

# On Nummelin splitting for continuous time Harris recurrent Markov processes and application to kernel estimation for multi-dimensional diffusions

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## Abstract

We introduce a sequence of stopping times that allow us to study an analogue of a life-cycle decomposition for a continuous time Markov process, which is an extension of the well-known splitting technique of Nummelin to the continuous time case. As a consequence, we are able to give deterministic equivalents of additive functionals of the process and to state a generalisation of Chen's inequality. We apply our results to the problem of non-parametric kernel estimation of the drift of multi-dimensional recurrent, but not necessarily ergodic, diffusion processes.

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## 1. Introduction

Consider a Harris recurrent strong Markov process  $X = (X_t)_{t \geq 0}$  with invariant measure  $\mu$ . If such a process has a recurrent point  $x_0$  (or more generally a recurrent atom), then it is possible to introduce a sequence of stopping times  $R_n$  – which is called *life-cycle decomposition* – such that

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1. For all  $n$ ,  $R_n < \infty$ ,  $R_{n+1} = R_n + R_1 \circ \theta_{R_n}$ . (Here,  $\theta$  denotes the shift operator.)
2.  $X_{R_n} = x_0$ .
3. For all  $n$ , the process  $(X_{R_n+t})_{t \geq 0}$  is independent of  $\mathcal{F}_{R_n}$ .

In this case, paths of the process can be decomposed into i.i.d. excursions  $[R_i, R_{i+1}[$ ,  $i = 1, 2, \dots$ , plus an initial segment  $[0, R_1]$ , and then limit theorems such as the ratio limit theorem for additive functionals of the process follow immediately as a direct application of the strong law of large numbers, in both the ergodic and the null recurrent case.

For general Harris processes, at least without further assumptions, recurrent atoms do not exist. However, for discrete time Harris chains, Athreya and Ney, see [1], and Nummelin, see [21], give a way of constructing a recurrent atom on an extended probability space, provided the transition operator of the chain satisfies a certain minorisation condition. This construction is called “splitting”. A well-known idea, see for instance Meyn and Tweedie [19,20], is to consider the discrete chain  $\tilde{X} = (\tilde{X}_n)_n$ , called the resolvent chain or R-chain, instead of the process in continuous time  $X$ . This resolvent chain is obtained when observing the process at independent exponential times. We propose to apply the splitting technique to this resolvent chain which is always possible. Hence, we use splitting at random times when sampling the process after independent exponential times. We then fill in the original process in between two successive exponential times. In other words, we construct bridges of the process  $X$  between exponential times such that at the exponential times, the splitting is satisfied. This construction is not evident since we want to preserve the Markov property of the process. It is for that reason that we have to change the structure of the “history” of the process. Actually, we construct a process  $Z_t$  taking values in  $E \times [0, 1] \times E$  together with a sequence of jump times  $T_n$  for this process such that at any time  $T_n$ , we know the present state of the process  $X_{T_n}$ , but also the future state  $X_{T_{n+1}}$ . Moreover, the following properties are fulfilled:

1. The first coordinate  $Z_t^1$  of  $Z_t$  has the same dynamics as the original process  $X_t$  starting from  $X_0 = x$  if we fix the initial condition  $Z_0^1 = x$ , but not  $Z_0^2$  and  $Z_0^3$ .
2. On each interval  $[T_n, T_{n+1}[$ ,  $Z_t^1$  and  $Z_t^3$  are constant, and the third coordinate  $Z_{T_n}^3$  represents a choice of  $X_{T_{n+1}}$  according to the splitting technique which has to be attained by the bridge (the process) between  $T_n$  and  $T_{n+1}$ .
3. The second coordinate is used only in order to model the splitting. It is not of further importance.

Then it is possible to define a sequence of stopping times  $(S_n, R_n)$  for this process which is a generalised life-cycle decomposition in the following sense:

1. For all  $n$ ,  $S_n < R_n < \infty$ ,  $S_{n+1} = S_n + S_1 \circ \theta_{S_n}$ ,  $R_n = \inf\{T_m : T_m > S_n\}$ .
2. For every  $n$ ,  $X_{R_n}$  is independent of  $\sigma\{X_s : s \leq S_n\}$  and  $\mathcal{L}(X_{R_n}) = \nu$  for some fixed probability measure  $\nu$ .
3. For all  $\mu$ -integrable functions  $f$  and for any initial measure  $\pi$ ,  $E_\pi(\int_{R_n}^{R_{n+1}} f(X_s) ds) = \mu(f)$  (up to a multiplicative constant).

Note that it is not possible to divide the path of the process into real i.i.d. excursions. We obtain the independence only after the waiting time  $R_n$  after  $S_n$ . Some special attention has to be paid to the initial segment  $\int_0^{R_1} f(X_s) ds$  — and it is for that reason that we have to introduce and to investigate functions that are called *special functions*.

As a consequence, we establish a generalisation of Chen’s inequality (compare also to [5], Lemma 1, (2.4)) for additive functionals of the Markov process (see Theorem 2.18) and get in

particular the existence of a deterministic equivalent of any integrable additive functional  $A$  of the process (see Corollary 2.19):

There exists a deterministic function  $t \mapsto v_t$  such that  $v_t \rightarrow \infty$  as  $t \rightarrow \infty$  such that for any  $\mu$ -integrable additive functional  $A$  of  $X$  and any initial measure  $\pi$ ,

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi (1/M \leq A_t/v_t \leq M) = 1. \quad (1.1)$$

Here, the rate of convergence of  $v_t$  to infinity is determined by the asymptotic behaviour of the process, and is given by  $v_t = t$  in the ergodic case.

Under some additional regularity assumptions, (1.1) can be strengthened to obtain weak convergence of martingales and additive functionals as indicated in [4] for the discrete case. This has been done in [11] for the continuous time case (see also Touati [24]) — still based on the splitting method. However, in [11] we have approximated the continuous time process by a sequence of processes converging to  $X$  which contain “discrete” parts, i.e. time intervals where the process is constant, and we have applied the splitting only to the approximating processes. This approach does not apply here since it does not work with the process itself.

Note that the asymptotic behaviour of additive functionals has been intensively studied for Markov chains (see Chen [4,5]) as well as for diffusions in dimension one, making use of techniques from analysis or of local time, see for instance Khasminskii [13,14] and also Csáki and Salminen [6]. The existence of a deterministic equivalent (1.1) for integrable additive functionals of Markov chains was established by Chen in [5], and then extended to the case of one-dimensional diffusions by Loukianova and Loukianov in [18]. However, the general case has not yet been studied before.

It was observed in [18] that the existence of a deterministic equivalent (1.1) is very useful in statistical inference for recurrent diffusion processes. In particular, concerning the rate of convergence, it permits to treat null recurrent processes like ergodic ones, replacing  $t$  by  $v_t$ . Using this method, the rate of convergence of parametric MLE of the drift was obtained in [18] for one-dimensional recurrent diffusions. We would like to stress that thanks to (1.1), the result of [18] can be extended to the multi-dimensional setting without further effort.

To illustrate a non-parametric application of our method, we treat in the second part of this paper the problem of kernel estimation for an unknown drift function  $b(\cdot)$  in the case of  $d$ -dimensional recurrent diffusions. Note that for this non-parametric problem, up to now, only the one-dimensional case has been intensively studied, using techniques that are strictly one-dimensional (local time); see for instance Kutoyants [16,17], Delattre, Hoffmann, Kessler [8]. On the other hand, the ergodic case has also been studied; see for instance Dalalyan and Reiss [7], but nothing seems to be known in the general multi-dimensional possibly null recurrent case.

Besides the fact of existence of a deterministic equivalent, the problem in the multi-dimensional case is getting some kind of uniform version of the ratio limit theorem without using local time techniques. This is done here using the life-cycle decomposition  $(S_n, R_n)$  of the continuous time process, paying particular attention to the initial segment by the use of special functions. The somewhat uniform version of the strong Chacon–Ornstein theorem is in some sense the main technical result of this second part of this article – interesting in its own right – given in Theorem 3.9.

As usual, in order to estimate, we assume a smoothness property at  $x_0$ , i.e. a Hölder condition of the following kind:

$$\sup_{x \in [x_0 - \delta, x_0 + \delta]} |b(x) - b(x_0)| \cdot |x - x_0|^{-\alpha} \leq \gamma, \quad (1.2)$$

for some  $\delta, \gamma > 0$  and some fixed  $\alpha \in (0, 1)$ . Then the Nadaraya–Watson estimator with bandwidth  $h = h_t = v_t^{-1/(2\alpha+d)}$  attains the rate  $v_t^{-\alpha/(2\alpha+d)}$ . This is [Theorem 3.6](#). Note that  $v_t$  depends on the asymptotic behaviour of the process and is not observable. It is possible, using our techniques, to replace  $v_t$  by  $V_t$  where  $V_t$  is some integrable additive functional of  $X$  (and hence observable) in order to get a random rate of convergence  $V_t^{-\alpha/(2\alpha+d)}$ . However, we are not yet able to replace  $v_t$  by  $V_t$  in the construction of the estimator itself. So our statistical result is more of theoretical interest. Note however that no results have been found before for the general multi-dimensional case.

