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# Vague convergence of locally integrable martingale measures†

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**Abstract**

In this paper, we introduce the concept of the vague convergence of locally integrable martingale measures in distribution, which is an organic combination of the vague convergence of Radon measures and the weak convergence of martingales in distribution. The conditions are provided for vague convergence of martingale measures. We also study the convergence of stochastic integrale with respect to martingale measures in distribution.

*Key words:* The characteristics of martingale measure; Martingale measure; Vague convergence.

## 0. Introduction

The purpose of this paper is to study the vague convergence of locally integrable martingale measures in distribution, which is an organic combination of the vague convergence of Radon measures and the weak convergence of martingales in distribution.

In Section 1, we will review the principal results of Walsh (1986): existence of a predictable (resp. optional) random measure which we will call the angle bracket (resp. square bracket) random measure of the orthogonal (resp. strongly orthogonal) martingale measure. We will prove the existence of compensator of the jump measure associated to a càdlàg martingale measure and study the relation of an orthogonal (resp. strongly orthogonal) martingale measure with independent increments and its characteristics. In Section 2, we will introduce the concept of the vague convergence of

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locally integrable martingale measures in distribution. The general theory of the limit theorems of semimartingales in Jacod and Shiryaev (1987) will be applied to study the limit theorems of martingale measures. We apply the characteristics of martingale measures to describe the convergence of martingale measures as the semimartingale case. The conditions will be provided for the vague convergence of stochastic integrals, which were introduced by Walsh (1986) and El Karoui and Méléard (1990) in Section 3.

## 1. Definition and basic properties of martingale measures

Let $E$ be a locally compact Hausdorff space with countable basis, $\mathscr{B}(E)$ the Borel $\sigma$-field on $E$ and $\mathscr{M}(E)$ the linear space formed by all Radon measures on $\mathscr{B}(E)$. Then there exists a metric function $d$ such that $(\mathscr{M}(E), d)$ is a separably complete space and $\mu_n \xrightarrow{v} \mu$ ($\mu_n$ vaguely converges to $\mu$ in $\mathscr{M}(E)$) is equivalent to $d(\mu_n, \mu) \to 0$ when $n \to \infty$ (see Yan, 1988, Theorem VI-4.8). Thus, $\mathscr{M}(E)$ is a Polish space with the topology of the vague convergence of measures.

**Definition 1.1.** Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a filtered probability space satisfying the "usual conditions".

(1) $\{M_t(A)$: $t \geq 0$, $A \in \mathscr{B}(E)\}$ is an $\mathscr{F}_t$-martingale measure if

(i) $M_0(A) = 0$ for all $A \in \mathscr{B}(E)$,

(ii) $\{M_t(A)\}_{t \geq 0}$ is a $\mathscr{F}_t$-locally integrable martingales for all $A \in \mathscr{B}(E)$,

(iii) for all $t > 0$, $M_t(\cdot)$ is a $L^2$-valued $\sigma$-finite measure (see El Karoui and Méléard, 1990).

(2) A martingale measure $M$ is said to be orthogonal if, for any two disjoint sets $A$ and $B$ in $\mathscr{B}(E)$, the martingales $\{M_t(A)\}_{t \geq 0}$ and $\{M_t(B)\}_{t \geq 0}$ are orthogonal, that is, $\langle M(A), M(B) \rangle = 0$.

(3) A martingale measure $M$ is said to be strongly orthogonal if, for any two disjoint sets $A$ and $B$ in $\mathscr{B}(E)$, the quadratic covariation of martingales $M(A)$ and $M(B)$, $[M(A), M(B)] = 0$.

It is clear that strong orthogonality implies orthogonality.

**Definition 1.2.** Let $M$ be an $\mathscr{F}_t$-martingale measure. $M$ is said to be an $R$-martingale measure if $M_t(\omega, \cdot)$ and $M_{t-}(\omega, \cdot)$ belong to $\mathscr{M}(E)$ for all $t > 0$, $\omega \in \Omega$.

**Definition 1.3.** If $M$ is a martingale measure and if, moreover, for all $A \in \mathscr{B}(E)$ the map $t \to M_t(A)$ is continuous, we will say that $M$ is continuous. If the map $t \to M_t(A)$ is càdlàg, we will say that $M$ is càdlàg.

In this paper, we study only càdlàg martingale measure.

**Definition 1.4.** Let $M$ and $N$ be two $\mathscr{F}_t$-martingale measures on the separable locally compact spaces $\boldsymbol{E}$ and $\boldsymbol{E}'$, respectively. If, for all $A \in \mathscr{B}(\boldsymbol{E})$ and $B \in \mathscr{B}(\boldsymbol{E}')$, $M(A)N(B)$ is an $\mathscr{F}_t$-martingale, then we say that $M$ and $N$ are orthogonal.

**Theorem 1.5.** (1) *If $M$ is an $\mathscr{F}_t$-orthogonal martingale measure, there exists a random $\sigma$-finite positive measure $\nu(\mathrm{d}s, \mathrm{d}x)$ on $\mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E})$, predictable, such that for all $A \in \mathscr{B}(\boldsymbol{E})$ the process $\{\nu([0, t] \times A)\}_{t \geq 0}$ is predictable and satisfies*

$$\nu([0, t] \times A) = \langle M(A) \rangle_t \quad P\text{-a.s.} \ \forall t > 0, \ A \in \mathscr{B}(\boldsymbol{E}).$$

*We denote $\nu = \langle M \rangle$.*

(2) *If $M$ is an $\mathscr{F}_t$-strongly orthogonal martingale measure, there exists a random $\sigma$-finite positive measure $\mu(\mathrm{d}s, \mathrm{d}x)$ on $\mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E})$, optional, such that for all $A \in \mathscr{B}(\boldsymbol{E})$ the process $\{\mu([0, t] \times A)\}_{t \geq 0}$ is optional and satisfies*

$$\mu([0, t] \times A) = [M(A)]_t \quad P\text{-a.s.} \ \forall t > 0, \ A \in \mathscr{B}(\boldsymbol{E}).$$

*If $\nu = \langle M \rangle$, then $\nu$ is the dual predictable projection of $\mu$. We denote $\mu = [M]$.*

**Proof.** (1) has been proved by Walsh (1986). The proof of (2) is exactly same as Walsh's proof. □

**Definition 1.6.** Let $M$ be an orthogonal martingale measure. $M$ is called integrable if $E\nu(\mathbb{R}_+ \times \boldsymbol{E}) < \infty$. $M$ is called locally integrable if there exist a sequence of compact subsets $K_n \uparrow \boldsymbol{E}$ and a sequence of stopping times $T_n \uparrow \infty$ such that $E\nu([0, T_n] \times K_n) < \infty$ for all $n \geq 1$.

