

New large deviation results for some super-Brownian processes

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Abstract

We give large deviation results for the super-Brownian excursion conditioned to have unit mass or unit extinction time and for super-Brownian motion with constant non-positive drift. We use a representation of these processes by a path-valued process, the so-called Brownian snake for which we state large deviation principles.

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1. Introduction

The present paper deals with several variants of super-Brownian motion: super-Brownian excursion normalized to have a unit mass or a unit duration and super-Brownian motion with (constant) non-positive drift. During the past few years, these processes have appeared many times in different contexts. Here we are interested in proving large deviation principles for certain renormalizations of these processes and applying them to get explicit estimates. The paper is organized as follows. In the present introduction we first specify the super-processes we are going to deal with, then we state most of our results and we will finish with brief bibliographical notes

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on large deviations in the context of super-processes. In Section 2 we recall some facts about the path-valued process called the Brownian snake which is our main tool for the proofs. In Section 3 we prove our first results. In Section 4 we state a large deviation result for the Brownian snake in a new context (under the excursion measure). Corollaries for ordinary super-Brownian motion are derived in Section 5. Then Section 6 deals with the situation where a drift is present either for a Brownian snake or for the associated super-process. Finally the Appendix supplies brief proofs of a few lemmas.

1.1. Description of certain super-Brownian processes

Loosely speaking the super-Brownian excursion (SBE) with unit mass is a super-Brownian motion conditioned to have a total mass equal to 1 and starting from infinitesimal mass. Its time-integral – called Integrated Super-Brownian Excursion (ISE) – notably appears as the scaling limit of lattice trees and of critical percolation clusters in high dimensions, see [1] for precise statements.

The super-Brownian excursion with unit duration is a super-Brownian motion conditioned to die at time 1 and starting from infinitesimal mass. It has been considered for instance by Abraham and Werner [2].

Super-Brownian motion (SBM) with (constant) drift can be seen as the scaling limit of a system of sub-critical branching particles. It has been shown to be the limit of rescaled contact processes [3] or rescaled Lotka–Volterra competing species models [4]. SBM with drift can also be seen as a SBM conditioned to survive a killing procedure of the trajectories.

All these super-processes can be defined by well-posed martingale problems. See for example [5] Section II.5 for SBM with drift, [6] for SBE with unit mass, and Appendix A.3 for SBE with unit duration. In the present paper we will rely heavily on the construction of these super-processes via path-valued processes called Brownian snakes. We will recall some facts about Brownian snakes in Section 2 and also how super-processes can be constructed from snakes. In our opinion this construction gives a more intuitive feeling of super-processes than martingale problems. However in order to keep this introduction short we postpone the introduction of snakes.

Denoting by $(Y_t)_{t \geq 0}$ one of the super-processes cited above, we are in particular interested in the range \mathcal{R} which is the union for $\varepsilon > 0$ of the closure of the union for $t \geq \varepsilon$ of the support of Y_t . We denote the radius of the range $R = \inf\{\rho; \forall t, Y_t(\{x, |x| > \rho\}) = 0\}$, the extinction time (or duration) $H = \inf\{t > 0, Y_t = 0\}$ and the total mass $Z = \int_0^\infty Y_t(1)dt$. The events $\{R \geq r\}$, $\{H \geq r\}$, $\{Z \geq r\}$ are not large deviation events for the ordinary SBM when $r \rightarrow +\infty$, since

$$\mathbb{P}(R \geq r) \sim \text{cst}/r^2, \quad \mathbb{P}(H \geq r) \sim \text{cst}/r, \quad \mathbb{P}(Z \geq r) \sim \text{cst}/\sqrt{r}, \quad (1)$$

but they will often be for the super-processes we are interested in. Numerous examples are given below concerning these events or some generalization.

1.2. Statement of the results

We start with the super-Brownian excursion with unit mass whose law is denoted $\mathbb{P}^{(Z=1)}$. We do not specify the starting point which does not interfere. Large deviations events arise naturally for this process. For instance, one can obtain by elementary arguments that

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2} \log \mathbb{P}^{(Z=1)}[H \geq r] = -\frac{1}{2}. \quad (2)$$

Results are harder to obtain when they also involve the spatial motion. It was first proved by [7] that

$$\lim_{r \rightarrow +\infty} \frac{1}{r^{4/3}} \log \mathbb{P}^{(Z=1)}[R \geq r] = -\frac{3}{2^{5/3}}. \quad (3)$$

This also follows from the large deviation principle obtained in [8]. This large deviation principle – which will be recalled in Section 2 as [Theorem 10](#) – is the starting point of this paper. We will first use it to derive corollaries in new settings and it will also be a key argument in the proofs of the new large deviation principles that we will formulate later on. A nice feature of our approach is that the rate functions are simple enough to allow the explicit computation of the asymptotic logarithmic rates for the probabilities of large deviation events. For instance the following proposition estimates the probability of large residual mass.

Proposition 1. *Let $\alpha \geq 0$, $\gamma \in (0, 1)$ and B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \overline{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta \geq 0$. Then*

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}^{(Z=1)} \left(\int_{\alpha\sqrt{r}}^{+\infty} Y_t(r^{3/4}B) dt \geq \gamma \right) = -L(\alpha, \gamma, \delta) \quad (4)$$

where

$$\begin{aligned} L(\alpha, \gamma, \delta) &= \frac{32^{-5/3} \delta^{4/3}}{(1-\gamma)^{1/3}} \quad \text{if } \alpha \leq 2^{-1/3} \delta^{2/3} (1-\gamma)^{1/3} \\ &= \frac{\alpha^2}{2(1-\gamma)} + \frac{\delta^2}{2\alpha} \quad \text{otherwise.} \end{aligned}$$

This result is in particular interesting for a domain B such as a cone which is invariant by scaling. For instance taking $B = \mathbb{R}^d$ we get, for $\gamma \in (0, 1)$,

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2} \log \mathbb{P}^{(Z=1)} \left[\int_r^{+\infty} Y_t(1) dt \geq \gamma \right] = -\frac{1}{2(1-\gamma)} \quad (5)$$

which is consistent with (2). We now state a result on the large values of the ratio radius/duration, more precisely R^2/H^α for $\alpha \in [0, 1]$. Very roughly speaking for a “non-normalized SBE” we expect R^2 , H , \sqrt{Z} to be of the same order of magnitude, because of the scaling properties. The following proposition confirms that large values of R^2/H are rare events.

Proposition 2. *For $\alpha \in [0, 1]$, let*

$$c(\alpha) = 2^{\frac{\alpha-5}{3-\alpha}} (1-\alpha)^{\frac{\alpha-1}{3-\alpha}} (3-\alpha)$$

and $c(1) = 1/2$. Then we have

$$\lim_{A \rightarrow +\infty} \frac{1}{A^{\frac{2}{3-\alpha}}} \log \mathbb{P}^{(Z=1)} \left(\frac{R^2}{H^\alpha} \geq A \right) = -c(\alpha). \quad (6)$$

Note that $c(0) = 3 \cdot 2^{-5/3}$ in accordance with (3).

Let us now state results concerning the super-Brownian excursion with unit duration whose law is denoted $\mathbb{P}^{[H=1]}$. Again the starting point is irrelevant.

Proposition 3. Let B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \overline{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta > 0$. Then

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2} \log \mathbb{P}^{|H=1|} [\mathcal{R} \cap rB \neq \emptyset] = -\frac{\delta^2}{2}. \quad (7)$$

Note that the exponential speed is different from the case of unit mass. For instance we obtain

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2} \log \mathbb{P}^{|H=1|} [R \geq r] = -\frac{1}{2} \quad (8)$$

which has to be compared to (3). Concerning the behavior of the total mass Z in this setting, a simple direct argument gives that, as $r \rightarrow +\infty$,

$$\mathbb{P}^{|H=1|} [Z \geq r] \sim 8\pi^2 r e^{-2\pi^2 r}. \quad (9)$$

The method we use to prove results on the SBE with unit duration such as Proposition 3 is to convert the conditioning by unit duration into a conditioning by unit mass. We explain how, at the beginning of Section 3.3. This technique can generate more results such as the following ones. First, along the same lines as in Proposition 2, we can examine the probability of large values of the ratio radius/mass, more precisely R^2/Z^β with $\beta \leq 1/2$.

Proposition 4. For $\beta \in [0, 1/2]$, we have

$$\lim_{A \rightarrow +\infty} \frac{1}{A^{\frac{1}{1+\beta}}} \log \mathbb{P}^{|H=1|} \left(\frac{R^2}{Z^\beta} \geq A \right) = -\frac{1}{2} \beta^{-\frac{\beta}{1+\beta}} (1 + \beta). \quad (10)$$

The above expression should be understood as $-1/2$ if $\beta = 0$; this confirms (8).

Now we look at the probability that a non-infinitesimal proportion γ of mass is put on a rescaled set. It happens that this probability is the same on the exponential scale as the probability considered in (7).

Proposition 5. Let $\gamma \in (0, 1)$ and B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \overline{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta > 0$. Then

$$\lim_{r \rightarrow +\infty} \frac{1}{r^2} \log \mathbb{P}^{|H=1|} \left(\frac{1}{Z} \int_0^1 Y_t(rB) dt \geq \gamma \right) = -\frac{\delta^2}{2}. \quad (11)$$

One may ask if the techniques used to derive the above results can also be applied to ordinary SBM, starting from a finite measure μ on \mathbb{R}^d . Formula (1) shows that its behavior is different from what we have seen for SBEs. Roughly speaking SBM is the succession of SBEs distributed according to an excursion measure called the canonical measure of the SBM (hence these excursions are not normalized as in the two cases we have considered before). Thus we are led to formulate a new large deviation principle under this excursion measure. This is Theorem 18 and as we want to formulate it in terms of Brownian snake we postpone its statement until Section 4. This theorem is a very natural generalization of Theorem 10 but working under an infinite measure requires some technical work. For the moment, as an example of an application, we can state the following corollary.

Proposition 6. Let $(Y_t)_{t \geq 0}$ be, under \mathbb{P}_μ a SBM starting from a finite compactly supported measure μ and B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \overline{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta > 0$. Then

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, Z \leq z \right] = -32^{-5/3} z^{-1/3} \delta^{4/3}. \quad (12)$$

Moreover, under the same hypothesis,

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, H \leq \gamma \sqrt{r} \right] = -\frac{\delta^2}{2\gamma}. \quad (13)$$

Theorem 18 can also be used to obtain some results on the SBM with (constant) non-positive drift. This is **Theorem 19** of Section 6.1. As usual the result is formulated for snakes and then applied to the associated super-process. Of course, the rate function has to be modified by addition of a term due to the drift but the reader will note that the renormalization of the process is also modified. Corollaries can be derived, for the SBM with (constant) non-positive drift such as the following.

Proposition 7. Let $\mathbb{P}_\mu^{[b]}$ denote the law of a SBM $(Y_t)_{t \geq 0}$ starting from a compactly supported measure μ , with constant non-positive drift $-2b$ and B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \bar{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta > 0$. Then

$$\lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} [\mathcal{R} \cap r B \neq \emptyset] = -2\delta\sqrt{b}. \quad (14)$$

We give here another application of **Theorem 19** concerning the probability that a big mass (or positive mass) is put on a rescaled set and/or on large times.

Proposition 8. Let $\gamma, \alpha \geq 0$ and B be a Borel subset of \mathbb{R}^d such that $\text{dist}(0, \bar{B}) = \text{dist}(0, \overset{\circ}{B}) = \delta \geq 0$. Then

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} \left(\int_{\alpha r}^{+\infty} Y_t(r B) dt > \gamma r \right) = \begin{cases} -2\gamma b^2 - 2\delta\sqrt{b} & \text{if } \alpha \leq \frac{\delta}{2\sqrt{b}} \\ -2\gamma b^2 - \frac{\delta^2}{2\alpha} - 2b\alpha & \text{otherwise.} \end{cases} \quad (15)$$

In view of the previous results one can also ask about the behavior of the total mass or the duration for a SBM with drift. These are easy results that we give for the sake of completeness. Note that the logarithmic rate in (17) is given by a particular case of **Proposition 8**.