## 2. Nummelin splitting for Markov processes in continuous time, Chen’s inequality and deterministic equivalents for additive functionals

Consider a probability space  $(\Omega, \mathcal{A}, (P_x)_x)$ , and on  $(\Omega, \mathcal{A}, (P_x)_x)$  a process  $X = (X_t)_{t \geq 0}$  which is strong Markov, taking values in a locally compact Polish space  $(E, \mathcal{E})$ , with càdlàg paths, and with  $X_0 = x$   $P_x$ -almost surely,  $x \in E$ . We write  $(P_t)_t$  for the semi-group of  $X$  and we suppose that  $X$  is recurrent in the sense of Harris, with invariant measure  $\mu$ , unique up to multiplication by a constant. Moreover, let  $(\mathcal{A}_t)_t$  be the filtration generated by  $X$ .

We impose the following regularity condition on the transition semi-group  $P_t$  of  $X$ :

- Assumption 2.1.** 1. The transition semi-group  $P_t$  of the process  $X$  is Feller.  
 2. There exists a sigma-finite positive measure  $\Lambda$  on  $(E, \mathcal{E})$  such that for every  $t > 0$ ,  $P_t(x, dy) = p_t(x, y)\Lambda(dy)$ , where  $(t, x, y) \mapsto p_t(x, y)$  is jointly measurable.

### 2.1. Preliminaries on additive functionals

We are interested in the asymptotic behaviour of integrable additive functionals of the process. We will show – using Nummelin splitting – that it is possible to find a deterministic function  $v_t$  associated with the process such that all integrable additive functionals are equivalent to  $v_t$  in probability.

We recall the definition of an additive functional:

**Definition 2.2.** An additive functional of the process  $X$  is an  $\bar{\mathbb{R}}_+$ -valued,  $(\mathcal{A}_t)_t$ -adapted process  $A = (A_t)_{t \geq 0}$  such that

1. Almost surely, the process is non-decreasing, right-continuous, having  $A_0 = 0$ .
2. For any  $s, t \geq 0$ ,  $A_{s+t} = A_t + A_s \circ \theta_t$  almost surely. Here,  $\theta$  denotes the shift operator.

Examples for additive functionals are  $A_t = \int_0^t f(X_s)ds$  where  $f$  is a positive measurable function. Such an additive functional is said to be *integrable* if  $\mu(f) < \infty$ .

It is well known that Harris recurrent Markov processes satisfy the ratio limit theorem (or Chacon–Ornstein limit theorem): For any positive,  $\mu$ -integrable functions  $f$  and  $g$  such that  $\mu(g) > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t f(X_s)ds}{\int_0^t g(X_s)ds} = \frac{\mu(f)}{\mu(g)} P_x\text{-almost surely } \forall x \in E.$$

We recall the notion of a special function (see also [\[23,3\]](#)):

**Definition 2.3.** A measurable function  $f : E \rightarrow \mathbb{R}_+$  is called special if for all bounded and positive measurable functions  $h$  such that  $\mu(h) > 0$ , the function

$$x \mapsto E_x \int_0^\infty \exp \left[ - \int_0^t h(X_s) ds \right] f(X_t) dt$$

is bounded.

Then the ratio limit theorem can be strengthened in the following way: Let  $f$  and  $g$  be two  $\mu$ -integrable, special functions having  $\mu(g) > 0$ . Then for any initial measures  $\pi_1$  and  $\pi_2$ ,

$$\lim_{t \rightarrow \infty} \frac{E_{\pi_1} \int_0^t f(X_s) ds}{E_{\pi_2} \int_0^t g(X_s) ds} = \frac{\mu(f)}{\mu(g)}.$$

This is the *strong Chacon–Ornstein limit theorem*; see also [3,22].

## 2.2. On Nummelin splitting in continuous time

The aim of this section is to construct – on an extended probability space – a process  $Z$  taking values in  $\mathcal{Z} := E \times [0, 1] \times E$  which admits a recurrent atom in a certain sense such that the first coordinate  $(Z_t^1)_t$  is a version of the original process  $(X_t)_t$ .

We start with some preliminary considerations: Introduce a sequence  $(\sigma_n)_{n \geq 1}$  of i.i.d.  $\exp(1)$ -waiting times, independent of the process  $X$  itself. Let  $T_0 := 0$ ,  $T_n := \sigma_1 + \dots + \sigma_n$  and  $\tilde{X}_n := X_{T_n}$ . Then it is well known that the chain  $\tilde{X} = (\tilde{X}_n)_n$  is recurrent in the sense of Harris and that its one-step transition kernel  $U^1(x, dy) := \int_0^\infty e^{-t} P_t(x, dy) dt$  satisfies a minorisation condition:

$$U^1(x, dy) \geq \alpha 1_C(x) \nu(dy), \quad (2.3)$$

where  $0 < \alpha < 1$ ,  $\mu(C) > 0$  and  $\nu$  is a probability measure equivalent to  $\mu(\cdot \cap C)$  (cf. [23, 11], Proposition 6.7, [19]). Note that by Assumption 2.1,  $U^1(x, dy) \ll \Lambda(dy)$ , with density  $u^1(x, y) := \int_0^\infty e^{-t} p_t(x, y) dt$ .

We follow the approach of Nummelin [21] in discrete time and define the following transition kernel  $Q((x, u), dy)$  from  $E \times [0, 1]$  to  $E$ :

$$Q((x, u), dy) = \begin{cases} \nu(dy) & \text{if } (x, u) \in C \times [0, \alpha] \\ \frac{1}{1-\alpha} (U^1(x, dy) - \alpha \nu(dy)) & \text{if } (x, u) \in C \times ]\alpha, 1] \\ U^1(x, dy) & \text{if } x \notin C. \end{cases} \quad (2.4)$$

Note that by construction,

$$\int_0^1 Q((x, u), dy) du = U^1(x, dy). \quad (2.5)$$

We now give the construction of  $Z_t = (Z_t^1, Z_t^2, Z_t^3)$  taking values in  $E \times [0, 1] \times E$ . The idea is the following: At any time  $T_n$ , knowing the position of the process  $X$  at that time, i.e. knowing the random variable  $X_{T_n}$ , we choose a uniform variable on  $[0, 1]$ , independently of the past. We use this uniform variable in order to realise a choice of the position  $X_{T_{n+1}}$  of the process at the next time  $T_{n+1}$  according to the splitting technique. But we choose this position  $X_{T_{n+1}}$  already at time  $T_n$ . Hence at time  $T_n$ , we dispose both of the positions  $X_{T_n}$  and  $X_{T_{n+1}}$ . Finally, letting time

evolve, on  $[T_n, T_{n+1}[$ , the first coordinate of  $Z$  represents a choice of the bridge of  $X$  from  $X_{T_n}$  to the already fixed position  $X_{T_{n+1}}$ . The first coordinate of  $Z$  will represent the evolution of the bridge, the second coordinate is the uniform variable used in order to realise the splitting, and the third coordinate represents the future value  $X_{T_{n+1}}$ . Here is the precise construction:

Let  $Z_0^1 = X_0 = x$ . Choose  $Z_0^2$  according to the uniform distribution  $U$  on  $[0, 1]$ . On  $\{Z_0^2 = u\}$ , choose  $Z_0^3 \sim Q((x, u), dx')$ . Then inductively in  $n \geq 0$ , on  $Z_{T_n} = (x, u, x')$ :

1. Choose a new jump time  $\sigma_{n+1}$  according to

$$e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt \quad \text{on } \mathbb{R}_+,$$

where we define  $0/0 := a/\infty := 1$ , for any  $a \geq 0$ , and put  $T_{n+1} := T_n + \sigma_{n+1}$ .

2. On  $\{\sigma_{n+1} = t\}$ , put  $Z_{T_n+s}^2 := u$ ,  $Z_{T_n+s}^3 := x'$  for all  $0 \leq s < t$ .
3. For every  $s < t$ , choose

$$Z_{T_n+s}^1 \sim \frac{p_s(x, y)p_{t-s}(y, x')}{p_t(x, x')} \Lambda(dy).$$

Choose  $Z_{T_n+s}^1 := x_0$  for some fixed point  $x_0 \in E$  on  $\{p_t(x, x') = 0\}$ . Moreover, given  $Z_{T_n+s}^1 = y$ , on  $s + u < t$ , choose

$$Z_{T_n+s+u}^1 \sim \frac{p_u(y, y')p_{t-s-u}(y', x')}{p_{t-s}(y, x')} \Lambda(dy').$$

Again, on  $\{p_{t-s}(y, x') = 0\}$ , choose  $Z_{T_n+s+u}^1 = x_0$ .