**Proposition 1.7.** *Let $M$ be an R-martingale measure. We denote by $M(\{t\} \times \mathrm{d}x) = M_t - M_{t-}$. Put*

$$\alpha(\mathrm{d}t, \mathrm{d}y) = \sum_{s > 0} I_{\{M(\{s\} \times \mathrm{d}x) \neq 0\}} \varepsilon_{(s, M(\{s\} \times \mathrm{d}s))}(\mathrm{d}t, \mathrm{d}y). \tag{1.1}$$

*It is called the random measure associated to the jumps of $M$, which is an integer-valued random measure on $\mathscr{B}_+ \times \mathscr{B}(\mathscr{M}(\boldsymbol{E}))$, where $\mathscr{B}(\mathscr{M}(\boldsymbol{E}))$ is the Borel $\sigma$-field of $\mathscr{M}(\boldsymbol{E})$. Then $\alpha$ has the dual predictable projection we denote it by $\beta$.*

**Proof.** It is the same as Proposition II-1.16 of Jacod and Shiryaev (1987). □

**Definition 1.8.** Let $M$ be an $\mathscr{F}_t$-adapted orthogonal R-martingale measure and let $\nu$, $\beta$ be the same as in Theorem 1.5 and Proposition 1.7, respectively. We say that $(\nu, \beta)$ is the characteristics of $M$.

Let $M$ be an orthogonal martingale measure, $\langle M\rangle = \nu$. We can construct a stochastic integral with respect to $M$ by the method which is used in the construction of Itô's integral (Walsh, 1986). Let us consider the set $\mathscr{S}$ which consists of all functions of the following form

$$h(\omega, t, x) = \sum_{i=1}^{n} h_i(\omega) I_{]s_i,t_i]}(t) I_{B_i}(x),$$

which satisfy $E(\int_{\mathbb{R}_+\times E} h^2(\omega, s, x)\nu(\mathrm{d}s, \mathrm{d}x)) < \infty$, where $B_i \in \mathscr{B}(\boldsymbol{E})$, $h_i$ are $\mathscr{F}_s$-measurable bounded functions and

$$L_\nu^2 = \left\{ f{:}\mathscr{P} \times \mathscr{B}(\boldsymbol{E}) \text{ measurable, } E\left( \int_{\mathbb{R}_+\times E} f^2 \nu(w, \mathrm{d}s, \mathrm{d}x) \right) < \infty \right\}.$$

If $h$ is a function in $\mathscr{S}$, it is easy to verify that we can define a martingale measure by

$$h\cdot M_t(A) = \sum_{i=1}^{n} h_i [M_{t_i\wedge t}(A\cap B_i) - M_{s_i\wedge t}(A\cap B_i)], \quad \forall A\in\mathscr{B}(\boldsymbol{E})$$

and $\langle h\cdot M\rangle = h^2(s, x)\nu(\mathrm{d}s, \mathrm{d}x)$. Since $\mathscr{S}$ is dense in $L_\nu^2$, the linear mapping

$$h \to \{h\cdot M_t(A),\ t\geq 0,\ A\in\mathscr{B}(\boldsymbol{E})\}$$

can be extended to $L_\nu^2$ as usual. If $f\in L_\nu^2$, $f\cdot M$ is called the stochastic integral with respect to $M$.

Let $f\in L_\nu^2$. Then $f\cdot M$ is a martingale measure. Moreover, if $M$ is continuous, $f\cdot M$ is continuous. If $f, g\in L_\nu^2$ and $A, B\in\mathscr{B}(\boldsymbol{E})$, we have

$$\langle f\cdot M(A), g\cdot M(B)\rangle = \int_0^{\cdot} \int_{A\cap B} f(s, x) g(s, x)\nu(\mathrm{d}s, \mathrm{d}x).$$

Let $M$ be an $R$-martingale measure. For any $f\in C_K(\mathbb{R}_+\times \boldsymbol{E})$, which is the space of all continuous functions defined on $\mathbb{R}_+\times \boldsymbol{E}$ vanishing outside a compact subset of $\mathbb{R}_+\times \boldsymbol{E}$, then $f\cdot M$ is still an $R$-martingale. Put $X = \int_0^{\cdot} \int_E f(s, x) M(\mathrm{d}s, \mathrm{d}x)$, we have $X$ is a real-valued martingale. Suppose that $\gamma$ is the random measure associated to the jumps of $X$ and $\lambda$ is the dual predictable projection of $\gamma$. For any $g\in C_K(\mathbb{R})$, we have

$$\int_0^{\cdot} \int_{\mathbb{R}} g(x)\gamma(\mathrm{d}s, \mathrm{d}x) = \int_0^{\cdot} \int_{\mathscr{M}(E)} g\left( \int_E f(s, x) y(\mathrm{d}x) \right) \alpha(\mathrm{d}s, \mathrm{d}y).$$

This implies

$$\int_0^{\cdot} \int_{\mathbb{R}} g(x)\lambda(\mathrm{d}s, \mathrm{d}x) = \int_0^{\cdot} \int_{\mathscr{M}(E)} g\left( \int_E f(s, x) y(\mathrm{d}x) \right) \beta(\mathrm{d}s, \mathrm{d}y). \tag{1.2}$$

**Definition 1.9.** Let $M$ be an $\mathscr{F}_t$-adapted martingale measure.

(a) $M$ is said to be with independent increments (MMII) if for all $0\leq s\leq t$ the random measure $M_t - M_s$ is independent from the $\sigma$-field $\mathscr{F}_s$.

(b) $M$ is said to be with stationary independent increments (MMIIS) if $M$ is MMII such that the distribution of the variable $M_t - M_s$ only depends on the difference $t-s$.

(c) If $M$ is an $R$-martingale measure, a time $t \geq 0$ is called a fixed time of discontinuity for $M$ if $P(M(\{t\} \times \mathrm{d}x) \neq 0) > 0$.

Since the set of fixed times of discontinuity of an $R$-martingale measure is at most countable, an $R$-martingale measure with stationary independent increments has no fixed time of discontinuity.

**Theorem 1.10.** (1) *Let $M$ be an $\mathscr{F}_t$-adapted orthogonal $R$-martingale measure. If $M$ is MMII, then there is a version $(\nu, \beta)$ of its characteristics that is deterministic.*

(2) *Suppose that $M$ is an $\mathscr{F}_t$-adapted strongly orthogonal $R$-martingale measure, then $M$ is MMII if and only if there exists a version $(\nu, \beta)$ of its characteristics that is deterministic.*

**Proof.** (1) Suppose that $M$ is MMII. Then $M(A)$ is a process with independent increments for all $A \in \mathscr{B}(\boldsymbol{E})$, hence $\nu([0,t] \times A)$ is a.s. deterministic. Next, $\alpha$ is a Poisson random measure, hence $\beta$ is a a.s. deterministic.

(2) It is sufficient to show that $M$ is MMII under $(\nu, \beta)$ being deterministic. We only show that if $A_1, \ldots, A_n$ are disjoint sets in $\mathscr{B}(\boldsymbol{E})$, then $X_t = (M_t(A_1), \ldots, M_t(A_n))$ is a $\mathbb{R}^n$-valued martingale with independent increments. Since $M$ is a strongly orthogonal martingale measure, $M(A_i)$, $M(A_j)$ $(i \neq j)$ have no common jumps. Let $\lambda$ and $\lambda_i$ denote the dual predictable projections of the random measures associated to the jumps of $X$ and $M(A_i)$, respectively. We have $\lambda = \sum_{i=1}^n \lambda_i$ and $\langle X^c \rangle = (a_{ij})$ are deterministic by the hypothesis and (1.2), where $a_{ii} = \langle M^c(A_i) \rangle$ and $a_{ij} = 0$, $i \neq j$. This implies $X$ is a $\mathbb{R}^n$-valued martingale with independent increments by Theorem II-4.15 in Jacod and Shiryaev (1987). □

## 2. Vague convergence of locally integrable martingale measures

The setting is as follows: for every $n \geq 1$ we consider a stochastic basis $\mathscr{B}^n = (\Omega^n, \mathscr{F}^n, \mathscr{F}_t^n, P^n)$, $E^n$ denotes the expectation with respect to $P^n$. All sets, variables, processes, martingale measures, etc. with the superscript $n$ are defined on $\mathscr{B}^n$, usually without mentioning. The stochastic measures that we will mention are all Radon stochastic measures on spaces what they are defined.

