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On an approximation problem for stochastic integrals where random time nets do not help

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Abstract

Given a geometric Brownian motion $S = (S_t)_{t\in[0,T]}$ and a Borel measurable function $g : (0,\infty) \to \mathbb{R}$ such that $g(S_T) \in L_2$, we approximate $g(S_T) - \mathbb{E}g(S_T)$ by

$$\sum_{i=1}^{n} v_{i-1}(S_{\tau_i} - S_{\tau_{i-1}})$$

where $0 = \tau_0 \leqslant \cdots \leqslant \tau_n = T$ is an increasing sequence of stopping times and the v_{i-1} are $\mathscr{F}_{\tau_{i-1}}$-measurable random variables such that $\mathbb{E}v_{i-1}^2(S_{\tau_i} - S_{\tau_{i-1}})^2 < \infty$ ($(\mathscr{F}_t)_{t\in[0,T]}$ is the augmentation of the natural filtration of the underlying Brownian motion). In case that g is not almost surely linear, we show that one gets a lower bound for the L_2-approximation rate of $1/\sqrt{n}$ if one optimizes over all nets consisting of $n+1$ stopping times. This lower bound coincides with the upper bound for all reasonable functions g in case deterministic time-nets are used. Hence random time nets do not improve the rate of convergence in this case. The same result holds true for the Brownian motion instead of the geometric Brownian motion.

Keywords: Approximation; Stochastic integrals; Random time nets

1. Introduction and result

The question, we are dealing with, arises from stochastic finance, where one is interested in the L_2-error which occurs while replacing a continuously adjusted portfolio by a

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discretely adjusted one. Assume a finite time horizon $T>0$ and a standard Brownian motion $B=(B_t)_{t\in[0,T]}$ defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $B_0\equiv 0$, continuous paths for all $\omega\in\Omega$, and $\mathcal{F}$ being the completion of $\sigma(B_t : t\in[0,T])$. Define $(\mathcal{F}_t)_{t\in[0,T]}$ to be the usual augmentation of the natural filtration generated by B and $S=(S_t)_{t\in[0,T]}$ to be the standard geometric Brownian motion

$$S_t := e^{B_t - t/2}.$$

Let us consider the discounted Black–Scholes model (with variance one for notational simplicity) and a Borel-measurable pay-off function $g:(0,\infty)\to\mathbb{R}$ such that $g(S_T)\in L_2$. If one wants to estimate the minimal quadratic hedging risk for $g(S_T)$, where the portfolio may be rebalanced at the time-knots $(\tau_i)_{i=0}^{n-1}$ coming from an increasing sequence of *stopping times*

$$0=\tau_0\leqslant\tau_1\leqslant\cdots\leqslant\tau_n=T,$$

then one is concerned with the optimization problem

$$\inf_{v_0,\dots,v_{n-1}}\left\|[g(S_T)-\mathbb{E}g(S_T)]-\sum_{i=1}^n v_{i-1}(S_{\tau_i}-S_{\tau_{i-1}})\right\|_{L_2}, \tag{1}$$

where the v_{i-1} are certain $\mathcal{F}_{\tau_{i-1}}$-measurable random variables (at first glance, one might replace $\mathbb{E}g(S_T)$ by $c\in\mathbb{R}$ in order to optimize over c as well, but one quickly checks that $c=\mathbb{E}g(S_T)$ is optimal because of the martingale setting). For equidistant nets questions of type (1) have been studied by Zhang [11], Gobet and Temam [6], and others. In [3], general deterministic not necessarily equidistant, nets were considered taking into account properties of g. In particular, it turned out that for each g, such that there are no $c_0,c_1\in\mathbb{R}$ with $g(S_T)=c_0+c_1S_T$ a.s., one has a lower rate of $1/\sqrt{n}$ for (1) if one optimizes over all deterministic time-nets of cardinality $n+1$ [3, Theorem 4.4, Lemma 4.9, Proof of Theorem 6.2]. Note, that $g(S_T)=c_0+c_1S_T$ a.s. implies a perfect approximation in (1).

The natural question arises what happens to the lower rate if we take random time-nets (in our understanding, always an increasing sequence of stopping times). It seems that the techniques from [3] do not apply in this case. On the other hand, Martini and Patry [10] identified the optimal strategy when one optimizes over random time-nets with a pre-given cardinality. Their included numerical example indicates an improvement of the approximation error by some factor compared to the case deterministic nets are used. However, a lower bound for the approximation rate was not considered. So the question was still open whether random time-nets improve the approximation rate. In the present paper, we give an answer to this problem as follows: firstly, one cannot achieve a rate better than $1/\sqrt{n}$, which is the same lower bound as for deterministic nets mentioned above. Secondly, for all reasonable g (see Theorem 1.2 and Remark 1.3) this lower bound is, up to a factor, the same as the upper bound obtained for deterministic nets. Hence one cannot take advantage from random time-nets in this case. To formulate our result we introduce, for a random variable $Z\in L_p$, $p\in[2,\infty)$, and $M=(M_t)_{t\in[0,T]}$ being either the Brownian motion $B=(B_t)_{t\in[0,T]}$ or the geometric Brownian motion $S=(S_t)_{t\in[0,T]}$, the approximation number

$$a_n^M(Z|L_p) := \inf\left\|[Z-\mathbb{E}Z]-\sum_{i=1}^n v_{i-1}(M_{\tau_i}-M_{\tau_{i-1}})\right\|_{L_p}, \tag{2}$$

where the infimum is taken over all sequences of stopping times $0 = \tau_0 \leqslant \tau_1 \leqslant \cdots \leqslant \tau_n = T$ and $\mathscr{F}_{\tau_{i-1}}$-measurable $v_{i-1} : \Omega \to \mathbb{R}$ with $v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}}) \in L_p$. As main result we get

Theorem 1.1. *Let M be either the Brownian motion B or the geometric Brownian motion S and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel function with $g(M_T) \in L_2$. If there are no constants $c_0, c_1 \in \mathbb{R}$ with $g(M_T) = c_0 + c_1 M_T$ a.s., then there is some $c > 0$ such that*

$$a_n^M(g(M_T)|L_2) \geqslant \frac{1}{c}\frac{1}{\sqrt{n}} \quad \text{for } n = 1, 2, \ldots .$$

The theorem is proved in Section 2. To discuss the upper bound $1/\sqrt{n}$ in some detail we need more notation. As before, let $M \in \{B, S\}$ and $g : \mathbb{R} \to \mathbb{R}$ be a Borel function with $g(M_T) \in L_2$. There is some $\varepsilon > 0$ such that

$$G(t,x) := \begin{cases} \mathbb{E}g(x + B_{T-t}) : M = B, \\ \mathbb{E}g(xS_{T-t}) : M = S \end{cases} \tag{3}$$

is a well-defined C^∞-function on $(-\varepsilon, T) \times \mathbb{R}$ and $(-\varepsilon, T) \times (0, \infty)$, respectively, (for a moment we extend B and S to $[0, T + \varepsilon]$) and satisfies there

$$\frac{\partial G}{\partial t} + \frac{\alpha^2}{2}\frac{\partial^2 G}{\partial x^2} = 0 \quad \text{with } \alpha(x) := \begin{cases} 1 : M = B, \\ x : M = S. \end{cases} \tag{4}$$

This is well-known where the argument for the extension by $(-\varepsilon, 0]$ can be found, for example, in [3, Lemma A.2]. By Itô's formula we deduce, as usual, that

$$g(M_T) = \mathbb{E}g(M_T) + \int_0^T \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u \text{ a.s.}$$

The following upper bound was proved in [3, Section 6].