Proposition 9. We have

$$\lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} [H \geq r] = -2b \quad (16)$$

moreover, as $r \uparrow +\infty$,

$$\mathbb{P}_\mu^{[b]} [Z \geq r] \sim \frac{|\mu|}{2\sqrt{2\pi} b^2 r^{3/2}} e^{2|\mu| b - 2b^2 r} \quad (17)$$

where $|\mu| = \mu(1)$ is the mass of the initial measure μ .

1.3. Bibliographical notes

Large deviations for super-processes have been the subject of many papers. In the present work the large deviation principles will always be stated in terms of snakes and then used

to obtain estimates on the associated super-processes. However in order to compare with the existing literature one has to see how the renormalization we do on snakes is translated for the corresponding super-processes. The reader will be able to check that the renormalization that appears in [Theorems 10 and 18](#) consists of rescaling a super-process $(Y_t)_{t \geq 0}$ into $(Y_t^{(r)})_{t \geq 0}$ defined by $Y_t^{(r)}(\phi) = \sqrt{r} Y_{\sqrt{r}t}(\phi(\cdot/r^{3/4}))$ and letting $r \rightarrow +\infty$. In other terms used in the literature, one can see that it amounts to multiplying the diffusion coefficient by $1/r$ and the branching rate by r .

In [\[9\]](#) the authors consider a renormalization of standard SBM of the form $Y_t^{(r)}(\phi) = r^{-d} Y_{r^{-1}t}(\phi(\cdot/r))$. They deal with large deviations related to a law of large numbers and the rate function is obtained as the Legendre transform of a log-Laplace functional. In [\[10\]](#) the renormalization consists in letting the branching rate tend to 0 and the authors obtain a Schilder type large deviation principle. In [\[11\]](#), it is the mass of standard SBM that is multiplied by $\varepsilon \downarrow 0$ and the initial measure is multiplied by $1/\varepsilon$ and by the branching property, the resulting large deviation principle can be seen as a Cramer type theorem. In [\[12\]](#) and [\[13\]](#), a large deviation principle is obtained for the occupation measure of SBM starting from the Lebesgue measure. A main tool is the analysis of the semi-linear equation associated with the log-Laplace functional of SBM. These results are continued in [\[14\]](#) where the case of SBM with immigration is studied. More special models can also be mentioned such as the SBM with super-Brownian immigration [\[15\]](#) or the single point catalytic SBM [\[16\]](#).

2. Some facts about the Brownian snake

In this paper we will use the following notations:

$\mathbb{R}_+ [0, +\infty)$.

c, c' denote constants whose values are unimportant and may change from line to line.

$\mu(\phi)$ integral of function ϕ with respect to measure μ .

$\text{Supp}(\mu)$ support of the measure μ .

A^C complement of the set A .

$\mathbf{1}_A$ or $\mathbf{1}(A)$ indicator function of set A .

$|x|$ for $x \in \mathbb{R}^d$, Euclidean norm of x .

$\mathcal{C}(X, Y)$ set of continuous functions from metric space X to metric space Y .

$\mathcal{M}_F(X)$ set of finite measures on the metric space X , equipped with the topology of weak convergence and its Borel σ -algebra.

\mathcal{D} set of real-valued indefinitely differentiable functions on \mathbb{R}^d with compact support.

$\text{dist}(0, B) \inf_{x \in B} |x|$.

$\text{Leb}(\cdot)$ Lebesgue measure.

Our methodology relies on the path-valued representation of super-processes, introduced by Le Gall. If we think of the measure-valued processes as the sum of the infinitesimal individual masses of branching particles, the associated path-valued process, called the Brownian snake, is a parametrization of the tree of the trajectories of these particles. A comprehensive treatment of the Brownian snake and its applications is given for instance in [\[17\]](#) but for the convenience of the reader we are going to recall briefly some definitions and properties of this process.

The Brownian snake takes its values in the set \mathcal{W} of all stopped paths (w, ζ) , where $\zeta \geq 0$ is called the lifetime of the path, and $w : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a continuous mapping which is constant on $[\zeta, +\infty)$. The Brownian snake can therefore be written as $(W_s, \zeta_s)_{s \geq 0}$ and $(\zeta_s)_{s \geq 0}$ is called the lifetime process. We consider in this paper several variants of the Brownian snake whose laws can be described in the following way.

- (D1) For the lifetime process $(\zeta_s)_{s \geq 0}$ several laws will be considered in this paper:
- (i) Brownian excursion normalized to have a specified length σ ,
 - (ii) Brownian excursion normalized to have a specified height h ,
 - (iii) reflecting Brownian motion stopped at time $\Sigma = \tau_4$ when its local time at 0 hits 4,
 - (iv) Brownian motion with constant non-positive drift $-b$ reflecting at 0 and stopped at time $\Sigma = \tau_4$ when its local time at 0 hits 4.
- (D2) On the contrary, the conditional distribution of $(W_s, \zeta_s)_{s \geq 0}$ knowing $(\zeta_s)_{s \geq 0}$ will always be the same, that is the law of an inhomogeneous Markov process whose transition kernels are described as follows: for every $s < s'$,
- $W_{s'}(u) = W_s(u)$ for $u \leq m := \inf_{r \in [s, s']} \zeta_r$; this property is called “snake property” in the sequel;
 - $(W_{s'}(m+t), 0 \leq t \leq \zeta_{s'} - m)$ is independent of W_s conditionally on $W_s(m)$ and has the law of a Brownian motion in \mathbb{R}^d , starting from $W_s(m)$ and stopped at time $\zeta_{s'} - m$.
- (D3) The initial value will most often be $\tilde{0}$, the constant path equal to $x \in \mathbb{R}^d$ and usually we consider the case $x = 0$. However it can also be $(w_0, \zeta_0) \in \mathcal{W}$ and in that case the lifetime process starts from ζ_0 .

According to the law of the lifetime as listed in (D1)(i), (ii), the corresponding laws of the Brownian snake, with initial path $\tilde{0}$, are respectively denoted (i) $\mathbb{N}^{(\sigma)}$ and (ii) $\mathbb{N}^{[h]}$. In the case (D1)(iii) [resp. (D1)(iv)] and with initial path \tilde{x} , let $(\alpha_i, \beta_i)_{i \in I}$ be the excursion intervals of (ζ_s) out of 0, up to time τ_4 and $(W^i)_{i \in I}$ be the corresponding “snake excursions” that is $W_s^i = W_{(\alpha_i+s) \wedge \beta_i}$. Let us also denote $(L_s^t)_{s \geq 0}$ the local time at level t up to time s of the lifetime process $(\zeta_s)_{s \geq 0}$. Then the random point measure

$$\sum_{i \in I} \delta_{(L_{\alpha_i}^0, W^i)}(d/dW) \quad (18)$$

is a Poisson measure with an intensity that can be written as $\mathbf{1}_{(0,4)}(l)d\mathbb{N}_x(dW)$ [resp. $\mathbf{1}_{(0,4)}(l)d\mathbb{N}_x^{[b]}(dW)$] and the infinite measure \mathbb{N}_x [resp. $\mathbb{N}_x^{[b]}$] is called the excursion measure out of x of the Brownian snake [resp. of the Brownian snake with drift $-b$]. Under \mathbb{N}_x , the “law” of the lifetime is distributed as the Itô measure of positive excursions of Brownian motion and the conditional law knowing the lifetime $(\zeta_s)_{s \geq 0}$ is as described before in (D2). Moreover denoting $\Sigma = \inf \{s' \geq 0; \forall s \geq s', \zeta_s = 0\}$ the time of return to 0 of the lifetime process we have simply, for a measurable $F : \mathcal{C}(\mathbb{R}_+, \mathcal{W}) \rightarrow \mathbb{R}_+$,

$$\mathbb{N}_x^{[b]}[F(W)] = \mathbb{N}_x \left[F(W) \exp \left(-\frac{b^2}{2} \Sigma \right) \right]. \quad (19)$$

The probability $\mathbb{N}^{(\sigma)}$ [respectively $\mathbb{N}^{[h]}$] is equal to the measure \mathbb{N}_0 conditioned on having a lifetime excursion length Σ equal to σ [resp. a lifetime excursion height H equal to h]. Well-known scaling properties of Brownian motion entail that the law $\mathbb{N}^{(\sigma)}$ is the law of $\theta_\sigma(W)$ under $\mathbb{N}^{(1)}(dW)$ where θ_α is the following scaling operator:

$$\theta_\alpha(W)_s(u) = \left(\alpha^{1/4} W_{\frac{s}{\alpha}} \left(\frac{u}{\alpha^{1/2}} \right), \alpha^{1/2} \zeta_{\frac{s}{\alpha}} \right). \quad (20)$$

It follows by the usual conditioning of an excursion by its length that, for any measurable test function $F : \mathcal{C}(\mathbb{R}_+, \mathcal{W}) \rightarrow \mathbb{R}_+$,

$$\mathbb{N}_0[F(W)] = \int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}[F(\theta_\sigma(W))]. \quad (21)$$

Given a Brownian snake $(W_s, \zeta_s)_{s \geq 0}$, we construct the associated super-process $(Y_t)_{t \geq 0}$ as the process taking its values in the set of measures on \mathbb{R}^d and defined by:

$$Y_t = \frac{1}{4} \int_0^\Sigma d_{(s)} L_s^t \delta_{W_s(\zeta_s)} \quad (22)$$

where the notation $d_{(s)} L_s^t$ means that we integrate with respect to the non-decreasing function $s \rightarrow L_s^t$. Almost surely, for all t , the support of Y_t is $\{W_s(\zeta_s); \zeta_s = t\}$ and the range is simply $\mathcal{R} = \{W_s(\zeta_s); s \geq 0\}$. The variables R, H defined in the introduction have convenient representations in terms of the underlying snake: $R = \sup_s |W_s(\zeta_s)|$, $H = \sup_s \zeta_s$. Note also that, due to the occupation times formula, we have for any measurable test function $\phi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}_+$,

$$\int_0^{+\infty} Y_t(\phi(t, \cdot)) dt = \frac{1}{4} \int_0^\Sigma \phi(\zeta_s, W_s(\zeta_s)) ds. \quad (23)$$

In particular we have that $Z = \Sigma/4$.

To construct (via (22)) a SBM starting from a Dirac measure, we use a Brownian snake whose lifetime process is a reflecting linear Brownian motion stopped when its local time at 0 hits 4 as in (D1)(iii); for the SBM with constant non-positive drift $-2b$ we use a Brownian snake whose lifetime process is distributed as in (D1)(iv); for the SBE normalized to have unit mass [resp. unit duration] we use a Brownian snake with lifetime process distributed as a Brownian excursion normalized to have length 4 as in (D1)(i) [resp. height 1 as in (D1)(ii)]. The coefficient 4 (also appearing in definition (22)) is an inevitable nuisance if we want the associated super-process to have the usual branching coefficient 1.

The previous definitions and scaling identities imply in particular that, for any measurable non-negative test function F ,

$$\mathbb{P}^{(Z=1)} [F(R, H)] = \mathbb{N}^{(1)} [F(\sqrt{2} R, 2H)], \quad (24)$$

$$\mathbb{P}^{[H=1]} [F(R, Z)] = \mathbb{N}^{[1]} [F(R, \Sigma/4)]. \quad (25)$$

Due to the excursion theory of the ordinary Brownian motion, the SBM $(Y_t)_{t \geq 0}$ (starting from an initial measure μ) can also be constructed using a Poisson point measure by the formula (see [17] Chap. IV):

$$\forall t > 0, \quad Y_t = \sum_i Y_t(W^i) \text{ where } \sum_i \delta_{W^i}(dW) \text{ has intensity } 4 \int \mu(dx) \mathbb{N}_x(dW). \quad (26)$$

The notation $Y_t(W^i)$ means that the process $(Y_t(W^i))_{t \geq 0}$ is constructed from W^i by (22). Similarly the SBM with non-positive drift $-2b$ can be constructed the same way, this time using the intensity $4 \int \mu(dx) \mathbb{N}_x^{[b]}(dW)$.

We insist that the results in this paper are only valid when Brownian motion in \mathbb{R}^d is the spatial motion of the super-processes or snakes considered.