4. At the jump time  $T_{n+1}$ , choose  $Z_{T_{n+1}}^1 := Z_{T_n}^3 = x'$ . Choose  $Z_{T_{n+1}}^2$  independently of  $Z_s$ ,  $s < T_{n+1}$ , according to the uniform law  $U$ . Finally, on  $\{Z_{T_{n+1}}^2 = u'\}$ , choose  $Z_{T_{n+1}}^3 \sim Q((x', u'), dx'')$ .

**Remark 2.4.** Let  $S$  be any random or deterministic time. Put  $T_1(S) := \inf\{T_m : T_m > S\}$ . Then, on  $Z_S = (x, u, x')$ , the construction of  $(Z_{(S+t) \wedge T_1(S)})_t$  is according to the same steps 1–4 as above. This will become clear in the proof of [Theorem 2.7](#).

Write  $\mathbb{F}$  for the filtration generated by  $Z$ . Moreover, let  $\mathbb{G}$  be the filtration generated by the first two coordinates  $(Z^1, Z^2)$  of  $Z$ . By abuse of notation, for any initial measure  $\pi$  on  $E$ , we shall write  $P_\pi$  for the unique probability measure under which  $Z$  starts from  $\pi(dx)U(du)Q((x, u), dx')$ .  $E_\pi$  is then the corresponding expectation. In the same way, we shall write  $P_x$  in the case where  $\pi = \delta_{\{x\}}$ .

By construction, we have the following:

**Remark 2.5.** For any  $n \geq 0$ , the strong Markov property holds with respect to  $T_n$ , i.e. for any  $f, g : \mathcal{Z} \rightarrow \mathbb{R}$  measurable and bounded, for any  $s > 0$  fixed,

$$E_\pi(g(Z_{T_n})f(Z_{T_n+s})) = E_\pi(g(Z_{T_n})E_{Z_{T_n}}(f(Z_s))).$$

The following first property is shown by a simple calculus.

**Proposition 2.6.** Under  $P_x$  we have: The sequence of jump times  $(T_n)_n$  is independent of the first coordinate process  $(Z_t^1)_t$ , and  $(T_n - T_{n-1})_{n \geq 1}$  are i.i.d.  $\exp(1)$ -variables. Moreover,  $T_{n+1} - T_n$  is independent of  $\mathcal{F}_{T_n-}$ .

**Theorem 2.7.**  $Z$  is a Markov process with respect to  $\mathbb{F}$ .

**Proof.** Let  $s, u > 0$ , let  $f, g : E \times [0, 1] \times E \rightarrow \mathbb{R}_+$  be positive, bounded and measurable. We fix an initial measure  $\pi$  on  $E$ .

(1) The existence of the process up to a possibly finite lifetime  $\zeta := \lim_n T_n$  is clear from general considerations (we refer the reader to the paper of Ikeda, Nagasawa, Watanabe [12], on the construction of Markov processes by piecing out). Due to Proposition 2.6, under  $P_x$ , the  $T_n$  are the jump times of a rate-1 Poisson process; hence  $\zeta = \infty$  almost surely.

(2) For any  $n \geq 0$ , conditioning on  $\mathcal{F}_{T_n}$ , and applying steps 1–3 of the construction, we arrive at

$$\begin{aligned} & E_\pi(g(Z_s)f(Z_{s+u})1_{\{T_n < s < s+u < T_{n+1}\}}) \\ &= E_\pi \left[ 1_{\{T_n < s\}} \int_{s+u-T_n}^{\infty} e^{-t} dt \int \frac{p_{s-T_n}(Z_{T_n}^1, y) p_{t-(s-T_n)}(y, Z_{T_n}^3)}{u^1(Z_{T_n}^1, Z_{T_n}^3)} g(y, Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy) \right. \\ &\quad \times \left. \int \frac{p_u(y, y') p_{t-(s+u-T_n)}(y', Z_{T_n}^3)}{p_{t-(s-T_n)}(y, Z_{T_n}^3)} f(y', Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy') \right] \\ &= E_\pi \left[ 1_{\{T_n < s\}} e^{-(s-T_n)} \int \frac{p_{s-T_n}(Z_{T_n}^1, y) u^1(y, Z_{T_n}^3)}{u^1(Z_{T_n}^1, Z_{T_n}^3)} g(y, Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy) \right. \\ &\quad \times \left. \int_u^{\infty} e^{-t} dt \int \frac{p_u(y, y') p_{t-u}(y', Z_{T_n}^3)}{u^1(y, Z_{T_n}^3)} f(y', Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy') \right] \\ &= E_\pi[g(Z_s)1_{\{T_n < s < T_{n+1}\}} E_{Z_s}(f(Z_u)1_{\{u < T_1\}})], \end{aligned}$$

since on  $\{s > T_n\}$ ,

$$\begin{aligned} & \mathcal{L}(Z_s; s < T_{n+1} | Z_{T_n}) 1_{\{s > T_n\}} \\ &= e^{-(s-T_n)} \frac{p_{s-T_n}(Z_{T_n}^1, y) u^1(y, Z_{T_n}^3)}{u^1(Z_{T_n}^1, Z_{T_n}^3)} \Lambda(dy) \delta_{(Z_{T_n}^2, Z_{T_n}^3)}(du, dx'). \end{aligned}$$

(3) Moreover we have

$$\begin{aligned} & E_\pi(g(Z_s)f(Z_{T_{n+1}})1_{\{T_n < s < T_{n+1}\}}) \\ &= E_\pi \left[ 1_{\{T_n < s\}} \int_{s-T_n}^{\infty} e^{-t} dt \int \frac{p_{s-T_n}(Z_{T_n}^1, y) p_{t-(s-T_n)}(y, Z_{T_n}^3)}{u^1(Z_{T_n}^1, Z_{T_n}^3)} g(y, Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy) \right. \\ &\quad \times \left. \int_0^1 du \int Q((Z_{T_n}^3, u), dx') f(Z_{T_n}^3, u, x') \right] \\ &= E_\pi \left[ 1_{\{T_n < s\}} e^{-(s-T_n)} \int \frac{p_{s-T_n}(Z_{T_n}^1, y) u^1(y, Z_{T_n}^3)}{u^1(Z_{T_n}^1, Z_{T_n}^3)} g(y, Z_{T_n}^2, Z_{T_n}^3) \Lambda(dy) \right. \\ &\quad \times \left. E_{(y, Z_{T_n}^2, Z_{T_n}^3)}(f(Z_{T_1})) \right] \\ &= E_\pi(g(Z_s)1_{\{T_n < s < T_{n+1}\}} E_{Z_s}(f(Z_{T_1}))). \end{aligned}$$

(4) Using (2) and (3), one shows easily that

$$E_{\pi}(g(Z_s)f(Z_{s+u})1_{\{T_n < s < T_{n+1} < s+u\}}) = E_{\pi}(g(Z_s)1_{\{T_n < s < T_{n+1}\}})E_{Z_s}(f(Z_u); u > T_1).$$

Hence we have the simple Markov property for the process  $Z$ . This finishes the proof. •

Note that the sequence  $(T_n)_n$  is no longer independent of the process  $Z$ . The  $T_n$  are the jump times of  $(Z^2, Z^3)$  by construction, and  $(Z_t^2, Z_t^3)_t$  is constant on every interval  $[T_n, T_{n+1}]$ . Note also that we have to keep the second and the third coordinates in order to get a real Markov process.

By construction, we have the following properties for the process  $Z$ :

**Proposition 2.8.** (a)  $A := C \times [0, \alpha] \times E$  is a recurrent atom for  $Z$  in the following sense: Let  $R := \inf\{n : Z_{T_n} \in A\}$ . Then  $\mathcal{L}(Z_{T_{R+1}}|Z_{T_R}^1, Z_{T_R}^2)$  is given by the measure  $\nu(dx) U(du) Q((x, u), dx')$ .

(b) We have equality in law:

$$\mathcal{L}((X_{T_n})_{n \geq 0} | X_0 = x) = \mathcal{L}((Z_{T_n}^1)_{n \geq 0} | Z_0^1 = x).$$

(c) We have equality in law:

$$\mathcal{L}((X_t)_{t \geq 0} | X_0 = x) = \mathcal{L}((Z_t^1)_{t \geq 0} | Z_0^1 = x).$$

**Proof.** (a) is evident by construction of  $Z$ .

(b) is evident since  $\int_0^1 Q((x, u), dx') = U^1(x, dx')$ .