Theorem 1.2. *Let $g : (0, \infty) \to \mathbb{R}$ be a Borel function such that $g(S_T) \in L_2$ and G be given by* (3) *for $M = S$. Assume that there is some $\theta \in [0, 1)$ such that*

$$\sup_{t \in [0,T)} (T-t)^\theta \left\| S_t^2 \frac{\partial^2 G}{\partial x^2}(t, S_t) \right\|_{L_2} < \infty. \tag{5}$$

Then there exists some $c > 0$ such that for each $n = 1, 2, \ldots$ there is a deterministic net $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} = T$ such that

$$\left\| [g(S_T) - \mathbb{E}g(S_T)] - \sum_{i=1}^n \frac{\partial G}{\partial x}(t_{i-1}^{(n)}, S_{t_{i-1}^{(n)}})(S_{t_i^{(n)}} - S_{t_{i-1}^{(n)}}) \right\|_{L_2} \leqslant \frac{c}{\sqrt{n}}.$$

Basic examples satisfying (5) are given in part (iii) of the following remark.

Remark 1.3. (i) If $g(S_T) \in L_2$, then $\mathbb{E}\sup_{t \leqslant b} |\frac{\partial G}{\partial x}(t, S_t)S_t|^2 < \infty$ for all $b \in [0, T)$ (cf. for example [3]) so that, in Theorem 1.2,

$$\mathbb{E}\left|\frac{\partial G}{\partial x}(t_{i-1}, S_{t_{i-1}})(S_{t_i} - S_{t_{i-1}})\right|^2 < \infty. \tag{6}$$

(ii) An analogue of Theorem 1.2 for $M = B$ follows from [3] as well (see [7]). Moreover, in [8] it is shown that $g(S_T) \in L_2$ without an additional assumption (like for example (5)) does not imply the conclusion of Theorem 1.2 (cf. Section 3).

(iii) Since $t \to \|S_t^2(\partial^2 G/\partial x^2)(t, S_t)\|_{L_2}$ is continuous and increasing on $[0, T)$ (see [2] where g is non-negative, which does not play any role for this assertion) it is not difficult to check that

$$\sup_{t\in[0,T)} (T-t)^\eta \int_0^t \left\| S_u^2 \frac{\partial^2 G}{\partial x^2}(u, S_u) \right\|_{L_2}^2 \mathrm{d}u < \infty \tag{7}$$

for some $\eta \in (0,1)$ implies (5) with $\theta = (\eta+1)/2 \in (1/2, 1)$ [1, Proof of Theorem 2.5]. Condition (7) is investigated in [1,5] in detail. For example, it is shown (under the normalization $T = 1$) in [5] that (7) is equivalent to

$$g(\exp(\cdot - 1/2)) \in (D_{1,2}(\mu), L_2(\mu))_{\eta,\infty},$$

where μ is the standard Gaussian measure on $\mathbb{R}$, $D_{1,2}(\mu)$ the Malliavin Sobolev space with respect to μ, and $(X_0, X_1)_{\eta,\infty}$ the real interpolation space with parameters (η, ∞) formed by the Banach spaces X_0 and X_1. This means, a minimal degree of smoothness of $h(x) = g(\exp(x - 1/2))$ implies (5) for some $\theta \in [0, 1)$. Basic examples for (5) are

$$\begin{aligned}
&g_1(x) := (x-K)^+ \quad \text{with } \theta = \tfrac{1}{4},\\
&g_2(x) := ((x-K)^+)^\alpha \quad \text{with } \theta = \frac{3}{4} - \frac{\alpha}{2},\\
&g_3(x) := \chi_{[K,\infty)}(x) \quad \text{with} \quad \theta = \frac{3}{4},\\
&g_4(x) := h_\alpha\left(\frac{T}{2} + \log x\right) \quad \text{with } \theta = \frac{3}{4} + \frac{\alpha}{2}
\end{aligned}$$

where $K > 0$, $\alpha \in (0, 1/2)$, and $h_\alpha(y) := y^{-\alpha}$ if $y > 0$ with $h_\alpha(y) := 0$ otherwise (see [11,6,3,1]).

(iv) The results in [3] are formulated for non-negative g because of their interpretation as pay-off function. The proofs are valid for general g, as used here, without modification.

The second upper bound, we want to recall, is taken from [4].

Theorem 1.4. *Let $g(y) := \int_0^y K(x)\,\mathrm{d}x$, $y \geqslant 0$, where $K : [0,\infty) \to \mathbb{R}$ is a Borel function integrable over compact intervals. Assume that $2 \leqslant p < q < \infty$* and

$$\mathbb{E}\left[\left| \int_0^{S_T} |K(x)|\,\mathrm{d}x \right|^2 + |K(S_T)|^q \right] < \infty. \tag{8}$$

Then there exists some $c > 0$ such that for $n = 1, 2, \ldots$ and $t_i^{(n)} := \mathrm{i}T/n$ one has that

$$\left\| [g(S_T) - \mathbb{E}g(S_T)] - \sum_{i=1}^n \frac{\partial G}{\partial x}(t_{i-1}^{(n)}, S_{t_{i-1}^{(n)}})(S_{t_i^{(n)}} - S_{t_{i-1}^{(n)}}) \right\|_{L_p} \leqslant \frac{c}{\sqrt{n}}.$$

The assumption of Theorem 1.4 is strictly stronger than that of Theorem 1.2. For example, g_1 from Remark 1.3(iii) falls into the setting of Theorem 1.4, but not g_2, g_3, and g_4. At the moment we do not see any major obstacles to adapt the setting of [4] to prove an analogue of Theorem 1.4 for the Brownian motion. However, this would exceed the scope of this paper and is not rigorously done yet. Combining Theorems 1.1, 1.2, and 1.4 we derive

Corollary 1.5. (i) *Let* $g : (0,\infty) \to \mathbb{R}$ *be as in Theorem* 1.2 *such that condition* (5) *is satisfied for some* $\theta \in [0,1)$. *If there are no constants* $c_0, c_1 \in \mathbb{R}$ *such that* $g(S_T) = c_0 + c_1 S_T$ *a.s.*, *then there is some* $c \geqslant 1$ *such that*

$$\frac{1}{c\sqrt{n}} \leqslant a_n^S(g(S_T)|L_2) \leqslant \frac{c}{\sqrt{n}}$$

for all $n = 1, 2, \ldots$. *The optimal rate is obtained by deterministic time-nets.*

(ii) *Let* $p \in [2,\infty)$ *and* $g(y) = \int_0^y K(x)\,\mathrm{d}x$ *be as in Theorem* 1.4 *such that condition* (8) *is satisfied for some* $q \in (p,\infty)$. *If* g *is not linear*, *then there is some* $c \geqslant 1$ *such that*

$$\frac{1}{c\sqrt{n}} \leqslant a_n^S(g(S_T)|L_p) \leqslant \frac{c}{\sqrt{n}}$$

for all $n = 1, 2, \ldots$. *The optimal rate is obtained by equidistant time-nets.*

