi} \sin(\Delta\beta(H, H')) (t' \log t' - t \log t - (t' - t) \log(t' - t)) + \cos(\Delta\beta(H, H')) t \right) & \text{if } H + H' = 1 \\ \frac{1}{2} \left( L_{11}(H, H') t^{H+H'} + L_{22}(H, H') t'^{H+H'} - L_{12}(H, H') (t' - t)^{H+H'} \right) & \text{if } H + H' \neq 1, \end{cases} \quad (7.2)$$

with

$$L_{11}(H, H') := \frac{L(H, H') \cos\left(\Delta\beta(H, H') - \frac{\pi}{2}(H + H')\right)}{\cos\left(\frac{\pi}{2}(H + H')\right)}, \quad (7.3)$$

$$L_{22}(H, H') := \frac{L(H, H') \cos\left(\Delta\beta(H, H') + \frac{\pi}{2}(H + H')\right)}{\cos\left(\frac{\pi}{2}(H + H')\right)} = L_{12}(H, H'), \quad (7.4)$$

and where

$$L(H, H') := \frac{(K(H) K(H'))^{1/2}}{K\left(\frac{H+H'}{2}\right)}, \quad (7.5)$$

$K(\cdot)$  is defined at (1.3),  $\Delta\beta(H, H') := \beta(H') - \beta(H)$ ,

$$\beta(H) := \begin{cases} \text{Arg}\left(a^+ e^{-i\left(H+\frac{1}{2}\right)\frac{\pi}{2}} + a^- e^{i\left(H+\frac{1}{2}\right)\frac{\pi}{2}}\right), & a^+ \neq a^-, \\ 0, & a^+ = a^- \neq 0, \end{cases} \quad (7.6)$$

where  $\text{Arg}(z) \in [0, 2\pi)$  is the principal argument of the complex number  $z \neq 0$  (note that  $L(H, H) = 1$  and  $L_{ij}(H, H', t, t') = L(H, H')$  for  $a^+ = a^- \neq 0$ ). In particular, for any  $a \neq 0$

$$\begin{aligned} \text{EX}_{(a,a)}(t) X_{(a,a)}(t') &= \text{EX}(t) X(t') = \frac{1}{2} L(H, H') (t^{H+H'} + t'^{H+H'} - (t' - t)^{H+H'}), \\ t &\leq t'. \end{aligned} \quad (7.7)$$

Note that the functions  $(H, H') \mapsto L(H, H')$ ,  $(H, H') \mapsto \Delta\beta(H, H')$  are analytic on  $(0, 1)^2$  and hence  $(H, H') \mapsto Q(H, H', t, t')$  is also analytic on  $(0, 1)^2$  for  $t' \neq t$  with possible exception on  $H + H' = 1$  where the denominator  $\cos(\frac{\pi}{2}(H + H'))$  in (7.3)–(7.4) vanishes. To show the analyticity of  $(H, H') \mapsto Q(H, H', t, t')$  on  $H + H' = 1$ , let  $\omega := H + H' - 1 \neq 0$ ,  $0 < t < t' < 1$  and rewrite  $Q(H, H', t, t')$  in (7.2) as

$$\begin{aligned} Q(H, H', t, t') &= \frac{1}{2} L(H, H') \cos(\Delta\beta(H, H')) (t^{1+\omega} + t'^{1+\omega} - (t' - t)^{1+\omega}) \\ &\quad - \frac{1}{2} L(H, H') \sin(\Delta\beta(H, H')) \\ &\quad \times \frac{\omega}{\tan\left(\frac{\omega\pi}{2}\right)} \left( t \left( \frac{t^\omega - 1}{\omega} \right) - t' \left( \frac{t'^\omega - 1}{\omega} \right) \right. \\ &\quad \left. + (t' - t) \left( \frac{(t' - t)^\omega - 1}{\omega} \right) \right). \end{aligned} \quad (7.8)$$

Then, the first term on the r.h.s. of (7.8) clearly is analytic in  $\omega$  at  $\omega = 0$  and the same conclusion holds for the second term because of the analyticity of the functions  $\omega \mapsto \frac{\omega}{\tan(\omega\pi/2)}$  and  $\omega \mapsto \omega^{-1}(t^\omega - 1)$  ( $t > 0$ ) in a neighborhood of  $\omega = 0$ . It is easy to check that the limit of (7.8) as  $\omega \rightarrow 0$  coincides with the definition of  $Q(H, H', t, t')$  in (7.2) for  $H + H' = 1$ . As a function of four variables,  $Q(H, H', t, t')$  is analytic on  $(0, 1)^4$  except for  $t = t'$  but  $Q$  is continuous on  $(0, 1)^4$ .

For  $0 < t < t' < 1$ , introduce the  $2 \times 2$ -matrix of the second order partial derivatives:

$$R^{(2)}(t, t') := \begin{pmatrix} R_{H, H'}^{(2)}(t, t') & R_{H, t'}^{(2)}(t, t') \\ R_{t, H'}^{(2)}(t, t') & R_{t, t'}^{(2)}(t, t') \end{pmatrix}, \quad (7.9)$$

whose elements are the corresponding partial derivatives of the function  $Q$  computed at  $H := H(t)$ ,  $H' := H(t')$ ,

$$\begin{aligned} R_{H, H'}^{(2)}(t, t') &:= \frac{\partial^2 Q}{\partial H \partial H'}(H(t), H(t'), t, t'), \dots, R_{t, t'}^{(2)}(t, t') \\ &:= \frac{\partial^2 Q}{\partial t \partial t'}(H(t), H(t'), t, t'). \end{aligned}$$

In particular, for all  $0 < t < t' < 1$ ,

$$R_{t, t'}^{(2)}(t, t') = \begin{cases} \frac{1}{\pi} L(H(t), H(t')) \sin(\beta(H(t')) - \beta(H(t))) (t' - t)^{-1} \\ \quad \text{if } H(t) + H(t') = 1 \\ -\frac{1}{2} L_{12}(H(t), H(t')) (H(t) + H(t')) \\ \quad (H(t) + H(t') - 1) (t' - t)^{H(t) + H(t') - 2} \quad \text{if } H(t) + H(t') \neq 1. \end{cases} \quad (7.10)$$