(c) Fix  $t > 0$ . Let  $\varphi : E \rightarrow \mathbb{R}_+$  be measurable and bounded. Then by construction of the process  $Z$ , and since  $Z_{T_n}^1$  is  $\mathcal{F}_{T_n-}$ -measurable,

$$\begin{aligned} E_x[\varphi(Z_t^1)] &= \sum_n E_x[\varphi(Z_t^1)1_{\{T_n \leq t < T_{n+1}\}}] \\ &= \sum_n E_x[E[\varphi(Z_t^1)1_{\{t < T_{n+1}\}} | \mathcal{F}_{T_n-}] 1_{\{T_n \leq t\}}] \\ &= \sum_n E_x \left( \left[ \int_0^1 du \int Q((Z_{T_n}^1, u), dx') \int_0^\infty e^{-s} 1_{\{t-T_n \leq s\}} ds \right. \right. \\ &\quad \times \left. \left. \int \frac{p_{t-T_n}(Z_{T_n}^1, y) p_{s-(t-T_n)}(y, x')}{u^1(Z_{T_n}^1, x')} \varphi(y) \Lambda(dy) \right] 1_{\{T_n \leq t\}} \right) \\ &= \sum_n E_x \left( \left[ \int_0^\infty e^{-s} 1_{\{t-T_n \leq s\}} ds \int p_{t-T_n}(Z_{T_n}^1, y) \varphi(y) \Lambda(dy) \right] 1_{\{T_n \leq t\}} \right) \\ &= \sum_n E_x \left[ E_{Z_{T_n}^1}(\varphi(X_{t-T_n}); t - T_n \leq T_1) 1_{\{T_n \leq t\}} \right] \\ &= E_x[\varphi(X_t)], \end{aligned}$$

since  $Z_{T_n}^1 \sim X_{T_n}$  by (b). Here, the fourth equality has been obtained using first that  $\int_0^1 du Q((Z_{T_n}^1, u), dx') = u^1(Z_{T_n}^1, x') \Lambda(dx')$  and then integrating against  $\Lambda(dx')$ . The equality of the processes can be shown in a similar way. •

**Remark 2.9.** Due to Proposition 2.8 (c), we can identify the process  $X$  as the first coordinate of the process  $Z$ . Hence we have imbedded  $X$  into a richer process  $Z$  which possesses a recurrent atom.



For our purpose we do not need the strong Markov property of  $Z$ . But it might be interesting to know conditions for  $Z$  being strong Markov. We impose the following additional conditions for that reason:

**Assumption 2.10.** 1. The measure  $\Lambda$  is non-atomic.

2. For any  $t > 0$ , for any  $y \in E$ ,  $x \mapsto p_t(x, y)$  is continuous.

3. For any  $T > 0$ ,  $y \in E$ , for any compact subset  $K \subset E$  such that  $y \notin K$ , there exists a constant  $C$ , such that

$$\sup_{x \in K} \sup_{t \leq T} p_t(x, y) \leq C.$$

4.  $x \mapsto u^1(x, y)$  is continuous in  $x \neq y$  and bounded on any compact set  $K$  such that  $y \notin K$ .

We are mainly interested in applications of our method to diffusion models, and in this situation, [Assumption 2.10](#) is quite natural:

**Example 2.11.** Suppose that the process  $X$  is a  $d$ -dimensional diffusion given as a strong solution of the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where  $W$  is a  $m$ -dimensional standard Brownian motion, where  $b$  and  $\sigma$  are bounded, having bounded derivatives of any order. Suppose moreover that the diffusion satisfies the uniform Hörmander condition (we refer the reader to [15] for details). Then by [15], Theorems 3.17 and 6.8, condition 2.10 is satisfied.

**Theorem 2.12.** Under [Assumptions 2.1](#) and [2.10](#), the strong Markov property holds for any stopping time  $S$  such that  $Z_S^1 \neq Z_S^3$  almost surely. Moreover,  $Z$  can be chosen to have càdlàg paths.

**Proof.** Note that due to [Assumptions 2.1](#) and [2.10](#), for any  $x \neq x'$ , for any  $t > 0$ , the bridge of the process  $X$  from  $X_0 = x$  to  $X_t = x'$  is Feller. Hence, for any  $n \geq 0$ ,  $(Z_{T_n+s}^1)_{s < T_{n+1}-T_n}$  can be chosen to have càdlàg paths, and then the trajectories of  $Z$  are càdlàg by construction.

Now, let  $S$  be any  $\mathbb{F}$ -stopping time such that  $Z_S^1 \neq Z_S^3$  almost surely. Let  $S_n$  be a sequence of stopping times taking values in a countable set such that  $S_n$  decreases to  $S$  as  $n \rightarrow \infty$ . Let  $f, g \in C_b(\mathcal{Z})$  be positive continuous functions such that  $x \mapsto f(x, u, x')$  vanishes at infinity for all fixed  $u, x'$ , and let  $s > 0$  be a fixed deterministic time. Let  $f_k$  be a sequence of positive functions such that  $f_k(x, u, x') \rightarrow f(x, u, x')1_{\{x \neq x'\}}$  for any  $(x, u, x')$ , as  $k \rightarrow \infty$ , such that  $f_k(x, u, x') \leq f(x, u, x')1_{\{d(x, x') > 1/k\}}$ , where  $d(\cdot, \cdot)$  is the distance on  $E$ . Then, due to the simple Markov property,

$$E_\pi(g(Z_{S_n})f_k(Z_{S_n+s})) = E_\pi(g(Z_{S_n})E_{Z_{S_n}}(f_k(Z_s))). \quad (2.6)$$

Clearly,  $Z_{S_n} = (Z_{S_n}^1, Z_{S_n}^2, Z_{S_n}^3) \rightarrow Z_S$ , by the path properties of  $Z$ . Moreover, since  $(Z^2, Z^3)$  is piecewise constant, we have almost surely: There exists some  $n_0$  such that for all  $n \geq n_0$ ,  $(Z_{S_n}^2, Z_{S_n}^3) = (Z_S^2, Z_S^3)$ . Recall that  $Z_S^3 \neq Z_S^1$ . Hence we have to show that for any sequence  $z_n = (x_n, u, x') \rightarrow z = (x, u, x') \in \mathcal{Z}$ , with  $x \neq x'$ , the corresponding expectations  $E_{z_n}(f_k(Z_s))$  converge to  $E_z(f_k(Z_s))$ . This is seen as follows. By construction of the process,

$$\begin{aligned} E_{z_n}(f_k(Z_s)) &= E_{z_n}(f_k(Z_s)1_{\{s < T_1\}}) + E_{z_n}(f_k(Z_s)1_{\{s > T_1\}}) \\ &= \int_s^\infty e^{-t} \frac{p_t(x_n, x')}{u^1(x_n, x')} \int_E \frac{p_s(x_n, y)p_{t-s}(y, x')}{p_t(x_n, x')} f_k(y, u, x') \Lambda(dy) dt \end{aligned}$$

$$+ \int_0^s e^{-t} \frac{p_t(x_n, x')}{u^1(x_n, x')} E_{x'}(f_k(Z_{s-t})) dt.$$

The second expression converges, using [Assumption 2.10](#) and dominated convergence. Now have a look at the first expression which equals

$$\frac{e^{-s}}{u^1(x_n, x')} \int_E p_s(x_n, y) u^1(y, x') f_k(y, u, x') \Lambda(dy).$$

Now,  $y \mapsto u^1(y, x') f_k(y, u, x') \in C_0(E)$ , the space of all continuous functions vanishing at infinity, due to [Assumption 2.10](#) and by construction of  $f_k$ . Then, due to the Feller property of  $X$ ,

$$\begin{aligned} \int_E p_s(x_n, y) u^1(y, x') f_k(y, u, x') \Lambda(dy) &= E_{x_n}(u^1(X_s, x') f_k(X_s, u, x')) \\ &\rightarrow E_x(u^1(X_s, x') f_k(X_s, u, x')) = \int_E p_s(x, y) u^1(y, x') f_k(y, u, x') \Lambda(dy). \end{aligned}$$

Then the assertion follows, using again the continuity of  $u^1(x, x')$  in  $x$ . Passing to the limit  $n \rightarrow \infty$  in (2.6) then yields

$$E_\pi(g(Z_S) f_k(Z_{S+s})) = E_\pi(g(Z_S) E_{Z_S}(f_k(Z_s))).$$

Note that almost surely,  $Z_{S+s}^1 \neq Z_{S+s}^3$ , and  $Z_s^1 \neq Z_s^3$ ; hence letting  $k \rightarrow \infty$ , dominated convergence yields

$$E_\pi(g(Z_S) f(Z_{S+s})) = E_\pi(g(Z_S) E_{Z_S}(f(Z_s))). \quad \bullet$$

### 2.3. Life-cycle decomposition in continuous time and applications

From now on, we will interpret  $X_t$  as the first coordinate of the process  $Z$ . Put

$$\begin{aligned} S_0 &:= 0, & R_0 &:= 0, & S_{n+1} &:= \inf\{T_m > R_n : Z_{T_m} \in A\}, \\ R_{n+1} &:= \inf\{T_m > S_{n+1}\}, & n &\geq 0. \end{aligned}$$

We shall write  $\mathbb{F}^X$  for the filtration generated by  $X$  interpreted as the first coordinate of  $Z$ .