**Definition 2.1.** Let $M^n$ and $M$ be martingale measures. We say that $M^n$ vague converges to $M$ in distribution and write $M^n \xrightarrow{v\mathscr{L}} M$ if, for all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$,

$$\int_0^{\cdot} \int_E f(s,x) M^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^{\cdot} \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x).$$

In this section, we shall only deal with locally integrable orthogonal martingale measures. For simplicity, we still call them martingale measures. In the following, we suppose $M$ is a martingale measure on a stochastic basis $\mathscr{B} = (\Omega, \mathscr{F}, \mathscr{F}_t, P)$.

**Theorem 2.2.** *Let $\boldsymbol{E} = \{a_1, \ldots, a_k\}$ and let $m^{n1}, \ldots, m^{nk}$ and $m^1, \ldots, m^k$ be a sequence of $k$ orthogonal local square integrable martingales. Put $M_t^n(A) = \sum_{i=1}^k m^{ni}\delta_{a_i}(A)$. If $(m^{n1}, \ldots, m^{nk}) \xrightarrow{\mathscr{L}} (m^1, \ldots, m^k)$ and*

$$\sup_n \sum_{i=1}^k E^n\left(\sup_{s \le N} |\Delta m_s^{ni}|\right) < \infty$$

*for all $N > 0$, then $M^n \xrightarrow{v\mathscr{L}} M$, where $M_t(A) = \sum_{i=1}^k m_t^i \delta_{a_i}(A)$.*

**Proof.** We know that $M^n$ and $M$ are martingale measures (see El Karoui and Méléard, 1990). We choose the discrete topology on $\boldsymbol{E}$. Then $\boldsymbol{E}$ is a separable locally compact space. Since

$$\sup_n \sum_{i=1}^k E\left(\sup_{s \le N} |\Delta m_s^{ni}|\right) < \infty$$

for all $N > 0$ and $(m^{n1}, \ldots, m^{nk}) \xrightarrow{\mathscr{L}} (m^1, \ldots, m^k)$, we have $\{(m^{n1}, \ldots, m^{nk})\}_{n \ge 1}$ is U.T. (see Jakuboski et al., 1989 or Mémin and Slominiski, 1991) and $(f(\cdot, a_i), m^{n1}, \ldots, m^{nk}) \xrightarrow{\mathscr{L}} (f(\cdot, a_i), m^1, \ldots, m^k)$ for all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$. Hence $\sum_{i=1}^k \int_0^\cdot f(s, a_i)\mathrm{d}m_s^{ni} \xrightarrow{\mathscr{L}} \sum_{i=1}^k \int_0^\cdot f(s, a_i)\mathrm{d}m_s^i$ (see Jakuboski et al., 1989). We have $M^n \xrightarrow{v\mathscr{L}} M$. □

**Theorem 2.3.** *Let $u^n$ and $u$ be $\boldsymbol{E}$-valued predictable càdlàg processes and let $m^n$ and $m$ be locally square integrable martingales. Put $M_t^n(A) = \int_0^t I_A(u_s^n)dm_s^n$, $M_t(A) = \int_0^t I_A(u_s)\mathrm{d}m_s$ for all $A \in \mathscr{B}(\boldsymbol{E})$. If $(u^n, m^n) \xrightarrow{\mathscr{L}} (u, m)$ and $\sup_n E^n(\sup_{s \le N} |\Delta m_s^n|) < \infty$, for all $N > 0$. We have $M^n \xrightarrow{v\mathscr{L}} M$.*

**Proof.** For all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, since $f$ is uniformly continuous and $(u^n, m^n) \xrightarrow{\mathscr{L}} (u, m)$, we have $(f(\cdot, u^n), m^n) \xrightarrow{\mathscr{L}} (f(\cdot, u), m)$. $\sup_n E^n(\sup_{s \le N} |\Delta m_s^n|) < \infty$ implies $\{m^n\}_{N \ge 1}$ is U.T. By Theorem 2.6 in Jakuboski et al. (1989), we get $\int_0^\cdot f(s, u_s^n)\mathrm{d}m_s^n \xrightarrow{\mathscr{L}} \int_0^\cdot f(s, u_s)\mathrm{d}m_s$. Thus,

$$\int_0^\cdot \int_E f(s, x) M^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_E f(s, x) M(\mathrm{d}s, \mathrm{d}x).$$

Hence, $M^n \xrightarrow{v\mathscr{L}} M$. □

**Theorem 2.4.** *Let $M^n$ and $M$ be martingale measures, $\langle M^n\rangle = v^n$, $\langle M\rangle = v$.*

(i) *Suppose for any compact subset $K$ of $\boldsymbol{E}$,*

$$\lim_{a\to\infty}\limsup_{n\to\infty} P^n(v^n([0,N]\times K) > a) = 0, \quad \forall N > 0 \tag{2.1}$$

*and*

$$\left(\int_0^{\cdot}\int_E I_{A_1}(s,x)M^n(\mathrm{d}s,\mathrm{d}x), \ldots, \int_0^{\cdot}\int_E I_{A_k}(s,x)M^n(\mathrm{d}s,\mathrm{d}x)\right)$$

$$\xrightarrow{\mathscr{L}}\left(\int_0^{\cdot}\int_E I_{A_1}(s,x)M(\mathrm{d}s,\mathrm{d}x), \ldots, \int_0^{\cdot}\int_E I_{A_k}(s,x)M(\mathrm{d}s,\mathrm{d}x)\right) \tag{2.2}$$

*for any compact set $B \subset \mathbb{R}_+ \times \boldsymbol{E}$ and any sequence $\{A_1, \ldots, A_k\}$ of $v$-continuous sets which are subsets of $B$. Then $M^n \xrightarrow{v\mathscr{L}} M$.*

(ii) *Let $M^n \xrightarrow{v\mathscr{L}} M$ and for all $\varepsilon > 0$,*

$$\lim_{m\to\infty}\limsup_{n\to\infty} P^n(v^n(A_m) > \varepsilon) = 0 \tag{2.3}$$

*for every sequence $\{A_n\}_{n\ge 1}$ of closed $v$-continuous sets such that $\lim_{m\to\infty} P(v(A_m) > \varepsilon) = 0$ for all $\varepsilon > 0$. Then we have $\int_0^{\cdot}\int_E I_A(s,x)M^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^{\cdot}\int_E I_A(s,x)M(\mathrm{d}s,\mathrm{d}x)$ for all $v$-continuous set $A$ which is subset of some compact set.*