Proof. Part (i) follows from Theorems 1.1 and 1.2. We turn to part (ii). Taking $n = 1$ in Theorem 1.4 gives $g(S_T) \in L_p$. Since g is not linear, but continuous, there do not exist constants $c_0, c_1 \in \mathbb{R}$ such that $g(S_T) = c_0 + c_1 S_T$ a.s. Consequently, Theorem 1.1 implies that

$$\frac{1}{c_{(1.1)}\sqrt{n}} \leqslant a_n^S(g(S_T)|L_2) \leqslant a_n^S(g(S_T)|L_p)$$

for all $n = 1, 2, \ldots$. Letting $t_i^{(n)} := \mathrm{i}T/n$ and

$$A_t^n := \sum_{i=1}^n \frac{\partial G}{\partial x}(t_{i-1}^{(n)}, S_{t_{i-1}^{(n)}})(S_{t_i^{(n)} \wedge t} - S_{t_{i-1}^{(n)} \wedge t})$$

we get a martingale $(A_t^n)_{t\in[0,T]}$ where one may use (6). Theorem 1.4 and $g(S_T) \in L_p$ imply $A_T^n \in L_p$ so that

$$\mathbb{E}\left|\frac{\partial G}{\partial x}(t_{i-1}^{(n)}, S_{t_{i-1}^{(n)}})(S_{t_i^{(n)} \wedge t} - S_{t_{i-1}^{n} \wedge t})\right|^p < \infty$$

for $t \in [0,T]$ and

$$a_n^S(g(S_T)|L_p) \leqslant \|[g(S_T) - \mathbb{E}g(S_T)] - A_T^n\|_{L_p} \leqslant \frac{c_{(1.4)}}{\sqrt{n}}. \qquad \square$$

2. Proof of Theorem 1.1

Before we turn to the Proof of Theorem 1.1 directly we start with some

Preparations: Sometimes we use $\mathbb{E}_\rho(\cdot) = \mathbb{E}(\cdot|\mathscr{F}_\rho)$ for ρ being a stopping time. To compute $a_n^M(Z|L_2)$ we recall that the optimal v_{i-1} are explicitly known once the time-net is chosen. In fact, for a sequence $\sigma = (\sigma_i)_{i=0}^n$ of stopping times $0 \leqslant \sigma_0 \leqslant \cdots \leqslant \sigma_n \leqslant T$ and $M \in \{B, S\}$, we exploit the Kunita–Watanabe type projection

$$P_\sigma^M : L_2 \to L_2 \quad \text{given by } P_\sigma^M Z := \sum_{i=1}^n v_{i-1}(\sigma, M)(M_{\sigma_i} - M_{\sigma_{i-1}})$$

with

$$v_{i-1}(\sigma, M) := \frac{\mathbb{E}(Z(M_{\sigma_i} - M_{\sigma_{i-1}})|\mathscr{F}_{\sigma_{i-1}})}{\mathbb{E}((M_{\sigma_i} - M_{\sigma_{i-1}})^2|\mathscr{F}_{\sigma_{i-1}})}\chi_{A_i}$$

and $A_i := \{\mathbb{E}((M_{\sigma_i} - M_{\sigma_{i-1}})^2 | \mathscr{F}_{\sigma_{i-1}}) \neq 0\}$ to get

$$\|Z - P_\sigma^M Z\|_{L_2} = \inf\left\{ \left\| Z - \sum_{i=1}^n v_{i-1}(M_{\sigma_i} - M_{\sigma_{i-1}}) \right\|_{L_2} \middle| \mathbb{E}v_{i-1}^2 (M_{\sigma_i} - M_{\sigma_{i-1}})^2 < \infty, v_{i-1} \text{ is } \mathscr{F}_{\sigma_{i-1}}\text{-measurable} \right\}.$$

In the Proof of Theorem 1.1, we want to restrict ourselves to sequences of stopping times $0 = \sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_n = T - \delta$, $\delta \in (0, T)$, with

$$\sup_{\omega, i} |\sigma_i(\omega) - \sigma_{i-1}(\omega)| \leqslant \frac{2T}{n}.$$

For this, we need the following two lemmas, where Lemma 2.2 confirms the intuition that the approximation is getting better when the time-net is refined.

Lemma 2.1. *Assume that* $\tau_0, \tau_1, \ldots, \tau_N : \Omega \to [0, T]$, $N \geqslant 1$, *are stopping times. Then*

$$\eta_l := \max\{\min\{\tau_{j_0}, \ldots, \tau_{j_{N-l}}\} | 0 \leqslant j_0 < \cdots < j_{N-l} \leqslant N\}$$

defines a sequence of stopping times $0 \leqslant \eta_0 \leqslant \eta_1 \leqslant \cdots \leqslant \eta_N \leqslant T$ *such that for all* $\omega \in \Omega$ *one has*

$$\{\tau_0(\omega), \ldots, \tau_N(\omega)\} = \{\eta_0(\omega), \ldots, \eta_N(\omega)\}.$$

The proof is obvious.

Lemma 2.2. *Let* $0 \leqslant \tau_0 \leqslant \cdots \leqslant \tau_n \leqslant T$ *and* $0 \leqslant \eta_0 \leqslant \cdots \leqslant \eta_N \leqslant T$ *be stopping times such that*

$$\{\tau_0(\omega), \ldots, \tau_n(\omega)\} \subseteq \{\eta_0(\omega), \ldots, \eta_N(\omega)\}$$

for all $\omega \in \Omega$. *Then, given* $Z \in L_2$ *and* $M \in \{B, S\}$, *one has that*

$$\inf \mathbb{E}\left(Z - \sum_{k=1}^N u_{k-1}(M_{\eta_k} - M_{\eta_{k-1}})\right)^2 \leqslant \inf \mathbb{E}\left(Z - \sum_{i=1}^n v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}})\right)^2$$

where the infima are taken over all $\mathscr{F}_{\eta_{k-1}}$*-measurable* u_{k-1} *and* $\mathscr{F}_{\tau_{i-1}}$*-measurable* v_{i-1} *such that*

$$\mathbb{E}u_{k-1}^2 (M_{\eta_k} - M_{\eta_{k-1}})^2 < \infty \quad \textit{and} \quad \mathbb{E}v_{i-1}^2 (M_{\tau_i} - M_{\tau_{i-1}})^2 < \infty.$$

Proof. Assume we are given v_{i-1}, $i = 1, \ldots, n$, as above. If we choose

$$u_{k-1} := \sum_{i=1}^n v_{i-1} \chi_{\{\tau_{i-1} \leqslant \eta_{k-1} < \tau_i\}}$$

for $k = 1, \ldots, N$, then it follows that u_{k-1} is $\mathscr{F}_{\eta_{k-1}}$-measurable. Since (η_k) is a refinement of (τ_i), it holds

$$\sum_{k=1}^N u_{k-1}(M_{\eta_k} - M_{\eta_{k-1}}) = \sum_{i=1}^n v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}}).$$

Moreover, one quickly checks that

$$\|u_{k-1}(M_{\eta_k} - M_{\eta_{k-1}})\|_{L_2} \leqslant \sum_{i=1}^{n} \|v_{i-1}(M_{\tau_i} - M_{\tau_{i-1}})\|_{L_2} < \infty. \qquad \square$$

Finally, the following lemma provides the necessary integrability properties (partially implicitly) needed in the Proof of Theorem 1.1.