Finally, define also for  $0 < t, t' < 1$ ,

$$A(t, t') := \frac{\partial^2 Q}{\partial H \partial H'}(H(t), H(t'), t, t') \quad (7.11)$$

$$B_m(t, t') := \begin{cases} (-1)^{m+1} \frac{(2m-2)! \left( \sum_{i=0}^q i^m a_i \right)^2}{\pi (m!)^2} L(H(t), H(t')) \\ \quad \times \sin(\beta(H(t')) - \beta(H(t))) \quad \text{if } H(t) + H(t') = 1 \\ (-1)^{m+1} \frac{\left( \sum_{i=0}^q i^m a_i \right)^2}{2(m!)^2} L_{12}(H(t), H(t')) \\ \quad \times \prod_{\ell=0}^{2m-1} (H(t) + H(t') - \ell) \quad \text{if } H(t) + H(t') \neq 1, \end{cases} \quad (7.12)$$

with  $\beta = 0$  if  $a^+ = a^-$ . Remark that  $A(t, t')$  and  $B_m(t, t')$  are bounded on  $(0, 1)^2$ .



**Lemma 4.** Under assumptions of Proposition 2 and for any  $0 < \varepsilon < 1/2$ ,  $[n\varepsilon] \leq k < k' \leq [(1 - \varepsilon)n]$ ,  $k' - k \geq 2q$ ,

$$\begin{aligned} & \text{Cov} \left( V_n^a X_{(a^+, a^-)} \left( \frac{k}{n} \right), V_n^a X_{(a^+, a^-)} \left( \frac{k'}{n} \right) \right) \\ &= \left( V_n^a H \left( \frac{k}{n} \right), V_n^a \frac{k}{n} \right) \times R^{(2)} \left( \frac{k}{n}, \frac{k'}{n} \right) \\ & \quad \times \left( V_n^a H \left( \frac{k'}{n} \right), V_n^a \frac{k'}{n} \right)^\top + \mathbf{1}_{\{m \geq 2\}} B_m \left( \frac{k}{n}, \frac{k'}{n} \right) \\ & \quad \times \frac{|k' - k|^{H(\frac{k}{n}) + H(\frac{k'}{n}) - 2m}}{n^{H(\frac{k}{n}) + H(\frac{k'}{n})}} + \lambda_n(k, k'). \end{aligned} \quad (7.13)$$

The remainder term  $\lambda_n(k, k')$  in (7.13) satisfies the following bound: for any  $\delta > 0$  there exist  $n_0 \in \mathbb{N}^*$ ,  $k_0 \in \mathbb{N}^*$  such that for any  $n > n_0$ ,  $|k - k'| > k_0$ ,  $[n\varepsilon] \leq k < k' \leq [(1 - \varepsilon)n]$

$$|\lambda_n(k, k')| \leq \delta \left( \frac{1}{n^{2(\eta \wedge m)}} + \frac{|k' - k|^{H(k/n) + H(k'/n) - 2m}}{n^{H(k/n) + H(k'/n)}} \right). \quad (7.14)$$

**Remark 11.** In (7.13)  $V_n^a \frac{k}{n} = \frac{1}{n} (\sum_{i=1}^q i a_i)$  or 0 depending on whether  $m = 1$  or  $m \geq 2$  holds. In particular, for  $a = a^*$  the quadratic form on the r.h.s. of (7.13) coincides with the term  $A(\frac{k}{n}, \frac{k'}{n}) V_n^{a^*} H(\frac{k}{n}) V_n^{a^*} H(\frac{k'}{n})$  in the expansion (4.7).

**Proof of Lemma 4.** For  $k, k' \in \{1, 2, \dots, n - q - 1\}$  and  $i, j \in \{0, 1, \dots, q\}$  denote

$$\begin{aligned} \delta_i &:= H \left( \frac{k}{n} + \frac{i}{n} \right) - H \left( \frac{k}{n} \right), \quad \delta'_j := H \left( \frac{k'}{n} + \frac{j}{n} \right) - H \left( \frac{k'}{n} \right), \\ \|\delta\| &:= \max_{1 \leq i \leq q} |\delta_i|, \quad \|\delta'\| := \max_{1 \leq j \leq q} |\delta'_j|. \end{aligned} \quad (7.15)$$

Assumption  $H(\cdot) \in \mathcal{C}^\eta(0, 1)$  implies that there exists  $C > 0$  not depending on  $k$  and  $n$  such that  $\max\{|\delta_i|, |\delta'_j|\} \leq C n^{-\eta \wedge 1}$ . Using the definition in (7.1) and the notation in (7.15),

$$\begin{aligned} & \mathbb{E} \left[ V_n^a X_{(a^+, a^-)} \left( \frac{k}{n} \right) V_n^a X_{(a^+, a^-)} \left( \frac{k'}{n} \right) \right] \\ &= \sum_{i,j=0}^q a_i a_j Q \left( H \left( \frac{k}{n} \right) + \delta_i, H \left( \frac{k'}{n} \right) + \delta'_j, \frac{k}{n} + \frac{i}{n}, \frac{k'}{n} + \frac{j}{n} \right). \end{aligned} \quad (7.16)$$

Similarly to (7.9), for any collection  $\mathbf{p} = (p_1, \dots, p_4) \in \mathbb{N}^4$  of integers and  $s \neq s'$ , introduce partial derivatives of order  $|\mathbf{p}| := p_1 + \dots + p_4$  of  $Q$  in (7.2):

$$R_{H^{p_1}, H^{p_2}, s^{p_3}, s'^{p_4}}^{(|\mathbf{p}|)}(s, s') := \frac{\partial^{|\mathbf{p}|} Q}{\partial H^{p_1} \partial H^{p_2} \partial s^{p_3} \partial s'^{p_4}}(H(s), H(s'), s, s'). \quad (7.17)$$