**Proof.** (i) Let $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$. Then there exists a compact subset $K$ of $\mathbb{R}_+ \times \boldsymbol{E}$ such that $K \supset \operatorname{supp}(f)$ and $v(\partial K) = 0$, $P$-a.s. by Lemma 4.3 in Kallenberg (1983). Suppose $a < f < b$, for all $m \ge 1$, then there exists $a = a_0 < a_1 < \cdots < a_k = b$ such that $a_i - a_{i-1} < 1/m$ and $v[(\partial\{(s,x)\colon f(s,x) = a_i\}) \cap K] = 0$, $P$-a.s. (Kallenberg, 1983, Lemma 4.3) for $i = 1, \ldots, k$. Put $A_i = \{(s,x)\colon a_{i-1} \le f(s,x) < a_i\} \cap K$, $i = 1, \ldots, k$, we have $A_i$ are disjoint and $v(\partial A_i) = 0$, $P$-a.s. Therefore, (2.2) holds for $A_i$. Put $f_m(s,x) = \sum_{i=1}^k a_{i-1} I_{A_i}(s,x)$. From (2.2) we have

$$\int_0^{\cdot}\int_E f_m(s,x)M^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^{\cdot}\int_E f_m(s,x)M(\mathrm{d}s,\mathrm{d}x). \tag{2.4}$$

Noting that $\sup|f - f_m| \le 1/m$, by Lenglart's inequality, we get

$$P^n\left(\sup_{t\le N}\left|\int_0^t\int_E [f(s,x) - f_m(s,x)]M^n(\mathrm{d}s,\mathrm{d}x)\right| > \varepsilon\right)$$

$$\le \frac{\delta}{\varepsilon^2} + P^n(v^n([0,N]\times K) > \delta m^2), \tag{2.5}$$

$$P\left(\sup_{t\le N}\left|\int_0^t\int_E[f(s,x)-f_m(s,x)]M(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)$$
$$\le\frac{\delta}{\varepsilon^2}+P(\nu([0,N]\times K)>\delta m^2) \tag{2.6}$$

for all $N>0$, $\varepsilon>0$ and $\delta>0$.

By the hypothesis and $\varepsilon>0$, $\delta>0$ are arbitrary, we have

$$\lim_{m\to\infty}P\left(\sup_{t\le N}\left|\int_0^t\int_E[f(s,x)-f_m(s,x)]M(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)=0$$

from (2.6). Hence, as $m\to\infty$, we get

$$\int_0^\cdot\int_E f_m(s,x)M(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^\cdot\int_E f(s,x)M(\mathrm{d}s,\mathrm{d}x).$$

(2.5) implies that

$$\lim_{m\to\infty}\limsup_{n\to\infty}P^n\left(\sup_{t\le N}\left|\int_0^t\int_E[f(s,x)-f_m(s,x)]M^n(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)=0.$$

We deduce $\int_0^\cdot\int_E f(s,x)M^n(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^\cdot\int_E f(s,x)M(\mathrm{d}s,\mathrm{d}x)$ by Theorem 4.2 in Billingsley (1968). Therefore $M^n\xrightarrow{v\mathscr{L}}M$.

(ii) Suppose $A$ is a Borel subset of some compact set $K$ and $\nu(\partial A)=0$, $P$-a.s. Since $\boldsymbol{E}$ is locally compact, we can suppose that $A\subset K^0$. We know that $\partial A$ is a compact subset from the hypothesis. For all $m\ge 1$, there are $x_1^m,\dots,x_{k_m}^m$ in $\partial A$ such that $\partial A\subset\bigcup_{i=1}^{k_m}S(x_i^m,r_m)$, where $0<r_m\downarrow 0$ and $S(x,r)$ is the open sphere with center $x$ and radius $r$. Put $E_m=K\backslash[A\cup\bigcup_{i=1}^{k_m}S(x_i^m,r_{ni})]$ and $G_m=K\backslash[(K\backslash A)\cup\bigcup_{i=1}^{k_m}S(x_i^m,r_m)]$, then $E_m$ and $G_m$ are closed sets and $E_m\bigcap G_m=\emptyset$. By using Lemma 3.4 in Kallenberg (1983), we can choose $r_m$ such that $\nu(\partial E_m)=0$, $P$-a.s. Hence, there exists $f_m\in C_K(\mathbb{R}_+\times\boldsymbol{E})$ such that $f_m(x)=1$ on $G_m$ and $f_m(x)=0$ on $E_m$. Since $M^n\xrightarrow{v\mathscr{L}}M$, we have

$$\int_0^\cdot\int_E f_m(s,x)M^n(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^\cdot\int_E f_m(s,x)M(\mathrm{d}s,\mathrm{d}x)$$

when $n\to\infty$. Further, noting that $|f_m(s,x)-I_A(s,x)|=0$ for $(s,x)\in E_m\cup G_m$ and $|f_m(s,x)-I_A(s,x)|\le 1$ for $(s,x)\in E_m^c\cap G_m^c$ we get

$$P^n\left(\sup_{t\le N}\left|\int_0^t\int_E[I_A(s,x)-f_m(s,x)]M^n(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)$$
$$\le\frac{\delta}{\varepsilon^2}+P^n(\nu^n(E_m^c\cap G_m^c)>\delta)$$
$$\le\frac{\delta}{\varepsilon^2}+P^n(\nu^n(\overline{E_m^c\cap G_m^c})>\delta),\quad\forall N>0,\ \varepsilon>0,\ \delta>0.$$

The condition (2.3) implies

$$\lim_{m\to\infty}\limsup_{n\to\infty} P^n\left(\sup_{t\le N}\left|\int_0^t\int_E [I_A(s,x)-f_m(s,x)]M^n(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)=0.$$

Since $\bigcap_m \overline{E_m \cap G_m} = \partial A$ and

$$\limsup_{m\to\infty} P\left(\sup_{t\le N}\left|\int_0^t\int_E [I_A(s,x)-f_m(s,x)]M(\mathrm{d}s,\mathrm{d}x)\right|>\varepsilon\right)$$

$$\le \varepsilon + \limsup_{m\to\infty} P(v(\overline{E_m\cap G_m})>\varepsilon^2),\quad \forall\varepsilon>0,$$

we have

$$\int_0^\cdot\int_E f_m(s,x)M(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^\cdot\int_E I_A(s,x)M(\mathrm{d}s,\mathrm{d}x).$$

Hence, we deduce $\int_0^\cdot\int_E I_A(s,x)M^n(\mathrm{d}s,\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_0^\cdot\int_E I_A(s,x)M(\mathrm{d}s,\mathrm{d}x)$ by Theorem 4.2 in Billingsley (1968). □

**Remark.** This is the extension of Theorem 5.2 in Thang (1991).

In the following, we only study the limit theorems of $R$-martingale measures.

Let $v^n, v\in\mathscr{M}(\boldsymbol{E})$ be random measures on $\mathscr{B}(\boldsymbol{E})$. We say that $v^n$ converges to $v$ in distribution and write $v^n\xrightarrow{\mathscr{L}}v$ if for any $f\in C_K(\boldsymbol{E})$, $\int_E f(x)v(\mathrm{d}x)\xrightarrow{\mathscr{L}}\int_E f(x)v(\mathrm{d}x)$.