Lemma 2.3. *Let $M \in \{B, S\}$. For a Borel function $g : \mathbb{R} \to \mathbb{R}$ such that $g(M_T) \in L_2$, $k, l \in \{0, 1, 2, \ldots\}$, $j \in \{1, 2\}$, and $b \in [0, T)$ one has that*

$$\mathbb{E} \sup_{0 \leqslant s \leqslant t \leqslant b} |M_t|^k |M_s|^l \left(\frac{\partial^j G}{\partial x^j}(s, M_s) \right)^2 < \infty,$$

where G is given by (3) *and* $0^0 := 1$.

Proof. For α defined in (4) and $1 < p, q < \infty$ with $1 = (1/p) + (1/q)$ we get

$$\begin{aligned}
&\mathbb{E} \sup_{0 \leqslant s \leqslant t \leqslant b} |M_t|^k |M_s|^l \left(\frac{\partial^j G}{\partial x^j}(s, M_s) \right)^2 \\
&\leqslant \left(\mathbb{E} \sup_{0 \leqslant s \leqslant t \leqslant b} \left| \frac{|M_t|^k |M_s|^l}{\alpha(M_s)^{2j}} \right|^p \right)^{1/p} \left(\mathbb{E} \sup_{0 \leqslant s \leqslant b} \left| \alpha(M_s)^j \, \frac{\partial^j G}{\partial x^j} \, (s, M_s) \right|^{2q} \right)^{1/q} \\
&\leqslant \left(\mathbb{E} \sup_{0 \leqslant s \leqslant b} \left| \frac{|M_s|^l}{\alpha(M_s)^{2j}} \right|^{2p} \right)^{1/2p} \left(\mathbb{E} \sup_{0 \leqslant t \leqslant b} |M_t|^{2kp} \right)^{1/2p} \\
&\quad \times \left(\mathbb{E} \sup_{0 \leqslant s \leqslant b} \left| \alpha(M_s)^j \frac{\partial^j G}{\partial x^j}(s, M_s) \right|^{2q} \right)^{1/q}
\end{aligned}$$

by Hölder's inequality. It is known that the first two factors are finite for all $1 < p < \infty$. Hence, we have to find a $1 < q < \infty$ such that the third factor is finite as well. We indicate the argument, but leave out some details because it should be standard. First we write

$$\alpha(x)^j \, \frac{\partial^j G}{\partial x^j} \, (s, x) = \overline{\mathbb{E}} \gamma(x, \overline{M}_{T-s}) p_s^{j,M}(\overline{B}_{T-s})$$

with $\gamma(x, y) := g(x + y)$ for $M = B$ and $\gamma(x, y) := g(xy)$ for $M = S$, where $\overline{M}$ and $\overline{B}$ are independent copies of M and B, and

$$p_s^{1,B}(\xi) = p_s^{1,S}(\xi) := \frac{\xi}{T - s},$$

$$p_s^{2,B}(\xi) := \frac{\xi^2}{(T - s)^2} - \frac{1}{T - s},$$

$$p_s^{2,S}(\xi) := \frac{\xi^2}{(T - s)^2} - \frac{\xi}{T - s} - \frac{1}{T - s},$$

see [9] and [3, Lemmas A.1 and A.2]. Letting $1<\beta'<2<\beta<\infty$ with $1=(1/\beta')+(1/\beta)$, we obtain

$$\mathbb{E}\sup_{0\leqslant s\leqslant b}\left|\alpha(M_s)^j\frac{\partial^j G}{\partial x^j}(s,M_s)\right|^{2q}\leqslant \sup_{0\leqslant s\leqslant b}(\overline{\mathbb{E}}|p_s^{j,M}(\overline{B}_{T-s})|^{\beta})^{2q/\beta} \times\mathbb{E}\sup_{0\leqslant s\leqslant b}(\overline{\mathbb{E}}|\gamma(M_s,\overline{M}_{T-s})|^{\beta'})^{2q/\beta'}.$$

Consequently, it suffices to verify that

$$\mathbb{E}\sup_{0\leqslant s\leqslant b}(\overline{\mathbb{E}}|\gamma(M_s,\overline{M}_{T-s})|^{\beta'})^{2q/\beta'}=\mathbb{E}\sup_{0\leqslant s\leqslant b}|\mathbb{E}(|g(M_T)|^{\beta'}|\mathscr{F}_s)|^{2q/\beta'}$$

is finite for an appropriate $1<q<\infty$. But this can be obtained by Doob's maximal inequality and the hyper-contraction property of the Ornstein–Uhlenbeck semi-group (the latter yields for $b\in[0,T)$ and a Borel function $h:\mathbb{R}\to\mathbb{R}$ such that $h(B_T)\in L_r$ for some $1<r<\infty$ some $r'\in(r,\infty)$ such that $\mathbb{E}(h(B_T)|\mathscr{F}_b)\in L_{r'}$). $\square$

Proof of Theorem 1.1. The strategy of our proof is as follows: after some preparations in step (a) we expand, in step (b), the integrand of the stochastic integral to be approximated into a zero and first order term ($h_0^1+h_1^1$ and $h_0^2+h_1^2$, respectively) and a corresponding remainder (h_2^1 and $h_2^2+h_3^2$, respectively). The Fact 2.4 shows that the dominating part in the approximation is the first order term. This leads to a lower bound for the approximation error under condition (9) for the time-nets. Condition (9) will be removed in step (c). Step (d) concludes the proof by verifying that the constant involved in the lower bound obtained in step (c) is positive when g is not almost surely linear.

(a) Let us first assume $\delta\in(0,T)$, $n\in\{1,2,\ldots\}$ with $n\geqslant 12T$, and a sequence of stopping times

$$0=\sigma_1\leqslant\cdots\leqslant\sigma_n=T-\delta\quad\text{such that}\quad\sup_{\omega,i}|\sigma_i(\omega)-\sigma_{i-1}(\omega)|\leqslant\frac{2T}{n}. \tag{9}$$

Recall that G is given by (3). By the Kunita–Watanabe projection we know that the optimal v_i in

$$\inf\left\{\left\|\int_0^{T-\delta}\frac{\partial G}{\partial x}(u,M_u)\,\mathrm{d}M_u-\sum_{i=1}^n v_{i-1}(M_{\sigma_i}-M_{\sigma_{i-1}})\right\|_{L_2}\right| \mathbb{E}v_{i-1}^2(M_{\sigma_i}-M_{\sigma_{i-1}})^2<\infty, v_{i-1}\text{ is }\mathscr{F}_{\sigma_{i-1}}\text{-measurable}\right\}$$

are given by

$$v_{i-1}^g(\sigma,M):=\frac{\mathbb{E}\left(\int_0^{T-\delta}\frac{\partial G}{\partial x}(u,M_u)\,\mathrm{d}M_u(M_{\sigma_i}-M_{\sigma_{i-1}})|\mathscr{F}_{\sigma_{i-1}}\right)}{\mathbb{E}((M_{\sigma_i}-M_{\sigma_{i-1}})^2|\mathscr{F}_{\sigma_{i-1}})}\chi_{A_i}$$

with $A_i:=\{\mathbb{E}((M_{\sigma_i}-M_{\sigma_{i-1}})^2|\mathscr{F}_{\sigma_{i-1}})\neq 0\}$. It should be noted that we may replace in $v_{i-1}^g(\sigma,M)$ the term $\int_0^{T-\delta}(\partial G/\partial x)(u,M_u)\,\mathrm{d}M_u$ by

$$\int_0^T\frac{\partial G}{\partial x}(u,M_u)\,\mathrm{d}M_u\quad\text{or}\quad\int_{\sigma_{i-1}}^{\sigma_i}\frac{\partial G}{\partial x}(u,M_u)\,\mathrm{d}M_u$$

so that $v_{i-1}^{g}(\sigma, M)$ only depends on σ_{i-1} and σ_i but not on whole time net and not on $T-\delta$ (as long as $\sigma_i \leqslant T-\delta$).