We have the following properties:

**Proposition 2.13.** (a) For any  $n \geq 1$ ,  $Z_{R_n+}$  is independent of  $\mathcal{G}_{S_n}$  and of  $\mathcal{F}_{S_n-}$ , and  $\mathcal{L}(Z_{R_n}^1 | \mathcal{G}_{S_n}) = \nu$ . In this sense, the sequence of  $\mathbb{F}$ -stopping times  $R_n$  is a life-cycle decomposition for the process  $Z$ .

(b)  $E(R_n - S_n | \mathcal{F}_{S_n-}) \leq \frac{1}{\alpha}$ , for all  $n \geq 1$ .

**Proof.** (a) is clear by construction. We show (b): Note that necessarily by (2.3),  $\nu \ll \Lambda$  and write  $\nu(dx') = \nu(x') \Lambda(dx')$ . Then by assumption, for all  $x \in C$ ,  $u^1(x, x') \geq \alpha \nu(x')$ ; hence  $\nu(x')/u^1(x, x') \leq 1/\alpha$ . Since  $Z_{S_n}^1$  is  $\mathcal{F}_{S_n-}$ -measurable, on  $\{Z_{S_n}^1 = x\}$ ,

$$\begin{aligned} E(R_n - S_n | \mathcal{F}_{S_n-}) &= \int_E \nu(x') \Lambda(dx') \int_0^\infty t e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt \\ &= \int_0^\infty \int_E \frac{\nu(x')}{u^1(x, x')} p_t(x, x') \Lambda(dx') t e^{-t} dt \\ &\leq \frac{1}{\alpha} \int_0^\infty t e^{-t} dt = \frac{1}{\alpha}, \end{aligned}$$

and this concludes the proof.  $\bullet$

**Remark 2.14.** Note that  $R_n$  is not a real life-cycle decomposition for  $X$ , i.e.  $X_{R_n+}$  is not independent of  $\sigma\{X_s : s < R_n\}$ . This is simply a fact by construction: For all  $t < R_n$ ,  $X_t = Z_t^1$  depends on  $Z_t^3$  and  $X_{R_n} = Z_t^3$  for  $S_n < t < R_n$ .

The following equality will be useful in what follows:

**Proposition 2.15.** Let  $f : E \rightarrow \mathbb{R}_+$  be a bounded measurable function. Then

$$E \left( \int_{R_n}^{S_{n+1}} f(X_s) ds \middle| \mathcal{F}_{S_n} \right) = E_{Z_{R_n}^1} \left( \int_0^{S_1} f(X_s) ds \right). \quad (2.7)$$

**Proof.** We use Markov's property twice, first with respect to  $\mathcal{F}_{R_n}$  and then in a second step with respect to  $\mathcal{F}_{S_n}$ . Since  $\mathcal{L}(Z_{R_n} | Z_{S_n}) = \delta_{Z_{S_n}^3} (dx) U(du) Q((x, u), dx')$ , this yields

$$\begin{aligned} & E \left( \int_{R_n}^{S_{n+1}} f(X_s) ds \middle| \mathcal{F}_{S_n} \right) \\ &= E \left[ \int_{E \times [0, 1] \times E} \mathcal{L}(Z_{R_n} | Z_{S_n})(dx, du, dx') E_{(x, u, x')} \int_0^{S_1} f(X_s) ds \right] \\ &= E \left[ E_{Z_{S_n}^3} \int_0^{S_1} f(X_s) ds \right] = E \left[ E_{Z_{R_n}^1} \int_0^{S_1} f(X_s) ds \right], \end{aligned}$$

since  $Z_{S_n}^3 = Z_{R_n}^1$ . •

Recall the definition of a special function 2.3. Take  $f$  a fixed bounded positive special function of the process  $X$ . Such functions always exist; see also Remark 2.21. Then we have:

**Proposition 2.16.** The functions

$$E \ni x \mapsto E_x \left( \int_0^{S_1} f(X_s) ds \right) \quad \text{and} \quad E \ni x \mapsto E_x \left( \int_0^{R_1} f(X_s) ds \right)$$

are bounded. Moreover, if the following additional assumption holds:

$$\sup_{x, x' \in E} \int_0^\infty t e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt < \infty, \quad (2.8)$$

then also the functions

$$\begin{aligned} \mathcal{Z} \ni (x, u, x') &\mapsto E_{(x, u, x')} \left( \int_0^{S_1} f(X_s) ds \right) \quad \text{and} \\ (x, u, x') &\mapsto E_{(x, u, x')} \left( \int_0^{R_1} f(X_s) ds \right) \end{aligned}$$

are bounded.

**Example 2.17.** The main application that we are interested in are diffusion models like those described in Example 2.11. In such models, condition (2.8) is satisfied. Note that the main problem is the explosion of  $p_t(x, x')$  near the diagonal  $x = x'$  when time  $t$  is small, and multiplication by the factor  $t$  acts in the “good sense” for our purpose.

**Proof.** First of all put

$$\tilde{S} := \inf\{T_n : n \geq 1, Z_{T_n}^1 \in C\}. \quad (2.9)$$

Since  $f$  is special, we have that

$$x \mapsto E_x \left( \int_0^\infty e^{-\int_0^t h(X_s) ds} f(X_t) dt \right) = E_x \left( \int_0^\infty e^{-\int_0^t h(Z_s^1) ds} f(Z_t^1) dt \right)$$

is bounded for any positive function  $h$  having  $\mu(h) > 0$ . Now take  $h = 1_C$ . Note that by Proposition 2.6,  $\tilde{S}$  is the first jump time of a Poisson process having jump rate  $1_C(Z_s^1)$ ; hence

$$E_x \left( \int_0^{\tilde{S}} f(Z_s^1) ds \right) = E_x \left( \int_0^\infty e^{-\int_0^t 1_C(Z_s^1) ds} f(Z_t^1) dt \right),$$

and this is bounded in  $x$ . Now let  $\tilde{S}_1, \tilde{S}_2, \dots, \tilde{S}_n, \dots$  be the successive visits of  $Z_{T_n}^1$  to  $C$ , i.e.

$$\tilde{S}_1 = \tilde{S}, \quad \tilde{S}_{n+1} := \inf\{T_m : T_m > \tilde{S}_n, Z_{T_m}^1 \in C\}, \quad \tilde{S}_0 := 0.$$

Moreover, write  $\tilde{R}_n := \inf\{T_m : T_m > \tilde{S}_n\}$ ,  $n \geq 0$ . Let  $K$  be a constant such that

$$\sup_x E_x \left( \int_0^{\tilde{S}} f(Z_s^1) ds \right) \leq K, \quad \sup_x f(x) \leq K.$$

Then

$$\begin{aligned} E_x \left( \int_0^{S_1} f(Z_s^1) ds \right) &= E_x \left( \int_0^{\tilde{S}_1} f(Z_s^1) ds \right) + \sum_{n \geq 1} E_x \left( \int_{\tilde{S}_n}^{\tilde{S}_{n+1}} f(Z_s^1) ds 1_{\{\tilde{S}_n < S_1\}} \right) \\ &\leq K + \sum_{n \geq 1} E_x \left( 1_{\{\tilde{S}_n < S_1\}} \left[ K E_{Z_{\tilde{S}_n}}(\tilde{R}_0) + E_{Z_{\tilde{S}_n}} \int_{\tilde{R}_0}^{\tilde{S}_1} f(Z_s^1) ds \right] \right) \\ &\leq K + \sum_{n \geq 1} E_x \left( 1_{\{\tilde{S}_n < S_1\}} \left[ \frac{K}{1-\alpha} + E_{Z_{\tilde{R}_n}^1} \int_0^{\tilde{S}_1} f(Z_s^1) ds \right] \right) \\ &\leq K + \frac{2K}{1-\alpha} \sum_{n \geq 1} P_x(\tilde{S}_n < S_1), \end{aligned}$$

where we used that

$$E_x(1_{\{\tilde{S}_n < S_1\}} E_{Z_{\tilde{S}_n}}(\tilde{R}_0)) \leq \frac{1}{1-\alpha},$$

since on  $(x, u) \in C \times [\alpha, 1]$ ,  $Q((x, u), dx') = \frac{1}{1-\alpha} (U^1(x, dx') - \alpha v(dx')) \leq \frac{1}{1-\alpha} U^1(x, dx')$ .