Without loss of generality, assume  $k' - k > 2q$  in the sequel. Using the Taylor expansion of order  $2m$  of  $Q$  in  $(H, H', s, s')$ , one obtains:

$$\mathbb{E} \left[ V_n^a X_{(a^+, a^-)} \left( \frac{k}{n} \right) V_n^a X_{(a^+, a^-)} \left( \frac{k'}{n} \right) \right] = \sum_{0 \leq |\mathbf{p}| \leq 2m} J_n(\mathbf{p}) + \sum_{|\mathbf{p}|=2m+1} \tilde{J}_n(\mathbf{p}), \quad (7.18)$$

where

$$J_n(\mathbf{p}) := \frac{1}{p_1! \cdots p_4!} R_{H^{p_1}, \dots, s'^{p_4}}^{(|\mathbf{p}|)} \left( \frac{k}{n}, \frac{k'}{n} \right) \sum_{i,j=0}^q a_i a_j \delta_i^{p_1} (\delta'_j)^{p_2} \left( \frac{i}{n} \right)^{p_3} \left( \frac{j}{n} \right)^{p_4},$$

$$\tilde{J}_n(\mathbf{p}) \leq C \sup_{I\left(\frac{k}{n}, \frac{k'}{n}\right)} |R_{H^{p_1}, \dots, s'^{p_4}}^{(|\mathbf{p}|)}(s, s')| \sum_{i,j=0}^q \left| \delta_i^{p_1} (\delta'_j)^{p_2} \left( \frac{i}{n} \right)^{p_3} \left( \frac{j}{n} \right)^{p_4} \right|$$

and where the set  $I\left(\frac{k}{n}, \frac{k'}{n}\right) \subset (0, 1)^4$  is defined by

$$I\left(\frac{k}{n}, \frac{k'}{n}\right) := \left\{ (H, H', s, s') : \left| H - H\left(\frac{k}{n}\right) \right| \leq \|\delta\|, \left| H' - H\left(\frac{k'}{n}\right) \right| \leq \|\delta'\|, \left| s - \frac{k}{n} \right| \leq \frac{q}{n}, \left| s' - \frac{k'}{n} \right| \leq \frac{q}{n}, s < s' \right\}.$$

Note that terms in (7.18) corresponding to  $|\mathbf{p}| = 0$  and  $|\mathbf{p}| = 1$  vanish because of the moment condition (2.1). In particular,

$$\sum_{|\mathbf{p}|=1} J_n(\mathbf{p}) = \sum_{i,j=0}^q a_i a_j \left( \delta_i R_H^{(1)} \left( \frac{k}{n}, \frac{k'}{n} \right) + \cdots + \frac{j}{n} R_{s'}^{(1)} \left( \frac{k}{n}, \frac{k'}{n} \right) \right) = 0.$$

In a similar way,

$$\begin{aligned} \sum_{|\mathbf{p}|=2} J_n(\mathbf{p}) &= \frac{1}{2} \left( 2 \left( \sum_{i=0}^q a_i \delta_i \right) \left( \sum_{j=0}^q a_j \delta'_j \right) R_{H, H'}^{(2)} \left( \frac{k}{n}, \frac{k'}{n} \right) \right. \\ &\quad \left. + \cdots + 2 \left( \sum_{i=0}^q a_i \frac{i}{n} \right) \left( \sum_{j=0}^q a_j \frac{j}{n} \right) R_{s, s'}^{(2)} \left( \frac{k}{n}, \frac{k'}{n} \right) \right) \\ &= \left( V_n^a H \left( \frac{k}{n} \right), V_n^a \frac{k}{n} \right) \times R^{(2)} \left( \frac{k}{n}, \frac{k'}{n} \right) \times \left( V_n^a H \left( \frac{k'}{n} \right), V_n^a \frac{k'}{n} \right)^\top, \end{aligned} \quad (7.19)$$

since  $\sum_{i=0}^q a_i \delta_i = V_n^a H \left( \frac{k}{n} \right), \dots, \sum_{j=0}^q a_j \frac{j}{n} = V_n^a \frac{k}{n}$ . Note that the partial derivatives in (7.17) involving arguments  $H$  and  $H'$  contributes to a factor  $\log\left(\frac{k'-k}{n}\right)$  at each differentiation while

each differentiation in  $s$  or  $s'$  contributes a factor  $\left|\frac{k'-k}{n}\right|^{-1}$ ; in other words,

$$\sup_{I\left(\frac{k}{n}, \frac{k'}{n}\right)} \left| R_{H^{p_1}, \dots, s'^{p_4}}^{(|p|)}(s, s') \right| \leq C \left( 1 + \left| \frac{k'-k}{n} \right|^{H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - p_3 - p_4} \right. \\ \left. \times \left| \log \left( \frac{k'-k}{n} \right) \right|^{p_1 + p_2} \right). \quad (7.20)$$

Also note

$$\left| V_n^a H\left(\frac{k}{n}\right) \right| \leq C n^{-(\eta \wedge m)}, \quad \left| V_n^a \frac{k}{n} \right| \leq C n^{-m} \quad 1 \leq k < n - q. \quad (7.21)$$

From (7.21) and (7.20) we obtain

$$\left| \sum_{|p|=2} J_n(p) \right| \leq C \left( \frac{1}{n^{2(\eta \wedge m)}} + \frac{|k'-k|^{H(k/n) + H(k'/n) - 2}}{n^{H(k/n) + H(k'/n)}} \mathbf{1}_{\{m=1\}} \right) \quad (7.22)$$

since  $V_n^a \frac{k}{n} = 0$  for  $m > 1$ . We also have from (7.2) (with  $\beta = 0$  if  $a^+ = a^-$ ),

$$J_n(0, 0, m, m) \mathbf{1}_{\{m \geq 2\}} = (-1)^{m+1} \frac{(2m-2)! \left( \sum_{i=0}^q a_i i^m \right)^2}{2(m!)^2} L \left( H\left(\frac{k}{n}\right), H\left(\frac{k'}{n}\right) \right) \\ \times \sin \left( \beta \left( H\left(\frac{k'}{n}\right) \right) \right) \\ - \beta \left( H\left(\frac{k}{n}\right) \right) \frac{|k-k'|^{1-2m}}{n} \quad \text{if } H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) = 1; \\ = (-1)^{m+1} \frac{\left( \sum_{i=0}^q a_i i^m \right)^2}{2(m!)^2} L_{12} \left( H\left(\frac{k}{n}\right), H\left(\frac{k'}{n}\right) \right) \\ \times \prod_{\ell=0}^{2m-1} \left( H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - \ell \right) \\ \times \frac{|k-k'|^{H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) - 2m}}{n^{H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right)}} \quad \text{if } H\left(\frac{k}{n}\right) + H\left(\frac{k'}{n}\right) \neq 1 \quad (7.23)$$

which coincides with the second expansion term on the r.h.s. of (7.13). It remains to prove that all the other terms in the Taylor expansion (7.18) are negligible with respect to the main terms in (7.13), in other words, to show the bound (7.14) on the remainder term  $\lambda_n(k, k')$ . Define

$$A_n := \frac{\log^2 n}{n^{2(\eta \wedge m)}} + \frac{|k'-k|^{H(k/n) + H(k'/n) - 2m}}{n^{H(k/n) + H(k'/n)}}.$$

