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Harnack inequality on configuration spaces: The coupling approach and a unified treatment

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Abstract

In this paper, we establish the dimension-free Harnack inequality on configuration spaces by using the coupling argument. Furthermore, a unified treatment is also used to prove the equivalence between the Harnack inequality on configuration space and that on the corresponding base space under a very mild condition.

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1. Introduction

The second quantization of a Markov semigroup, which can be realized by an independent particle system on the configuration space (cf. [16, (5.3)]), is a fundamental model in the study of infinite-dimensional analysis. Second quantization is a powerful tool used in quantum field theory for describing the many-particle systems. We refer to [10,15] for physical background of the second quantization semigroup. A key point for the study of this model is to characterize the second quantization semigroup by using properties of the base process. For known results concerning particle systems on configuration spaces, we refer to [13,14,23,5] for functional inequalities and exponential convergence, [8] for the Feller and strong Feller properties, and [24] for small time behaviors. In this paper, we aim to establish the dimension-free Harnack inequality in the sense of [17] on configuration spaces. See e.g. [22] and the references therein for applications of the dimension-free Harnack inequality.

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Let us first introduce the basic framework. Let $(M, \mathscr{F})$ be a measurable space, and let $P(x, \mathrm{d}y)$ be a transition probability on M. Then

$$Pf(x) := \int_M f(y)\, P(x, \mathrm{d}y), \quad x \in M,\, f \in \mathscr{B}_b(M)$$

gives rise to a Markov operator P. Here and in what follows, $\mathscr{B}_b(M)$ (resp. $\mathscr{B}_b^+(M)$) denotes the set of all bounded measurable (resp. bounded non-negative measurable) functions on M. Denote by δ_x the Dirac measure concentrated at $x \in M$. Consider the infinite configuration space

$$\Gamma := \left\{ \gamma = \sum_{i=1}^{\infty} \delta_{x_i};\ x_i \in M \right\}$$

equipped with the σ-field induced by $\{\gamma \mapsto \gamma(A);\ A \in \mathscr{F}\}$. In many cases, one may be restricted to the locally finite configuration space

$$\Gamma_0 := \big\{ \gamma \in \Gamma;\ \gamma(K) < \infty \text{ for compact } K \subset M \big\},$$

on which the induced σ-field coincides with the Borel σ-field of the vague topology. See [1] for more details on analysis and geometry on configuration spaces. Let $\mathscr{B}_b(\Gamma)$ be the set of all bounded measurable functions on Γ, and $\mathscr{B}_b^+(\Gamma)$ the set of all non-negative elements in $\mathscr{B}_b(\Gamma)$. The Markov operator P^Γ considered in this paper is

$$P^\Gamma F(\gamma) := \int_{M^{\mathbb{N}}} F\left(\sum_{i=1}^{\infty} \delta_{z_i} \right) \prod_{i=1}^{\infty} P(x_i, \mathrm{d}z_i), \qquad \gamma = \sum_{i=1}^{\infty} \delta_{x_i},\, F \in \mathscr{B}_b(\Gamma).$$

The central purpose of this paper is to discuss the dimension-free Harnack inequality for P^Γ. It is well-known that the coupling argument, developed in [2,20], is quite efficient for proving a Harnack inequality (see also [9,11,22]). Therefore, it is natural for us to investigate the Harnack inequality for P^Γ via the coupling approach. Since the underlying space Γ is infinite-dimensional, on the other hand, we can use techniques in infinite-dimensional analysis to handle this problem. A unified treatment will be presented (see Section 3 below).

In order to formulate the Harnack inequality for P^Γ, we need a distance-like function on $\Gamma \times \Gamma$. Let

$$I : M^{\mathbb{N}} \to \Gamma, \qquad (x_i)_{i \geq 1} \mapsto \sum_{i=1}^{\infty} \delta_{x_i}.$$

For any non-negative function φ on $M \times M$, define

$$\varphi^\Gamma(\gamma, \eta) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(x_i, y_i);\ (x_i)_{i\geq 1} \in I^{-1}(\gamma), (y_i)_{i \geq 1} \in I^{-1}(\eta) \right\}, \quad \gamma, \eta \in \Gamma. \quad (1.1)$$

The organization of this paper is as follows. The Harnack inequality for P^Γ is established in Section 2 via a coupling approach. We first present a general result, and then as an application diffusion processes on the configuration space over a Riemannian manifold are considered. The results are also applied to study the strong Feller property, hyper-bounded property, and entropy-cost inequality. In Section 3, a unified treatment is used to prove that under a very mild condition P^Γ satisfies the Harnack inequality iff so does P. Finally, we present a fundamental property of φ^Γ in Section 4.

2. Coupling approach

This section is devoted to establish the Harnack inequality for P^Γ via a coupling argument. We first present a general result and then apply it to the diffusions on the configuration space over a Riemannian manifold as a special case.

2.1. A general result

Theorem 2.1. *Assume that for any $x, y \in M$ there exist an M-valued random variable $\xi_{x,y}$ and a $[0,\infty)$-valued random variable $R_{x,y}$ such that*

$$Pf(x) = \mathbb{E} f(\xi_{x,y}), \qquad Pf(y) = \mathbb{E}\big[R_{x,y} f(\xi_{x,y})\big], \quad f \in \mathscr{B}_b(M). \tag{2.1}$$

(1) *Let $p \in (1,\infty)$ and $\varphi_p : M \times M \to [0,\infty)$ such that*

$$\big(\mathbb{E} R_{x,y}^{p/(p-1)}\big)^{p-1} \le \mathrm{e}^{\varphi_p(x,y)}, \quad x, y \in M. \tag{2.2}$$

Then

$$(P^\Gamma F(\gamma))^p \le \big(P^\Gamma F^p(\eta)\big)\mathrm{e}^{\varphi_p^\Gamma(\gamma,\eta)}, \quad F \in \mathscr{B}_b^+(\Gamma),\ \gamma, \eta \in \Gamma.$$

(2) *Let φ be a non-negative function on $M \times M$ such that*

$$\mathbb{E}[R_{x,y} \log R_{x,y}] \le \varphi(x,y), \quad x, y \in M. \tag{2.3}$$

Then

$$P^\Gamma \log F(\gamma) \le \log P^\Gamma F(\eta) + \varphi^\Gamma(\gamma,\eta), \quad 1 \le F \in \mathscr{B}_b(\Gamma), \gamma, \eta \in \Gamma.$$