The starting point for the results in the present paper is the main theorem of [8] recalled below. This is a generalization of the Schilder Theorem to the Brownian snake. In that case we suppose that the lifetime process is a Brownian excursion conditioned to have length 1 and the initial point is 0; that is we work under $\mathbb{N}^{(1)}$. We use the notation $\hat{w} = w(\zeta)$ for the “terminal point” of a stopped path $(w, \zeta) \in \mathcal{W}$. We denote by \mathcal{P}^σ the set of (deterministic) continuous functions

$(W, \zeta) : s \rightarrow (W_s, \zeta_s)$ from $[0, \sigma]$ to \mathcal{W} such that $\zeta_0 = \zeta_\sigma = 0$, $W_s(0) = 0$ for $s \in [0, \sigma]$ and having the *snake property* i.e. $\forall s < t$, $W_t(u) = W_s(u)$ if $u \leq \inf_{v \in [s, t]} \zeta_v$. The space \mathcal{P}^σ is endowed with the (complete) metric

$$d((W, \zeta), (W', \zeta')) = \sup_{0 \leq s \leq \sigma} |\zeta_s - \zeta'_s| + \sup_{0 \leq s \leq \sigma} \sup_{r \geq 0} |W_s(r) - W'_s(r)|. \quad (27)$$

The subset \mathcal{H}^σ of \mathcal{P}^σ consists of the elements such that $s \rightarrow \zeta_s$ and $s \rightarrow \hat{W}_s = W_s(\zeta_s)$ are both absolutely continuous functions. The relevant derivatives are then denoted by $s \rightarrow \dot{\zeta}_s$ and $s \rightarrow \dot{\hat{W}}_s$. We use below the convention $\frac{0}{0} = 0$. Concerning the vocabulary of large deviation theory, the reader can refer for instance to [18].

Theorem 10 ([8] Theorem 1). *The laws under $\mathbb{N}^{(1)}$ of*

$$\left((W_s^{(r)}, \zeta_s^{(r)}) \right)_{s \geq 0} = \left(\frac{1}{r^{3/4}} W_s(\sqrt{r} \cdot), \frac{1}{\sqrt{r}} \zeta_s \right)_{s \geq 0} \quad (28)$$

satisfy, when $r \rightarrow +\infty$, a large deviation principle with speed r and good rate function J_1 defined by

$$J_1(W, \zeta) = \frac{1}{2} \int_0^1 \dot{\zeta}_s^2 ds + \frac{1}{4} \int_0^1 \frac{|\dot{\hat{W}}_s|^2}{|\dot{\zeta}_s|} ds \quad (29)$$

if $(W_s, \zeta_s) \in \mathcal{H}^1$ and $+\infty$ otherwise.

3. First proofs

In this section we begin with the proofs of the results concerning the SBE conditioned to have unit mass i.e. constructed from a Brownian snake whose lifetime is a Brownian excursion conditioned to have length 4. Next we state a result linking Brownian snakes having lifetime excursions which are conditioned respectively to have fixed length or unit height. Then this result is used to prove the results concerning the SBE conditioned to have unit duration.

3.1. Proof of Proposition 1

Using successively definition (22) and occupation times formula (23), the scaling property (20) and finally definition (28) of $(W^{(r)}, \zeta^{(r)})$ we obtain:

$$\begin{aligned} & \mathbb{P}^{(Z=1)} \left(\int_{\alpha\sqrt{r}}^{+\infty} Y_t(r^{3/4} B) dt \geq \gamma \right) \\ &= \mathbb{N}^{(4)} \left(\text{Leb} \left\{ s \in [0, 4]; \hat{W}_s \in r^{3/4} B, \zeta_s \geq \alpha\sqrt{r} \right\} \geq 4\gamma \right) \\ &= \mathbb{N}^{(1)} \left(\text{Leb} \left\{ s \in [0, 1]; \hat{W}_s \in r^{3/4} \frac{1}{\sqrt{2}} B, \zeta_s \geq \frac{\alpha}{2}\sqrt{r} \right\} \geq \gamma \right) \\ &= \mathbb{N}^{(1)} \left(\text{Leb} \left\{ s \in [0, 1]; \hat{W}_s^{(r)} \in \frac{1}{\sqrt{2}} B, \zeta_s^{(r)} \geq \frac{\alpha}{2} \right\} \geq \gamma \right). \end{aligned} \quad (30)$$

This latter quantity can obviously be bounded in the following way

$$\mathbb{N}^{(1)} \left((W^{(r)}, \zeta^{(r)}) \in \Lambda_o \right) \leq (30) \leq \mathbb{N}^{(1)} \left((W^{(r)}, \zeta^{(r)}) \in \Lambda_c \right)$$

where

$$\begin{aligned} \Lambda_c &= \left\{ (W_s, \zeta_s)_{s \in [0,1]} \in \mathcal{P}^1; \text{Leb} \left\{ s \in [0, 1]; \hat{W}_s \in \frac{1}{\sqrt{2}} \bar{B}, \zeta_s \geq \frac{\alpha}{2} \right\} \geq \gamma \right\} \\ \Lambda_o &= \left\{ (W_s, \zeta_s)_{s \in [0,1]} \in \mathcal{P}^1; \text{Leb} \left\{ s \in [0, 1]; \hat{W}_s \in \frac{1}{\sqrt{2}} \overset{\circ}{B}, \zeta_s > \frac{\alpha}{2} \right\} > \gamma \right\}. \end{aligned}$$

It is not hard to see that Λ_c is a closed subset of \mathcal{P}^1 and Λ_o an open subset of \mathcal{P}^1 . By Theorem 10 it suffices to check that the infimum of J_1 over Λ_o and the infimum of J_1 over Λ_c are both equal to $L(\alpha, \gamma, \delta)$.

We first prove that, for every $(W, \zeta) \in \Lambda_o$, we have $J_1(W, \zeta) \geq L(\alpha, \gamma, \delta)$. For such $(W, \zeta) = (W_s, \zeta_s)_{s \in [0,1]} \in \Lambda_o$, let a be the first time where $\hat{W}_s \in \frac{1}{\sqrt{2}} \bar{B}$ and $\zeta_s \geq \frac{\alpha}{2}$ and b be the last time before 1 where these two properties are satisfied. We have $b > a + \gamma$. From (W, ζ) , we construct $(\tilde{W}, \tilde{\zeta}) = (\tilde{W}_s, \tilde{\zeta}_s)_{s \in [0,1]} \in \Lambda_c$ by the following procedure. On any excursion interval $[\lambda, \lambda']$ of the lifetime over its future infimum up to time a – that is an excursion interval of $(\zeta_s - \inf_{u \in [s,a]} \zeta_u, 0 \leq s \leq a)$ out of 0 – we force the new lifetime $\tilde{\zeta}$ to be constant and also the corresponding paths \tilde{W} i.e. for $s \in [\lambda, \lambda']$ we set $\tilde{\zeta}_s = \zeta_\lambda$ and $\tilde{W}_s = W_\lambda$. We do similarly after time b for the excursions over the (past) infimum. We get a new lifetime $(\tilde{\zeta}_s)_{s \in [0,1]}$ which is non-decreasing up to a and non-increasing after b . Trivially $J_1(\tilde{W}, \tilde{\zeta}) \leq J_1(W, \zeta)$ (the convention $0/0 = 0$ is adopted in definition (29)). Consequently in the search for the infimum we can restrict to snake excursions having a lifetime non-decreasing up to a time a and non-increasing after b . For such excursions the Cauchy–Schwarz inequality implies

$$\int_0^a \dot{\zeta}_s^2 ds \geq \frac{1}{a} \left(\int_0^a \dot{\zeta}_s ds \right)^2 = \frac{1}{a} \zeta_a^2 \quad (31)$$

$$\int_0^a \frac{|\dot{W}_s|^2}{|\dot{\zeta}_s|} ds \geq \frac{\left| \int_0^a \dot{W}_s ds \right|^2}{\int_0^a |\dot{\zeta}_s| ds} = \frac{|\hat{W}_a|^2}{\zeta_a} \geq \frac{\delta^2}{2\zeta_a}. \quad (32)$$

Similarly,

$$\int_b^1 \dot{\zeta}_s^2 ds \geq \frac{1}{1-b} \zeta_b^2, \quad \int_b^1 \frac{|\dot{W}_s|^2}{|\dot{\zeta}_s|} ds \geq \frac{\delta^2}{2\zeta_b}.$$

We have obtained the following lower bound: if $(W, \zeta) \in \Lambda_o$ then

$$J_1(W, \zeta) \geq \frac{1}{2} \left(\frac{1}{a} \zeta_a^2 + \frac{1}{1-b} \zeta_b^2 \right) + \frac{1}{4} \left(\frac{\delta^2}{2\zeta_a} + \frac{\delta^2}{2\zeta_b} \right) \quad (33)$$

and we recall that necessarily, $\zeta_a, \zeta_b \geq \alpha/2$ and $0 < a \leq a + \gamma \leq b < 1$. Now we want to minimize the right-hand side of (33) with respect to the four variables a, b, ζ_a, ζ_b satisfying the constraints $\zeta_a, \zeta_b \geq \alpha/2$ and $0 < a \leq a + \gamma \leq b < 1$. Concerning the minimization with respect to a and b the minimum is attained for $b = a + \gamma$ and a taking the value

$$a^* = (1 - \gamma) \frac{\zeta_a}{\zeta_a + \zeta_b}. \quad (34)$$

For these values of a and b we have

$$\frac{1}{a}\zeta_a^2 + \frac{1}{1-b}\zeta_b^2 = \frac{(\zeta_a + \zeta_b)^2}{1-\gamma}.$$

So, the right-hand side of (33) is at least

$$\inf_{\zeta_a, \zeta_b \geq \alpha/2} \left\{ \frac{(\zeta_a + \zeta_b)^2}{2(1-\gamma)} + \frac{\delta^2}{8\zeta_a} + \frac{\delta^2}{8\zeta_b} \right\} = \inf_{\zeta \geq \frac{\alpha}{2}} \left\{ 2\frac{\zeta^2}{1-\gamma} + \frac{\delta^2}{4\zeta} \right\} \\ = L(\alpha, \gamma, \delta).$$

Indeed the first equality is straightforward by arguing on the lines of constant values for $\zeta_a + \zeta_b$. For the last equality the minimum is reached at ζ^* given by

$$\zeta^* = \max \left(\frac{\delta^{2/3} (1-\gamma)^{1/3}}{2^{4/3}}, \frac{\alpha}{2} \right). \quad (35)$$

Now that we have proved that the infimum of J_1 over Λ_o is at least $L(\alpha, \gamma, \delta)$, we will construct $(\tilde{W}, \tilde{\zeta}) \in \Lambda_c$ such that $J_1(\tilde{W}, \tilde{\zeta}) = L(\alpha, \gamma, \delta)$. First we fix $q \in \overline{B}$ with Euclidean norm δ . We consider ζ^* given by (35) above and $a^* = (1-\gamma)/2$ in accordance with (34) when $\zeta_a = \zeta_b = \zeta^*$. We define $(\tilde{W}, \tilde{\zeta}) = (\tilde{W}_s, \tilde{\zeta}_s)_{s \in [0,1]}$ as “piecewise linear” in the following way. The lifetime $(\tilde{\zeta}_s)_{s \in [0,1]}$ is the piecewise linear continuous function given by $\tilde{\zeta}_s = \frac{s\zeta^*}{a^*}$ if $0 \leq s \leq a^*$, by $\tilde{\zeta}_s = \zeta^*$ if $a^* \leq s \leq a^* + \gamma$ and finally by $\tilde{\zeta}_s = \frac{(1-s)\zeta^*}{1-\gamma-a^*}$ if $a^* + \gamma \leq s \leq 1$. For every $s \in [0, 1]$ we set

$$\tilde{W}_s(u) = \frac{\min(u, \tilde{\zeta}_s)}{\zeta^*} \frac{q}{\sqrt{2}}.$$

This path-valued function $(\tilde{W}, \tilde{\zeta})$ is constructed so that all the inequalities previously shown to get a lower bound on J_1 are in fact equalities so that we get $J_1(\tilde{W}, \tilde{\zeta}) = L(\alpha, \gamma, \delta)$ which ends the proof of the proposition. \square