**Theorem 2.5.** *Let $M^n$ and $M$ be $R$-martingale measures, $\langle M^n\rangle = v^n$, $\langle M\rangle = v$, $\beta^n$ and $\beta$ be the dual predictable projections of the random measures associated to the jumps of $M^n$ and $M$, respectively. $M$ has no fixed time of discontinuity and MMII. Suppose that*

(i) $v^n\xrightarrow{\mathscr{L}}v$,

(ii) *For each $f\in C_K(\mathbb{R}_+\times\boldsymbol{E})$,*

$$\int_0^t\int_{\mathscr{M}(E)}\left[\int_E f(s,x)y(\mathrm{d}x)\right]^2\beta^n(\mathrm{d}s,\mathrm{d}y)<\infty,$$

$$\int_0^t\int_{\mathscr{M}(E)}\left[\int_E f(s,x)y(\mathrm{d}x)\right]^2\beta(\mathrm{d}s,\mathrm{d}y)<\infty$$

*and*

$$\lim_{a\to\infty}\limsup_{n\to\infty}P^n\left\{\int_0^t\int_{\mathscr{M}(E)}\left[\int_E f(s,x)y(\mathrm{d}x)\right]^2 I_{\{|\int_E f(s,x)y(\mathrm{d}x)|>a\}}\beta^n(\mathrm{d}s,\mathrm{d}y)>\delta\right\}=0$$

*for all $t>0$, $\delta>0$,*

(iii) *For all* $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$ *and* $g \in C_0^+(\mathbb{R})$

$$\int_0^t \int_{\mathcal{M}(E)} g\left( \int_E f(s,x) y(\mathrm{d}x) \right) \beta^n(\mathrm{d}s, \mathrm{d}y) \xrightarrow{P} \int_0^t \int_{\mathcal{M}(E)} g\left( \int_E f(s,x) y(\mathrm{d}x) \right) \beta(\mathrm{d}s, \mathrm{d}y)$$

*for all* $t > 0$. $g \in C_0^+(\mathbb{R})$ *means that* $g$ *is a continuous function which is* 0 *around* 0 *and outside a compact subset of* $\mathbb{R}$.

*Then* $M^n \xrightarrow{v\mathscr{L}} M$.

**Proof.** Since $M$ is MMII, we know that $v$ and $\beta$ are deterministic by Theorem 1.9. Put $X^n = \int_0^{\cdot} \int_E f(s,x) M^n(\mathrm{d}s, \mathrm{d}x)$, $X = \int_0^{\cdot} \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x)$ for all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, then $X^n$ and $X$ are square integrable martingales and $X$ is independent increments without fixed time of discontinuity. Let $\lambda^n$ and $\lambda$ be the dual predictable projections of the random measures associated to the jumps of $X^n$ and $X$, respectively. For all $g \in C_0^+(\mathbb{R})$, we have $g \cdot \lambda^n \xrightarrow{\mathscr{L}} g \cdot \lambda$ by the condition (iii) and (1.2). And condition (i) implies

$$\langle X^n \rangle_t = \int_0^t \int_E f^2(s,x) v^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{P} \int_0^t \int_E f^2(s,x) v(\mathrm{d}s, \mathrm{d}x) = \langle X \rangle_t$$

for all $t > 0$. (ii) means that $\lim_{a \to \infty} \limsup_{n \to \infty} P^n(x^2 I_{\{|x| > a\}} \cdot \lambda_t^n > \delta) = 0$ for all $t > 0$, $\delta > 0$. Hence, $X^n \xrightarrow{\mathscr{L}} X$ by Theorem VIII-2.18 in Jacod and Shiryaev (1987). □

**Theorem 2.6.** *Let* $M^n$ *and* $M$ *be R-martingale measures with independent increments and* $M$ *have no fixed time of discontinuity,* $v^n$, $v$, $\beta^n$, $\beta$ *be random measures given in Theorem* 2.5. *Suppose that for each* $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$ *and for all* $t > 0$,

$$\lim_{a \to \infty} \limsup_{n \to \infty} \int_0^t \int_{\mathcal{M}(E)} \left[ \int_E f(s,x) y(\mathrm{d}x) \right]^2 I_{\{|\int_E f(s,x) y(\mathrm{d}x)| > a\}} \beta^n(\mathrm{d}s, \mathrm{d}y) = 0.$$

*Then* $M^n \xrightarrow{v\mathscr{L}} M$ *if and only if*

(i) $v^n \xrightarrow{\mathscr{L}} v$,

(ii) *For all* $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, $g \in C_0^+(\mathbb{R})$ *and* $t > 0$,

$$\int_0^t \int_{\mathcal{M}(E)} g\left( \int_E f(s,x) y(\mathrm{d}x) \right) \beta^n(\mathrm{d}s, \mathrm{d}y) \to \int_0^t \int_{\mathcal{M}(E)} g\left( \int_E f(s,x) y(\mathrm{d}x) \right) \beta(\mathrm{d}s, \mathrm{d}y).$$

**Proof.** For each $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, put

$$X^n = \int_0^{\cdot} \int_E f(s,x) M^n(\mathrm{d}s, \mathrm{d}x), \qquad X = \int_0^{\cdot} \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x),$$

then $X^n$ and $X$ are square integrable martingales with independent increments and $X$ has no fixed time of discontinuity. Let $\lambda^n$, $\lambda$ be the same as in the proof of Theorem

2.5. Under the condition $\lim_{a\to\infty}\limsup_{n\to\infty}|x|^2 I_{\{|x|>a\}}\cdot\lambda_t^n = 0$ for all $t>0$, $X^n \xrightarrow{\mathscr{L}} X$ if and only if $\langle X^n\rangle_t \to \langle X\rangle_t$ and $g\cdot\lambda_t^n \to g\cdot\lambda_t$ for all $t>0$ and $g\in C_0^+(\mathbb{R})$ by Theorem VII-3.7 in Jacod and Shiryaev (1987). Theorem is proven. □

**Theorem 2.7.** *Let $M^n$ be R-martingale measures and $M$ be a continuous MMII, $\langle M^n\rangle = v^n$, $\langle M\rangle = v$. Suppose that $v^n \xrightarrow{\mathscr{L}} v$, $P^n(\sup_{s\le N}|M^n(\{s\}\times K)|>a)\to 0$, as $n\to\infty$, for all $a>0$, $N>0$ and compact set $K$ and $|M^n(\{s\}\times K)|\le C$, where $|M^n(\{s\}\times K)|$ denotes the total variation of random measure $M^n(\{s\}\times \mathrm{d}x)$ on $K$. Then $M^n \xrightarrow{v\mathscr{L}} M$.*

**Proof.** Since $v^n \xrightarrow{\mathscr{L}} v$ and $v$ is deterministic, then

$$\int_0^t\int_E f(s,x)v^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{P} \int_0^t\int_E f(s,x)v(\mathrm{d}s,\mathrm{d}x)$$

for all $t>0$ and $0\le f\in C_K(\mathbb{R}_+\times \boldsymbol{E})$. Suppose that $X^n$, $X$, $\lambda^n$ are the same as in the proof of Theorem 2.6, then

$$\langle X^n\rangle_t = \int_0^t\int_E f^2(s,x)v^n(\mathrm{d}s,\mathrm{d}x) \xrightarrow{P} \int_0^t\int_E f^2(s,x)v(\mathrm{d}s,\mathrm{d}x) = \langle X\rangle_t \tag{2.7}$$

for all $t>0$. And

$$g\cdot\lambda_t^n \le C I_{\{|x|>a\}}\cdot\lambda_t^n = C\lambda^n([0,t]\times\{|x|>a\}) \tag{2.8}$$

for all $g\in C_0^+(\mathbb{R})$, where $a>0$, $C>0$ such that $g\le C$ and $g(x)=0$ for $|x|\le a$.