Proof. (1) It suffices to prove the statement for $\varphi_p^\Gamma(\gamma,\eta) < \infty$. Fix any $(x_i)_{i\ge1} \in I^{-1}(\eta)$ and $(y_i)_{i\ge1} \in I^{-1}(\gamma)$ such that

$$\sum_{i=1}^\infty \varphi_p(x_i, y_i) < \infty.$$

By (2.1) and (2.2), we may construct a family of independent $M \times [0,\infty)$-valued random variables $\{(\xi_i, R_i);\ i \ge 1\}$ such that for any $i \ge 1$ and $f \in \mathscr{B}_b(M)$,

$$Pf(x_i) = \mathbb{E} f(\xi_i), \qquad Pf(y_i) = \mathbb{E}\big[R_i f(\xi_i)\big], \qquad \big(\mathbb{E} R_i^{p/(p-1)}\big)^{p-1} \le \mathrm{e}^{\varphi_p(x_i,y_i)}.$$

Now it holds from Hölder's inequality that

$$\begin{aligned}
\Big(P^\Gamma F(\gamma)\Big)^p &= \left(\mathbb{E}\left[\left(\prod_{i=1}^\infty R_i\right) F\left(\sum_{i=1}^\infty \delta_{\xi_i}\right)\right]\right)^p \\
&\le \left(\mathbb{E} F^p\left(\sum_{i=1}^\infty \delta_{\xi_i}\right)\right)\left(\mathbb{E}\left[\prod_{i=1}^\infty R_i\right]^{p/(p-1)}\right)^{p-1} \\
&= \big(P^\Gamma F^p(\eta)\big)\prod_{i=1}^\infty \Big(\mathbb{E} R_i^{p/(p-1)}\Big)^{p-1} \\
&\le \big(P^\Gamma F^p(\eta)\big)\prod_{i=1}^\infty \mathrm{e}^{\varphi_p(x_i,y_i)}
\end{aligned}$$

$$= \left(P^{\Gamma} F^p(\eta)\right) \exp\left[\sum_{i=1}^{\infty} \varphi_p(x_i, y_i)\right].$$

This completes the proof of the first assertion by minimizing in $(x_i)_{i\geq 1} \in I^{-1}(\eta)$ and $(y_i)_{i\geq 1} \in I^{-1}(\gamma)$.

(2) By (2.1) and (2.3), for any fixed $(x_i)_{i\geq 1} \in I^{-1}(\eta)$ and $(y_i)_{i\geq 1} \in I^{-1}(\gamma)$, we may construct a sequence of independent $M \times [0, \infty)$-valued random variables $\{(\xi_i, R_i);\ i \geq 1\}$ such that

$$Pf(x_i) = \mathbb{E} f(\xi_i), \qquad Pf(y_i) = \mathbb{E}[R_i f(\xi_i)], \qquad \mathbb{E}[R_i \log R_i] \leq \varphi(x_i, y_i)$$

for all $i \geq 1$ and $f \in \mathscr{B}_b(M)$. By the Young inequality (see e.g. [3, Lemma 2.4]), we obtain

$$\begin{aligned}
P^{\Gamma} \log F(\gamma) &= \mathbb{E}\left[\left(\prod_{i=1}^{\infty} R_i\right) \log F\left(\sum_{i=1}^{\infty} \delta_{\xi_i}\right)\right] \\
&\leq \log \mathbb{E} F\left(\sum_{i=1}^{\infty} \delta_{\xi_i}\right) + \mathbb{E}\left[\left(\prod_{i=1}^{\infty} R_i\right) \log\left(\prod_{j=1}^{\infty} R_j\right)\right] \\
&= \log P^{\Gamma} F(\eta) + \sum_{j=1}^{\infty} \mathbb{E}\left[\left(\prod_{i=1}^{\infty} R_i\right) \log R_j\right] \\
&= \log P^{\Gamma} F(\eta) + \sum_{j=1}^{\infty} \mathbb{E}[R_j \log R_j] \\
&\leq \log P^{\Gamma} F(\eta) + \sum_{j=1}^{\infty} \varphi(x_j, y_j),
\end{aligned}$$

which finishes the proof since $(x_i)_{i\geq 1} \in I^{-1}(\eta)$ and $(y_i)_{i\geq 1} \in I^{-1}(\gamma)$ are arbitrary. □

This result implies the Harnack inequality for a large number of Markov semigroups of independent particle systems such that the semigroup P_t of the single particle process satisfies a Harnack inequality derived from the coupling argument, see e.g. [20,9,11] for Harnack inequalities associated to various SDEs and SPDEs using coupling.

2.2. *Diffusion processes on the configuration space over a Riemannian manifold*

As applications of Theorem 2.1, in this subsection we consider the particular case where the base process is a diffusion on a Riemannian manifold. We will assume throughout this subsection that M is a complete connected Riemannian manifold and let P_t be the semigroup generated by $L := \Delta + Z$ for some C^1-vector field Z. To establish the Harnack inequality, we shall assume that the curvature of L is bounded below, i.e. there exists a constant $K \in \mathbb{R}$ such that

$$\mathrm{Ric}(X, X) - \langle \nabla_X Z, X\rangle \geq -K|X|^2, \quad X \in TM. \tag{2.4}$$

Let ρ be the Riemannian distance on M. According to [17], this condition implies the dimension-free Harnack inequality

$$\begin{aligned}
(P_t f(x))^p &\leq P_t f^p(y) \\
&\quad \times \exp\left[\frac{Kp\rho(x, y)^2}{2(p-1)(1 - \mathrm{e}^{-2Kt})}\right], \quad x, y \in M,\ t > 0,\ f \in \mathscr{B}_b^+(M)
\end{aligned} \tag{2.5}$$

for all $p \in (1, \infty)$. Indeed, according to [18,21], (2.4) is equivalent to (2.5) for all/some $p \in (1, \infty)$, as well as the log-Harnack inequality

$$P_t \log f(x) \le \log P_t f(y) + \frac{K\rho(x, y)^2}{2(1 - \mathrm{e}^{-2Kt})}, \quad x, y \in M, 1 \le f \in \mathscr{B}_b(M).$$

When $Z = \nabla V$ for some $V \in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is infinite, where $\mathrm{d}x$ stands for the Riemannian volume measure, P_t is symmetric in $L^2(\mu)$. Moreover, P_t^Γ is a realization of the second quantization of P_t (see [16, (5.3)]), which is symmetric in $L^2(\pi_\mu)$, where (and in the sequel) π_μ stands for the Poisson measure with intensity μ.