We used an equality in the spirit of (2.7) (with  $R_n$  and  $S_n$  replaced by  $\tilde{R}_n$  and  $\tilde{S}_n$ ) in order to obtain the second inequality. Note that we have to cut the integral over  $[\tilde{S}_n, \tilde{S}_{n+1}]$  into two pieces  $[\tilde{S}_n, \tilde{R}_n]$  and  $[\tilde{R}_n, \tilde{S}_{n+1}]$  in order to be able to apply (2.7).

Now,

$$P_x(\tilde{S}_n < S_1) = P_x(\tilde{S}_{n-1} < S_1, Z_{\tilde{S}_n}^2 > \alpha),$$

and then, conditioning with respect to  $\mathcal{F}_{\tilde{S}_{n-1}}$  and using that  $Z_{\tilde{S}_n}^2$  is independent of  $\mathcal{F}_{\tilde{S}_{n-1}}$ , we get inductively in  $n$  that

$$P_x(\tilde{S}_n < S_1) = P_x(\tilde{S}_{n-1} < S_1) \cdot (1-\alpha) = (1-\alpha)^n.$$

Hence,  $x \mapsto E_x(\int_0^{S_1} f(Z_s^1) ds)$  is bounded. Finally,

$$E_x \left( \int_0^{R_1} f(Z_s^1) ds \right) \leq E_x \left( \int_0^{S_1} f(Z_s^1) ds \right) + K E_x(R_1 - S_1),$$

which is still bounded since  $E_x(R_1 - S_1) \leq 1/\alpha$ . This yields the first assertion.

Moreover, write  $z := (x, u, x')$ . Then

$$\begin{aligned} E_z \left( \int_0^{S_1} f(Z_s^1) ds \right) &\leq K \cdot E_z(T_1) + E_z \left( \int_{T_1}^{S_1} f(Z_s^1) ds \right) \\ &\leq K \sup_z E_z(T_1) + E_{x'} \left( \int_0^{S_1} f(Z_s^1) ds \right), \end{aligned}$$

where we used once more the formula (2.7). Note that

$$E_z(T_1) = \int_0^\infty t e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt,$$

and this is bounded using assumption (2.8). Finally,  $E_{x'} \int_0^{S_1} f(Z_s^1) ds$  is bounded, using the first part of the proof. •

We are now able to formulate Chen's inequality (compare to [5], lemma 1, (2.4)) for the process  $Z$ . This is the main result of this section.

**Theorem 2.18.** *Suppose that Assumption 2.1 is satisfied. Take a fixed positive, bounded  $\mu$ -integrable special function  $f$  of  $X$ . Let  $\pi$  be an arbitrary initial measure. Then we have for any fixed  $s > 0$ ,*

$$\begin{aligned} E_\pi \left( \int_0^t f(X_u) du 1_{\{\int_0^t f(X_u) du \geq s\}} \right) \\ \leq P_\pi \left( \int_0^t f(X_u) du \geq s \right) \left[ C + s + E_v \left( \int_0^t f(X_u) du \right) \right]. \end{aligned} \quad (2.10)$$

Here,  $C$  is a constant that depends on the special function  $f$  such that

$$\sup_x E_x \int_0^{R_1} f(X_u) du + \sup_x f(x) \leq C.$$

**Proof.** Write  $A_t := \int_0^t f(X_s) ds$  and put  $\sigma_s := \inf\{t : A_t \geq s\}$ . Then

$$\begin{aligned} E_\pi \left( \int_0^t f(X_u) du 1_{\{\int_0^t f(X_u) du \geq s\}} \right) \\ = E_\pi(A_t; \sigma_s \leq t) \leq s P_\pi(\sigma_s \leq t) + E_\pi \left( \int_{\sigma_s}^t f(X_u) du; \sigma_s \leq t \right). \end{aligned}$$

But, using the Markov property of  $X$  with respect to  $\mathcal{F}_{\sigma_s}$  (recall that the original process  $X$  is strong Markov!), we get

$$E_\pi \left( \int_{\sigma_s}^t f(X_u) du; \sigma_s \leq t \right) \leq E_\pi \left( \left[ E_{X_{\sigma_s}} \int_0^t f(X_u) du \right]; \sigma_s \leq t \right). \quad (2.11)$$

In order to evaluate  $E_{X_{\sigma_s}} \int_0^t f(X_u) du$ , we now use the splitting technique inside the expectation  $E_{X_{\sigma_s}}$  and interpret  $X_u$  as the first coordinate of  $Z$ . Hence,

$$\begin{aligned} E_{X_{\sigma_s}} \int_0^t f(X_u) du &= E_{X_{\sigma_s}} \int_0^t f(Z_u^1) du \\ &\leq E_{X_{\sigma_s}} \left[ \int_0^{R_1} f(Z_u^1) du + \int_{R_1}^t f(Z_u^1) du 1_{\{R_1 < t\}} \right]. \end{aligned}$$

The two integrals in the last expression will be treated separately: First of all, using the property of the special function (see [Proposition 2.16](#)), we get immediately that

$$E_{X_{\sigma_s}} \int_0^{R_1} f(Z_u^1) du \leq C.$$

Moreover,

$$E_{X_{\sigma_s}} \left[ \int_{R_1}^t f(Z_u^1) du 1_{\{R_1 < t\}} \right] \leq E_{X_{\sigma_s}} \left[ E_{Z_{R_1}} \int_0^t f(Z_u^1) du \right] = E_v \int_0^t f(Z_u^1) du,$$

since  $Z_{R_1}^1 \sim v$ . Replacing this in (2.11) yields the result. •

As a corollary of Chen's inequality, we get the existence of deterministic equivalents for additive functionals of the process:

**Corollary 2.19.** *Grant [Assumption 2.1](#). Let  $g$  be a fixed special function of  $X$  with  $\mu(g) > 0$ , fix some probability  $m$  and put*

$$v_t := E_m \int_0^t g(X_s) ds.$$

*Then for any other  $\mu$ -integrable function  $h$  with  $\mu(h) > 0$  and any probability measure  $\pi$ ,*

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left( v_t/M \leq \int_0^t h(X_s) ds \leq v_t \cdot M \right) = 1.$$

**Proof.** Let  $f$  be the fixed special function of [Theorem 2.18](#) and suppose w.l.o.g. that  $\mu(f) > 0$ . Then due to (2.10),

$$P_\pi \left( \int_0^t f(X_s) ds \geq v_t/M \right) \geq \frac{E_\pi \int_0^t f(X_u) du / v_t - 1/M}{C/v_t + 1/M + E_v \int_0^t f(X_s) ds / v_t}.$$

Then, by the strong Chacon–Ornstein theorem, and since  $v_t \rightarrow \infty$  as  $t \rightarrow \infty$ ,

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left( \int_0^t f(X_s) ds \geq v_t/M \right) = 1.$$

Moreover, by Markov's inequality and the strong Chacon–Ornstein theorem,

$$P_\pi \left( \int_0^t f(X_s) ds > v_t \cdot M \right) \leq \frac{E_\pi \int_0^t f(X_s) ds}{v_t M} \rightarrow \frac{\mu(f)}{M \mu(g)} \quad \text{as } t \rightarrow \infty,$$

and then the assertion follows on letting tend  $M \rightarrow \infty$ .

The general assertion follows then by the ratio limit theorem, since

$$\frac{\int_0^t h(X_s) ds}{\int_0^t f(X_s) ds} \rightarrow \frac{\mu(h)}{\mu(f)}$$

almost surely as  $t \rightarrow \infty$ . •

Sometimes, the following equality will be useful:

**Proposition 2.20.** *Let  $f : E \rightarrow \mathbb{R}_+$  be a measurable function such that  $\mu(|f|) < \infty$ . Then, up to multiplication by a constant,*

$$E_v \int_0^{R_1} f(X_s) ds = C \mu(f),$$

where the constant  $C$  depends only on the process, not on  $f$ .