3.2. Proof of Proposition 2

We follow the lines of the previous proof. We set $r = A^{2/(3-\alpha)}$. We use successively (24), definition (28) of $(W^{(r)}, \zeta^{(r)})$, Theorem 10 and finally we compute the infimum with arguments similar to the previous proof. We obtain:

$$\begin{aligned} \frac{1}{A^{\frac{2}{3-\alpha}}} \log \mathbb{P}^{(Z=1)} \left(\frac{R^2}{H^\alpha} \geq A \right) &= \frac{1}{r} \log \mathbb{N}^{(1)} \left(\frac{2R^2(W)}{2^\alpha H^\alpha(W)} \geq r^{\frac{3}{2}-\frac{\alpha}{2}} \right) \\ &= \frac{1}{r} \log \mathbb{N}^{(1)} \left(\frac{R^2(W^{(r)})}{H^\alpha(W^{(r)})} \geq 2^{\alpha-1} \right) \\ &\rightarrow -\inf \left\{ J_1(W, \zeta); (W, \zeta) \in \mathcal{H}^1, \frac{R^2(W)}{H^\alpha(W)} \geq 2^{\alpha-1} \right\} \\ &= -\inf \left\{ 2H^2 + \frac{R^2}{2H}; H > 0, R > 0, \frac{R^2}{H^\alpha} \geq 2^{\alpha-1} \right\} \\ &= -\inf \left\{ 2H^2 + \frac{2^{\alpha-2}}{H^{1-\alpha}}; H > 0 \right\} = c(\alpha). \end{aligned}$$

From the last equality $c(\alpha)$ can be computed exactly to give the value announced in the statement of the proposition. \square

3.3. Exchanging conditioning

We first compare $\varphi_{\Sigma|H}$ and $\varphi_{H|\Sigma}$ which denote respectively the density of the length Σ [resp. the height H] under the Itô measure of positive excursions of Brownian motion conditioned on $H = 1$ [resp. $\Sigma = 1$]. This is an elementary lemma using only scaling properties. For the convenience of the reader we give a proof in the [Appendix](#).

Lemma 11. *We have, for $x > 0$,*

$$\varphi_{H|\Sigma}(x) = \frac{\sqrt{2\pi}}{x^4} \varphi_{\Sigma|H}\left(\frac{1}{x^2}\right). \quad (36)$$

We apply this result to the conditionings of the Brownian snake and obtain the following lemma.

Lemma 12. *For any measurable non-negative test function F ,*

$$\begin{aligned} & \mathbb{N}^{(1)}[F(W_s(u), u \leq \zeta_s; s \geq 0)] \\ &= \mathbb{N}^{(1)}\left[\frac{2H}{\sqrt{2\pi}} F\left(\frac{1}{\sqrt{H}} W_{sH^2}(uH), u \leq \frac{1}{H} \zeta_{sH^2}; s \geq 0\right)\right]. \end{aligned} \quad (37)$$

Proof. We use the notations $\varphi_{\Sigma|H}$ and $\varphi_{H|\Sigma}$ in the sense they have in the previous lemma, relatively to the Brownian excursion of the lifetime. We write

$$\begin{aligned} & \mathbb{N}^{(1)}[F(W_s(u), u \leq \zeta_s; s \geq 0)] \\ &= \int_0^{+\infty} \varphi_{\Sigma|H}(\sigma) d\sigma \mathbb{N}_0[F(W_s(u), u \leq \zeta_s; s \geq 0) | \Sigma = \sigma, H = 1] \\ &= \int_0^{+\infty} \varphi_{\Sigma|H}(\sigma) d\sigma \mathbb{N}_0\left[F\left(\sigma^{1/4} W_{\frac{s}{\sigma}}\left(\frac{u}{\sqrt{\sigma}}\right), u \leq \sqrt{\sigma} \zeta_{\frac{s}{\sigma}}; s \geq 0\right) \middle| \Sigma = 1, H = \frac{1}{\sqrt{\sigma}}\right] \\ &= \int_0^{+\infty} \varphi_{\Sigma|H}\left(\frac{1}{h^2}\right) \frac{2dh}{h^3} \mathbb{N}_0\left[F\left(\frac{1}{\sqrt{h}} W_{sh^2}(uh), u \leq \frac{1}{h} \zeta_{sh^2}; s \geq 0\right) \middle| \Sigma = 1, H = h\right] \\ &= \int_0^{+\infty} dh \varphi_{H|\Sigma}(h) \varphi(h) \mathbb{N}_0\left[F\left(\frac{1}{\sqrt{h}} W_{sh^2}(uh), u \leq \frac{1}{h} \zeta_{sh^2}; s \geq 0\right) \middle| \Sigma = 1, H = h\right] \\ &= \mathbb{N}_0\left[\varphi(H) F\left(\frac{1}{\sqrt{H}} W_{sH^2}(uH), u \leq \frac{1}{H} \zeta_{sH^2}; s \geq 0\right) \middle| \Sigma = 1\right] \end{aligned}$$

where

$$\varphi(h) = \frac{\varphi_{\Sigma|H}\left(\frac{1}{h^2}\right)}{\varphi_{H|\Sigma}(h)} \frac{2}{h^3}.$$

The second equality is due to scaling (see (20)); other equalities are straightforward conditioning or change of variables. We conclude the proof of the lemma by noticing that, according to the previous lemma, $\varphi(h) = 2h/\sqrt{2\pi}$. Note that Formula (37) typically implies that

$$\mathbb{N}^{(1)}[F(R, \Sigma)] = \frac{2}{\sqrt{2\pi}} \mathbb{N}^{(1)}\left[H \cdot F\left(\frac{R}{\sqrt{H}}, \frac{1}{H^2}\right)\right]. \quad \square \quad (38)$$

3.4. Proof of Proposition 3

It suffices to show that

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{N}^{(1)} \left[\exists s, \hat{W}_s \in \sqrt{r} B \right] = -\frac{\delta^2}{2}. \quad (39)$$

From (37), we deduce that, for any $r > 0$,

$$\begin{aligned} \mathbb{N}^{(1)} \left[\exists s, \hat{W}_s \in \sqrt{r} B \right] &= \frac{2}{\sqrt{2\pi}} \mathbb{N}^{(1)} \left[H \mathbf{1}_{\left\{ \exists s, \frac{1}{\sqrt{H}} \hat{W}_s \in \sqrt{r} B \right\}} \right] \\ &= \frac{2\sqrt{r}}{\sqrt{2\pi}} \mathbb{N}^{(1)} \left[H(W^{(r)}) \mathbf{1}_{\left\{ \exists s, \hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})} B \right\}} \right]. \end{aligned}$$

The desired result will follow from the following facts

$$\begin{aligned} \liminf_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{N}^{(1)} \left[H(W^{(r)}) \mathbf{1}_{\left\{ \exists s, \hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})} \overset{\circ}{B} \right\}} \right] \\ \geq -\inf \left\{ J_1(W); \exists s, \hat{W}_s \in \sqrt{H(W)} \overset{\circ}{B} \right\} \end{aligned} \quad (40)$$

$$\begin{aligned} \limsup_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{N}^{(1)} \left[H(W^{(r)}) \mathbf{1}_{\left\{ \exists s, \hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})} \bar{B} \right\}} \right] \\ \leq -\inf \left\{ J_1(W); \exists s, \hat{W}_s \in \sqrt{H(W)} \bar{B} \right\} \end{aligned} \quad (41)$$

and the fact that the right-hand sides in (40) and (41) are both equal to $-\delta^2/2$. Let us start with the lim inf. Let $\eta > 0$. We write

$$\begin{aligned} \frac{1}{r} \log \mathbb{N}^{(1)} \left[H(W^{(r)}) \mathbf{1}_{\left\{ \exists s, \hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})} \overset{\circ}{B} \right\}} \right] \\ \geq \frac{1}{r} \log \eta + \frac{1}{r} \log \mathbb{N}^{(1)} \left[H(W^{(r)}) > \eta, \exists s, W_s^r \in \sqrt{H(W^{(r)})} \overset{\circ}{B} \right]. \end{aligned}$$

We apply the large deviation principle to the open set $\{W; H(W) > \eta, \exists s, W_s \in \sqrt{H(W)} \overset{\circ}{B}\}$. Then we let $\eta \rightarrow 0$.

For the lim sup we also easily get rid of the $H(W^{(r)})$ factor, for instance by applying the Hölder inequality with an exponent approaching 1. Then we use again the large deviation principle, this time for the closed set $\{W; \exists s, \hat{W}_s \in \sqrt{H(W)} \bar{B}\}$.

Now, let us compute the infima on the right-hand sides of (40) and (41). Let $(W, \zeta) \in \mathcal{P}^1$ be a snake excursion such that at time T , we have $\hat{W}_T \in \sqrt{H(W)} \overset{\circ}{B}$ so that $|\hat{W}_T| \geq \delta \sqrt{H(W)} \geq \delta \sqrt{\zeta_T}$. Suppose that, as in the proof of Proposition 1, we eliminate all excursions of the lifetime over its future infimum up to time T , and similarly after time T all excursions over the (past) infimum, forcing the lifetime to be constant on these intervals. We get a new lifetime process $\tilde{\zeta}$ which is non-decreasing up to T and non-increasing after T . The corresponding snake excursion $(\tilde{W}, \tilde{\zeta})$ satisfies $J_1(\tilde{W}, \tilde{\zeta}) \leq J_1(W, \zeta)$. We want to prove that $J_1(\tilde{W}, \tilde{\zeta})$ is greater than $\delta^2/2$. As in (32), the Cauchy–Schwarz inequality implies

$$\frac{1}{4} \int_0^T \frac{|\dot{\tilde{W}}_s|^2}{|\dot{\tilde{\zeta}}_s|} ds \geq \frac{|\hat{\tilde{W}}_T|^2}{4 \tilde{\zeta}_T} \geq \frac{\delta^2}{4},$$

the last inequality being justified by $|\hat{W}_T| = |\hat{W}_T| \geq \delta\sqrt{\zeta_T} = \delta\sqrt{\tilde{\zeta}_T}$. With similar bound on $[T, 1]$, we obtain as desired that $J_1(W, \zeta) \geq \delta^2/2$. Conversely, it is easy to construct snake excursion functions with piecewise linear “tent-like” lifetime and spatial motion such that $\exists s, \hat{W}_s \in \sqrt{H(W)} \bar{B}$ and such that the corresponding values of J_1 approach $\delta^2/2$. This concludes the proof of the proposition. \square

3.5. Proof of Proposition 4

This proposition is an easy corollary of Proposition 2. Using (25) and (38), we obtain:

$$\begin{aligned} \mathbb{P}^{|H|=1} \left(\frac{R^2}{Z^\beta} \geq A \right) &= \mathbb{N}^{|1|} \left(\frac{R^2}{\Sigma^\beta} \geq \frac{A}{4^\beta} \right) \\ &= \frac{2}{\sqrt{2\pi}} \mathbb{N}^{(1)} \left[H \mathbf{1} \left(\frac{R^2}{H^{1-2\beta}} \geq 2^{-2\beta} A \right) \right]. \end{aligned}$$

Using the same argument as in the proof of Proposition 3, we can get rid of the factor H in the latter expression. We are left with an expression identical to the one treated in the proof of Proposition 2 with $\alpha = 1 - 2\beta$. We conclude that

$$\lim_{A \rightarrow +\infty} \frac{1}{A^{\frac{1}{1+\beta}}} \log \mathbb{P}^{|H|=1} \left(\frac{R^2}{Z^\beta} \geq A \right) = c(1 - 2\beta)$$

which leads to the announced result. \square

3.6. Proof of Proposition 5

Using successively the occupation times formula (23), Lemma 12 and definition (28) of $W^{(r)}$, we obtain

$$\begin{aligned} \mathbb{P}^{|H|=1} \left(\frac{1}{Z} \int_0^1 Y_t(\sqrt{r}B) dt \geq \gamma \right) &= \mathbb{N}^{|1|} \left(\frac{1}{\Sigma} \int_0^\Sigma \mathbf{1}_{\{\hat{W}_s \in \sqrt{r}B\}} ds \geq \gamma \right) \\ &= \mathbb{N}^{(1)} \left[\frac{2H}{\sqrt{2\pi}} \mathbf{1} \left(H^2 \int_0^{1/H^2} \mathbf{1}_{\{\frac{1}{\sqrt{H}} \hat{W}_{sH^2} \in \sqrt{r}B\}} ds \geq \gamma \right) \right] \\ &= \frac{2\sqrt{r}}{\sqrt{2\pi}} \mathbb{N}^{(1)} \left[H(W^{(r)}) \mathbf{1} \left(\int_0^1 \mathbf{1}_{\{\hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})}B\}} ds \geq \gamma \right) \right]. \end{aligned}$$