Recall that

$$\{\rho^2\}^\Gamma(\gamma, \eta) = \inf\left\{\sum_{i=1}^\infty \rho(x_i, y_i)^2;\ (x_i)_{i\ge1} \in I^{-1}(\gamma), (y_i)_{i\ge1} \in I^{-1}(\eta)\right\}, \quad \gamma, \eta \in \Gamma.$$

Throughout this subsection, we shall simply set

$$\begin{aligned}\rho^\Gamma(\gamma, \eta) &:= \left(\{\rho^2\}^\Gamma(\gamma, \eta)\right)^{1/2}\\ &= \inf\left\{\left(\sum_{i=1}^\infty \rho(x_i, y_i)^2\right)^{1/2};\ (x_i)_{i\ge1} \in I^{-1}(\gamma), (y_i)_{i\ge1} \in I^{-1}(\eta)\right\}\end{aligned}$$

if there is no confusion. When $Z = \nabla V$ for some $V \in C^2(M)$ with $\mu(M) = \infty$ mentioned above, it is proved in [12] that ρ^Γ is the intrinsic distance for the Dirichlet form associated to the semigroup P_t^Γ.

Theorem 2.2. *Assume that (2.4) holds. For any $p \in (1, \infty)$,*

$$\left(P_t^\Gamma F(\gamma)\right)^p \le \left(P_t^\Gamma F^p(\eta)\right) \exp\left[\frac{Kp\rho^\Gamma(\gamma, \eta)^2}{2(p-1)(1 - \mathrm{e}^{-2Kt})}\right] \tag{2.6}$$

holds for all $\gamma, \eta \in \Gamma$, $t > 0$, and $F \in \mathscr{B}_b^+(\Gamma)$. Moreover, the log-Harnack inequality

$$P_t^\Gamma \log F(\eta) \le \log P_t^\Gamma F(\gamma) + \frac{K\rho^\Gamma(\gamma, \eta)^2}{2(1 - \mathrm{e}^{-2Kt})}, \quad \gamma, \eta \in \Gamma \tag{2.7}$$

holds for all $t > 0$ and $F \in \mathscr{B}_b(\Gamma)$ with $F \ge 1$.

Remark 2.3. Let μ be an infinite measure on M. According to Proposition 4.1 below, if

$$\lim_{r\downarrow0} \sup_{x\in M} \mu\big(\{y \in M;\ \rho(x, y) < r\}\big) = 0$$

and there exist some constants $C \ge 1$ and $N \in \mathbb{N}$ and a fixed point o in M such that

$$\mu\big(\{x \in M;\ \rho(x, o) < n + 1\}\big) \le C\mu\big(\{x \in M;\ \rho(x, o) < n\}\big) < \infty$$

holds for all $n > N$, then $\rho^\Gamma(\gamma, \eta) = \infty$ for $(\pi_\mu \times \pi_\mu)$-a.e. $(\gamma, \eta) \in \Gamma \times \Gamma$. Therefore, under this circumstance Theorem 2.2 is trivial in the sense of $(\pi_\mu \times \pi_\mu)$-a.e.

Since the Harnack inequality implies the strong Feller property (see e.g. [22, Theorem 4.4(2)]), the following result is a direct consequence of Theorem 2.2.

Corollary 2.4. *For any $t > 0$, P_t^Γ has the ρ^Γ-strong Feller property, i.e.*

$$\lim_{\rho^\Gamma(\gamma,\eta)\to 0} P_t^\Gamma F(\eta) = P_t^\Gamma F(\gamma), \quad \gamma \in \Gamma,\ F \in \mathscr{B}_b(\Gamma).$$

Remark 2.5. When $L = \Delta$ on $\mathbb{R}^d$, the ρ^Γ-Feller property is confirmed in [8, Theorem 7.1] on the smaller space Γ_0. Therefore, Corollary 2.4 provides a much stronger and more general assertion.

Proof of Theorem 2.2. Fix an arbitrary $t > 0$. According to Theorem 2.1, for any $x, y \in M$ it suffices to construct a random variable $(\xi_{x,y}, R_{x,y})$ on $M \times [0, \infty)$ such that

$$P_t f(x) = \mathbb{E} f(\xi_{x,y}), \qquad P_t f(y) = \mathbb{E}\big[R_{x,y} f(\xi_{x,y})\big], \quad f \in \mathscr{B}_b(M), \tag{2.8}$$

$$\left(\mathbb{E} R_{x,y}^{p/(p-1)}\right)^{p-1} \le \exp\left[\frac{Kp\rho(x,y)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right], \tag{2.9}$$

and

$$\mathbb{E}[R_{x,y}\log R_{x,y}] \le \frac{K\rho(x,y)^2}{2(1-\mathrm{e}^{-2Kt})}. \tag{2.10}$$

We will adopt the coupling by parallel displacement introduced in [2], which goes back to [7]. As explained in [2, Section 3], due to [19, Chapter 2] we may simply assume that M does not have cut-locus so that the parallel displacement

$$P_{x,y} : T_x M \to T_y M$$

along the minimal geodesic from x to y is smooth in (x, y).

Now, let B_t be the d-dimensional Brownian motion and let X_s solve the Itô differential equation in the sense of [6]:

$$\mathrm{d}^{\text{Itô}} X_s = \sqrt{2} u_s \mathrm{d}B_s + Z(X_s)\mathrm{d}s, \qquad X_0 = x,$$

where u_t is the horizontal lift of X_s on the frame bundle $O(M)$. Then

$$P_t f(x) = \mathbb{E} f(X_t), \quad f \in \mathscr{B}_b(M). \tag{2.11}$$

On the other hand, let Y_s solve the equation

$$\mathrm{d}^{\text{Itô}} Y_s = \sqrt{2} P_{X_s,Y_s} u_s \mathrm{d}B_s + Z(Y_s)\mathrm{d}s - h_s \nabla\rho(X_s, \bullet)(Y_s)\mathrm{d}s, \qquad Y_0 = y,$$

where

$$h_s := 1_{\{s<\tau\}} \frac{2K\mathrm{e}^{-Ks}\rho(x,y)}{1-\mathrm{e}^{-2Kt}}, \qquad \tau := \inf\{s \ge 0;\ X_s = Y_s\}.$$