We shall prove below that there exist  $\beta_1 > 0$ ,  $\beta_2 > 0$  independent of  $n, k, k'$  such that for any  $0 < \varepsilon < 1/2$ ,  $n \in \mathbb{N}^*$  and  $[n\varepsilon] \leq k < k' \leq [(1 - \varepsilon)n]$ ,

$$J_n(\mathbf{p}) \leq \delta_n \Lambda_n, \quad 2 < |\mathbf{p}| \leq 2m, \quad \mathbf{p} \neq (0, 0, m, m), \quad (7.24)$$

$$\tilde{J}_n(\mathbf{p}) \leq \delta'_n \Lambda_n, \quad |\mathbf{p}| = 2m + 1 \quad (7.25)$$

where

$$\delta_n(k', k) := \left(\frac{1}{n}\right)^{\beta_1} + \left(\frac{|k' - k|}{n}\right)^{\beta_2}, \quad \delta'_n(k', k) := \delta_n(k', k) + \frac{1}{|k' - k|}.$$

The statement of the lemma including (7.14) follows from (7.18), (7.19), (7.24) and (7.25).

Let us prove (7.24). Define

$$U_n(\mathbf{p}) := \sum_{i=0}^q a_i \delta_i^{p_1} \left(\frac{i}{n}\right)^{p_3}, \quad U'_n(\mathbf{p}) := \sum_{j=0}^q a_j (\delta'_j)^{p_2} \left(\frac{j}{n}\right)^{p_4}. \quad (7.26)$$

We claim that for any  $\eta > 0$  and  $m \geq 1$

$$|U_n(\mathbf{p})| \leq C \left( \frac{1}{n^{m \vee p_3}} + \frac{1}{n^{\eta + p_3}} \right) \quad (7.27)$$

and a similar bound holds for  $U'_n(\mathbf{p})$ . Let us check that (7.27) and (7.20) imply (7.24). Indeed, from the above relations and the definition of  $J_n(\mathbf{p})$  we obtain

$$\begin{aligned} |J_n(\mathbf{p})| &\leq \left| R_{H^{p_1}, \dots, t^{p_4}}^{(|\mathbf{p}|)} \left( \frac{k}{n}, \frac{k'}{n} \right) \right| |U_n(\mathbf{p})| |U'_n(\mathbf{p})| \\ &\leq C (\log n)^{p_1 + p_2} \left| \frac{k' - k}{n} \right|^{H(k/n) + H(k'/n) - p_3 - p_4} \\ &\quad \times \left( \frac{1}{n^{m \vee p_3}} + \frac{1}{n^{\eta + p_3}} \right) \left( \frac{1}{n^{m \vee p_4}} + \frac{1}{n^{\eta + p_4}} \right) \\ &\leq C (\log n)^{p_1 + p_2} (I_{11} + I_{12} + I_{21} + I_{22}), \end{aligned}$$

where

$$\begin{aligned} I_{11} &:= \left| \frac{k' - k}{n} \right|^{H(k/n) + H(k'/n) - p_3 - p_4} \frac{1}{n^{2m}}, \\ I_{12} &:= \left| \frac{k' - k}{n} \right|^{H(k/n) + H(k'/n) - p_3 - p_4} \frac{1}{n^{(m \vee p_4) + \eta + p_3}}, \\ I_{21} &:= \left| \frac{k' - k}{n} \right|^{H(k/n) + H(k'/n) - p_3 - p_4} \frac{1}{n^{(m \vee p_3) + \eta + p_4}}, \\ I_{22} &:= \left| \frac{k' - k}{n} \right|^{H(k/n) + H(k'/n) - p_3 - p_4} \frac{1}{n^{2\eta + p_3 + p_4}}. \end{aligned}$$

It suffices to prove (7.24) for  $I_{ij}$ ,  $i, j = 1, 2$ .

Let  $H := (H(k/n) + H(k'/n))/2$ ,  $j = |k' - k|$ ,  $p := p_3 + p_4$ . For  $I_{11}$ , (7.24) follows from

$$\begin{aligned} \left(\frac{j}{n}\right)^{2H-p} \frac{1}{n^{2m}} &= \left(\frac{j}{n}\right)^{2H} \left(\frac{n}{j}\right)^p \frac{1}{n^{2m}} \\ &\leq \left(\frac{j}{n}\right)^{2H} \left(\frac{n}{j}\right)^{2m-1} \frac{1}{n^{2m}} = \left(\frac{j^{2H-2m}}{n^{2H}}\right) \left(\frac{j}{n}\right) \leq \Lambda_n \delta_{11n}, \\ \delta_{11,n} &:= \frac{j}{n}, \end{aligned} \quad (7.28)$$

which is immediate from  $0 < p \leq 2m - 1$  and  $1 \leq j < n$ .