**Proof.** For  $n \geq 1$ , let  $\xi_n := \int_{R_{n-1}}^{R_n} f(X_s) ds$ . As usual, we interpret  $X$  as the first coordinate of  $Z$ . Then  $\xi_0, \xi_2, \xi_4, \dots, \xi_{2n}$  are i.i.d., and the same is true for  $\xi_1, \xi_3, \xi_5, \dots$ . Hence by the strong law of large numbers,

$$\lim_n \frac{\int_0^{R_n} f(X_s) ds}{n} = \lim_n \left( \frac{\xi_0 + \xi_2 + \dots}{n} + \frac{\xi_1 + \xi_3 + \dots}{n} \right) = E_v \int_0^{R_1} f(X_s) ds \quad (2.12)$$

almost surely. Now, let  $g$  be another  $\mu$ -integrable function such that  $\mu(g) > 0$ ; then the ratio limit theorem gives

$$\lim_n \frac{\int_0^{R_n} f(X_s) ds}{\int_0^{R_n} g(X_s) ds} = \frac{\mu(f)}{\mu(g)},$$

and on the other hand, by (2.12),

$$\lim_n \frac{\int_0^{R_n} f(X_s) ds}{\int_0^{R_n} g(X_s) ds} = \frac{E_v \int_0^{R_1} f(X_s) ds}{E_v \int_0^{R_1} g(X_s) ds},$$

and thus the assertion, putting  $C := (E_v \int_0^{R_1} g(X_s) ds) / \mu(g)$ . •

Note that we use heavily the existence of special functions throughout this section. So we close this section with some remarks on special functions.

**Remark 2.21.** Any positive measurable function  $f$  is called *special* for the discrete chain  $\bar{X}$  (defined at the beginning of Section 2.2) if

$$x \mapsto E_x \left( \sum_{n=1}^{\infty} (1 - h(\bar{X}_1)) \cdots (1 - h(\bar{X}_{n-1})) f(\bar{X}_n) \right)$$

is bounded in  $x$  for any bounded, positive measurable function  $h$  such that  $\mu(h) > 0$  (see [23]). Now, by [11], (5.29), page 59, we know that

$$E_x \left( \sum_{n=1}^{\infty} (1 - h(\bar{X}_1)) \cdots (1 - h(\bar{X}_{n-1})) f(\bar{X}_n) \right) = E_x \left( \int_0^{\infty} e^{-\int_0^t h(X_s) ds} f(X_t) dt \right). \quad (2.13)$$

As a consequence, any special function of the chain  $\bar{X}$  is also a special function of the process  $X$  and vice versa.

The following is a direct consequence of (2.13) and of a known result of [23], exercise 4.11, chapter 6, page 215:

**Corollary 2.22.** *Suppose that the transition operator  $P_t$  of  $X$  is strongly Feller. Then any positive bounded function  $f$  having compact support is special.*

**Proof.** By dominated convergence, since  $P_t$  is strongly Feller, also the transition kernel  $U^1 = \int_0^\infty e^{-t} P_t dt$  of  $\bar{X}$  is strongly Feller. Then by [23], all positive bounded functions having compact support are special for  $\bar{X}$ , and hence for  $X$ . •

### 3. Application to kernel estimation in multi-dimensional diffusion models

Let  $X$  be a diffusion process in dimension  $d$  given as solution of the following stochastic differential equation:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad (3.14)$$

where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$  are supposed to be bounded, such that a strong solution to (3.14) exists, and where  $W$  is an  $m$ -dimensional standard Brownian motion.

We assume the following:

- Assumption 3.1.**
1.  $X$  is recurrent in the sense of Harris with invariant measure  $\mu$ .
  2. The invariant measure admits a continuous Lebesgue density  $p$  which is strictly positive everywhere:  $\mu(dx) = p(x)\lambda(dx)$ , where  $\lambda$  denotes Lebesgue's measure on  $\mathbb{R}^d$ .
  3. The transition semi-group of the diffusion satisfies [Assumption 2.1](#) and condition (2.8) with  $\Lambda(dx) = \lambda(dx)$ .

**Example 3.2.** Note that [Assumption 3.1](#), 3, is satisfied if the diffusion is elliptic, and if  $b$  and  $\sigma$  are bounded, having bounded derivatives of any order; see for instance [25], page 5, and [15].

**Remark 3.3.** (a) A one-dimensional diffusion

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

is Harris recurrent with invariant measure equivalent to Lebesgue's measure if the function

$$S(x) := \int_0^x s(y)dy, \quad s(y) = \exp\left(-\int_0^y \frac{2b}{\sigma^2}(v)dv\right)$$

satisfies

$$\lim_{x \rightarrow -\infty} S(x) = -\infty, \quad S(0) = 0, \quad \lim_{x \rightarrow \infty} S(x) = \infty.$$

See for example the monograph of Has'minskii [9]. See also Khas'minskii [13,14] for more concrete examples of null recurrent one-dimensional diffusions, and Höpfner and Kutoyants [10] for a statistical study of parametric and semiparametric models of one-dimensional null recurrent diffusions.

(b) Conditions of recurrence for multi-dimensional diffusions are much less studied (and much more difficult). We refer the reader to Bhattacharya, see [2], who gives some generalisations of the above criterion to the multi-dimensional case.



We observe trajectories of the diffusion continuously in time. Our aim is to estimate the unknown drift function  $b$  at some fixed point  $x_0 \in \mathbb{R}^d$ . As usual, we assume a smoothness property at  $x_0$ , i.e. a Hölder condition of the following kind:

$$\sup_{x \in [x_0 - \delta, x_0 + \delta]} |b(x) - b(x_0)| \cdot |x - x_0|^{-\alpha} \leq \gamma, \quad (3.15)$$

for some  $\delta, \gamma > 0$  and some fixed  $\alpha \in (0, 1)$ .

We will use a Nadaraya–Watson type kernel estimator: Let  $\varphi$  be a kernel, i.e.,  $\varphi$  is a continuous positive function having compact support such that  $\int_{\mathbb{R}^d} \varphi(x) dx = 1$ . For any bandwidth  $h > 0$ , put  $\varphi_h(y) := \varphi(\frac{y-x_0}{h})/h^d$ . Let  $v_t$  be the deterministic equivalent of  $X$  of 2.19 and let  $h_t := v_t^{-\frac{1}{2\alpha+d}} \wedge \delta \rightarrow 0$  as  $t \rightarrow \infty$ . Note that  $h_t^d v_t \rightarrow \infty$ . We define

$$\hat{b}_t := \frac{\int_0^t \varphi_{h_t}(X_s) dX_s}{\int_0^t \varphi_{h_t}(X_s) ds}, \quad \text{where we define } \frac{a}{0} := 0. \quad (3.16)$$

**Remark 3.4.** Note that the definition of our estimator makes use of the knowledge of  $v_t$ . However,  $v_t$  is not observable, so the definition of this estimator is only of theoretical interest for the moment. If one restricts attention to sub-models, for example to the sub-model of ergodic diffusions (where  $v_t$  could be replaced by  $t$ ) or the sub-model of null recurrent diffusions where one has regular variation at 0 of resolvents of the diffusion

$$E_x \left( \int_0^\infty e^{-\frac{1}{t}s} g(X_s) ds \right) \sim t^\alpha \frac{1}{l(t)} \mu(g),$$

as  $t \rightarrow \infty$  (and in this case  $v_t$  can be replaced by  $t^\alpha/l(t)$ ), then the definition of the estimator makes perfect sense.

Note that due to Remark 2.21, all positive bounded functions having compact support are special functions of the diffusion. As a consequence, any of the functions  $\varphi_{h_t}$  is special. Moreover, we have the following:

**Remark 3.5.** The set  $C$  of condition (2.3) can be chosen to be compact. As a consequence,  $1_C$  is a special function.

**Proof.** Since  $\mu(C) > 0$  and since  $\mu \sim \lambda$ , by the properties of Lebesgue's measure,  $C$  must contain a compact set of positive measure. Then it suffices to replace  $C$  in (2.3) by this compact set. Again, thanks to Remark 2.21, all positive bounded functions having compact support are special functions. Hence  $1_C$  is special. •

Then the following is our main theorem:

**Theorem 3.6.** Suppose that  $p(x_0) > 0$ . Let  $r_t := v_t^{\frac{\alpha}{2\alpha+d}}$  and let  $\pi$  be an arbitrary initial measure. Then  $r_t$  is an upper rate of convergence, i.e.

$$\lim_{K \rightarrow \infty} \limsup_{t \rightarrow \infty} P_\pi(r_t |\hat{b}_t - b(x_0)| > K) = 0. \quad (3.17)$$

**Corollary 3.7.** Let  $g$  be a fixed special function of the diffusion, i.e. a bounded function having compact support. Put

$$V_t := \int_0^t g(X_s) ds, \quad R_t := V_t^{\frac{\alpha}{2\alpha+d}}.$$

Then 3.6 remains true when replacing  $r_t$  by  $R_t$ , which is an observable quantity.

**Proof.** This follows immediately from the fact that  $V_t/v_t$  is bounded in probability; see Corollary 2.19. •

In the sequel, we are going to give a proof of this theorem.