Then the equation for Y_s has a unique solution and $X_s = Y_s$ holds for $s \ge \tau$. By Itô's formula and the second variational formula (cf. [2] and references within), we have

$$\mathrm{d}\rho(X_s, Y_s) \le K\rho(X_s, Y_s)\mathrm{d}s - h_s \mathrm{d}s.$$

Applying the Gronwall lemma to the above inequality, it holds that

$$\rho(X_s, Y_s) \le \frac{\mathrm{e}^{Ks}(\mathrm{e}^{-2Ks} - \mathrm{e}^{-2Kt})\rho(x,y)}{1-\mathrm{e}^{-2Kt}}, \quad 0 \le s \le t.$$

In particular, one has $\rho(X_t, Y_t) = 0$, and so $X_t = Y_t =: \xi_{x,y}$. To formulate $P_t f(y)$ using $\xi_{x,y}$, rewrite the equation for Y_s as

$$\mathrm{d}^{\text{Itô}} Y_s = \sqrt{2} P_{X_s,Y_s} u_s \mathrm{d}\tilde{B}_s + Z(Y_s)\mathrm{d}s, \qquad Y_0 = y,$$

where

$$\tilde{B}_s := B_s - \frac{1}{\sqrt{2}} \int_0^s h_r (P_{X_r,Y_r} u_r)^{-1} \nabla \rho(X_r, \bullet)(Y_r)\, \mathrm{d}r, \quad s \geq 0.$$

By the Girsanov theorem, $(\tilde{B}_s)_{s \in [0,t]}$ is a d-dimensional Brownian motion under the probability measure $R_{x,y}\mathbb{P}$, where

$$R_{x,y} := \exp\left[\frac{1}{\sqrt{2}} \int_0^t \langle h_s (P_{X_s,Y_s} u_s)^{-1} \nabla \rho(X_s, \bullet)(Y_s), \mathrm{d}B_s\rangle - \frac{1}{4}\int_0^t h_s^2\, \mathrm{d}s\right].$$

Therefore, due to (2.11) and $\xi_{x,y} = X_t = Y_t$, we obtain (2.8).

In order to get (2.9) and (2.10), we first observe that

$$\int_0^t h_s^2\, \mathrm{d}s \leq \frac{4K^2 \rho(x, y)^2}{(1 - \mathrm{e}^{-2Kt})^2} \int_0^t \mathrm{e}^{-2Ks}\, \mathrm{d}s = \frac{2K\rho(x, y)^2}{1 - \mathrm{e}^{-2Kt}}.$$

Let

$$M_t = \frac{1}{\sqrt{2}} \int_0^t \left\langle h_s (P_{X_s,Y_s} u_s)^{-1} \nabla \rho(X_s, \bullet)(Y_s), \mathrm{d}B_s\right\rangle.$$

Then

$$R_{x,y} = \exp\left[M_t - \frac{1}{2}\langle M\rangle_t\right]$$

and

$$\langle M\rangle_t = \frac{1}{2}\int_0^t h_s^2\, \mathrm{d}s \leq \frac{K\rho(x, y)^2}{1 - \mathrm{e}^{-2Kt}}.$$

Thus,

$$\begin{aligned}
\left(\mathbb{E} R_{x,y}^{p/(p-1)}\right)^{p-1} &= \left(\mathbb{E}\exp\left[\frac{p}{p-1}M_t - \frac{p^2}{2(p-1)^2}\langle M\rangle_t + \frac{p}{2(p-1)^2}\langle M\rangle_t\right]\right)^{p-1} \\
&\leq \left(\mathbb{E}\exp\left[\frac{p}{p-1}M_t - \frac{p^2}{2(p-1)^2}\langle M\rangle_t\right]\right)^{p-1} \\
&\quad \times \exp\left[\frac{p}{2(p-1)^2} \cdot \frac{K\rho(x, y)^2}{1 - \mathrm{e}^{-2Kt}} \cdot (p-1)\right] \\
&\leq \exp\left[\frac{Kp\rho(x, y)^2}{2(p-1)(1 - \mathrm{e}^{-2Kt})}\right],
\end{aligned}$$

where in the last step we have used the fact that $\exp\left[\frac{p}{p-1}M_t - \frac{p^2}{2(p-1)^2}\langle M\rangle_t\right]$ is a supermartingale. Furthermore, since

$$\begin{aligned}
\log R_{x,y} &= M_t - \frac{1}{2}\langle M\rangle_t \\
&= \frac{1}{\sqrt{2}}\int_0^t \left\langle h_s (P_{X_s,Y_s} u_s)^{-1} \nabla \rho(X_s, \bullet)(Y_s), \mathrm{d}\tilde{B}_s\right\rangle + \frac{1}{4}\int_0^t h_s^2\, \mathrm{d}s,
\end{aligned}$$

we arrive at

$$\mathbb{E}[R_{x,y}\log R_{x,y}] = \mathbb{E}_{R_{x,y}\mathbb{P}}[\log R_{x,y}] = \frac{1}{4}\mathbb{E}_{R_{x,y}\mathbb{P}}\left[\int_0^t h_s^2\,\mathrm{d}s\right] \le \frac{K\rho(x,y)^2}{2(1-\mathrm{e}^{-2Kt})}.$$

The proof is now completed. □

We can also use the Harnack type inequality to describe the hyper-bounded property and the entropy-cost inequality. Denote by $\|P_t^\Gamma\|_{p\to q}$ the operator norm from $L^p(\pi_\mu)$ to $L^q(\pi_\mu)$. Recall that ρ^Γ is a non-negative measurable function on $\Gamma\times\Gamma$ (cf. [12]). For a measurable function $F\ge 0$ on Γ with $\pi_\mu(F)=1$, let $\mathscr{C}(F\pi_\mu,\pi_\mu)$ be the class of all couplings of $F\pi_\mu$ and π_μ, and let

$$W_2^{\rho^\Gamma}(F\pi_\mu,\pi_\mu) = \inf_{\Pi\in\mathscr{C}(F\pi_\mu,\pi_\mu)}\left(\int_{\Gamma\times\Gamma}\rho^\Gamma(\gamma,\eta)^2\,\Pi(\mathrm{d}\gamma,\mathrm{d}\eta)\right)^{1/2}$$

be the transportation-cost from $F\pi_\mu$ to π_μ induced by the cost-function $(\rho^\Gamma)^2$.