Repeating the same argument as in the proof of Proposition 3, the latter expression has the same asymptotic behavior, in the exponential scale, as

$$\mathbb{N}^{(1)} \left[\int_0^1 \mathbf{1}_{\{\hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})}B\}} ds \geq \gamma \right]. \quad (42)$$

Mimicking the proofs of Propositions 3 and 1 we get

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \left\{ \mathbb{N}^{(1)} \left[\int_0^1 \mathbf{1}_{\{\hat{W}_s^{(r)} \in \sqrt{H(W^{(r)})}B\}} ds \geq \gamma \right] \right\}$$

$$\begin{aligned}
&= -\inf \left\{ J_1(W); \int_0^1 \mathbf{1}_{\{\hat{W}_s \in \sqrt{H(W)}B\}} ds \geq \gamma \right\} \\
&= -\inf \left\{ \frac{1}{2} \left(\frac{1}{a} H^2 + \frac{1}{1-b} H^2 \right) + \frac{(\delta \sqrt{H})^2}{2H}, 0 < a \leq a + \gamma \leq b < 1, H > 0 \right\} \\
&= -\frac{\delta^2}{2}
\end{aligned}$$

which gives the sought-after result. \square

3.7. Proof of (9)

Using successively (25), (38) and an integration by parts, we get

$$\begin{aligned}
\mathbb{P}^{|H|=1}[Z \geq r] &= \mathbb{N}^{(1)}(\Sigma \geq 4r) \\
&= \frac{2}{\sqrt{2\pi}} \mathbb{N}^{(1)}[H \mathbf{1}_{\{H^{-2} \geq 4r\}}] \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{1/2\sqrt{r}} \varphi_{H|\Sigma}(y) y dy \\
&= \frac{2}{\sqrt{2\pi}} \left[F_H^{(\Sigma=1)}\left(\frac{1}{2\sqrt{r}}\right) \frac{1}{2\sqrt{r}} - \int_0^{\frac{1}{2\sqrt{r}}} F_H^{(\Sigma=1)}(y) dy \right]
\end{aligned}$$

where $F_H^{(\Sigma=1)}$ is the distribution function of H under $\mathbb{N}^{(1)}$. The following expansion can be found for instance in [19]:

$$F_H^{(\Sigma=1)}(y) = \sum_{n=-\infty}^{+\infty} (1 - 4y^2 n^2) e^{-2y^2 n^2}.$$

We use Lemma 21 of the Appendix and note that the integral term in the expression above is negligible. Finally we get (9). \square

4. A large deviation principle for the Brownian snake under excursion measure

In this section we obtain a (partial) large deviation principle for the Brownian snake considered under the excursion measure. After some technical preliminaries we state the result and prove it in three steps.

4.1. Preliminaries on the rate function

Let \mathcal{P} be the union over $\sigma > 0$ of all the sets \mathcal{P}^σ defined in Section 2. The space \mathcal{P} is endowed with the metric

$$d((W, \zeta), (W', \zeta')) = |\Sigma(W) - \Sigma(W')| + \sup_{s \geq 0} |\zeta_s - \zeta'_s| + \sup_{s \geq 0} \sup_{r \leq \zeta_s \vee \zeta'_s} |W_s(r) - W'_s(r)|$$

so that it induces on every space \mathcal{P}^σ the metric (27) used before. The notation \mathcal{H} refers to the elements of \mathcal{P} for which $s \rightarrow \zeta_s$ and $s \rightarrow \hat{W}_s = W_s(\zeta_s)$ are absolutely continuous functions. We first extend – in the obvious way – J_1 , defined on \mathcal{P}^1 to a rate function defined on all of \mathcal{P} . In the sequel an element $(W, \zeta) \in \mathcal{P}$ is often simply denoted W .

Lemma 13. Let us call J the function on \mathcal{P} defined by

$$J(W) = \frac{1}{2} \int_0^{\Sigma(W)} \dot{\zeta}_s^2 ds + \frac{1}{4} \int_0^{\Sigma(W)} \frac{|\dot{W}_s|^2}{|\dot{\zeta}_s|} ds \quad (43)$$

if $W \in \mathcal{H}$ and $+\infty$ if not. Then,

(1) the function J is invariant under the scaling transformation defined in (20):

$$\forall \alpha > 0, \quad J(\theta_\alpha(W)) = J(W); \quad (44)$$

(2) for all $L, \sigma_1 > 0$, the set $\{W \in \mathcal{P}; J(W) \leq L, \Sigma(W) \leq \sigma_1\}$ is compact;

(3) the function J is lower semi-continuous that is, for every sequence (W_n) converging to W_0 in \mathcal{P} ,

$$\liminf_{n \rightarrow +\infty} J(W_n) \geq J(W_0).$$

Proof. Assertion (1) amounts to trivial changes of variables in the definition of J . Assertion (2) was obtained in [8] when J is restricted to \mathcal{P}^1 (thus equals J_1). Now if (W_n) is a sequence in \mathcal{P} such that $J(W_n) \leq L$ and $\Sigma(W_n) \leq \sigma_1$, we write $W_n = \theta_{\Sigma(W_n)}(W_n^1)$ where $W_n^1 \in \mathcal{P}^1$. By (1) we have $J(W_n^1) \leq L$, so we can extract a converging subsequence $(W_{n_k}^1)$. A further extraction can also make $\Sigma(W_{n_k})$ converge and we finally use the continuity of the scaling operators to conclude that (W_{n_k}) converge. Thus assertion (2) is proved. Assertion (3) follows using a proof by contradiction. \square

4.2. Preliminaries on the Brownian snake

In this subsection we estimate the moments of the Hölder variation of the Brownian snake.

Lemma 14. Let $(\zeta_s)_{s \in [0,1]}$ be a Brownian excursion normalized to have length 1. There exists a constant c such that, for all $0 < s < t < 1$ and all $q \geq 1$,

$$\mathbb{E} \left[\sup_{[u,v] \subset [s,t]} |\zeta_u - \zeta_v|^q \right] \leq c^q q^{q/2} |s - t|^{q/2}.$$

Proof. Such a normalized Brownian excursion can be constructed by $\zeta_t = (1-t) \left| B_{\frac{t}{1-t}} \right|$ where B is a Brownian motion in \mathbb{R}^3 , starting from 0, as is proved for instance in [20] p. 42. Elementary inequalities show that

$$|\zeta_u - \zeta_v| \leq \left| B_{\frac{u}{1-u}} - B_{\frac{v}{1-v}} \right| + |t - s| \left| B_{\frac{v}{1-v}} \right|. \quad (45)$$

We first restrict to $[s, t] \subset [0, 1/2]$. We note that

$$\begin{aligned} \sup_{[u,v] \subset [s,t]} \left| B_{\frac{u}{1-u}} - B_{\frac{v}{1-v}} \right| &= \sup_{[u,v] \subset [\frac{s}{1-s}, \frac{t}{1-t}]} |B_u - B_v| \\ &\leq 2 \sup_{h \leq \frac{t}{1-t} - \frac{s}{1-s}} \left| B_{h+\frac{s}{1-s}} - B_{\frac{s}{1-s}} \right| \\ &\stackrel{(d)}{=} 2 \sqrt{\frac{t}{1-t} - \frac{s}{1-s}} \sup_{h \leq 1} |B_h| \end{aligned}$$

where the latter equality in law is simply a Brownian scaling. So, to treat the first term in (45), it suffices to prove that $\mathbb{E}(\sup_{u \leq 1} |B_u|^q) \leq c^q q^{q/2}$. But this is a consequence of the finiteness, for small $\alpha > 0$, of $\mathbb{E}[\exp(\alpha \sup_{u \leq 1} |B_u|^2)]$. The latter point follows for instance from the theory of large deviations of the Brownian motion. The second term in (45) is easier. The case $[s, t] \subset [1/2, 1]$ is obtained by time-reversal and the general case is a consequence of the previous two particular cases. \square

Lemma 15. *There exists a constant c such that, for all $0 < s < t < 1$ and all $k \geq 1$,*

$$\mathbb{N}^{(1)} \left[|\hat{W}_s - \hat{W}_t|^k \right] \leq c^k k^{3k/4} |s - t|^{k/4}.$$

Proof. We recall that the Brownian snake we consider has a spatial motion distributed as a Brownian motion (B_s) in \mathbb{R}^d . By (D2) of Section 2, $W_s(\cdot)$ and $W_t(\cdot)$ are two Brownian trajectories which coincide up to time $\inf_{[s,t]} \zeta$ and after that time they evolve independently for durations $\zeta_s - \inf_{[s,t]} \zeta$ and $\zeta_t - \inf_{[s,t]} \zeta$ respectively. Thus,

$$\mathbb{N}^{(1)} \left[|\hat{W}_s - \hat{W}_t|^k |(\zeta_s, s \geq 0) \right] = \left| \zeta_s + \zeta_t - 2 \inf_{[s,t]} \zeta \right|^{k/2} \mathbb{E}(|B_1|^k).$$

But it is elementary that $\mathbb{E}(|B_1|^k) \leq c^k k^{k/2}$ and we note that obviously

$$\left| \zeta_s + \zeta_t - 2 \inf_{[s,t]} \zeta \right| \leq 2 \sup_{[u,v] \subset [s,t]} |\zeta_u - \zeta_v|$$

so that the announced result follows from Lemma 14. \square

Lemma 16. *For $(W_s, \zeta_s)_{s \in [0,1]} \in \mathcal{P}^1$ and $\gamma \in (0, 1/4)$, set*

$$|\hat{W}|_\gamma = \sup_{0 \leq s < t \leq 1} \frac{|\hat{W}_s - \hat{W}_t|}{|s - t|^\gamma}.$$

Then there exists a constant c such that for all k large enough

$$\mathbb{N}^{(1)} \left[|\hat{W}|_\gamma^k \right] \leq c^k k^{3k/4}.$$

Proof. The argument is classical. Let D_m be the regular subdivision of $[0, 1]$ with stepsize 2^{-m} . We set $Z_m = \sup\{|\hat{W}_s - \hat{W}_t|; s, t \in D_m, |s - t| = 2^{-m}\}$. But $\sup_{|s-t| \leq 2^{-m}} |\hat{W}_s - \hat{W}_t| \leq 2 \sum_{i \geq m} Z_i$. For $W \in \mathcal{P}^1$,

$$\begin{aligned} |\hat{W}|_\gamma &= \sup_{m \geq 0} \sup_{2^{-m-1} \leq |s-t| \leq 2^{-m}} \frac{|\hat{W}_s - \hat{W}_t|}{|s - t|^\gamma} \\ &\leq \sup_{m \geq 0} 2^{(m+1)\gamma} 2 \sum_{i \geq m} Z_i \leq 2^{\gamma+1} \sum_{i \geq 0} 2^{i\gamma} Z_i. \end{aligned}$$

Passing to the k -norm, we get

$$\mathbb{N}^{(1)} \left[|\hat{W}|_\gamma^k \right]^{1/k} \leq c \sum_{i \geq 0} 2^{i\gamma} \mathbb{N}^{(1)} \left[Z_i^k \right]^{1/k}. \quad (46)$$

But using Lemma 15, we have $\mathbb{N}^{(1)} \left[Z_i^k \right] \leq 2^i c^k k^{3k/4} (2^{-i})^{k/4}$. Injecting this bound in (46) leads to the desired result. \square

Lemma 17. For $(W_s, \zeta_s)_{s \in [0,1]} \in \mathcal{P}^1$ and $\gamma \in (0, 1/4)$, set

$$|\zeta|_{2\gamma} = \sup_{0 \leq s < t \leq \sigma} \frac{|\zeta_s - \zeta_t|}{|s - t|^{2\gamma}}.$$

Then there exists a constant c such that for all k large enough

$$\mathbb{N}^{(1)} \left[|\zeta|_{2\gamma}^k \right] \leq c^k k^{k/2}.$$

Proof. This is proved similarly to Lemma 16, using the bound given by Lemma 14. \square

4.3. Statement of a large deviation principle

We are now ready to examine the large deviation of the Brownian snake under its excursion measure.

