

# A revisit to $W_2^n$ -theory of super-parabolic backward stochastic partial differential equations in $\mathbb{R}^{d\star}$

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## Abstract

Backward stochastic partial differential equations of parabolic type with variable coefficients are considered in the whole Euclidean space. Improved existence and uniqueness results are given in the Sobolev space  $H^n (=W_2^n)$  under weaker assumptions than those used by X. Zhou [X. Zhou, A duality analysis on stochastic partial differential equations, J. Funct. Anal. 103 (1992) 275–293]. As an application, a comparison theorem is obtained.

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## 1. Introduction

In this paper, we consider the Cauchy problem for backward stochastic partial differential equations (BSPDEs) in divergence form

$$\begin{cases} dp(t, x) = -\{\partial_{x^i}[a^{ij}(t, x)\partial_{x^j}p(t, x) + \sigma^{ik}(t, x)q^k(t, x)] + b^i(t, x)\partial_{x^i}p(t, x) \\ \quad - c(t, x)p(t, x) + v^k(t, x)q^k(t, x) + F(t, x)\}dt + q^k(t, x)dW_t^k, \\ (t, x) \in [0, T] \times \mathbb{R}^d, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}^d, \end{cases} \quad (1.1)$$

and in non-divergence form

$$\begin{cases} dp(t, x) = -[a^{ij}(t, x)\partial_{x^i x^j}^2 p(t, x) + b^i(t, x)\partial_{x^i}p(t, x) - c(t, x)p(t, x) \\ \quad + \sigma^{ik}(t, x)\partial_{x^i}q^k(t, x) + v^k(t, x)q^k(t, x) + F(t, x)]dt + q^k(t, x)dW_t^k, \\ (t, x) \in [0, T] \times \mathbb{R}^d, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

where  $W \triangleq \{W_t^k; t \geq 0\}$  is a  $d_1$ -dimensional Wiener process generating a natural filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ . The coefficients  $a, b, c, \sigma, v$  and the free term  $F$  and the terminal condition  $\phi$  are all random functions. An adapted solution of Eq. (1.1) or (1.2) is a  $\mathcal{P} \times B(\mathbb{R}^d)$ -measurable function pair  $(p, q)$  satisfying Eq. (1.1) or (1.2) under some appropriate sense, where  $\mathcal{P}$  is the predictable  $\sigma$ -algebra generated by  $\{\mathcal{F}_t\}_{t \geq 0}$ .

BSPDEs, a natural extension of BSDEs (see e.g. [5,14]), originally arise in the optimal control of processes with incomplete information, as adjoint equations (usually in the form of (1.1)) of Duncan–Mortensen–Zakai filtration equations (see e.g. [3,13,17,18,22]). In [11], an adapted version of stochastic Feynman–Kac formula is derived involving BSPDEs (in the form of (1.2)), which has been found useful in mathematical finance. A class of fully nonlinear BSPDEs, the so-called backward stochastic Hamilton–Jacobi–Bellman equations, are also introduced in the study of controlled non-Markovian processes by Peng [15]. For more aspects of BSPDEs, we refer to e.g. [2,6,19–21].

In [23], a  $W_2^n$ -theory of the Cauchy problem for BSPDEs of type (1.1) was established by the finite-dimensional approximation (Galerkin’s method) and a duality analysis on stochastic PDEs. Those results are basically complete however not refined due to a strong requirement on the coefficients. More specifically, the theory requires the boundedness of the derivatives of the coefficients up to the order  $n$ , to reach the regularity that  $p \in H^{n+1}$  and  $q \in H^n$  with respect to  $x$ . Comparing to the counterpart theory of PDEs, we believe that this requirement is not natural.

In this paper, we establish an improved  $W_2^n$ -theory of the Cauchy problem for BSPDEs of type (1.1) and (1.2). First we refine the existence and uniqueness result first given by Hu and Peng [8] concerning backward stochastic evolution equations in Hilbert spaces. Then we use it to prove the existence and uniqueness of the weak solution (see Definition 2.1) of Eq. (1.1). Following this result, we obtain the existence, uniqueness and regularity of the strong solution (see Definition 2.1) of Eq. (1.2), under much weaker assumptions on the coefficients than those used by Zhou [23], by applying some classical techniques from the theory of PDEs instead of duality analysis. Our improvements are natural and substantial. Our results require weaker conditions—the boundedness of the derivatives of the coefficients up to the order  $n - 1$ , to reach the regularity that  $p \in H^{n+1}$  and  $q \in H^n$  with respect to  $x$ , which coincides with the classical theory of PDEs. As an application of our results, we prove a comparison theorem for the strong solution of Eq. (1.2), which, in some sense, improves the results obtained by Ma and Yong [12].