Corollary 2.6. *Assume that (2.4) holds for $Z=\nabla V$ for some $V\in C^2(M)$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is infinite.*

(1) *For any $t>0$ and $p\in(1,\infty)$, we have*

$$\|P_t^\Gamma\|_{p\to\infty} \le \left(\mathrm{ess}_{\pi_\mu}\inf_{\gamma\in\Gamma}\int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right]\pi_\mu(\mathrm{d}\eta)\right)^{-1/p} \tag{2.12}$$

and

$$\|P_t^\Gamma\|_{p\to q} \le \left(\int_\Gamma \frac{\pi_\mu(\mathrm{d}\gamma)}{\left(\int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right]\pi_\mu(\mathrm{d}\eta)\right)^{q/p}}\right)^{1/q}, \quad q\in[1,\infty). \tag{2.13}$$

(2) *For any $t>0$,*

$$\int_\Gamma (P_t^\Gamma F)\log P_t^\Gamma F\,\mathrm{d}\pi_\mu \le \frac{K}{2(1-\mathrm{e}^{-2Kt})}W_2^{\rho^\Gamma}(F\pi_\mu,\pi_\mu)^2, \quad F\ge 0, \pi_\mu(F)=1.$$

Remark 2.7. In the situation of Corollary 2.6. Similarly as in Remark 2.3, if

$$\lim_{r\downarrow 0}\sup_{x\in M}\mu\big(\{y\in M;\ \rho(x,y)<r\}\big)=0$$

and there exist some constants $C\ge 1$ and $N\in\mathbb{N}$ and a fixed point o in M such that

$$\mu\big(\{x\in M;\ \rho(x,o)<n+1\}\big)\le C\mu\big(\{x\in M;\ \rho(x,o)<n\}\big)<\infty$$

holds for all $n>N$, then Corollary 2.6(1) is nothing but trivial since $\rho^\Gamma=\infty(\pi_\mu\times\pi_\mu)$-a.e. implies that the right-hand sides of both (2.12) and (2.13) become infinities.

It is straightforward to obtain Corollary 2.6 from the Harnack inequality (2.6) and the log-Harnack inequality (2.7) (see e.g. [4, proof of Lemma 4.2] and [22, proof of Proposition 4.6]). For the sake of completeness and reader's convenience, we include here a simple proof.

Proof of Corollary 2.6. (1) Let F be a non-negative measurable function on Γ such that $\|F\|_p \le 1$. According to Theorem 2.2, one has

$$\big(P_t^\Gamma F(\gamma)\big)^p \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right] \le P_t^\Gamma F^p(\eta), \quad \gamma,\eta\in\Gamma.$$

Integrating both sides w.r.t. $\pi_\mu(\mathrm{d}\eta)$ in the above inequality and noting that π_μ is an invariant probability measure of P_t^Γ, we get

$$\big(P_t^\Gamma F(\gamma)\big)^p \int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right] \pi_\mu(\mathrm{d}\eta) \le \pi_\mu(F^p) \le 1, \quad \gamma\in\Gamma,$$

which yields that

$$P_t^\Gamma F(\gamma) \le \left(\int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right] \pi_\mu(\mathrm{d}\eta)\right)^{-1/p}, \quad \gamma\in\Gamma.$$

Therefore,

$$\begin{aligned}
\|P_t^\Gamma\|_{p\to\infty} &= \sup_{F\ge 0,\|F\|_p\le 1} \|P_t^\Gamma F\|_\infty \\
&= \sup_{F\ge 0,\|F\|_p\le 1} \mathrm{ess}_{\pi_\mu} \sup_{\gamma\in\Gamma} |P_t^\Gamma F(\gamma)| \\
&\le \left(\mathrm{ess}_{\pi_\mu} \inf_{\gamma\in\Gamma} \int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right] \pi_\mu(\mathrm{d}\eta)\right)^{-1/p},
\end{aligned}$$

and

$$\begin{aligned}
\|P_t^\Gamma\|_{p\to q} &= \sup_{F\ge 0,\|F\|_p\le 1} \left(\int_\Gamma |P_t^\Gamma F(\gamma)|^q \,\pi_\mu(\mathrm{d}\gamma)\right)^{1/q} \\
&\le \left(\int_\Gamma \frac{\pi_\mu(\mathrm{d}\gamma)}{\left(\int_\Gamma \exp\left[-\frac{Kp\rho^\Gamma(\gamma,\eta)^2}{2(p-1)(1-\mathrm{e}^{-2Kt})}\right] \pi_\mu(\mathrm{d}\eta)\right)^{q/p}}\right)^{1/q}.
\end{aligned}$$

(2) Let $F\ge 0$ such that $\pi_\mu(F)=1$. Applying (2.7) to $P_t^\Gamma F$ in place of F, it follows that

$$P_t^\Gamma\{\log P_t^\Gamma F\}(\gamma) \le \log P_t^\Gamma\{P_t^\Gamma F\}(\eta) + \frac{K\rho^\Gamma(\gamma,\eta)^2}{2(1-\mathrm{e}^{-2Kt})}.$$

Taking integral for both sides w.r.t. $\Pi\in\mathscr{C}(F\pi_\mu,\pi_\mu)$ and using the symmetry of P_t^Γ, we arrive at

$$\begin{aligned}
\int_\Gamma (P_t^\Gamma F)\log P_t^\Gamma F \,\mathrm{d}\pi_\mu &= \int_\Gamma F P_t^\Gamma\{\log P_t^\Gamma F\}\,\mathrm{d}\pi_\mu \\
&= \int_{\Gamma\times\Gamma} P_t^\Gamma\{\log P_t^\Gamma F\}(\gamma)\,\Pi(\mathrm{d}\gamma,\mathrm{d}\eta) \\
&\le \int_{\Gamma\times\Gamma} \left\{\log P_t^\Gamma\{P_t^\Gamma F\}(\eta) + \frac{K\rho^\Gamma(\gamma,\eta)^2}{2(1-\mathrm{e}^{-2Kt})}\right\} \Pi(\mathrm{d}\gamma,\mathrm{d}\eta)
\end{aligned}$$

$$= \int_\Gamma \log P_t^\Gamma\{P_t^\Gamma F\}\,\mathrm{d}\pi_\mu + \frac{K}{2(1-\mathrm{e}^{-2Kt})}\int_{\Gamma\times\Gamma}\rho^\Gamma(\gamma,\eta)^2\,\Pi(\mathrm{d}\gamma,\mathrm{d}\eta).$$