(b) Now we decompose $(\partial G/\partial x)(t, M_t)$. This is done differently in the case of the Brownian motion and the geometric Brownian motion. In order to distinguish between the two cases we denote G in the case of the Brownian motion by G_1 and in the case of the geometric Brownian motion by G_2. From (4) it follows that

$$\frac{\partial^2 G_1}{\partial x \partial t} + \frac{1}{2}\frac{\partial^3 G_1}{\partial x^3} = 0 \quad \text{and} \quad \frac{\partial^2 G_2}{\partial x \partial t} + \frac{x^2}{2}\frac{\partial^3 G_2}{\partial x^3} + x\frac{\partial^2 G_2}{\partial x^2} = 0$$

on $[0,T) \times \mathbb{R}$ and $[0,T) \times (0,\infty)$, respectively. For $0 \leqslant s \leqslant t < T$ Itô's formula yields, a.s.,

$$\begin{aligned}
\frac{\partial G_1}{\partial x}(t, B_t) &= \frac{\partial G_1}{\partial x}(s, B_s) + \int_s^t \frac{\partial^2 G_1}{\partial x^2}(u, B_u)\,\mathrm{d}B_u \\
&= \frac{\partial G_1}{\partial x}(s, B_s) + \frac{\partial^2 G_1}{\partial x^2}(s, B_s)(B_t - B_s) \\
&\quad + \left[\int_s^t \frac{\partial^2 G_1}{\partial x^2}(u, B_u)\,\mathrm{d}B_u - \frac{\partial^2 G_1}{\partial x^2}(s, B_s)(B_t - B_s)\right] \\
&=: (h_0^1 + h_1^1 + h_2^1)(s,t)
\end{aligned}$$

and, a.s.,

$$\begin{aligned}
\frac{\partial G_2}{\partial x}(t, S_t) &= \frac{\partial G_2}{\partial x}(s, S_s) + \int_s^t \frac{\partial^2 G_2}{\partial x^2}(u, S_u)\,\mathrm{d}S_u - \int_s^t S_u \frac{\partial^2 G_2}{\partial x^2}(u, S_u)\,\mathrm{d}u \\
&= \frac{\partial G_2}{\partial x}(s, S_s) + \frac{\partial^2 G_2}{\partial x^2}(s, S_s)(S_t - S_s) \\
&\quad + \left[\int_s^t \frac{\partial^2 G_2}{\partial x^2}(u, S_u)\,\mathrm{d}S_u - \frac{\partial^2 G_2}{\partial x^2}(s, S_s)(S_t - S_s)\right] \\
&\quad - \int_s^t S_u \frac{\partial^2 G_2}{\partial x^2}(u, S_u)\,\mathrm{d}u \\
&=: (h_0^2 + h_1^2 + h_2^2 + h_3^2)(s,t).
\end{aligned}$$

We obtain two-parameter processes $(h_i^k(s,t))_{(s,t)\in\Delta}$ with index-set

$$\Delta := \{(s,t) \mid 0 \leqslant s \leqslant t < T\}$$

such that $h_i^k(s,t)$ is $\mathscr{F}_t$-measurable and where we may suppose that all trajectories are continuous on Δ. Assume stopping times $0 \leqslant \sigma \leqslant \tau < T$ and that h is one of the above h_i^k. Defining $Z = (Z_u)_{u\in[0,T]}$ by

$$Z_u := h(\sigma, u)\chi_{\{\sigma < u \leqslant \tau\}}$$

we get that Z is adapted and that all trajectories are left side continuous and have right limits. Now we estimate

$$P(h(\sigma,\cdot), M; \sigma, \tau) := \mathbb{E}_\sigma\left(\int_\sigma^\tau Z_u\,\mathrm{d}M_u - \frac{\mathbb{E}_\sigma(\int_\sigma^\tau Z_v\,\mathrm{d}M_v(M_\tau - M_\sigma))}{\mathbb{E}_\sigma(M_\tau - M_\sigma)^2}\chi_A(M_\tau - M_\sigma)\right)^2$$

with $A := \{\mathbb{E}_\sigma(M_\tau - M_\sigma)^2 \neq 0\}$.

Fact 2.4. For $\delta \in (0,T)$, $\varepsilon \in (0,1/3)$, and stopping times $0 \leqslant \sigma \leqslant \tau \leqslant T-\delta$ with $\tau - \sigma \leqslant \varepsilon$ one has

$$\mathbb{E} \int_\sigma^\tau h(\sigma,u)^2 \alpha(M_u)^2 \, du < \infty$$

if $h = h_i^1$, $M = B$, $\alpha(x) = 1$ or $h = h_i^2$, $M = S$, and $\alpha(x) = x$. Moreover, a.s. it holds that

(i) $P(h_1(\sigma,\cdot), M; \sigma, \tau) \geqslant \frac{1}{c}(\frac{\partial^2 G}{\partial x^2}(\sigma, M_\sigma))^2 \mathbb{E}_\sigma(\langle M\rangle_\tau - \langle M\rangle_\sigma)^2$,

(ii) $P(h_2^1(\sigma,\cdot), B; \sigma, \tau) \leqslant 4\varepsilon^2 \mathbb{E}_\sigma \sup_{\sigma\leqslant v\leqslant\tau} (\frac{\partial^2 G_1}{\partial x^2}(v, B_v) - \frac{\partial^2 G_1}{\partial x^2}(\sigma, B_\sigma))^2$,

(iii) $P(h_2^2(\sigma,\cdot), S; \sigma, \tau) \leqslant \frac{3\varepsilon^2}{1-3\varepsilon} \mathbb{E}_\sigma \sup_{\sigma\leqslant v\leqslant\tau} (\frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma, S_\sigma))^2 S_v^4$,

(iv) $P(h_3^2(\sigma,\cdot), S; \sigma, \tau) \leqslant \varepsilon^3 \mathbb{E}_\sigma \sup_{\sigma\leqslant v\leqslant u\leqslant\tau} (S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v, S_v))^2$,

where $c > 0$ is an absolute constant, $h_1 := h_1^1$ if $M = B$, $h_1 := h_1^2$ if $M = S$, and $\langle M\rangle_t = \int_0^t \alpha(M_u)^2 \, du$.