Theorem 18. As $r \rightarrow +\infty$, the laws under \mathbb{N}_0 of $W^{(r)}$ satisfy a partial large deviation principle with speed r and rate function J in the following sense:

- for every open subset U of \mathcal{P} ,

$$\liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[W^{(r)} \in U \right] \geq -\inf_U J \quad (47)$$

- for every closed subset $F \subset \{W \in \mathcal{P}; \Sigma(W) \leq \sigma_1\} \setminus \{0\}$ with $\sigma_1 > 0$,

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[W^{(r)} \in F \right] \leq -\inf_F J. \quad (48)$$

Proof. As the reader can check, thanks to the definition of the metric on \mathcal{P} , supposing that 0 does not belong to the closed set F implies the boundedness of $\mathbb{N}_0 [W^{(r)} \in F]$.

For A a Borel subset of \mathcal{P} , Formula (21) gives:

$$\mathbb{N}_0(W^{(r)} \in A) = \int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in A). \quad (49)$$

By the contraction principle (see [18] 4.2.1) applied to the continuous function θ_σ and to the large deviation principle given in Theorem 10, we have,

- for every open subset U of \mathcal{P} ,

$$\liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}^{(1)} \left[\theta_\sigma(W^{(r)}) \in U \right] \geq -I(\sigma, U) \quad (50)$$

- for every closed subset K of \mathcal{P} ,

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}^{(1)} \left[\theta_\sigma(W^{(r)}) \in K \right] \leq -I(\sigma, K) \quad (51)$$

where, for $A \subset \mathcal{P}$,

$$I(\sigma, A) = \inf \{J_1(\tilde{W}); \tilde{W} \in \mathcal{P}^1 \text{ and } \theta_\sigma(\tilde{W}) \in A\} = \inf_{A \cap \mathcal{P}^\sigma} J.$$

For the last equality, we have used (44). As a consequence, for any $A \subset \mathcal{P}$, we have

$$\inf_{\sigma > 0} I(\sigma, A) = \inf_{W \in A} J(W). \quad (52)$$

We now want to use Inequality (50) [respectively Inequality (51)] in Formula (49) to get, via Eq. (52), the desired result (47) [respectively (48)]. This is a kind of Laplace method for the integral in (49). We do it in three steps.

Step 1. We first prove (47). For shortness we set $\tilde{I} = \inf_{\sigma > 0} I(\sigma, U) = \inf_U J$ and fix $\varepsilon > 0$. Let $\sigma_0 > 0$ be such that $I(\sigma_0, U) \leq \tilde{I} + \varepsilon$. We can find $W_0 \in \mathcal{P}^1$ such that $\theta_{\sigma_0}(W_0) \in U$ and $J_1(W_0) \leq I(\sigma_0, U) + \varepsilon$. Then, for σ in a neighbourhood of σ_0 , say $\sigma \in]\sigma_0 - \eta, \sigma_0 + \eta[$, we have $\theta_\sigma(W_0) \in U$ because U is open. Therefore,

$$I(\sigma, U) \leq J_1(W_0) \leq \tilde{I} + 2\varepsilon. \quad (53)$$

But trivially,

$$\mathbb{N}_0(W^{(r)} \in U) \geq \int_{\sigma_0 - \eta}^{\sigma_0 + \eta} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in U).$$

Then

$$\begin{aligned} \liminf_{r \uparrow +\infty} \left\{ \frac{1}{r} \log \mathbb{N}_0 \left[W^{(r)} \in U \right] + (\tilde{I} + 3\varepsilon) \right\} &= \liminf_{r \uparrow +\infty} \frac{1}{r} \log \left(\mathbb{N}_0 \left[W^{(r)} \in U \right] e^{r(\tilde{I} + 3\varepsilon)} \right) \\ &\geq \liminf_{r \uparrow +\infty} \frac{1}{r} \log \left(\int_{\sigma_0 - \eta}^{\sigma_0 + \eta} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in U) e^{r(\tilde{I} + 3\varepsilon)} \right) \\ &\geq 0 \end{aligned}$$

which proves (47) by letting ε tend to 0. The last inequality follows from Fatou's lemma

$$\begin{aligned} \liminf_{r \uparrow +\infty} \int_{\sigma_0 - \eta}^{\sigma_0 + \eta} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in U) e^{r(\tilde{I} + 3\varepsilon)} \\ \geq \int_{\sigma_0 - \eta}^{\sigma_0 + \eta} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \liminf_{r \uparrow +\infty} \left\{ \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in U) e^{r(\tilde{I} + 3\varepsilon)} \right\} \end{aligned}$$

and

$$\liminf_{r \uparrow +\infty} \left\{ \mathbb{N}^{(1)}(\theta_\sigma(W^{(r)}) \in U) e^{r(\tilde{I} + 3\varepsilon)} \right\} = +\infty$$

because of (50) and (53).

Step 2. We now prove (48) under a certain hypothesis on F . We first consider a closed subset F such that

$$F \subset \{W; \sigma_2 \leq \Sigma(W) \leq \sigma_1\} \quad \text{for certain } \sigma_1, \sigma_2 > 0. \quad (54)$$

We note that $\Sigma(W^{(r)}) = \Sigma(W)$ and rewrite (49):

$$\begin{aligned} \frac{1}{r} \log \mathbb{N}_0 \left[W^{(r)} \in F \right] &= \frac{1}{r} \log \int_{\sigma_2}^{\sigma_1} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)}(W^{(r)} \in \theta_\sigma^{-1}(F \cap \mathcal{P}^\sigma)) \\ &\leq \frac{1}{r} \log \left[\frac{\sigma_1 - \sigma_2}{2\sqrt{2\pi} (\sigma_1)^{3/2}} \mathbb{N}^{(1)} \left(W^{(r)} \in \tilde{F} \right) \right] \end{aligned}$$

where

$$\tilde{F} = \bigcup_{\sigma_1 \leq \sigma \leq \sigma_2} \theta_\sigma^{-1}(F \cap \mathcal{P}^\sigma).$$

By the continuity of $(\sigma, W) \rightarrow \theta_\sigma(W)$ and the closedness of F we obtain that \tilde{F} is closed (use a sequential criterion). Therefore we can apply (51) to claim that the lim sup of the right-hand side

in the above inequality is lower than $\inf_{\tilde{F}} J_1 = \inf_F J$. This completes the proof of (48) when F satisfies (54). Note that (54) holds when F is a closed ball of $\mathcal{P} \setminus \{0\}$, say of center \tilde{W} and radius R . Indeed, if not, we can find $W^n \in F$ with $\sigma_n = \Sigma(W^n) \downarrow 0$. Since $\sigma_n \leq \tilde{\sigma} = \Sigma(\tilde{W})$ for large n , we have

$$\begin{aligned} R &\geq d(W^n, \tilde{W}) = |\tilde{\sigma} - \sigma_n| + \sup_s |\tilde{\zeta}_s - \zeta_s^n| + \sup_{s,u} |\tilde{W}_s(u) - W_s^n(u)| \\ &\geq \tilde{\sigma} - \sigma_n + \sup_{\sigma_n \leq s \leq \tilde{\sigma}} |\tilde{\zeta}_s| + \sup_{\sigma_n \leq s \leq \tilde{\sigma}, u \geq 0} |\tilde{W}_s(u)|. \end{aligned}$$

Using the continuity of $(\tilde{W}, \tilde{\zeta})$ and passing to the limit $n \rightarrow +\infty$, we get

$$R \geq \tilde{\sigma} + \sup_{0 \leq s \leq \tilde{\sigma}} |\tilde{\zeta}_s| + \sup_{0 \leq s \leq \tilde{\sigma}, u \geq 0} |\tilde{W}_s(u)| = d(\tilde{W}, 0)$$

which contradicts $0 \notin F$.

We easily deduce that (48) holds if F is a compact subset of $\{W \neq 0; \Sigma(W) \leq \sigma_1\}$ for a certain $\sigma_1 > 0$. It suffices to cover F by a finite number of balls of small radius and to apply (48) to each of them.

Step 3. The proof of (48) will be complete if we prove exponential tightness that is, for every $\sigma_1 > 0$, there exist a constant c and, for every $L > 0$ a compact subset K_L of \mathcal{P} contained in $\{\Sigma \leq \sigma_1\}$ such that

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[W^{(r)} \notin K_L, \Sigma(W^{(r)}) \leq \sigma_1 \right] \leq -cL. \quad (55)$$

For fixed $\sigma_1 > 0$ and $\gamma \in (0, 1/4)$ and using the notation of Lemmas 16 and 17, we set, for $L > 0$,

$$K_L = \left\{ W \in \mathcal{P}; |\hat{W}|_\gamma^{4/3} \leq L, |\zeta|_{2\gamma}^2 \leq L, \Sigma(W) \leq \sigma_1 \right\}. \quad (56)$$

This set is compact as a consequence of the Arzelà–Ascoli Theorem. We note that

$$|\hat{W}^{(r)}|_\gamma = \frac{1}{r^{3/4}} |\hat{W}|_\gamma \quad \text{and} \quad |\zeta(W^{(r)})|_{2\gamma} = \frac{1}{\sqrt{r}} |\zeta(W)|_{2\gamma}.$$

The sought-after result (55) will follow from the existence of c such that, for large r ,

$$\mathbb{N}_0 \left[|\hat{W}|_\gamma^{4/3} \geq rL, \Sigma \leq \sigma_1 \right] \leq e^{-crL} \quad (57)$$

and

$$\mathbb{N}_0 \left[|\zeta|_{2\gamma}^2 \geq rL, \Sigma \leq \sigma_1 \right] \leq e^{-crL}. \quad (58)$$

It follows from Lemma 16 that, for c sufficiently small, $\mathbb{N}^{(1)} \left(e^{c|\hat{W}|_\gamma^{4/3}} \right) < +\infty$ which implies that, for every L , $\mathbb{N}^{(1)} \left(|\hat{W}|_\gamma^{4/3} \geq L \right) \leq c' e^{-cL}$. The left-hand side of (57) can be rewritten using the usual conditioning by the length (21). We note that for $\sigma > 0$, $|\theta_\sigma(\hat{W})|_\gamma = \sigma^{\frac{1}{4}-\gamma} |\hat{W}|_\gamma$ and we use the previous upper bound to get, for $r \geq 1$:

$$\text{l.h.s of (57)} = \int_0^{\sigma_1} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} \mathbb{N}^{(1)} \left(|\hat{W}|_\gamma^{4/3} \geq \frac{rL}{\sigma^{\frac{1}{3}-\frac{4\gamma}{3}}} \right)$$

$$\begin{aligned}
&\leq c' \int_0^{\sigma_1} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} e^{-c \frac{rL}{\sigma^{\frac{1}{3}-\frac{4\gamma}{3}}}} \\
&\leq c' \left(\int_0^{\sigma_1} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} e^{-\frac{c}{2} \frac{L}{\sigma^{\frac{1}{3}-\frac{4\gamma}{3}}}} \right) e^{-\frac{c}{2} \frac{rL}{\sigma_1^{\frac{1}{3}-\frac{4\gamma}{3}}}}.
\end{aligned}$$

This implies (57). The proof of (58) is similar. The proof of Theorem 18 is complete. \square

5. Applications to ordinary SBM

In this section we use Theorem 18 proved in the previous section to derive some corollaries for the ordinary SBM, more precisely Proposition 6 stated in the introduction.