This completes the proof by optimizing in $\Pi \in \mathscr{C}(F\pi_\mu, \pi_\mu)$ and noting that

$$\int_\Gamma \log P_t^\Gamma\{P_t^\Gamma F\}\,\mathrm{d}\pi_\mu \le \log\int_\Gamma P_t^\Gamma\{P_t^\Gamma F\}\,\mathrm{d}\pi_\mu = \log\pi_\mu(F) = \log 1 = 0. \quad \square$$

3. A unified treatment

Let $p \in (1,\infty)$, $\varphi_p : M\times M \to [0,\infty)$, and C_p be a non-negative constant. In this section, a unified treatment will be used to discuss the relationship between the Harnack inequality on base space

$$(Pf(x))^p \le \left(Pf^p(y)\right)\mathrm{e}^{C_p\varphi_p(x,y)}, \quad f\in\mathscr{B}_b^+(M),\ x,y\in M \tag{3.1}$$

and that on the corresponding configuration space

$$\left(P^\Gamma F(\gamma)\right)^p \le \left(P^\Gamma F^p(\eta)\right)\mathrm{e}^{C_p\varphi_p^\Gamma(\gamma,\eta)}, \quad F\in\mathscr{B}_b^+(\Gamma),\ \gamma,\eta\in\Gamma. \tag{3.2}$$

Theorem 3.1. (1) *First,* (3.1) *implies* (3.2).
(2) *Conversely, if there exists a sequence* $\{(x_i,y_i)\in M\times M;\ i\ge 1\}$ *such that* $\sum_{i=1}^\infty\varphi_p(x_i,y_i) < \infty$*, then* (3.2) *implies* (3.1).

Remark 3.2. If φ_p is a distance on M, then the condition in Theorem 3.1(2) is automatically fulfilled (simply set $x_i = y_i = o$ for $i\ge 1$, where o is any fixed point in M).

Proof of Theorem 3.1. (1) (a) For a measurable function F on Γ, let

$$\hat F(\mathbf{x}) = F\left(\sum_{i=1}^\infty \delta_{x_i}\right), \quad \mathbf{x} = (x_i)_{i\ge1}\in M^{\mathbb{N}}.$$

Then $\hat F$ is measurable on $M^{\mathbb{N}}$ equipped with the product σ-field $\mathscr{F}^{\mathbb{N}}$. Indeed, we only need to check this for $F(\gamma) = \gamma(A)$, $A\in\mathscr{F}$. But for this F,

$$\hat F(\mathbf{x}) = \sum_{i=1}^\infty \delta_{x_i}(A) = \sum_{i=1}^\infty 1_A(x_i)$$

is clearly measurable on $M^{\mathbb{N}}$.

(b) Fix any $\mathbf{x} = (x_i)_{i\ge1}\in I^{-1}(\gamma)$ and $\mathbf{y} = (y_i)_{i\ge1}\in I^{-1}(\eta)$. Since

$$P^{\otimes\mathbb{N}}\hat F(\mathbf{x}) = P^\Gamma F(\gamma), \qquad P^{\otimes\mathbb{N}}\hat F^p(\mathbf{y}) = P^\Gamma F^p(\eta),$$

it suffices to prove that

$$\left(P^{\otimes\mathbb{N}}\hat F(\mathbf{x})\right)^p \le \left(P^{\otimes\mathbb{N}}\hat F^p(\mathbf{y})\right)\exp\left[C_p\sum_{i=1}^\infty\varphi_p(x_i,y_i)\right]. \tag{3.3}$$

(c) By a standard approximation argument, we only need to prove (3.3) for

$$\hat F(\mathbf{x}) = f(x_{i_1},\ldots,x_{i_n}), \quad \mathbf{x} = (x_i)_{i\ge1}\in M^{\mathbb{N}},$$

where $f \in C_b(M^n)$, $1 \le i_1 < i_2 < \cdots < i_n$, $n \in \mathbb{N}$. Noting that φ_p is non-negative, it remains to prove that

$$\big(P^{\otimes n} f(x_{i_1}, \ldots, x_{i_n})\big)^p \le \big(P^{\otimes n} f^p(y_{i_1}, \ldots, y_{i_n})\big) \exp\left[C_p \sum_{k=1}^{n} \varphi_p(x_{i_k}, y_{i_k})\right]. \tag{3.4}$$

We shall prove this inequality by iterating in n.

(d) If $n = 1$, (3.4) follows immediately from (3.1). Assume that (3.4) holds for $n = m$. By (3.1) and the assumption, we obtain

$$\begin{aligned}
&\Big(P^{\otimes(m+1)} f(x_{i_1}, \ldots, x_{i_m}, x_{i_{m+1}})\Big)^p = \big(P\big\{P^{\otimes m} f(x_{i_1}, \ldots, x_{i_m}, \bullet)\big\}(x_{i_{m+1}})\big)^p \\
&\quad \le P\big\{P^{\otimes m} f(x_{i_1}, \ldots, x_{i_m}, \bullet)\big\}^p(y_{i_{m+1}}) \times \mathrm{e}^{C_p \varphi_p(x_{i_{m+1}}, y_{i_{m+1}})} \\
&\quad \le P\left\{P^{\otimes m} f^p(y_{i_1}, \ldots, y_{i_m}, \bullet) \exp\left[C_p \sum_{k=1}^{m} \varphi_p(x_{i_k}, y_{i_k})\right]\right\}(y_{i_{m+1}}) \\
&\qquad \times \mathrm{e}^{C_p \varphi_p(x_{i_{m+1}}, y_{i_{m+1}})} \\
&\quad = P^{\otimes(m+1)} f(y_{i_1}, \ldots, y_{i_m}, y_{i_{m+1}}) \times \exp\left[C_p \sum_{k=1}^{m+1} \varphi_p(x_{i_k}, y_{i_k})\right].
\end{aligned}$$

That is, (3.4) holds for $n = m + 1$. Therefore, the first assertion follows.