In this work, we mainly focus on the linear BSPDE under the super-parabolicity condition (Assumption 2.2). There are other results discussing BSPDEs under more general assumptions. For instance, Ma and Yong [12] study the linear BSPDE under the parabolicity (or degenerate) condition only. To compensate the degeneracy, their results require more assumptions on the coefficients, the free term and the terminal condition. Moreover, Hu et al. [7] and Tang [19] study some semi-linear BSPDEs by various approaches.

This paper is organized as follows. In Section 2, we present our main results (Theorems 2.1–2.3), and prove Theorem 2.3. In Section 3, we discuss backward stochastic evolution equations in Hilbert spaces, and then prove Theorem 2.1. In Section 4, we complete the proof of Theorem 2.2. Finally in Section 5, we prove a comparison theorem for the strong solution of Eq. (1.2).

## 2. Main results

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  be a complete filtered probability space on which is defined a  $d_1$ -dimensional Wiener process  $W = \{W_t; t \geq 0\}$  such that  $\{\mathcal{F}_t\}_{t \geq 0}$  is the natural filtration generated by  $W$ , augmented by all the  $P$ -null sets in  $\mathcal{F}$ . Fix a positive number  $T$ . Denote by  $\mathcal{P}$  the  $\sigma$ -algebra of predictable sets on  $\Omega \times (0, T)$  associated with  $\{\mathcal{F}_t\}_{t \geq 0}$ .

For the sake of convenience, we denote

$$D_i = \partial_{x^i}, \quad D_{ij} = \partial_{x^i x^j}^2, \quad i, j = 1, \dots, d,$$

and for any multi-index  $\alpha = (\alpha_1, \dots, \alpha_d)$

$$D^\alpha = (\partial_{x^1})^{\alpha_1} (\partial_{x^2})^{\alpha_2} \dots (\partial_{x^d})^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

Moreover, denote by  $Du$  and  $D^2u$  respectively the gradient and the Hessian matrix for the function  $u$  defined on  $\mathbb{R}^d$ . We will also use the summation convention.

Throughout the paper, by saying that a vector-valued or matrix-valued function belongs to a function space (for instance,  $Du \in L^2(\mathbb{R}^d)$ ), we mean all the components belong to that space.

Let  $n$  be an integer. Let  $H^n = H^n(\mathbb{R}^d)$  ( $n \neq 0$ ) be the Sobolev space  $W_2^n(\mathbb{R}^d)$ . We denote

$$H^0 = L^2 = H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d), \\ \mathbb{H}^n = \mathbb{H}^n(\mathbb{R}^d) = L^2(\Omega \times (0, T), \mathcal{P}, H^n).$$

In addition, denote  $\|\cdot\|_n = \|\cdot\|_{H^n}$ . Moreover, for a function  $u$  defined on  $\Omega \times (0, T) \times \mathbb{R}^d$ , we denote

$$\|u\|_n^2 = E \int_0^T \|u(t, \cdot)\|_n^2 dt.$$

The same notations will be used for vector-valued and matrix-valued functions, and in this case we denote  $|u|^2 = \sum_i |u^i|^2$  and  $|u|^2 = \sum_{ij} |u^{ij}|^2$ , respectively.

Let us now turn to the notions of solutions to Eqs. (1.1) and (1.2).

**Definition 2.1.** A function pair  $(p, q) : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^{d_1}$  is called

- (i) a weak solution of Eq. (1.1), if  $p \in \mathbb{H}^1$  and  $q \in \mathbb{H}^0$ , such that for every  $\eta \in H^1$  (or  $C_0^\infty(\mathbb{R}^d)$ ) and almost every  $(\omega, t) \in \Omega \times [0, T]$ , it holds that

$$\begin{aligned} \int_{\mathbb{R}^d} p(t, x) \eta(x) dx &= \int_{\mathbb{R}^d} \phi(x) \eta(x) dx + \int_t^T \int_{\mathbb{R}^d} \{D_i [a^{ij}(t, x) D_j p(t, x) \\ &\quad + \sigma^{ik}(t, x) q^k(t, x)] + b^i(t, x) D_i p(t, x) - c(t, x) p(t, x) \} \eta(x) dx dt \end{aligned}$$

$$\begin{aligned}
& + v^k(t, x)q^k(t, x) + F(t, x)\} \eta(x) dx dt \\
& - \int_t^T \int_{\mathbb{R}^d} q^k(t, x) \eta(x) dx dW_t^k;
\end{aligned} \tag{2.1}$$

(ii) a strong solution of Eq. (1.2), if  $p \in \mathbb{H}^2$ ,  $q \in \mathbb{H}^1$  and  $p \in C([0, T], L^2(\mathbb{R}^d))(a.s.)$  such that for all  $t \in [0, T]$  and a.s.  $\omega \in \Omega$ , it holds that

$$\begin{aligned}
p(t, x) = & \phi(x) + \int_t^T [a^{ij}(t, x) D_{ij} p(t, x) + b^i(t, x) D_i p(t, x) - c(t, x) p(t, x) \\
& + \sigma^{ik}(t, x) D_i q^k(t, x) + v^k(t, x) q^k(t, x) + F(t, x)] dt \\
& - \int_t^T q^k(t, x) dW_t^k
\end{aligned} \tag{2.2}$$

for a.e.  $x \in \mathbb{R}^d$ .