### 3.1. Preliminaries

We start with some results that will be needed later. First of all, we have a kind of uniform version of the Chacon–Ornstein theorem.

**Proposition 3.8.** *Let  $f$  be a positive, bounded, continuous function. Then*

$$\lim_{t \rightarrow \infty} \frac{1}{v_t} E_\pi \int_0^t \varphi_{h_t}(X_s) f(X_s) ds = cp(x_0) f(x_0),$$

for some constant depending only on the choice of  $C$  and  $\alpha$  and of the function  $g$  of Corollary 2.19.

**Proof.** The proof is given in several steps.

1. We start by showing the following result: Let  $N_t := \sum_{n \geq 1} 1_{\{S_n \leq t\}}$ . Put  $R_t := t + R_1 \circ \theta_t = \inf\{R_n : R_n > S_{N_t+1}\}$ . Then we have

$$\lim_{t \rightarrow \infty} \frac{1}{v_t} E_\pi \int_0^{R_t} \varphi_{h_t}(X_s) f(X_s) ds = cp(x_0) f(x_0). \quad (3.18)$$

This is shown as follows: Using Proposition 2.20, we have

$$\begin{aligned} & E_\pi \int_0^{R_t} \varphi_{h_t}(X_s) f(X_s) ds \\ &= E_\pi \left( \int_0^{R_1} \varphi_{h_t}(X_s) f(X_s) ds \right) + \sum_{n \geq 1} E_\pi \left( 1_{\{S_n \leq t\}} \int_{R_n}^{R_{n+1}} (\varphi_{h_t} f)(X_s) ds \right) \\ &= E_\pi \left( \int_0^{R_1} \varphi_{h_t}(X_s) f(X_s) ds \right) + E_\pi(N_t) E_\nu \left( \int_{R_n}^{R_{n+1}} (\varphi_{h_t} f)(X_s) ds \right) \\ &= E_\pi \left( \int_0^{R_1} \varphi_{h_t}(X_s) f(X_s) ds \right) + E_\pi(N_t) \mu(\varphi_{h_t} f), \end{aligned}$$

where we used that  $X_{R_n+}$  is independent of  $\{S_n \leq t\}$ .

Now note that due to our assumptions,  $\varphi$  is of compact support. Write  $C := \|\varphi\|_\infty$  and write  $I := 1_{x_0+h_0\text{supp}(\varphi)}$ . Then we can write that  $\varphi_{h_t}(X_s) f(X_s) \leq \frac{C}{h_t^d} \|f\|_0 I(X_s)$ , where  $\|f\|_0$  is the supremum of the continuous function  $f$  on  $x_0 + h_0\text{supp}(\varphi)$ . As a consequence, using that  $I$  is a special function,

$$E_\pi \left( \int_0^{R_1} \varphi_{h_t}(X_s) f(X_s) ds \right) \leq \frac{C}{h_t^d} \|f\|_0 \sup_x E_x \int_0^{R_1} I(X_s) ds,$$

and this tends to zero after having divided by  $v_t$  since  $h_t^d v_t \rightarrow \infty$ .

Moreover,  $E_\pi(N_t)/v_t$  converges to a constant depending only on the process and on  $g$ , the function used in order to build  $v_t$ . Hence

$$\lim_{t \rightarrow \infty} \frac{E_\pi \int_0^{R_t} \varphi_{h_t}(X_s) f(X_s) ds}{v_t} = \lim_{t \rightarrow \infty} \frac{E_\pi(N_t)}{v_t} \cdot \lim_{t \rightarrow \infty} \mu(\varphi_{h_t} f) = cp(x_0) f(x_0).$$

This gives (3.18).

2. We have

$$\begin{aligned} & E_\pi \int_0^{R_t} \varphi_{h_t}(X_s) f(X_s) ds - E_\pi \int_0^t \varphi_{h_t}(X_s) f(X_s) ds \\ &= E_\pi \int_t^{R_t} \varphi_{h_t}(X_s) f(X_s) ds \\ &= E_\pi \left( E_{Z_t} \int_0^{R_1} \varphi_{h_t}(X_s) f(X_s) ds \right) \\ &\leq C \|f\|_0 \frac{1}{h_t^d} E_\pi \left( E_{Z_t} \int_0^{R_1} I(X_s) ds \right) \\ &\leq C \|f\|_0 \left[ \sup_z E_z \int_0^{R_1} I(X_s) ds \right] \frac{1}{h_t^d}, \end{aligned}$$

where  $z = (x, u, x')$ , and this tends to zero after having divided by  $v_t$  since  $h_t^d v_t \rightarrow \infty$ . This finishes our proof. •

The following theorem is the main theorem of this section and gives some kind of uniform existence of a deterministic equivalent.

**Theorem 3.9.** *We have*

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left( \frac{1}{M} \leq \frac{1}{v_t} \int_0^t \varphi_{h_t}(X_s) ds \leq M \right) = 1.$$

**Proof.** First of all,

$$P_\pi \left( \int_0^t \varphi_{h_t}(X_s) ds > M v_t \right) \leq \frac{1}{M} \frac{1}{v_t} E_\pi \left( \int_0^t \varphi_{h_t}(X_s) ds \right),$$

and then the assertion follows thanks to Proposition 3.8.

Moreover, as in the proof of 3.8, there exists a constant independent of  $t$ , such that

$$\sup_z E_z \int_0^{R_1} \varphi \left( \frac{X_s - x_0}{h_t} \right) ds < C.$$

Then we have, applying Chen's inequality (2.10),

$$\begin{aligned} P_\pi \left( \frac{v_t}{M} \leq \int_0^t \varphi_{h_t}(X_s) ds \right) &= P_\pi \left( \frac{v_t h_t^d}{M} \leq \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds \right) \\ &\geq \frac{E_\pi \int_0^t \varphi_{h_t}(X_s) ds / v_t - 1/M}{\frac{C}{v_t h_t^d} + 1/M + E_v \int_0^t \varphi_{h_t}(X_s) ds / v_t}. \end{aligned}$$

Now write  $a_t := E_\pi \int_0^t \varphi_{h_t}(X_s) ds / v_t$  and  $b_t := E_v \int_0^t \varphi_{h_t}(X_s) ds / v_t$ . Then we know by [Proposition 3.8](#), since  $p(x_0) > 0$ , that  $\lim_{t \rightarrow \infty} a_t = \lim_{t \rightarrow \infty} b_t > 0$ ; hence

$$\liminf_{t \rightarrow \infty} P_\pi \left( \frac{v_t}{M} \leq \int_0^t \varphi_{h_t}(X_s) ds \right) \geq \frac{\lim a_t - \frac{1}{M}}{\frac{1}{M} + \lim b_t},$$

and the assertion follows, letting tend  $M \rightarrow \infty$ . •

We have the following corollary of [Theorem 3.9](#):

**Corollary 3.10.** *We have*

$$\lim_{K \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left( -K \leq \frac{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s}{\sqrt{h_t^d v_t}} \leq K \right) = 1.$$

**Proof.**

$$P_\pi \left( \left| \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s \right| \geq K \sqrt{h_t^d v_t} \right) \leq \frac{E_\pi \int_0^t \varphi^2 \left( \frac{X_s - x_0}{h_t} \right) \sigma^2(X_s) ds}{K^2 h_t^d v_t},$$

and then the assertion follows from [Proposition 3.8](#). •

### 3.2. Proof of [Theorem 3.6](#)

We are now able to give the proof of [Theorem 3.6](#):

Let  $t$  be sufficiently large such that  $x_0 + h_t \text{supp } \varphi \subset [x_0 - \delta, x_0 + \delta]$  and such that  $h_t = v_t^{-\frac{1}{2\alpha+d}}$ . We have clearly that

$$\begin{aligned} |\hat{b}_t - b(x_0)| &\leq \frac{\left| \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s \right|}{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds} + \frac{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) |b(X_s) - b(x_0)| ds}{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds} \\ &\leq \frac{\left| \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s \right|}{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds} + \gamma h_t^\alpha \\ &= \frac{\left| \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s \right|}{\sqrt{v_t h_t^d}} \cdot \frac{v_t h_t^d}{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds} \cdot \frac{1}{\sqrt{v_t h_t^d}} + \gamma h_t^\alpha \\ &= \frac{\left| \int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) \sigma(X_s) dW_s \right|}{r_t} \cdot \frac{r_t^2}{\int_0^t \varphi \left( \frac{X_s - x_0}{h_t} \right) ds} \cdot \frac{1}{r_t} + \gamma h_t^\alpha. \end{aligned}$$

By construction,  $h_t^\alpha r_t = 1$ . Hence the assertion follows from [Theorem 3.9](#) and from [Corollary 3.10](#).

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