The basic reason for the lower estimate in Theorem 1.1 is the lower estimate from the above item (i). We postpone the proof of the fact and see first how we can use it. From $\sup_i |\sigma_i - \sigma_{i-1}| \leqslant \frac{2T}{n} =: \varepsilon < 1/3$ $(n \geqslant 12T)$, $P(h_0^2(\sigma_{i-1},\cdot), S; \sigma_{i-1}, \sigma_i) = 0$ a.s., $(a+b+c+d)^2 \geqslant (a^2/2) - 4(b^2+c^2+d^2)$, and Fact 2.4 we derive that

$$\begin{aligned}
& n\left\| \int_0^{T-\delta} \frac{\partial G_2}{\partial x}(u, S_u)\, dS_u - \sum_{i=1}^n v_{i-1}^g(\sigma, S)(S_{\sigma_i} - S_{\sigma_{i-1}}) \right\|_{L_2}^2 \\
& \geqslant n \sum_{i=1}^n \mathbb{E}\left[\frac{1}{2} P(h_1^2(\sigma_{i-1},\cdot), S; \sigma_{i-1}, \sigma_i) - 4P(h_2^2(\sigma_{i-1},\cdot), S; \sigma_{i-1}, \sigma_i) \right. \\
& \quad \left. -4P(h_3^2(\sigma_{i-1},\cdot), S; \sigma_{i-1}, \sigma_i) - 4P(h_0^2(\sigma_{i-1},\cdot), S; \sigma_{i-1}, \sigma_i) \right] \\
& \geqslant \frac{n}{2c} \mathbb{E}\left[\sum_{i=1}^n \left(\frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 \mathbb{E}_{\sigma_{i-1}}(\langle S\rangle_{\sigma_i} - \langle S\rangle_{\sigma_{i-1}})^2 \right] \\
& \quad - n^2 \frac{12(2T/n)^2}{1-3(2T/n)} \mathbb{E} \sup_i \sup_{\sigma_{i-1}\leqslant v\leqslant\sigma_i} \left(\frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 S_v^4 \\
& \quad - 4n^2 \left(\frac{2T}{n} \right)^3 \mathbb{E} \sup_i \sup_{\sigma_{i-1}\leqslant v\leqslant u\leqslant\sigma_i} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v, S_v) \right)^2 \\
& \geqslant \frac{1}{2c} \mathbb{E}\left[\sum_{i=1}^n \left| \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right| (\langle S\rangle_{\sigma_i} - \langle S\rangle_{\sigma_{i-1}}) \right]^2 \\
& \quad - 96T^2 \mathbb{E} \sup_i \sup_{\sigma_{i-1}\leqslant v\leqslant\sigma_i} \left(\frac{\partial^2 G_2}{\partial x^2}(v, S_v) - \frac{\partial^2 G_2}{\partial x^2}(\sigma_{i-1}, S_{\sigma_{i-1}}) \right)^2 S_v^4 \\
& \quad - \frac{32T^3}{n} \mathbb{E} \sup_{0\leqslant v\leqslant u\leqslant T-\delta} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v, S_v) \right)^2,
\end{aligned}$$

where we have used that $n \geqslant 12T$. Assuming a sequence of stopping times $\sigma^{(n)} = (\sigma_i^{(n)})_{i=0}^n$ satisfying condition (9), we get by Lemma 2.3 and Lebesgue's dominated convergence that the second and the third term are converging to zero as $n \to \infty$, so that

$$\begin{aligned}
&\liminf_{n\to\infty} \sqrt{n} \left\| \int_0^{T-\delta} \frac{\partial G_2}{\partial x}(u, S_u)\,\mathrm{d}S_u - \sum_{i=1}^n v_{i-1}^{(n)}(S_{\sigma_i^{(n)}} - S_{\sigma_{i-1}^{(n)}}) \right\|_{L_2} \\
&\geqslant \sqrt{\frac{1}{2c}} \left\| \int_0^{T-\delta} \left| \frac{\partial^2 G_2}{\partial x^2}(u, S_u) \right| \mathrm{d}\langle S \rangle_u \right\|_{L_2}
\end{aligned}$$

by Fatou's lemma with $v_{i-1}^{(n)} := v_{i-1}^g(\sigma^{(n)}, S)$. In the same way one shows

$$\begin{aligned}
&\liminf_{n\to\infty} \sqrt{n} \left\| \int_0^{T-\delta} \frac{\partial G_1}{\partial x}(u, B_u)\,\mathrm{d}B_u - \sum_{i=1}^n v_{i-1}^{(n)}(B_{\sigma_i^{(n)}} - B_{\sigma_{i-1}^{(n)}}) \right\|_{L_2} \\
&\geqslant \sqrt{\frac{1}{2c}} \left\| \int_0^{T-\delta} \left| \frac{\partial^2 G_1}{\partial x^2}(u, B_u) \right| \mathrm{d}u \right\|_{L_2}
\end{aligned}$$

for $v_{i-1}^{(n)} := v_{i-1}^g(\sigma^{(n)}, B)$.

(c) Now take sequences of stopping times $\tau^{(n)} = (\tau_i^{(n)})_{i=0}^n$ with

$$0 = \tau_0^{(n)} \leqslant \cdots \leqslant \tau_n^{(n)} = T.$$

Stopping additionally at $\frac{kT}{n}$, $k = 1, \ldots, n-1$, we get a new sequence $\eta^{(2n-1)} = (\eta_k^{(2n-1)})_{k=0}^{2n-1}$ according to Lemma 2.1. Taking $\delta \in (0, T)$ and $\sigma_k^{(2n-1)} := \eta_k^{(2n-1)} \wedge (T - \delta)$ we get sequences of stopping times $\sigma^{(2n-1)} = (\sigma_k^{(2n-1)})_{k=0}^{2n-1}$ with

$$0 = \sigma_0^{(2n-1)} \leqslant \cdots \leqslant \sigma_{2n-1}^{(2n-1)} = T - \delta$$

and

$$\sup_{\omega,k} |\sigma_k^{(2n-1)}(\omega) - \sigma_{k-1}^{(2n-1)}(\omega)| \leqslant \frac{T}{n} \leqslant \frac{2T}{2n-1}$$

which is condition (9). By Lemma 2.2 and step (b) we derive

$$\begin{aligned}
&\liminf_n \sqrt{n} \left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u - \sum_{i=1}^n v_{i-1}^g(\tau^{(n)}, M)(M_{\tau_i^{(n)}} - M_{\tau_{i-1}^{(n)}}) \right\|_{L_2} \\
&\geqslant \liminf_n \sqrt{n} \left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u - \sum_{i=1}^{2n-1} v_{i-1}^g(\eta^{(2n-1)}, M)(M_{\eta_i^{(2n-1)}} - M_{\eta_{i-1}^{(2n-1)}}) \right\|_{L_2} \\
&\geqslant \liminf_n \sqrt{n} \left\| \int_0^{T-\delta} \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u - \sum_{i=1}^{2n-1} v_{i-1}^g(\sigma^{(2n-1)}, M)(M_{\sigma_i^{(2n-1)}} - M_{\sigma_{i-1}^{(2n-1)}}) \right\|_{L_2} \\
&\geqslant \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2c}} \left\| \int_0^{T-\delta} \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| \mathrm{d}\langle M \rangle_u \right\|_{L_2}.
\end{aligned}$$

(The last term is finite because of Lemma 2.3.)