(2) According to the assumption, one can fix a sequence $\{(x_i, y_i) \in M \times M;\ i \ge 1\}$ with

$$\sum_{i=1}^{\infty} \varphi_p(x_i, y_i) < \infty.$$

Obviously, it suffices to prove (3.1) for $f \in \mathscr{B}_b^+(M)$ with $f \le 1$. Now, let $f \in \mathscr{B}_b^+(M)$ satisfy $f \le 1$. For any $x, y \in M$, let $x_0 = x$, $y_0 = y$. Taking

$$F(\gamma) = \mathrm{e}^{\gamma(\log f)}, \quad \gamma = \sum_{i=0}^{\infty} \delta_{x_i}, \qquad \eta = \sum_{i=0}^{\infty} \delta_{y_i}$$

in (3.2), and noting that

$$\begin{aligned}
&P^{\Gamma} F(\gamma) = \prod_{i=0}^{\infty} P\big\{\mathrm{e}^{\log f}\big\}(x_i) = \prod_{i=0}^{\infty} P f(x_i), \\
&P^{\Gamma} F^p(\eta) = \prod_{i=0}^{\infty} P\big\{\mathrm{e}^{p \log f}\big\}(y_i) = \prod_{i=0}^{\infty} P f^p(y_i), \qquad \varphi_p^{\Gamma}(\gamma, \eta) \le \sum_{i=0}^{\infty} \varphi_p(x_i, y_i),
\end{aligned}$$

we arrive at

$$\begin{aligned}
\prod_{i=0}^{\infty} (P f(x_i))^p &= \Big(P^{\Gamma} F(\gamma)\Big)^p \le P^{\Gamma} F^p(\eta) \exp\left[C_p \sum_{i=0}^{\infty} \varphi_p(x_i, y_i)\right] \\
&= \left(\prod_{i=0}^{\infty} P f^p(y_i)\right)\left(\prod_{i=0}^{\infty} \mathrm{e}^{C_p \varphi_p(x_i, y_i)}\right) \\
&= \prod_{i=0}^{\infty} \Big(\big(P f^p(y_i)\big)\mathrm{e}^{C_p \varphi_p(x_i, y_i)}\Big).
\end{aligned}$$

Then there must exist some $i_1 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ such that

$$(Pf(x_{i_1}))^p \le \big(Pf^p(y_{i_1})\big)e^{C_p\varphi_p(x_{i_1}, y_{i_1})}.$$

Similarly, taking

$$F(\gamma) = e^{\gamma(\log f)}, \quad \gamma = \sum_{i\in\mathbb{N}_0\setminus\{i_1\}} \delta_{x_i}, \qquad \eta = \sum_{i\in\mathbb{N}_0\setminus\{i_1\}} \delta_{y_i}$$

in (3.2), we get

$$\prod_{i\in\mathbb{N}_0\setminus\{i_1\}} (Pf(x_i))^p \le \prod_{i\in\mathbb{N}_0\setminus\{i_1\}} \Big(\big(Pf^p(y_i)\big)e^{C_p\varphi_p(x_i, y_i)}\Big).$$

Then there must exist some $i_2 \in \mathbb{N}_0 \setminus \{i_1\}$ such that

$$(Pf(x_{i_2}))^p \le \big(Pf^p(y_{i_2})\big)e^{C_p\varphi_p(x_{i_2}, y_{i_2})}.$$

Repeating this argument, we conclude that

$$(Pf(x_i))^p \le \big(Pf^p(y_i)\big)e^{C_p\varphi_p(x_i, y_i)}$$

holds for all $i \in \mathbb{N}_0$. Particularly, taking $i = 0$ gives (3.1). $\square$

We conclude this section by pointing out that we also have a similar result concerning the log-Harnack inequality. Let φ be a non-negative function on $M \times M$, and C be a non-negative constant. Consider

$$P\log f(x) \le \log Pf(y) + C\varphi(x, y), \quad x, y \in M,\ 1 \le f \in \mathscr{B}_b(M) \tag{3.5}$$

and

$$P^\Gamma \log F(\gamma) \le \log P^\Gamma F(\eta) + C\varphi^\Gamma(\gamma, \eta), \quad \gamma, \eta \in \Gamma,\ 1 \le F \in \mathscr{B}_b(\Gamma). \tag{3.6}$$

Theorem 3.3. (1) *First, (3.5) implies (3.6).*

(2) *Conversely, if there exists a sequence* $\{(x_i, y_i) \in M \times M;\ i \ge 1\}$ *such that* $\sum_{i=1}^\infty \varphi(x_i, y_i) < \infty$*, then (3.6) implies (3.5).*

The proof is similar to that of Theorem 3.1, and therefore, we omit it here.

4. About φ^Γ

In this section, we will present a fundamental property of φ^Γ. Recall that φ is a non-negative function on $M \times M$, and φ^Γ is defined by (1.1).

For $x \in M$ and $r > 0$, let

$$B^\varphi(x, r) = \{y \in M;\ \varphi(y, x) < r\}.$$

For simplicity, set $B_n^\varphi = B^\varphi(o, n)$, where $n \in \mathbb{N}$ and o is a fixed point in M. Let μ be an infinite measure on $(M, \mathscr{F})$ such that $\mu(B_n^\varphi) < \infty$ for all $n \in \mathbb{N}$.

In the following proposition, we will use the following two assumptions:

(A1) $\lim_{r\to 0} \sup_{x\in M} \mu\big(B^\varphi(x, r)\big) = 0$;

(A2) there exist some constants $C \ge 1$ and $N \in \mathbb{N}$ such that $\mu(B_{n+1}^\varphi) \le C\mu(B_n^\varphi)$ holds for all $n > N$.

A sufficient condition for (A2) is that μ has the volume doubling property with doubling constant C, i.e.

$$\mu\big(B^{\varphi}(x,2r)\big) \le C\mu\big(B^{\varphi}(x,r)\big), \quad x \in M, r > 0.$$

Indeed, it follows from the volume doubling property that for all $n \in \mathbb{N}$

$$\mu(B^{\varphi}_{n+1}) \le C\mu\left(B^{\varphi}_{(n+1)/2}\right) \le C\mu(B^{\varphi}_n).$$

On the other hand, (A2) is strictly weaker than the volume doubling property. For instance, for $\mu(\mathrm{d}x) = \mathrm{e}^x \mathrm{d}x$ on $\mathbb{R}$ with $o = 0$ and φ being the Euclidean distance, we have

$$\sup_{n\ge 1} \frac{\mu\big(B^{\varphi}_{n+1}\big)}{\mu\big(B^{\varphi}_n\big)} = \sup_{n\ge 1} \frac{\mathrm{e}^{n+1} - \mathrm{e}^{-n-1}}{\mathrm{e}^n - \mathrm{e}^{-n}} = \sup_{n\ge 1} \frac{\mathrm{e} - \mathrm{e}^{-2n-1}}{1 - \mathrm{e}^{-2n}} \le \frac{\mathrm{e}}{1 - \mathrm{e}^{-2}} < 4.$$

Then (A2) holds with $C = 4$ and $N = 1$. However, observing that

$$\sup_{r>0} \frac{\mu\big(B^{\varphi}(x,2r)\big)}{\mu\big(B^{\varphi}(x,r)\big)} = \sup_{r>0} \frac{\mathrm{e}^{x+2r} - \mathrm{e}^{x-2r}}{\mathrm{e}^{x+r} - \mathrm{e}^{x-r}} = \sup_{r>0}\big(\mathrm{e}^r + \mathrm{e}^{-r}\big) = \infty, \quad x \in M,$$

μ does not satisfy the volume doubling condition.