Next, consider  $I_{22}$ . Assume first that  $\eta < m$ . Observe that in this case,  $\Lambda_n \geq n^{-2\eta}$  if  $j \geq n^{(\eta-H)/(m-H)}$  and  $\Lambda_n \geq j^{2H-2m} n^{-2H}$  if  $j \leq n^{(\eta-H)/(m-H)}$ . Thus in the case  $j \geq n^{(\eta-H)/(m-H)}$ , the bound (7.24) for  $I_{22}$  translates to

$$\left(\frac{j}{n}\right)^{2H-p} \frac{1}{n^{2\eta+p}} \leq \frac{1}{n^{2\eta}} \delta_{22,n}, \quad \text{with } \delta_{22,n} := \frac{j^{2H-p}}{n^{2H}} = \frac{1}{j^p} \left(\frac{j}{n}\right)^{2H} \leq \left(\frac{j}{n}\right)^{2H} \quad (7.29)$$

since  $j \geq 1$ . Next, let  $1 \leq j \leq n^{(\eta-H)/(m-H)}$ ,  $p \geq 1$ , then the bound (7.24) for  $I_{22}$  follows by

$$\begin{aligned} \left(\frac{j}{n}\right)^{2H-p} \frac{1}{n^{2\eta+p}} &\leq \delta_{22,n} \frac{j^{2H-2m}}{n^{2H}}, \\ \text{with } \delta_{22,n} &:= \frac{j^{2m-p}}{n^{2\eta}} = \left(\frac{j}{n^{\frac{\eta-H}{m-H}}}\right)^{2m-p} n^{-\beta_{22}} \leq n^{-\beta_{22}}, \end{aligned} \quad (7.30)$$

where  $\beta_{22} := \frac{2H(m-\eta)+p(\eta-H)}{m-H} > 0$ . Next, consider  $I_{22}$  for  $\eta \geq m$ . Then the first term in the definition of  $\Lambda_n$  is negligible with respect to the second term and the corresponding relation reduces to

$$\frac{j^{2H-p}}{n^{2\eta+2H}} \leq \delta_{22,n} \frac{j^{2H-2m}}{n^{2H}}, \quad \text{with } \delta_{22,n} := \frac{j^{2m-p}}{n^{2\eta}} \leq \left(\frac{j}{n}\right)^{2m}. \quad (7.31)$$

Next, consider  $I_{12}$ . Accordingly, we need to show

$$\left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4)+\eta+p_3}} \leq \delta_n \left( \frac{1}{n^{2(\eta \wedge m)}} + \frac{j^{2H-2m}}{n^{2H}} \right), \quad (7.32)$$

where  $\delta_n$  satisfies the bound in (7.26). Let  $\eta < m$  then (7.32) becomes

$$\left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4)+\eta+p_3}} \leq \delta_n \left( \frac{1}{n^{2\eta}} + \frac{j^{2H-2m}}{n^{2H}} \right). \quad (7.33)$$

Let  $j > n^{\frac{\eta-H}{m-H}}$ . The first term on the r.h.s. of (7.33) dominates the second one and (7.32) for  $2H \leq p_3 + p_4$  follows from

$$\left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4)+\eta+p_3}} \leq \delta'_{12,n} \frac{1}{n^{2\eta}}, \quad \text{with } \delta'_{12,n} := n^{-\beta'_{12}}, \quad \beta'_{12} > 0,$$

where

$$\begin{aligned}\beta'_{12} &:= 2H - \eta + (m \vee p_4) - p_4 + \frac{\eta - H}{m - H}(p_3 + p_4 - 2H) \\ &= (m - H)^{-1} \times \begin{cases} (p_3 + p_4 - m)(\eta - H) + H(m - \eta), & p_4 \geq m, \\ (m - p_4 + H)(m - \eta) + p_3(\eta - H), & p_4 < m. \end{cases}\end{aligned}$$

Thus  $\beta'_{12} \geq H(m - \eta)/(m - H) > 0$ . On the other hand, if  $p_3 + p_4 < 2H < 2$  then the first inequality in (7.33) holds with  $\delta_n = \delta'_{12,n}$  where

$$\delta'_{12,n} := \left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4) - \eta + p_3}} \leq n^{-(m-\eta)},$$

with  $m - \eta > 0$ , since  $m \geq 2$ ,  $p_4 < 2$  and  $(m \vee p_4) - \eta + p_3 \geq m - \eta > 0$ .

Next, let  $j \leq n^{\frac{\eta-H}{m-H}}$ . Then the second term on the r.h.s. of (7.33) dominates the first one and (7.32) follows from

$$\begin{aligned}\left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4) + \eta + p_3}} &\leq \delta''_{12,n} \frac{j^{2H-2m}}{n^{2H}}, \\ \text{with } \delta''_{12,n} &:= \frac{j^{2m-p_3-p_4}}{n^{\eta+(m \vee p_4)-p_4}} \leq n^{-\beta''_{12}},\end{aligned}$$

where

$$\begin{aligned}\beta''_{12} &:= \eta + (m \vee p_4) - p_4 - \left(\frac{\eta - H}{m - H}\right)(2m - p_3 - p_4) \\ &= (m - H)^{-1} \times \begin{cases} (p_3 + p_4 - m)(\eta - H) + H(m - \eta), & p_4 \geq m, \\ (m - p_4 + H)(m - \eta) + p_3(\eta - H), & p_4 < m, \end{cases}\end{aligned}$$

and therefore  $\beta''_{12} = \beta'_{12}$ , implying  $\beta''_{12} > 0$  as above.

It remains to consider  $I_{12}$  for  $\eta \geq m$ . In this case the second term on the r.h.s. of (7.33) dominates the first one for all  $1 \leq j \leq n$  and (7.32) follows from

$$\begin{aligned}\left(\frac{j}{n}\right)^{2H-p_3-p_4} \frac{1}{n^{(m \vee p_4) + \eta + p_3}} &\leq \delta'''_{12,n} \frac{j^{2H-2m}}{n^{2H}}, \\ \text{with } \delta'''_{12,n} &:= \frac{j^{2m-p_3-p_4}}{n^{\eta+(m \vee p_4)-p_4}} \leq \left(\frac{j}{n}\right)^{2m-p_3-p_4} \leq \left(\frac{j}{n}\right)\end{aligned}$$

since  $p_3 + p_4 < 2m$  and  $\eta + (m \vee p_4) - 2m + p_3 = (\eta - m) + ((m \vee p_4) - m) + p_3 \geq 0$ . This proves (7.32) or (7.24) for  $J_{12}$ . Since consideration of  $I_{21}$  is completely analogous, the proof of (7.24) is now complete.