Since $\lim_{n\to\infty} P^n(\sup_{s\le N}|\Delta X_s^n|>a)=0$ is equivalent to

$$\lim_{n\to\infty} P^n(\lambda^n([0,t]\times\{|x|>a\})>\varepsilon)=0,\quad \forall\varepsilon>0$$

(see Jacod and Shiryaev, 1987, Lemma VI-4.22). And since $|\Delta X_t^n| \le C_1\sup_{s\le t}|M^n(\{s\}\times K)|\le C_1 C$ and

$$P^n\left(\sup_{s\le N}|\Delta X_t^n|>a\right)\le P^n\left(\sup_{s\le N} C_1|M^n(\{s\}\times K)|>a\right),$$

where $\sup_x|f(x)|\le C_1$, $K$ is a subset of $\mathbb{R}_+\times\boldsymbol{E}$ and $\operatorname{supp}\{f\}\subset K$, we have $g\cdot\lambda_N^n \xrightarrow{P} 0$ for all $N>0$ by the hypothesis and (2.8). Hence, $X^n \xrightarrow{\mathscr{L}} X$ by (2.7) and Theorem VIII-3.11 in Jacod and Shiryaev (1987) □

By Theorem 2.7, we immediately get the following corollary.

**Corollary 2.8.** *Let $\boldsymbol{E}=\{a_1,\dots,a_k\}$ and let $m^{n1},\dots,m^{nk}$ (resp. $m^1,\dots,m^k$) be $k$ orthogonal (resp. orthogonal continuous) locally integrable martingales. $|\Delta m^{ni}|\le b$, $\langle m^{ni}\rangle = C^{ni}$, $\langle m^i\rangle = C^i$, $i=1,\dots,k$, $n\ge 1$, $\lambda^{ni}$ be the dual predictable projections* *of the random measures associated to the jumps of* $m^{ni}$. *Put*

$$M^n(A) = \sum_{i=1}^{k} m^{ni}\delta_{a_i}(A), \qquad M(A) = \sum_{i=1}^{k} m^i \delta_{a_i}(A).$$

*Suppose that*

(i) $C_t^{ni} \xrightarrow{P} C_t^i$, $i = 1, \dots, k$ *and* $\lim_{t\to\infty} \sup_n P^n(\sum_{i=1}^k C_t^{ni} > 0) = 0$ *for all* $t > 0$.

(ii) $\lim_{n\to\infty} P^n(\lambda^{ni}([0,t] \times \{|x| > a\} > \varepsilon) = 0$ *for all* $\varepsilon > 0$, *and* $a > 0$.

*Then* $M^n \xrightarrow{v\mathscr{L}} M$.

**Theorem 2.9.** *Let* $M^n$ *be strongly orthogonal R-martingale measures and let* $M$ *be a continuous MMII.* $[M^n] = \mu^n$, $\langle M \rangle = v$. *Suppose* $|M^n(\{t\} \times K)| \le C$ *for all* $t > 0$, $n \ge 1$ *and compact set* $K \subset \mathbb{R}_+ \times \boldsymbol{E}$. *We have the equivalence between*

(i) $M^n \xrightarrow{v\mathscr{L}} M$,

(ii) $\mu^n \xrightarrow{\mathscr{L}} v$,

(iii) $v^n \xrightarrow{\mathscr{L}} v$ *and* $\lim_{n\to\infty} P^n(\lambda_f^n([0,t] \times \{|x| > a\}) > \varepsilon) = 0$ *for all* $t > 0$, $a > 0$, $\varepsilon > 0$, $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, *where* $v^n$ *is the dual predictable projection of* $\mu^n$, $\lambda_f^n$ *is the dual predictable projection of the random measure associated to jumps of* $\int_0^{\cdot} \int_E f(s,x) M^n(\mathrm{d}s, \mathrm{d}x)$.

**Proof.** For all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, assume that $X^n$ and $X$ are the same as in the proof of Theorem 2.7. We have $|\Delta X^n| \le C$ by the hypothesis. Under this condition, we deduce equivalence between:

(a) $X^n \xrightarrow{\mathscr{L}} X$,

(b) $[X^n]_t \xrightarrow{P} \langle X \rangle_t$, for all $t > 0$,

(c) $\langle X^n \rangle_t \xrightarrow{P} \langle X \rangle_t$ and $\lim_{n\to\infty} P^n(\lambda_f^n([0,t] \times \{|x| > a\}) > \varepsilon) = 0$ for all $t > 0$, $a > 0$, $\varepsilon > 0$ by Theorem VIII-3.11 in Jacod and Shiryaev (1987). Theorem is proved. □

**Theorem 2.10.** *Let* $M^n$ *and* $M$ *be the same as in Theorem* 2.9. *Suppose that, for all* $\delta > 0$, $t > 0$, $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$,

$$\lim_{a\to\infty} \limsup_{n\to\infty} P^n \left\{ \int_0^t \int_{\mathscr{M}(E)} \left| \int_E f(s,x) y(\mathrm{d}x) \right| I_{\{|\int_E f(s,x)y(\mathrm{d}x)| > a\}} \beta^n(\mathrm{d}s, \mathrm{d}y) > \delta \right\} = 0. \quad (2.9)$$

*We have the equivalence between*

(i) $M^n \xrightarrow{v\mathscr{L}} M$,

(ii) $\mu^n \xrightarrow{\mathscr{L}} v$.

**Proof.** For all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, assume that $X^n$ and $X$ are the same as in the proof of Theorem 2.7. (2.9) means that $\lim_{a\to\infty} \limsup_{n\to\infty} P^n(|x| I_{\{|x|>a\}} \cdot \lambda_t^n > \delta) = 0$ for all $t > 0$, $\delta > 0$. Under this condition, we deduce equivalence between

(a) $X^n \xrightarrow{\mathscr{L}} X$,

(b) $[X^n]_t \xrightarrow{P} \langle X \rangle_t$ for all $t > 0$

by Theorem VIII-3.12 in Jacod and Shiryaev (1987). □

**Corollary 2.11.** *Let $M^n$ be R-martingale measures with independent increments and $M$ be a continuous MMII. Suppose that*

$$\lim_{n\to\infty} P^n\left(\sup_{s\le N} |M^n(\{s\} \times K)| > \varepsilon\right) = 0$$

*for all $N > 0$, $a > 0$ and compact set $K \subset \mathbb{R}_+ \times \boldsymbol{E}$* and

$$\int_0^t \int_{\mathscr{M}(E)} \left[\int_E f(s, x) y(\mathrm{d}x)\right]^2 I_{\{|\int_E f(s,x)y(\mathrm{d}x)| > \varepsilon\}} \beta(\mathrm{d}s, \mathrm{d}y) \xrightarrow{P} 0, \quad \forall \varepsilon > 0,\ t > 0.$$