(d) Assuming

$$\left\| \int_0^{T-\delta} \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| \mathrm{d}\langle M \rangle_u \right\|_{L_2} = 0$$

would imply $(\partial^2 G/\partial x^2)(T_0, M_{T_0}) = 0$ a.s. for some $T_0 \in (0, T)$ and, by the arguments of [3, Lemma 4.8], the existence of constants $c_0, c_1 \in \mathbb{R}$ such that $g(M_T) = c_0 + c_1 M_T$ a.s. But this is a contradiction to our assumption. Consequently,

$$\left\| \int_0^{T-\delta} \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| \mathrm{d}\langle M \rangle_u \right\|_{L_2} > 0. \tag{10}$$

To derive Theorem 1.1 it remains to observe that inequality (10) guarantees that $a_n^M(g(M_T)|L_2) > 0$ for all n: in fact, in this case we find $\tau^{(n)} = (\tau_i^{(n)})_{i=0}^n$ which realize $a_n^M(g(M_T)|L_2)$ up to a factor $1 + \varepsilon$ with $\varepsilon > 0$ and apply step (c) to these nets to end up with the conclusion of the theorem.

Assuming now $a_{n_0}^M(g(M_T)|L_2) = 0$ for some n_0 would give nets $\tau^{(l)}$, $0 = \tau_0^{(l)} \leqslant \cdots \leqslant \tau_{n_0}^{(l)} = T$, with

$$\left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u - \sum_{i=1}^{n_0} v_{i-1}^g(\tau^{(l)}, M)(M_{\tau_i^{(l)}} - M_{\tau_{i-1}^{(l)}}) \right\|_{L_2} \leqslant \frac{1}{2^l}$$

for $l = 1, 2, \ldots$. Stopping again additionally at $\frac{kT}{l}$, $k = 1, \ldots, l-1$, and finally at $T - \delta$ as in step (c), we get a new sequence $\sigma^{(l)}$ with

$$0 = \sigma_0^{(l)} \leqslant \cdots \leqslant \sigma_{n_0+l-1}^{(l)} = T - \delta$$

and $|\sigma_k^{(l)}(\omega) - \sigma_{k-1}^{(l)}(\omega)| \leqslant T/l$ for all k and ω. Repeating steps (b) and (c) gives

$$\begin{aligned}
0 &= \liminf_{l\to\infty} \frac{\sqrt{n_0 + l - 1}}{2^l} \\
&\geqslant \liminf_{l\to\infty} \sqrt{n_0 + l - 1} \left\| \int_0^T \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u \right. \\
&\quad \left. - \sum_{i=1}^{n_0} v_{i-1}^g(\tau^{(l)}, M)(M_{\tau_i^{(l)}} - M_{\tau_{i-1}^{(l)}}) \right\|_{L_2} \\
&\geqslant \liminf_{l\to\infty} \sqrt{n_0 + l - 1} \left\| \int_0^{T-\delta} \frac{\partial G}{\partial x}(u, M_u)\,\mathrm{d}M_u \right. \\
&\quad \left. - \sum_{i=1}^{n_0+l-1} v_{i-1}^g(\sigma^{(l)}, M)(M_{\sigma_i^{(l)}} - M_{\sigma_{i-1}^{(l)}}) \right\|_{L_2} \\
&\geqslant \sqrt{\frac{1}{2c}} \left\| \int_0^{T-\delta} \left| \frac{\partial^2 G}{\partial x^2}(u, M_u) \right| \mathrm{d}\langle M \rangle_u \right\|_{L_2}
\end{aligned}$$

which contradicts (10). □

Proof of Fact 2.4. (i) First we remark that

$$\begin{aligned}
&\mathbb{E}\int_\sigma^\tau h_1(\sigma,u)^2\alpha(M_u)^2\,\mathrm{d}u \\
&\quad = \mathbb{E}\int_\sigma^\tau \left(\frac{\partial^2 G}{\partial x^2}(\sigma, M_\sigma)(M_u - M_\sigma)\right)^2 \alpha(M_u)^2\,\mathrm{d}u < \infty,
\end{aligned}$$

where one can use Lemma 2.3. Moreover, it is easy to see that it is enough to prove assertion (i) with $(\partial^2 G/\partial x^2)(\sigma, M_\sigma)$ replaced by 1 on both sides. By Itô's formula we get

$$\mathbb{E}_\sigma(M_\tau - M_\sigma)^3 = 3\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\,\mathrm{d}\langle M\rangle_u \text{ a.s.}$$

and

$$\mathbb{E}_\sigma(M_\tau - M_\sigma)^4 = 6\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\,\mathrm{d}\langle M\rangle_u \text{ a.s.}$$

Consequently, by Hölder's inequality,

$$\begin{aligned}
\left(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\,\mathrm{d}\langle M\rangle_u\right)^2 &= \frac{1}{9}\,(\mathbb{E}_\sigma(M_\tau - M_\sigma)^3)^2 \\
&\leqslant \frac{1}{9}\,\mathbb{E}_\sigma(M_\tau - M_\sigma)^2\mathbb{E}_\sigma(M_\tau - M_\sigma)^4 \\
&= \frac{2}{3}\,\mathbb{E}_\sigma(M_\tau - M_\sigma)^2\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\mathrm{d}\langle M\rangle_u \text{ a.s.}
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{3}\,\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\,\mathrm{d}\langle M\rangle_u \leqslant \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\,\mathrm{d}\langle M\rangle_u \\
- \frac{(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\,\mathrm{d}\langle M\rangle_u)^2}{\mathbb{E}_\sigma(M_\tau - M_\sigma)^2}\chi_A \text{ a.s.}
\end{aligned}$$

where $A := \{\mathbb{E}_\sigma(M_\tau - M_\sigma)^2 \neq 0\}$. On the other hand, the Burkholder–Davis–Gundy and Doob's maximal inequality give, a.s.,

$$\begin{aligned}
\frac{1}{c}\,\mathbb{E}_\sigma(\langle M\rangle_\tau - \langle M\rangle_\sigma)^2 \leqslant \mathbb{E}_\sigma \sup_{\sigma\leqslant u\leqslant\tau} (M_u - M_\sigma)^4 \leqslant d\,\mathbb{E}_\sigma(M_\tau - M_\sigma)^4 \\
= 6d\,\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\mathrm{d}\langle M\rangle_u
\end{aligned}$$

for absolute $c, d > 0$ so that

$$\begin{aligned}
\frac{1}{18cd}\,\mathbb{E}_\sigma(\langle M\rangle_\tau - \langle M\rangle_\sigma)^2 \leqslant \mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)^2\mathrm{d}\langle M\rangle_u \\
- \frac{(\mathbb{E}_\sigma \int_\sigma^\tau (M_u - M_\sigma)\,\mathrm{d}\langle M\rangle_u)^2}{\mathbb{E}_\sigma(M_\tau - M_\sigma)^2}\chi_A \text{ a.s.}
\end{aligned}$$

and the assertion follows.