Let us prove the claim (7.27). Write  $\delta_i = \delta_{0i} + \Gamma_i$ , where for  $i \in \{0, \dots, q\}$  and  $r = \{0, \dots, [\eta]\}$ ,

$$\Gamma_i := \sum_{r=1}^{[\eta]} \frac{H_r}{r!} \left(\frac{i}{n}\right)^r, \quad H_r := H^{(r)} \left(\frac{k}{n}\right).$$

By assumption  $C(\eta)$ ,

$$\max_i (|\delta_{0i}|) \leq C n^{-\eta}, \quad \max_r (|H_r|) \leq C. \quad (7.34)$$

From binomial expansion,

$$U_n(\mathbf{p}) = \sum_{\tau_0 + \dots + \tau_{[\eta]} = p_1} \binom{p_1}{\tau_0, \dots, \tau_{[\eta]}} \left( \prod_{r=1}^{[\eta]} \frac{H_r^{\tau_r}}{r!} \right) \sum_{i=0}^q a_i \delta_{0i}^{\tau_0} \left( \frac{i}{n} \right)^{p_3 + \sum_{r=1}^{[\eta]} r \tau_r},$$

where  $\tau_i \in \mathbb{N}$ ,  $i = 0, 1, \dots, [\eta]$  are nonnegative integers. According to (2.1), for any  $s = 0, 1, \dots$

$$\left| \sum_{i=0}^q a_i \left( \frac{i}{n} \right)^s \right| \leq C n^{-s} \mathbf{1}(s \geq m).$$

Therefore from (7.34) it follows that

$$\left| \sum_{i=0}^q a_i \delta_{0i}^{\tau_0} \left( \frac{i}{n} \right)^{p_3 + \sum_{r=1}^{[\eta]} r \tau_r} \right| \leq C \begin{cases} n^{-\eta \tau_0 - p_3 - \sum_{r=1}^{[\eta]} r \tau_r}, & \tau_0 \geq 1, \\ n^{-p_3 - \sum_{r=1}^{[\eta]} r \tau_r}, & \tau_0 = 0 \text{ and } p_3 + \sum_{r=1}^{[\eta]} r \tau_r \geq m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that for  $\tau_0 = 0$ , we have  $\sum_{r=1}^{[\eta]} r \tau_r \geq p_1$ . Hence,

$$n^{-p_3 - \sum_{r=1}^{[\eta]} r \tau_r} \mathbf{1}\left(\tau_0 = 0, p_3 + \sum_{r=1}^{[\eta]} r \tau_r \geq m\right) \leq (n^{-m}) \wedge (n^{-p_1 - p_3}) \leq n^{-(m \vee p_3)}.$$

Therefore

$$\left| \sum_{i=0}^q a_i \delta_{0i}^{\tau_0} \left( \frac{i}{n} \right)^{p_3 + \sum_{r=1}^{[\eta]} r \tau_r} \right| \leq C \left( n^{-\eta - p_3} + n^{-(m \vee p_3)} \right),$$

proving (7.27) and (7.24).

It remains to prove (7.25). Let  $\tilde{U}_n(\mathbf{p}) := \sum_{i=0}^q |a_i \delta_i^{p_1} \left( \frac{i}{n} \right)^{p_3}|$ . Since  $\|\delta\| \leq C n^{-(\eta \wedge 1)}$ , see (7.34), it follows that

$$\tilde{U}_n(\mathbf{p}) \leq C n^{-(\eta \wedge 1) p_1 - p_3} \quad (7.35)$$

and a similar bound holds for  $\tilde{U}'_n(\mathbf{p}) := \sum_{j=0}^q |a_j (\delta'_j)^{p_2} \left( \frac{j}{n} \right)^{p_4}|$ . Relations (7.35) and (7.20) imply

$$\tilde{J}_n(\mathbf{p}) \leq C \left( 1 + \left| \frac{k' - k}{n} \right|^{2H-p} \times \left| \log \left( \frac{k' - k}{n} \right) \right|^{2m+1-p} \right) n^{-(\eta \wedge 1)(2m+1-p)-p} \quad (7.36)$$

where  $p = p_3 + p_4 \in \{0, 1, \dots, 2m+1\}$  and  $|\mathbf{p}| = 2m+1$ . Relation (7.25) now can be verified directly, by inspecting the cases  $\eta < 1$ ,  $\eta \geq 1$ ,  $p = 0, 1, 2m+1$  and  $1 < p < 2m+1$  separately. This ends the proof of Lemma 4.  $\square$

**Proof of Proposition 1.** Note first that the result of Lemma 4 cannot be used for establishing Proposition 1 because Proposition 1 treats the case  $k$  close to  $k'$  which is excluded by the proof of Lemma 4 (the second derivative  $R_{t,t'}$  does not exist for  $t = t'$ ).

For any  $[n\varepsilon] \leq k \leq [(1 - \varepsilon)n]$  similarly to (7.16) and using the notation in (7.15) with  $k' = k + \ell$  we rewrite the variance  $\mathcal{V}_n(k) := \text{Var} \left( V_n^a X_{(a^+, a^-)} \left( \frac{k}{n} \right) \right)$  as

$$\mathcal{V}_n(k) = \frac{1}{2} \sum_{i,j=0}^q a_i a_j Q \left( H \left( \frac{k}{n} \right) + \delta_i, H \left( \frac{k}{n} \right) + \delta_j, \frac{k}{n} + \frac{i}{n}, \frac{k}{n} + \frac{j}{n} \right). \quad (7.37)$$

Next, we use a Taylor expansion of  $Q$  given by two analytic expressions in (7.2); see also (7.8). In order to avoid cumbersome notation, we shall assume in the rest of the proof that the function  $H(\cdot)$  is separated from  $1/2$ , i.e.,  $\sup_{t \in (0,1)} |H(t) - 1/2| > 0$ . Then the second expression for  $Q$  in (7.2) applies, viz.,  $Q(H, H', t, t') = (\frac{1}{2})(Q_{11}(H, H', t) + Q_{22}(H, H', t') - Q_{12}(H, H', t, t'))$ , where

$$\begin{aligned} Q_{11}(H, H', t) &:= L_{11}(H, H')t^{H+H'}, & Q_{22}(H, H', t) &:= L_{22}(H, H')t^{H+H'}, \\ Q_{12}(H, H', t, t') &:= L_{12}(H, H')|t' - t|^{H+H'}. \end{aligned}$$