5.1. Proof of (12)

By Theorem 18, we easily prove that

$$\begin{aligned}
&\lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, \Sigma \leq 4z \right] \\
&= -\inf \left\{ J(W); \exists s, \hat{W}_s \in B, \Sigma(W) \leq 4z \right\}.
\end{aligned} \tag{59}$$

The same result also holds under \mathbb{N}_x instead of \mathbb{N}_0 . The usual argument shows that this infimum coincides with

$$\inf \left\{ \frac{2H^2}{\sigma} + \frac{R^2}{2H}; H > 0, R \geq \delta, \sigma \leq 4z \right\} = 3.2^{-5/3} z^{-1/3} \delta^{4/3}. \tag{60}$$

Using the Poisson representation (26), the following trivial upper bound can be given:

$$\begin{aligned}
\mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, Z \leq z \right] &\leq \mathbb{P} \left[\exists i; \mathcal{R}(W^i) \cap r^{3/4} B \neq \emptyset, \Sigma(W^i) \leq 4z \right] \\
&= 1 - \exp \left\{ - \int \mu(dx) \mathbb{N}_x \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, \Sigma \leq 4z \right] \right\}.
\end{aligned}$$

As $r \uparrow +\infty$ this expression is equivalent to the integral term which tends to 0 according to (59) and (60) and we obtain,

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, Z \leq z \right] \leq -3.2^{-5/3} z^{-1/3} \delta^{4/3}. \tag{61}$$

Conversely, for $\alpha \in (0, 1)$, we start with an obvious lower bound:

$$\begin{aligned}
&\mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, Z \leq z \right] \\
&\geq \mathbb{P} \left[\exists i; \mathcal{R}(W^i) \cap r^{3/4} B \neq \emptyset, \Sigma(W^i) \leq 4\alpha z, \sum_{j \neq i} \Sigma(W^j) \leq 4(1-\alpha)z \right] \\
&= \mathbb{P} \left[\exists i; \mathcal{R}(W^i) \cap r^{3/4} B \neq \emptyset, \Sigma(W^i) \leq 4\alpha z \right] \mathbb{P}_\mu [Z \leq (1-\alpha)z].
\end{aligned}$$

For the last equality we have use the classical results on the law of a Poisson measure conditioned to have an atom in a certain set (see [21] chap. 10). Note that the latter factor does not depend on r . For the first factor, we can use again (59) and (60). Therefore, passing to the lim inf we get

$$\liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, Z \leq z \right] \geq -3.2^{-5/3} (\alpha z)^{-1/3} \delta^{4/3}. \quad (62)$$

Letting $\alpha \uparrow 1$, the combination of (62) and (61) completes the proof. \square

5.2. Proof of (13)

The Poisson representation (26) gives

$$\begin{aligned} \mathbb{P}_\mu \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, H \leq \gamma \sqrt{r} \right] \\ = \mathbb{P}_\mu \left[H \leq \gamma \sqrt{r} \right] \left(1 - \exp \left\{ - \int \mu(dx) \mathbb{N}_x \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, H \leq \gamma \sqrt{r} \right] \right\} \right). \end{aligned}$$

Therefore it is enough to prove that (13) holds with \mathbb{P}_μ replaced by \mathbb{N}_x or even \mathbb{N}_0 . But conditioning by the height of the lifetime excursion, one obtains

$$\begin{aligned} \mathbb{N}_0 \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset, H \leq \gamma \sqrt{r} \right] &= \int_0^{\gamma \sqrt{r}} \frac{dh}{2h^2} \mathbb{N}^{|h|} \left[\mathcal{R} \cap r^{3/4} B \neq \emptyset \right] \\ &= \int_0^{\gamma \sqrt{r}} \frac{dh}{2h^2} \mathbb{P}^{H=1} \left[\mathcal{R} \cap \frac{r^{3/4}}{\sqrt{h}} B \neq \emptyset \right] \\ &= \frac{1}{\sqrt{r}} \int_0^\gamma \frac{du}{2u^2} \mathbb{P}^{H=1} \left[\mathcal{R} \cap \sqrt{\frac{r}{u}} B \neq \emptyset \right]. \end{aligned}$$

Using (7) in the previous expression leads to the announced result. \square

6. Applications to the case with drift

In this section we apply the previous results to the situation where a drift is added, first to a Brownian snake and then to a SBM. We give here the proof of Propositions 7 and 8 stated in the introduction. For the convenience of the reader a sketch of proof for the easier Proposition 9 is given at the end of this section.

6.1. Large deviations for a Brownian snake with drift

We recall that $\mathbb{N}_0^{[b]}(dW)$ denotes the excursion measure of the Brownian snake with drift $-b$ as defined above Formula (19). Note that in the following theorem we introduce a new re-normalization of the Brownian snake denoted $(W^{[r]}, \zeta^{[r]})$ which is different from the one used in Theorem 10, denoted there $(W^{(r)}, \zeta^{(r)})$.

Theorem 19. *The laws under $\mathbb{N}_0^{[b]}(dW)$ of*

$$(W^{[r]}, \zeta^{[r]}) = \left(\frac{1}{r} W_{rs}(r \cdot), \frac{1}{r} \zeta_{rs} \right)_{s \geq 0} \quad (63)$$

satisfy as $r \rightarrow +\infty$ a large deviation principle with rate function $W \rightarrow J(W) + \frac{b^2}{2} \Sigma(W)$ in the following way:

- for every open subset U of \mathcal{P} ,

$$\liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} \left[W^{[r]} \in U \right] \geq - \inf_U \left(J + \frac{b^2}{2} \Sigma \right) \quad (64)$$

- for every closed subset K of \mathcal{P} not containing 0,

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} \left[W^{[r]} \in K \right] \leq -\inf_K \left(J + \frac{b^2}{2} \Sigma \right).$$

Proof. Under \mathbb{N}_0 and recalling that spatial motion is Brownian motion, the following scaling identity holds:

$$\mathbb{N}_0[F(W)] = \frac{1}{\sqrt{\alpha}} \mathbb{N}_0[F(\theta_\alpha(W))] \quad (65)$$

for $F: \mathcal{C}(\mathbb{R}_+, \mathcal{W}) \rightarrow \mathbb{R}_+$ measurable test function and $\alpha > 0$. It follows that, for any measurable subset A of \mathcal{P} ,

$$\begin{aligned} \mathbb{N}_0^{[b]} \left[W^{[r]} \in A \right] &= \mathbb{N}_0 \left[\mathbf{1}_{\{W^{[r]} \in A\}} e^{-\frac{b^2}{2} \Sigma(W)} \right] \\ &= \frac{1}{\sqrt{r}} \mathbb{N}_0 \left[\mathbf{1}_{\{\theta_r(W)^{[r]} \in A\}} e^{-\frac{b^2}{2} \Sigma(\theta_r(W))} \right] \\ &= \frac{1}{\sqrt{r}} \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in A\}} e^{-\frac{b^2}{2} r \Sigma(W^{(r)})} \right]. \end{aligned} \quad (66)$$

We have used

$$\Sigma(\theta_r(W)) = r \Sigma(W) = r \Sigma(W^{(r)}) \quad \text{and} \quad \theta_r(W)^{[r]} = W^{(r)}.$$

In expression (66) the multiplicative factor $1/\sqrt{r}$ does not interfere with exponential speed. For the remaining term, taking into account the large deviation principle for $W^{(r)}$ given in Theorem 18 and remembering the Varadhan–Laplace Lemma, Theorem 19 seems natural. Moreover this expression (66) shows that the proof of the theorem consists of checking the following facts:

- for every open subset U of \mathcal{P} ,

$$\liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in U\}} e^{-\frac{b^2}{2} r \Sigma(W^{(r)})} \right] \geq -\inf_U \left(J + \frac{b^2}{2} \Sigma \right) \quad (67)$$

- for every $\sigma_1 > 0$ and every compact subset K of \mathcal{P} not containing 0 and contained in $\{W \in \mathcal{P}; \Sigma(W) \leq \sigma_1\}$ we have,

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in K\}} e^{-\frac{b^2}{2} r \Sigma(W^{(r)})} \right] \leq -\inf_K \left(J + \frac{b^2}{2} \Sigma \right) \quad (68)$$

- for every $\sigma_1 > 0$, there exist a constant c and, for every $L > 0$ a compact subset K_L of \mathcal{P} contained in $\{\Sigma \leq \sigma_1\}$ such that

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \notin K_L, \Sigma(W^{(r)}) \leq \sigma_1\}} e^{-\frac{b^2}{2} r \Sigma(W^{(r)})} \right] \leq -c L \quad (69)$$

- for every $\sigma_1 > 0$, we have

$$\limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{\Sigma(W^{(r)}) \geq \sigma_1\}} e^{-\frac{b^2}{2} r \Sigma(W^{(r)})} \right] \leq -\sigma_1 \frac{b^2}{2}. \quad (70)$$

The last inequality (70) is trivial. The previous one (69) is a consequence of (55).

Now let us prove (67) for U open subset of \mathcal{P} . Let $y \neq 0$ belong to U and $\eta > 0$ be such that $B(y, \eta) \subset U$. Then $\Sigma(W) \leq \Sigma(y) + \eta$ if W belongs to $B(y, \eta)$. Hence,

$$\begin{aligned} \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in U\}} \exp \left(-r \frac{b^2}{2} \Sigma(W^{(r)}) \right) \right] &\geq \mathbb{N}_0 \left[W^{(r)} \in U \right] \exp - \left(r \frac{b^2}{2} (\Sigma(y) + \eta) \right) \\ &\geq \mathbb{N}_0 \left[W^{(r)} \in B(y, \eta) \right] \exp - \left(r \frac{b^2}{2} (\Sigma(y) + \eta) \right). \end{aligned}$$

Therefore, using Theorem 18 we get

$$\begin{aligned} \liminf_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in U\}} \exp \left(-r \frac{b^2}{2} \Sigma(W^{(r)}) \right) \right] &\geq - \inf_{B(y, \eta)} J - \frac{b^2}{2} (\Sigma(y) + \eta) \\ &\geq -J(y) - \frac{b^2}{2} (\Sigma(y) + \eta). \end{aligned}$$

We let $\eta \downarrow 0$ and minimize over $y \in U$ to get (67).

Now we prove (68) for K compact subset of \mathcal{P} not containing 0 and contained in $\{\Sigma \leq \sigma_1\}$. For any $\eta > 0$, the compact set K can be covered by a finite number of closed balls of radius η , not containing 0:

$$K \subset \bigcup_{i=1}^N \overline{B}(y_i, \eta).$$

Then

$$\begin{aligned} \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in K\}} e^{-r \frac{b^2}{2} \Sigma(W^{(r)})} \right] &\leq \sum_{i \leq N} \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in \overline{B}(y_i, \eta)\}} e^{-r \frac{b^2}{2} \Sigma(W^{(r)})} \right] \\ &\leq \sum_{i \leq N} \mathbb{N}_0 \left[W^{(r)} \in \overline{B}(y_i, \eta) \right] e^{-r \frac{b^2}{2} (\Sigma(y_i) - \eta)}. \end{aligned}$$

We use Theorem 18 (more precisely the case of closed balls) to get:

$$\begin{aligned} \limsup_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0 \left[\mathbf{1}_{\{W^{(r)} \in K\}} \exp \left(-r \frac{b^2}{2} \Sigma(W^{(r)}) \right) \right] &\leq - \inf_{i \leq N} \left(\inf_{\overline{B}(y_i, \eta)} J - \frac{b^2}{2} (\Sigma(y_i) - \eta) \right) \\ &\leq - \inf_{i \leq N} \inf_{\overline{B}(y_i, \eta)} \left(J + \frac{b^2}{2} \Sigma \right) + b^2 \eta \\ &= - \inf \left\{ \left(J + \frac{b^2}{2} \Sigma \right) (W); W \in \bigcup_{i \leq N} \overline{B}(y_i, \eta) \right\} + b^2 \eta \\ &\xrightarrow{\eta \rightarrow 0} - \inf_K \left(J + \frac{b^2}{2} \Sigma \right). \end{aligned} \tag{71}$$

For the last convergence we have used the lower semi-continuity of J and the continuity of $W \rightarrow \Sigma(W)$. This completes the proof of Theorem 19. \square

6.2. Proof of Proposition 7

Due to the Poisson representation (26), Formula (14) follows from

$$\lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} [\mathcal{R} \cap r B \neq \emptyset] = -2 \delta \sqrt{b}.$$