Recall that π_μ denotes the Poisson measure with intensity μ.

Proposition 4.1. *Assume that both* (A1) *and* (A2) *hold. Then* $\varphi^{\Gamma}(\gamma,\eta) = \infty$ *for* $(\pi_\mu \times \pi_\mu)$*-a.e.* $(\gamma,\eta) \in \Gamma \times \Gamma$.

Proof. (1) Let

$$\varphi(x,\gamma) = \inf_{y\in \mathrm{supp}\gamma} \varphi(x,y), \quad x \in M, \gamma \in \Gamma,$$

and

$$D(\gamma,\varepsilon) = \big\{x \in M;\ \varphi(x,\gamma) \ge \varepsilon\big\}, \quad \gamma \in \Gamma, \varepsilon \in (0,1).$$

Noting that

$$B^{\varphi}_n \setminus \left(\bigcup_{x\in(\mathrm{supp}\gamma)\cap B^{\varphi}_{n+1}} B^{\varphi}(x,\varepsilon)\right) \subset D(\gamma,\varepsilon),$$

it holds that

$$\begin{aligned}\mu\big(D(\gamma,\varepsilon)\big) &\ge \mu(B^{\varphi}_n) - \mu\left(\bigcup_{x\in(\mathrm{supp}\gamma)\cap B^{\varphi}_{n+1}} B^{\varphi}(x,\varepsilon)\right) \\ &\ge \mu(B^{\varphi}_n) - \gamma(B^{\varphi}_{n+1}) \sup_{x\in M} \mu\big(B^{\varphi}(x,\varepsilon)\big) \qquad (4.1)\end{aligned}$$

for all $\gamma \in \Gamma$, $\varepsilon \in (0,1)$, and $n \in \mathbb{N}$.

(2) Recall that the Poisson measure π_μ has the characteristic functional

$$\int_{\Gamma} \mathrm{e}^{\mathrm{i}\gamma(f)}\,\pi_\mu(\mathrm{d}\gamma) = \exp\left[\int_M \left(\mathrm{e}^{\mathrm{i}f} - 1\right)\mathrm{d}\mu\right], \quad f \in L^1(\mu) \cap L^{\infty}(\mu). \qquad (4.2)$$

Since $\mu(B_n^\varphi) < \infty$ for all $n \in \mathbb{N}$ and $\mu(B_n^\varphi) \to \infty$ as $n \to \infty$, we deduce from (4.2) that for any $t \in \mathbb{R}$

$$\begin{aligned}\int_\Gamma \exp\Big\{\mathrm{i}t\frac{\gamma(B_n^\varphi)}{\mu(B_n^\varphi)}\Big\}\pi_\mu(\mathrm{d}\gamma) &= \exp\left[\int_M \left(\exp\Big\{\frac{\mathrm{i}t}{\mu(B_n^\varphi)}1_{B_n^\varphi}\Big\} - 1\right)\mathrm{d}\mu\right] \\ &= \exp\left[\mu(B_n^\varphi)\left(\exp\Big\{\frac{\mathrm{i}t}{\mu(B_n^\varphi)}\Big\} - 1\right)\right] \\ &\to \mathrm{e}^{\mathrm{i}t} \quad \text{as } n \to \infty,\end{aligned}$$

which implies that $\frac{\gamma(B_n^\varphi)}{\mu(B_n^\varphi)} \to 1$ in law and so in probability π_μ. Then we can choose a subsequence $\{n_k;\ k \geq 1\} \subset \mathbb{N}$ such that $\frac{\gamma(B_{n_k}^\varphi)}{\mu(B_{n_k}^\varphi)} \to 1$ as $k \to \infty$ for π_μ-a.e. $\gamma \in \Gamma$. Thus, there exist $N_0 \in \mathbb{N}$ and $\Gamma' \subset \Gamma$ with $\pi_\mu(\Gamma') = 1$ such that

$$\gamma(B_{n_k}^\varphi) \leq 2\mu(B_{n_k}^\varphi), \quad k > N_0, \gamma \in \Gamma'. \tag{4.3}$$

(3) Due to (A2), we obtain from (4.1) and (4.3) that

$$\begin{aligned}\mu\big(D(\gamma,\varepsilon)\big) &\geq \mu(B_{n_k}^\varphi) - 2\mu(B_{n_k+1}^\varphi)\sup_{x\in M}\mu(B^\varphi(x,\varepsilon)) \\ &\geq \mu(B_{n_k}^\varphi) - 2C\mu(B_{n_k}^\varphi)\sup_{x\in M}\mu(B^\varphi(x,\varepsilon))\end{aligned}$$

for all $\gamma \in \Gamma'$, $\varepsilon \in (0,1)$ and $k > N \vee N_0$. By the assumption (A1), one can choose $\varepsilon_0 \in (0,1)$ such that

$$\sup_{x\in M}\mu(B^\varphi(x,\varepsilon_0)) \leq \frac{1}{4C}.$$

Then we arrive at

$$\mu\big(D(\gamma,\varepsilon_0)\big) \geq \frac{1}{2}\mu(B_{n_k}^\varphi), \quad \gamma \in \Gamma',\ k > N \vee N_0.$$

Letting $k \to \infty$, we get $\mu\big(D(\gamma,\varepsilon_0)\big) = \infty$ for π_μ-a.e. $\gamma \in \Gamma$. Therefore, we conclude that $\eta\big(D(\gamma,\varepsilon_0)\big) = \infty$ for $(\pi_\mu \times \pi_\mu)$-a.e. $(\gamma,\eta) \in \Gamma \times \Gamma$. Now the desired assertion follows immediately by noting that

$$\varphi^\Gamma(\gamma,\eta) \geq \varepsilon_0\eta\big(D(\gamma,\varepsilon_0)\big), \quad \gamma,\eta \in \Gamma. \quad \square$$

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