Accordingly,  $\mathcal{V}_n(k) = \frac{1}{2}(\mathcal{V}_{n11}(k) + \mathcal{V}_{n22}(k) - \mathcal{V}_{n12}(k))$ , where  $\mathcal{V}_{nij}$  are defined as in (7.37) with  $Q$  replaced by  $Q_{ij}$ ,  $1 \leq i \leq j \leq 2$ . Note that  $(H, H', t) \in (0, 1)^3 \mapsto Q_{11}(H, H', t)$  is analytic except for  $H + H' = 1$  and the last case is excluded by our assumption that  $H(\cdot)$  is separated from  $1/2$ . Using the last fact, the Taylor expansion of  $Q_{11}$  as in (7.18) in the proof of Lemma 4 and the bounds (7.21), it follows that for all  $n$  large enough,

$$\sup_{[n\varepsilon] \leq k \leq [(1-\varepsilon)n]} |\mathcal{V}_{n11}(k)| \leq C(\varepsilon) \frac{1}{n^{2(\eta \wedge m)}} \quad (7.38)$$

with  $C(\varepsilon) \geq 0$  not depending on  $n$ . Clearly, the same bound (7.38) holds for  $\mathcal{V}_{n22}$ .

It remains to estimate  $\mathcal{V}_{n12}$ . For  $i, j \in \{0, \dots, q\}$ , we have

$$\begin{aligned} \left| \frac{j-i}{n} \right|^{2H\left(\frac{k}{n}\right) + \delta_i + \delta_j} &= \left| \frac{j-i}{n} \right|^{2H\left(\frac{k}{n}\right)} \exp \left\{ (\delta_i + \delta_j) \log \left| \frac{j-i}{n} \right| \right\} = \left| \frac{j-i}{n} \right|^{2H\left(\frac{k}{n}\right)} \\ &\quad + O \left( \frac{\log n}{n^{2H\left(\frac{k}{n}\right) + \eta \wedge 1}} \right). \end{aligned}$$

Here and in the rest of the proof, we denote  $O(\cdot)$  a bounded function depending on  $H(\cdot)$ ,  $\varepsilon$  and  $q$  only, and not depending on  $k$  or  $n$ . Next, since  $(H, H') \mapsto L_{12}(H, H')$  is analytic for  $H + H' \neq 1$ , so

$$L_{12} \left( H \left( \frac{k}{n} \right) + \delta_i, H \left( \frac{k}{n} \right) + \delta_j \right) = 1 + O \left( \frac{1}{n^{\eta \wedge 1}} \right).$$

Therefore,

$$\mathcal{V}_{n12}(k) = C(H(k/n), a) n^{-2H\left(\frac{k}{n}\right)} + O \left( \frac{\log n}{n^{2H\left(\frac{k}{n}\right) + \eta \wedge 1}} \right) \quad (7.39)$$

where  $C(H, a) = -n^{2H} \sum_{i,j=0}^q a_i a_j \left| \frac{j-i}{n} \right|^{2H} = 2\text{Var} \left( V^a B_H(0) \right)$  does not depend on  $n$  and does not vanish for  $H \in (0, 1)$ ; see Section 2.2.



Using (7.38) and (7.39), we obtain

$$\begin{aligned}\mathcal{V}_n(k) &= -\frac{1}{2} C(H(k/n), a) n^{-2H(\frac{k}{n})} + O\left(\frac{1}{n^{2(\eta \wedge m)}}\right) + O\left(\frac{\log n}{n^{2H(\frac{k}{n}) + \eta \wedge 1}}\right) \\ &= \text{Var}\left(V_n^a B_{H(\frac{k}{n})}\left(\frac{k}{n}\right)\right) \left(1 + O\left(\frac{\log n}{n^{\eta \wedge 1}}\right) + O\left(\frac{1}{n^{2(\eta \wedge m) - 2H(\frac{k}{n})}}\right)\right).\end{aligned}$$

Hence the bound of Proposition 1 can be deduced.  $\square$

**Proof of Corollary 3.** Fact (i) is immediate from Proposition 1. By Property 1,  $E(V_n^{a*} X_{(a^+, a^-)}(k/n))^2 \sim (4 - 4^{H(k/n)})n^{-2H(k/n)}$  and hence  $E(V_n^{a*} X_{(a^+, a^-)}(k/n))^2 \geq Cn^{-2H(k/n)}$  for some  $C > 0$  independent of  $1 \leq k \leq n - 2$  and  $n$ . Use the last fact, Proposition 2 (4.7), (4.8), boundedness of  $A(t, t')$  and  $B(t, t')$ , and  $|V_n^{a*} H(\frac{k}{n})| \leq Cn^{-(\eta \wedge 2)}$  to conclude inequality (4.9) for  $|k - k'| > k_0$  and  $k_0$  large enough; for  $|k - k'| \leq k_0$  (4.9) is trivial since the l.h.s. of (4.9) is bounded by 1.

It remains to prove fact (ii). Let  $k' - k =: \tilde{k} \geq 1$ ,  $H := H(k/n)$ ,  $H' := H(k'/n)$ . Let first  $\tilde{k} \leq n^{1-\alpha}$ . Then in view of (4.9) it suffices to check that

$$\frac{1}{\tilde{k}^{4-H-H'}} \leq \frac{Cn^{\theta(1-\alpha)}}{\tilde{k}^\gamma}, \quad \frac{1}{n^{2(\eta \wedge 2) - H - H'}} \leq \frac{Cn^{\theta(1-\alpha)}}{\tilde{k}^\gamma}. \quad (7.40)$$