Item (ii) is left to the reader so that we turn to item

(iii) We let $A(u) := \frac{\partial^2 G_2}{\partial x^2}(u, S_u) - \frac{\partial^2 G_2}{\partial x^2}(\sigma \wedge u, S_{\sigma\wedge u})$ for $u \in [0, T)$ and obtain $\mathbb{E}\int_0^{T-\delta} A(u)^2 S_u^2 \, du < \infty$ by Lemma 2.3. For $N \in \{1, 2, \ldots\}$ we define

$$\tau_N := \inf\left\{ u \in [\sigma, \tau] \middle| \left| \int_\sigma^{\sigma\vee u} A(v)\, dS_v \right| > N \right\} \wedge \tau,$$

where $\inf \emptyset := \infty$, and estimate $\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee u} A(v)\, dS_v\right]^2 S_u^2 \, du$ from above. We have that

$$\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee u} A(v)\, dS_v\right]^2 S_u^2 \, du \leqslant \mathbb{E}_\sigma \int_0^{\varepsilon} \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)\, dS_v\right]^2 S_{(\sigma+u)\wedge\tau_N}^2 \, du \text{ a.s.}$$

For $u \in [0, T]$ Itô's formula implies, a.s.,

$$\begin{aligned}
&\mathbb{E}_\sigma \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)\, dS_v\right]^2 S_{(\sigma+u)\wedge\tau_N}^2 \\
&\quad = \mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} A(t)^2 S_t^4 \, dt + \mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left[\int_\sigma^{\sigma\vee t} A(v)\, dS_v\right]^2 S_t^2 \, dt \\
&\qquad + 4\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left[\int_\sigma^{\sigma\vee t} A(v)\, dS_v\right] A(t) S_t^3 \, dt \\
&\quad \leqslant 3\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} A(t)^2 S_t^4 \, dt + 3\,\mathbb{E}_\sigma \int_\sigma^{(\sigma+u)\wedge\tau_N} \left[\int_\sigma^{\sigma\vee t} A(v)\, dS_v\right]^2 S_t^2 \, dt \\
&\quad \leqslant 3\mathbb{E}_\sigma \int_\sigma^{\tau_N} A(t)^2 S_t^4 \, dt + 3\,\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee t} A(v)\, dS_v\right]^2 S_t^2 \, dt
\end{aligned}$$

where we used $|ab| \leqslant (1/2)(a^2 + b^2)$. As a result, a.s.,

$$\begin{aligned}
&\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee u} A(v)\, dS_v\right]^2 S_u^2 \, du \\
&\quad \leqslant \mathbb{E}_\sigma \int_0^{\varepsilon} \left[\int_\sigma^{(\sigma+u)\wedge\tau_N} A(v)\, dS_v\right]^2 S_{(\sigma+u)\wedge\tau_N}^2 \, du \\
&\quad \leqslant 3\varepsilon\mathbb{E}_\sigma \int_\sigma^{\tau_N} A(t)^2 S_t^4 \, dt + 3\varepsilon\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee t} A(v)\, dS_v\right]^2 S_t^2 \, dt,
\end{aligned}$$

which implies, a.s.,

$$\begin{aligned}
\mathbb{E}_\sigma \int_\sigma^{\tau_N} \left[\int_\sigma^{\sigma\vee u} A(v)\, dS_v\right]^2 S_u^2 \, du &\leqslant \frac{3\varepsilon}{1-3\varepsilon}\, \mathbb{E}_\sigma \int_\sigma^{\tau_N} A(t)^2 S_t^4 \, dt \\
&\leqslant \frac{3\varepsilon^2}{1-3\varepsilon}\, \mathbb{E}_\sigma \sup_{\sigma\leqslant t\leqslant\tau} A(t)^2 S_t^4
\end{aligned}$$

where we remark that Lemma 2.3 ensures that

$$\mathbb{E} \sup_{\sigma\leqslant t\leqslant\tau} A(t)^2 S_t^4 < \infty.$$

Letting $N \to \infty$ implies $\mathbb{E}\int_\sigma^\tau [\int_\sigma^{\sigma\vee u} A(v)\,\mathrm{d}S_v]^2 S_u^2\,\mathrm{d}u < \infty$ and

$$\mathbb{E}_\sigma \int_\sigma^\tau \left[\int_\sigma^{\sigma\vee u} A(v)\,\mathrm{d}S_v\right]^2 S_u^2\,\mathrm{d}u \leqslant \frac{3\varepsilon^2}{1-3\varepsilon}\,\mathbb{E}_\sigma \sup_{\sigma\leqslant t\leqslant\tau} A(t)^2 S_t^4 \text{ a.s.}$$

Finally, from $h_2^2(\sigma,\sigma\vee u) = \int_\sigma^{\sigma\vee u} A(v)\,\mathrm{d}S_v$, $u\in[0,T-\delta]$, a.s. we derive that $\mathbb{E}\int_\sigma^\tau [h_2^2]^2(\sigma,u) S_u^2\,\mathrm{d}u < \infty$ and, a.s.,

$$\begin{aligned} P(h_2^2(\sigma,\cdot),S;\sigma,\tau) &\leqslant \mathbb{E}_\sigma \int_\sigma^\tau [h_2^2]^2(\sigma,u) S_u^2\,\mathrm{d}u \\ &= \mathbb{E}_\sigma \int_\sigma^\tau \left[\int_\sigma^{\sigma\vee u} A(v)\,\mathrm{d}S_v\right]^2 S_u^2\,\mathrm{d}u \\ &\leqslant \frac{3\varepsilon^2}{1-3\varepsilon}\,\mathbb{E}_\sigma \sup_{\sigma\leqslant t\leqslant\tau} A(t)^2 S_t^4. \end{aligned}$$

(iv) The last inequality follows from, a.s.,

$$\begin{aligned} P(h_3^2(\sigma,\cdot),S;\sigma,\tau)) &\leqslant \mathbb{E}_\sigma \int_\sigma^\tau [h_3^2]^2(\sigma,u) S_u^2\,\mathrm{d}u \\ &= \mathbb{E}_\sigma \int_\sigma^\tau \left[\int_\sigma^u S_v \frac{\partial^2 G_2}{\partial x^2}(v,S_v)\,\mathrm{d}v\right]^2 S_u^2\,\mathrm{d}u \\ &\leqslant \varepsilon\mathbb{E}_\sigma \int_\sigma^\tau \int_\sigma^u \left(\frac{\partial^2 G_2}{\partial x^2}(v,S_v)\right)^2 S_v^2\,\mathrm{d}v S_u^2\,\mathrm{d}u \\ &\leqslant \varepsilon^3 \mathbb{E}_\sigma \sup_{\sigma\leqslant v\leqslant u\leqslant\tau} \left(S_u S_v \frac{\partial^2 G_2}{\partial x^2}(v,S_v)\right)^2. \qquad \square \end{aligned}$$

3. Concluding remarks

(i) For the sake of clarity we restricted ourselves to the Brownian motion and the geometric Brownian motion as underlying diffusions. It might be possible to extend the considerations to the setting considered in [1], where each of the cases, the Brownian motion and the geometric Brownian motion, was considered more general.

(ii) Letting M be the Brownian motion or the geometric Brownian motion, for future work the following problem could be of interest: what are the sequences $\beta = (\beta)_{n=1}^\infty$ with $\beta_n \downarrow 0$ and $\sup_n \sqrt{n}\beta_n = \infty$ such that there exists a function $g = g_{\beta,M}$ with $g(M_T)\in L_2$ and

$$a_n^M(g(M_T)|L_2) \geqslant \beta_n \quad \text{for } n = 1,2,\ldots \tag{11}$$

(recall that $a_n^M(\cdot|L_2)$ was introduced in (2)). If we would restrict ourselves to *deterministic* time-nets $(\tau_i)_{i=0}^n$ in the definition of $a_n^M(\cdot|L_2)$, then the problem is solved: as shown in [8] for all $\beta_n \downarrow 0$ and $M\in\{B,S\}$ there is a function $g_{\beta,M}$ such that (11) is satisfied (for deterministic nets). Since the techniques from [8] completely relay on the deterministic structure of the time-nets, the problem seems to be open for random nets.

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