*Then $M^n \xrightarrow{v\mathscr{L}} M$ if and only if $v^n \xrightarrow{\mathscr{L}} v$.*

**Proof.** By Theorem 2.7, it is sufficient to prove necessity. Suppose that $M^n \xrightarrow{v\mathscr{L}} M$. We have

$$\int_0^{\cdot} \int_E f(s, x) M^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^{\cdot} \int_E f(s, x) M(\mathrm{d}s, \mathrm{d}x)$$

for all $f \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$. Assume that $X^n$ and $X$ are the same as in the proof of Theorem 2.6, then $X^n$, $X$ are square integrable martingales with independent increments and $X$ is continuous. When $n \to \infty$, we have

$$\sup_{t\le N} \left| \int_0^t \int_E f^2(s, x) v^n(\mathrm{d}s, \mathrm{d}x) - \int_0^t \int_E f^2(s, x) v(\mathrm{d}s, \mathrm{d}x) \right| \to 0$$

for all $t > 0$. Hence $\lim_{n\to\infty} \int_0^\infty \int_E f^2(s,x) v^n(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \int_E f^2(s,x) v(\mathrm{d}s, \mathrm{d}x)$. That is, $\lim_{n\to\infty} \int_0^\infty \int_E f(s,x)\, v^n(\mathrm{d}s, \mathrm{d}x) = \int_0^\infty \int_E f(s,x) v(\mathrm{d}s, \mathrm{d}x)$ for all $f \ge 0$ in $C_K(\mathbb{R}_+ \times \boldsymbol{E})$. To drop the condition $f \ge 0$, we can consider $f = f^+ - f^-$ and notice that $\int_0^\infty \int_E f^\pm (s,x) v(\mathrm{d}s, \mathrm{d}x)$ is deterministic, we have $v^n \xrightarrow{\mathscr{L}} v$. □

## 3. Convergence of stochastic integrals in distribution

Let $f^n$ and $f$ be real-valued measurable functions on $(\mathbb{R}_+ \times \boldsymbol{E}, \mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E}))$ and $v$ is a Radon measure on $\mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E})$. We will say that $f^n$ converges continuously to

$f$($\nu$-a.s.) and write "$f^n \xrightarrow{\text{c.c.}} f(\nu - \text{a.s.})$" if there exists a $\nu$-null set $B \in \mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E})$ such that, if $(t, x) \notin B$ then $f^n(t_n, x_n) \to f(t, x)$ whenever $(t_n, x_n) \to (t, x)$.

Clearly, if $f^n$ converges uniformly to a continuous $f$, then $f^n \xrightarrow{\text{c.c.}} f$($\nu$-a.s.) for all Radon measure $\nu$ on $\mathscr{B}_+ \times \mathscr{B}(\boldsymbol{E})$.

**Theorem 3.1.** *Let $f^n \in L^2_{\nu^n}$ and $f \in L^2_\nu$ be real-valued measurable functions. Under the conditions of Theorem 2.7, if $f^n \xrightarrow{\text{c.c.}} f$($\nu$-a.s.) and if $\{f^n\}$ are uniformly bounded, we have $f^n \cdot M^n \xrightarrow{v\mathscr{L}} f \cdot M$.*

*In particular, if $f^n$ and $f$ are functions on $\mathbb{R}_+ \times \boldsymbol{E}$ vanishing outside a common compact set $K \subset \mathbb{R}_+ \times \boldsymbol{E}$, we have*

$$\int_0^\cdot \int_E f^n(s, x) M^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_E f(s, x) M(\mathrm{d}s, \mathrm{d}x).$$

**Proof.** Note that $\langle f^n \cdot M^n \rangle = (f^n)^2 \cdot \nu^n$, $\langle f \cdot M \rangle = f^2 \cdot \nu$. Since $g$ is uniformly continuous for all $g \in C_K(\mathbb{R}_+ \times \boldsymbol{E})$, we have $(gf^n)^2 \xrightarrow{\text{c.c.}} (gf)^2$($\nu$-a.s.) and $(gf^n)^2$, $(gf)^2$ vanish outside the compact set Supp($g$). As $\nu$ is stochastic contiunous, we deduce that

$$\int_0^\cdot \int_E (gf^n)^2 \nu^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{P} \int_0^\cdot \int_E (gf)^2 \nu(\mathrm{d}s, \mathrm{d}x)$$

by Lemma 6.2 in Kasahara and Watanaba (1986). For any compact set $K \subset \mathbb{R}_+ \times \boldsymbol{E}$, $N > 0$,

$$\begin{aligned}\sup_{s \le N} |(f^n \cdot M^n)(\{s\} \times K)| &= \sup_{s \le N} \left| \int_K f^n(s, x) M^n(\{s\} \times \mathrm{d}x) \right| \\ &\le C \sup_{s \le N} |M^n(\{s\} \times K)|,\end{aligned}$$

where $C$ is a constant, this implies that

$$P^n\left( \sup_{s \le N} |(f^n \cdot M^n)(\{s\} \times K)| > \varepsilon \right) \le P^n\left( \sup_{s \le N} |M^n(\{s\} \times K)| > \frac{\varepsilon}{C} \right) \to 0$$

by the hypothesis. We get $f^n \cdot M^n \xrightarrow{v\mathscr{L}} f \cdot M$ by Theorem 2.7. $\square$

We next relax the assumption that $(f^n, n \ge 1)$ vanish outside a compact set. Let $K_1 \subset K_2 \subset \cdots$ be a compact exhaustion of $\boldsymbol{E}$ with $K_n \subset K^\circ_{n+1}$ $(n \ge 1)$.

**Theorem 3.2.** *Let $f^n \in L^2_{\nu^n}$ and $f \in L^2_\nu$ be uniformly bounded real-valued functions. Under the conditions of Theorem 2.7, if $f^n \xrightarrow{\text{c.c.}} f$($\nu$-a.s.) and if for all $\varepsilon > 0$, $t > 0$,*

$$\lim_{k \to \infty} \limsup_{n \to \infty} P^n\left( \int_0^t \int_{E \setminus K_k} f^n(s, x)^2 \nu^n(\mathrm{d}s, \mathrm{d}x) > \varepsilon \right) = 0. \tag{3.1}$$

*Then* $f^n \cdot M^n \xrightarrow{v\mathscr{L}} f \cdot M$ *and*

$$X^n = \int_0^\cdot \int_E f^n(s, x) M^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{\mathscr{L}} \int_0^\cdot \int_E f(s, x) M(\mathrm{d}s, \mathrm{d}x) = X.$$

**Proof.** It is clear that $f^n \cdot M^n \xrightarrow{v\mathscr{L}} f \cdot M$ by Theorem 3.1. We only prove the second conclusion.