The second inequality in (7.40) holds for  $\tilde{k} \leq n^{1-\alpha}$  because of (4.10) and  $\min(2(\eta \wedge 2) - 2H, 2(\eta \wedge 2) - 2H') \leq 2(\eta \wedge 2) - H - H'$ . The first inequality in (7.40) holds trivially since  $\gamma - (4 - H - H') < 0$ . Next, let  $\tilde{k} > n^{1-\alpha}$ . Then using (4.9) it suffices to check the inequalities

$$\frac{1}{\tilde{k}^{4-H-H'}} \leq \frac{C}{n^{(\gamma-\theta)(1-\alpha)}}, \quad \frac{1}{n^{2(\eta \wedge 2) - H - H'}} \leq \frac{C}{n^{(\gamma-\theta)(1-\alpha)}}$$

which easily follow from (4.10), similarly as above.  $\square$

**Proof of Proposition 3.** Part (i) is immediate from Theorem 1(i) and Corollary 3(i). Part (ii) follows from Theorem 1(ii) and Corollary 3, by taking  $\theta = 0$  and  $\gamma > 1/2$  sufficiently close to  $1/2$  (in such case the inequality  $\gamma < 4 - 2H(t)$  of (4.10) is satisfied since  $H(t) < 1$ ). To prove (iii), use Corollary 3 and Theorem 1(iii) with  $\theta = 0$  and  $\gamma > 1/2$  which can be chosen under (4.11), with  $\mu = (2 - 3\alpha)/4$  and condition (4.12). Note that the lower bound (4.12) implies  $(\eta \wedge 1) > (2 - 3\alpha)/4$  and (4.11) implies  $2((\eta \wedge 2) - \sup_{t \in (0,1)} H(t)) > (2 - 3\alpha)/4$ , or condition  $\kappa \geq \mu$  of Theorem 1(iii). Parts (iv) and (v) follows similarly from Corollary 3, Theorem 1(iv) (with  $\theta = 0$ ,  $\kappa \geq \mu_1 = (1 - 3\alpha)/4$  satisfied from (4.11)) and Theorem 2 (with  $\alpha \geq 1 - 2\kappa$  and  $(1 + 2(\eta \wedge 2))^{-1} \geq 1 - 2(\eta \wedge 1)$  for all  $\eta > 0$ ).  $\square$

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## References

- [1] A. Ayache, A. Benassi, S. Cohen, J. Lévy Véhel, Regularity and Identification of Generalized Multifractal Gaussian Processes, in: Lecture Notes in Math., vol. 1857, Springer, Berlin, 2005, pp. 290–312.

- [2] A. Ayache, C. El-Nouty, The small ball behavior of a non stationary increments process: the multifractional Brownian motion, Preprint CMLA 2004-08, 2004. <http://www.cmla.enscachan.fr/documentation/prepublications/2004.html>.
- [3] A. Ayache, J. Lévy Véhel, On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion, *Stochastic Process. Appl.* 111 (2004) 119–156.
- [4] J.-M. Bardet, Testing for the presence of self-similarity of Gaussian time series having stationary increments, *J. Time Ser. Anal.* 21 (2000) 497–516.
- [5] J.-M. Bardet, D. Surgailis, Measuring the roughness of random paths by increment ratios, *Bernoulli* 17 (2011) 749–780.
- [6] J.-M. Bardet, D. Surgailis, Nonparametric estimation of the local Hurst function of multifractional Gaussian processes, 2011. Preprint available on: <http://arxiv.org/abs/1010.2895>.
- [7] J.-M. Bardet, D. Surgailis, Moment bounds and central limit theorems for Gaussian subordinated arrays, *J. Multivariate Anal.* 114 (2013) 457–473.
- [8] A. Begyn, Quadratic variations along irregular subdivisions for Gaussian process, *Electron. J. Probab.* (2005) 691–717.
- [9] A. Begyn, Asymptotic expansion and central limit theorem for quadratic variations of Gaussian processes, *Bernoulli* 13 (2007) 712–753.
- [10] A. Begyn, Functional limit theorems for generalized quadratic variations of Gaussian processes, *Stochastic Process. Appl.* 117 (2007) 1848–1869.
- [11] A. Benassi, P.R. Bertrand, S. Cohen, J. Istas, Identification of the Hurst index of a step fractional Brownian motion, *Stat. Inference Stoch. Process.* 3 (2000) 101–111.
- [12] A. Benassi, S. Cohen, J. Istas, Identifying the multifractal function of a Gaussian process, *Statist. Probab. Lett.* 39 (1998) 337–345.
- [13] A. Benassi, S. Jaffard, D. Roux, Elliptic Gaussian random processes, *Rev. Mat. Iberoam.* 13 (1997) 19–90.
- [14] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.
- [15] J.-F. Coeurjolly, Identification of multifractional Brownian motion, *Bernoulli* 11 (2005) 987–1008.
- [16] J.-F. Coeurjolly, Erratum: identification of multifractional Brownian motion, J.-F. Coeurjolly, *Bernoulli* 12 (2006) 381–382.
- [17] X. Guyon, J. Leon, Convergence en loi des  $H$ -variations d'un processus gaussien stationnaire, *Ann. Inst. Henri Poincaré Probab. Stat.* 25 (1989) 265–282.
- [18] J. Istas, G. Lang, Quadratic variations and estimation of the local Hölder index of a Gaussian process, *Ann. Inst. Henri Poincaré Probab. Stat.* 33 (1997) 407–436.
- [19] C. Lacaux, Real harmonizable multifractional Lévy motions, *Ann. Inst. Henri Poincaré Probab. Stat.* 40 (2004) 259–277.
- [20] R. Peltier, J. Lévy Véhel, Multifractional Brownian motion: definition and preliminary results. Preprint INRIA, 1995. Available on: <http://hal.inria.fr/docs/00/07/40/45/PDF/RR-2645.pdf>.
- [21] G. Samorodnitsky, M. Taqqu, *Stable Non-Gaussian Random Variables*, Chapman and Hall, London, 1994.
- [22] S. Stoev, M. Taqqu, How rich is the class of multifractional Brownian motions? *Stochastic Process. Appl.* 116 (2006) 200–221.
- [23] D. Surgailis, Nonhomogeneous fractional integration and multifractional processes, *Stochastic Process. Appl.* 118 (2008) 171–198.
- [24] D. Surgailis, G. Teyssière, M. Vaičiulis, The increment ratio statistic, *J. Multivariate Anal.* 99 (2008) 510–541.