This is an easy consequence of Theorem 19:

$$\begin{aligned} \lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} [\mathcal{R} \cap r B \neq \emptyset] &= \lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} [\mathcal{R}(W^{[r]}) \cap B \neq \emptyset] \\ &= -\inf \left\{ J(W) + \frac{b^2}{2} \Sigma(W); \mathcal{R}(W) \cap B \neq \emptyset \right\} \\ &= -\inf \left\{ \frac{2H^2}{\sigma} + \frac{\delta^2}{2H} + \frac{b^2}{2} \sigma; H, \sigma > 0 \right\}. \end{aligned}$$

The last equality follows from the method used in the proofs of Propositions 1 and 2. We are left with easy optimization on variables H and σ which leads to (14). \square

6.3. Proof of Proposition 8

We first work under the excursion measure. We use the usual sequence of arguments: occupation times formula (23), definition (63) of the rescaled snake, Theorem 19 and the usual technique to simplify the infimum:

$$\begin{aligned} &\frac{1}{r} \log \mathbb{N}_0^{[b]} \left(\int_{\alpha r}^{+\infty} Y_t(r B) dt > \gamma r \right) \\ &= \frac{1}{r} \log \mathbb{N}_0^{[b]} \left[\text{Leb} \left(\{s \geq 0; \hat{W}_s^{[r]} \in B, \zeta_s^{[r]} \geq \alpha\} \right) > 4\gamma \right] \\ &\rightarrow -\inf \left\{ J + \frac{b^2}{2} \Sigma, \text{Leb} \left(\{s \geq 0; \hat{W}_s \in B, \zeta_s \geq \alpha\} \right) \geq 4\gamma \right\} \\ &= -\inf \left\{ \frac{2H^2}{\sigma - 4\gamma} + \frac{\delta^2}{2H} + \frac{b^2}{2} \sigma, H \geq \alpha, \sigma > 4\gamma \right\} = -I(\gamma). \end{aligned} \quad (72)$$

Computing the last infimum $I(\gamma)$ gives the right-hand side of (15). Note that

$$I(\gamma) = I(0) + 2\gamma b^2. \quad (73)$$

Let us now denote

$$\psi_r(W) = \int_{\alpha r}^{+\infty} Y_t(W)(r B) dt$$

so that we are, by the Poisson representation (26), interested in the behavior of $\sum_i \psi_r(W^i)$ where $\sum_i \delta_{W^i}(dW)$ has intensity $4 \int \mu(dx) \mathbb{N}_x^{[b]}(dW)$. Since

$$\begin{aligned} \mathbb{P}_\mu^{[b]} \left[\sum_i \psi_r(W^i) > \gamma r \right] &\geq \mathbb{P}_\mu^{[b]} \left[\exists i; \psi_r(W^i) > \gamma r \right] \\ &= 1 - \exp \left\{ -4 \int \mu(dx) \mathbb{N}_x^{[b]} [\psi_r(W) > \gamma r] \right\}, \end{aligned} \quad (74)$$

it is straightforward to obtain that

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} \left(\int_{\alpha r}^{+\infty} Y_t(r B) dt > \gamma r \right) \geq -I(\gamma). \quad (75)$$

Moreover when $\gamma = 0$, Inequality (74) is an equality and we deduce

$$\lim_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} \left(\int_{\alpha r}^{+\infty} Y_t(r B) dt > 0 \right) = -I(0). \quad (76)$$

From now on we will suppose $\gamma > 0$ and we are interested in proving the upper bound converse to (75). We notice that, for $k \leq n$ fixed integers, the event $\{\sum_i \psi_r(W^i) > \gamma r\}$ is included in the union of

$$\#\{i; \psi_r(W^i) > 0\} \geq k \quad (77)$$

and of the (finite) union of the events

$$\left\{ \exists i_1, \dots, i_k \text{ distinct; } \forall q, \psi_r(W^{i_q}) \geq \frac{m_q}{n} \gamma r \right\} \quad (78)$$

where m_1, \dots, m_k vary in $\{0, \dots, n\}$ so that $m_1 + \dots + m_k \geq n - k$. The definition of Poisson measure and the limit (72) for $\gamma = 0$ yield

$$\limsup_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} \left[\#\{i; \psi_r(W^i) > 0\} \geq k \right] \leq -kI(0).$$

We will take k so large that the probability of (77) becomes negligible with respect to the remaining events that we now estimate. It follows from (72) that

$$\begin{aligned} & \limsup_{r \rightarrow +\infty} \frac{1}{r} \log \mathbb{P}_\mu^{[b]} \left[\left\{ \exists i_1, \dots, i_k \text{ distinct; } \forall q, \psi_r(W^{i_q}) \geq \frac{m_q}{n} \gamma r \right\} \right] \\ & \leq - \sum_{q=1}^k I\left(\frac{m_q}{n} \gamma\right) \mathbf{1}_{\{m_q \neq 0\}} \\ & = - \frac{2\gamma b^2}{n} \sum_{q=1}^k m_q - I(0) \#\{q, m_q \neq 0\} \\ & \leq -2\gamma b^2 \left(1 - \frac{k}{n}\right) - I(0), \end{aligned}$$

the equality being justified by (73). Letting $n \rightarrow +\infty$ we obtain the desired upper bound. \square

6.4. Proof of Proposition 9

The proof of (16) is very similar to the previous ones and reduces to

$$\lim_{r \uparrow +\infty} \frac{1}{r} \log \mathbb{N}_0^{[b]} [H \geq r] = - \inf \left\{ \frac{2H^2}{\sigma} + \frac{R^2}{2H} + \frac{b^2}{2} \sigma; H \geq 1, R, \sigma > 0 \right\}.$$

We now pass to the proof of (17). First remark that the law of Z under $\mathbb{P}_\mu^{[b]}$ is the same as the law of Z under $\mathbb{P}_{|\mu| \delta_0}^{[b]}$. In that case we know that, for any test function F ,

$$\mathbb{E}_{|\mu|_{\delta_0}}^{[b]} [F(Z)] = \mathbb{E} \left[F \left(\frac{\tau_{4|\mu|}}{4} \right) e^{2|\mu| b - \frac{b^2}{2} \tau_{4|\mu|}} \right]$$

where $\tau_{4|\mu|}$ is the hitting time of $4|\mu|$ for the local time at 0 of a reflecting Brownian motion. The density of this variable is well known and we obtain

$$\mathbb{P}_{|\mu|_{\delta_0}}^{[b]} [Z \geq r] = \int_{4r}^{+\infty} e^{2|\mu| b - \frac{b^2}{2} x} \frac{2|\mu|}{\sqrt{2\pi} x^{3/2}} e^{-\frac{2|\mu|^2}{x}} dx.$$

Then (17) follows since we are in the situation treated by the following elementary lemma.

Lemma 20. *If $\beta > 0$, $\gamma \in \mathbb{R}$ and $\lim_{+\infty} g = 1$ then, as $r \rightarrow +\infty$,*

$$\int_r^{+\infty} \frac{e^{-\beta x}}{x^\gamma} g(x) dx \sim \frac{e^{-\beta r}}{\beta r^\gamma}.$$

Appendix

A.1. Proof of Lemma 11

Let ψ be an arbitrary test function and n denote the Itô measure of positive Brownian excursions (as defined for instance in [22] chap. XII). Using the well-known conditioning by the height H and the scaling property, we get

$$\begin{aligned} n[\psi(H, \Sigma)] &= \int_0^{+\infty} \frac{dh}{2h^2} n[\psi(h, \Sigma) | H = h] \\ &= \int_0^{+\infty} \frac{dh}{2h^2} n[\psi(h, h^2 \Sigma) | H = 1] \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{dh dx}{2h^2} \varphi_{\Sigma|H}(x) \psi(h, h^2 x) \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{d\sigma dy}{2\sigma^{3/2} y^4} \varphi_{\Sigma|H} \left(\frac{1}{y^2} \right) \psi(\sqrt{\sigma} y, \sigma) \end{aligned}$$

where the last equality follows from the change of variables $y = 1/\sqrt{x}$, $\sigma = x h^2$. But we can resume the above calculation, conditioning this time with respect to the length Σ and we get

$$\begin{aligned} n[\psi(H, \Sigma)] &= \int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} n[\psi(H, \sigma) | \Sigma = \sigma] \\ &= \int_0^{+\infty} \frac{d\sigma}{2\sqrt{2\pi} \sigma^{3/2}} n[\psi(\sqrt{\sigma} H, \sigma) | \Sigma = 1] \\ &= \int_0^{+\infty} \int_0^{+\infty} \frac{d\sigma dy}{2\sqrt{2\pi} \sigma^{3/2}} \varphi_{H|\Sigma}(y) \psi(\sqrt{\sigma} y, \sigma). \end{aligned}$$

The comparison of the results of both calculations gives the announced formula. \square

A.2. On the maximum of the normalized excursion

Lemma 21. *Let*

$$F(y) = \sum_{n=-\infty}^{+\infty} (1 - 4y^2 n^2) e^{-2y^2 n^2}.$$

Then, as $y \rightarrow 0$,

$$F(y) \sim \frac{\sqrt{2}\pi^{5/2}}{y^3} e^{-\frac{\pi^2}{2y^2}}.$$

Proof. We introduce the classical Jacobi θ function:

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi x n^2} \text{ satisfying } \theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right).$$

We get

$$\begin{aligned} F(y) &= -\frac{\pi^{3/2}}{\sqrt{2}} \frac{1}{y^3} \theta'\left(\frac{\pi}{2y^2}\right) \\ &= \frac{\pi^{5/2}}{\sqrt{2}} \frac{1}{y^3} \sum_{n=-\infty}^{+\infty} n^2 e^{-\frac{\pi^2 n^2}{2y^2}} \end{aligned}$$

which leads to the sought-after result. \square

A.3. Definition by martingale problems

The law of super-Brownian excursion with unit duration starting from x is the unique law such that the canonical process satisfies the following martingale problem:

$$\left\{ \begin{array}{l} \forall \phi \in \mathcal{D}, \forall t \in (0, 1), \\ Y_t(\phi) = \int_0^t Y_s \left(\frac{\Delta}{2} \phi \right) ds + \int_0^t \left(\frac{1}{Y_s(1)} - \frac{2}{1-s} \right) Y_s(\phi) ds + M_t(\phi) \\ \quad \text{where the local martingale } M(\phi) \text{ is such that} \\ \langle M(\phi) \rangle_t = \int_0^t Y_s(\phi^2) ds \\ \text{and} \\ \lim_{h \rightarrow 0} Y_h(1) = 0, \quad \lim_{h \rightarrow 0} \frac{Y_h}{Y_h(1)} = \delta_x \quad \text{a.s.} \end{array} \right.$$

Sketch of proof. Using the arguments of [6] Section 8, it suffices to prove that the law of SBM starting from μ conditioned to have a duration equal to 1 – that is conditioned to die at time 1 – denoted $\mathbb{P}_\mu^{|H=1|}$ is the unique solution of the martingale problem

$$\left\{ \begin{array}{l} \forall \phi \in \mathcal{D}, \forall t \in [0, 1), \\ Y_t(\phi) = \mu(\phi) + \int_0^t Y_s \left(\frac{\Delta}{2} \phi \right) ds + \int_0^t \left(\frac{1}{Y_s(1)} - \frac{2}{1-s} \right) Y_s(\phi) ds + M_t(\phi) \\ \quad \text{where the local martingale } M(\phi) \text{ is such that} \\ \langle M(\phi) \rangle_t = \int_0^t Y_s(\phi^2) ds. \end{array} \right.$$

We start with the law \mathbb{P}_μ of the ordinary SBM and notice that under this law the duration H has the density $\frac{2\mu(1)}{h^2} \exp(-2\mu(1)/h)$. It follows that, for $t \in (0, 1)$ on the σ -algebra of events prior to t , the law $\mathbb{P}_\mu^{|H=1|}$ has a density with respect to \mathbb{P}_μ which is given by

$$Z_t = \frac{e^{2\mu(1)}}{\mu(1)} \frac{Y_t(1)}{(1-t)^2} e^{-\frac{2Y_t(1)}{1-t}}.$$

Using the Girsanov Theorem the expression of the above martingale problem follows easily from the martingale problem defining the ordinary SBM. \square

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