Let $\varphi_k$, $k = 1, 2, \ldots$, be continuous functions such that $I_{K_k} \le \varphi_k \le I_{K_{k+1}}$ and define

$$X^{nk} = \int_0^\cdot \int_E \varphi_k(x) f^n(s, x) M^n(\mathrm{d}s, \mathrm{d}x), \qquad X^k = \int_0^\cdot \int_E \varphi_k(x) f(s, x) M(\mathrm{d}s, \mathrm{d}x).$$

For every $k \ge 0$, since

$$\langle X^{nk} \rangle_t = \int_0^t \int_E \varphi_k^2(x) f^{n2}(s, x) \nu^n(\mathrm{d}s, \mathrm{d}x) \xrightarrow{P} \int_0^t \int_E \varphi_k^2(x) f^2(s, x) \nu(\mathrm{d}s, \mathrm{d}x) = \langle X^k \rangle_t$$

by Lemma 6.2 in Kasahara and Watanaba (1986) for all $t > 0$ and $\langle X^{nk} \rangle_t$, $\langle X^k \rangle_t$ are increasing processes and $\langle X^k \rangle_t$ is deterministic, we have $\langle X^{nk} \rangle \xrightarrow{\mathscr{L}} \langle X^k \rangle$. But again,

$$P^n\left( \sup_{s \le N} |\Delta X_s^{nk}| > \varepsilon \right) \le P^n\left( \sup_{s \le N} |M^n(\{s\} \times K_{k+1})| > \frac{\varepsilon}{C} \right)$$

this implies that $\lim_{n \to \infty} P^n(\sup_{s \le N} |\Delta X_s^{nk}| > \varepsilon) = 0$ for all $N > 0$, $\varepsilon > 0$ and $|\Delta X^{nk}| \le C_1$ by the hypothesis. Hence $X^{nk} \xrightarrow{\mathscr{L}} X^k$ by Theorem VIII-3.11 in Jacod and Shiryaev (1987). By Lenglart inequality, Theorem 4.2 in Billingsley (1968) and condition (3.1), as in the proof of Theorem 2.4, we finish the proof of the Theorem. □

**Theorem 3.3.** *Let* $f^n$ *and* $f$ *be as in Theorem* 3.2, *but we drop* (3.1), *instead we assume that for every* $N > 0$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} \sup\{ |f^n(s, x)| : 0 \le s \le N,\ x \notin K_k \} = 0. \tag{3.2}$$

*Put* $X^n = \int_0^\cdot \int_E f^n(s,x) M^n(\mathrm{d}s, \mathrm{d}x)$, $X^{nk} = \int_0^\cdot \int_{E \setminus K_k} f^n(s,x) M^n(\mathrm{d}s, \mathrm{d}x)$. *Suppose that for all* $t > 0$, $\varepsilon > 0$,

$$\lim_{k \to \infty} \limsup_{n \to \infty} P^n(\langle X^{nk} \rangle_t > \varepsilon) = 0. \tag{3.3}$$

*Under the conditions of Theorem* 2.7, *we have* $X^n \xrightarrow{\mathscr{L}} \int_0^\cdot \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x)$.

**Proof.** This theorem is proven with the same method as in Kasahara and Watanaba (1986). Let us describe quickly the principal steps of the proof.

Let $\varphi_k$ be as in the proof of Theorem 3.2 and put

$$\xi_k^n = \int_0^T \int_E [\varphi_{k+1}(x) - \varphi_k(x)] f^{n2}(s, x) \nu^n(\mathrm{d}s, \mathrm{d}x), \quad \forall k, n \geq 1$$

and for all $k \geq 1$,

$$a_k = \int_0^T \int_E [\varphi_{k+1}(x) - \varphi_k(x)] f^2(s, x) \nu(\mathrm{d}s, \mathrm{d}x),$$

then $\xi_k^n \xrightarrow{P} a_k$ as $n \to \infty$ by the hypothesis and $\sum_{k=1}^{\infty} a_k \leq \int_0^T \int_E f^2(\mathrm{d}s, x) \nu(\mathrm{d}s, \mathrm{d}x) < \infty$ (recall that $f \in L_\nu^2$). By Lemma 6.7 in Kasahara and Watanaba (1986), there exist $k_1 \leq k_2 \leq \cdots \to \infty$ such that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P^n\left( \sum_{k=m}^{k_n} \xi_k^n \geq \varepsilon \right) = 0, \quad \forall \varepsilon > 0 \tag{3.4}$$

and

$$\sum_{k=1}^{k_n} \xi_k^n \xrightarrow{P} \sum_{k=1}^{\infty} a_k, \ n \to \infty. \tag{3.5}$$

Let $f_n^{(1)}(t, x) = f^n(t, x) I_{K_{k_n}}(x), f_n^{(2)} = f^n - f_n^{(1)}$. Define

$$W^n = \int_0^\cdot \int_E f_n^{(2)}(s, x) M^n(\mathrm{d}s, \mathrm{d}x), \qquad Y^n = \int_0^\cdot \int_E f_n^{(1)}(s, x) M^n(\mathrm{d}s, \mathrm{d}x).$$

Observe that, if $m \leq k_n$ then

$$\int_0^T \int_{E \setminus K_{m+1}} |f_n^{(1)}(s, x)|^2 \nu^n(\mathrm{d}s, \mathrm{d}x)$$
$$\leq \int_0^T \int_E [\varphi_{k_n+1}(x) - \varphi_m(x)] f^{n2}(s, x) \nu^n(\mathrm{d}s, \mathrm{d}x) \leq \sum_{k=m}^{k_n} \xi_k^n.$$

Thus (3.4) implies that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P^n\left( \int_0^T \int_{E \setminus K_{m+1}} |f_n^{(1)}(s, x)|^2 \nu^n(\mathrm{d}s, \mathrm{d}x) > \varepsilon \right) = 0 \ \forall \varepsilon > 0. \tag{3.6}$$

Since, if $k_n \geq m > 1$,

$$\begin{aligned} |\langle W^n \rangle_t - \langle X^{nm} \rangle_t| &\leq |\langle W^n, W^n - X^{nm} \rangle_t| + |\langle W^n - X^{nm}, X^{nm} \rangle_t| \\ &\leq \langle W^n \rangle_t^{1/2} \langle W^n - X^{nm} \rangle_t^{1/2} + \langle X^{nm} \rangle_t^{1/2} \langle W^n - X^{nm} \rangle_t^{1/2} \\ &\leq 2 \left( \sum_{k=1}^{k_n} \xi_k^n \right)^{1/2} \left( \sum_{k=m-1}^{k_n} \xi_k^n \right)^{1/2}. \end{aligned}$$

Therefore, by (3.4) and (3.5), we easily see that for all $\varepsilon > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} P^n(|\langle W^n \rangle_t - \langle X^{nm} \rangle_t| > \varepsilon) = 0. \tag{3.7}$$

Combining (3.7) and (3.3) we observe that for all $\varepsilon > 0$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} P^n(\langle W\rangle_t > \varepsilon) = 0.$$

This means that $W^n \xrightarrow{\mathscr{L}} 0$. By (3.6) and the conditions of Theorem 3.2, we have

$$Y^n \xrightarrow{\mathscr{L}} \int_0^{\cdot} \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x).$$

Hence, $X^n = Y^n + W^n \xrightarrow{\mathscr{L}} \int_0^{\cdot} \int_E f(s,x) M(\mathrm{d}s, \mathrm{d}x)$. □

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