Now fix some constants  $K \in (1, \infty)$  and  $\kappa \in (0, 1)$ .

**Assumption 2.1.** The given functions  $a, b, c, \sigma, v$  and  $F$  are  $\mathcal{P} \times B(\mathbb{R}^d)$ -measurable with values in the set of real symmetric  $d \times d$  matrices,  $\mathbb{R}^d$ ,  $\mathbb{R}$ ,  $\mathbb{R}^{d \times d_1}$ ,  $\mathbb{R}^{d_1}$ , and  $\mathbb{R}$ , respectively. The real function  $\phi$  is  $\mathcal{F}_T \times B(\mathbb{R}^d)$ -measurable.

**Assumption 2.2.** We assume the super-parabolicity condition, i.e.,

$$\kappa I + (\sigma^{ik})(\sigma^{ik})^* \leq 2(a^{ij}) \leq KI, \quad \forall(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d.$$

Then we have the following result concerning the existence and uniqueness of the weak solution of Eq. (1.1). The proof of this theorem will be given in Section 3.

**Theorem 2.1.** Let the functions  $a^{ij}, b^i, c, \sigma^{ik}$  and  $v^k$  satisfy Assumptions 2.1 and 2.2, and be bounded by  $K$ . Suppose

$$F \in \mathbb{H}^{-1}, \quad \phi \in L^2(\Omega, \mathcal{F}_T, L^2).$$

Then Eq. (1.1) has a unique weak solution  $(p, q)$  in the space  $\mathbb{H}^1 \times \mathbb{H}^0$  such that  $p \in C([0, T], L^2)(a.s.)$ , and

$$\|p\|_1^2 + \|q\|_0^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_0^2 \leq C(\|F\|_{-1}^2 + E\|\phi\|_0^2), \tag{2.3}$$

where the constant  $C = C(K, \kappa, T)$ .

To investigate the (strong) solution of Eq. (1.2), we need, in addition, the following

**Assumption 2.3.** There exists a function  $\gamma : [0, \infty) \rightarrow [0, \infty)$  such that  $\gamma$  is continuous and increasing,  $\gamma(r) = 0$  if and only if  $r = 0$ , and for any  $(\omega, t) \in \Omega \times [0, T]$  and any  $x, y \in \mathbb{R}^d$ ,

$$|a(\omega, t, x) - a(\omega, t, y)| + |\sigma(\omega, t, x) - \sigma(\omega, t, y)| \leq \gamma(|x - y|). \tag{2.4}$$

Then we have the following theorem, whose proof will be given in Section 4.

**Theorem 2.2.** Let Assumptions 2.1–2.3 be satisfied. Assume that the functions  $b^i, c$  and  $v^k$  are bounded by  $K$ . Suppose

$$F \in \mathbb{H}^0, \quad \phi \in L^2(\Omega, \mathcal{F}_T, H^1).$$

Then Eq. (1.2) has a unique strong solution  $(p, q)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$  such that  $p \in C([0, T], L^2) \cap L^\infty([0, T], H^1)(a.s.)$ , and moreover,

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C(\|F\|_0^2 + E\|\phi\|_1^2), \quad (2.5)$$

where the constant  $C$  depends only on  $K, \kappa, T$  and the function  $\gamma$ .

With the aid of Theorem 2.2, we can obtain the following.

**Theorem 2.3.** Let Assumptions 2.1 and 2.2 be satisfied. Let  $n$  be a positive integer. Assume that for any multi-index  $\alpha$  s.t.  $|\alpha| \leq n$ ,

$$\begin{aligned} \text{ess sup}_{\Omega \times [0, T] \times \mathbb{R}^d} (|D^\alpha a| + |D^\alpha b| + |D^\alpha c| + |D^\alpha \sigma| + |D^\alpha v|) &\leq K, \\ F \in \mathbb{H}^n, \quad \phi &\in L^2(\Omega, \mathcal{F}_T, H^{n+1}). \end{aligned} \quad (2.6)$$

Then Eq. (1.2) has a unique strong solution  $(p, q)$  such that

$$p \in \mathbb{H}^{n+2}, \quad q \in \mathbb{H}^{n+1}, \quad p \in C([0, T], H^n) \cap L^\infty([0, T], H^{n+1})(a.s.),$$

with the estimate

$$\|p\|_{n+2}^2 + \|q\|_{n+1}^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_{n+1}^2 \leq C(\|F\|_n^2 + E\|\phi\|_{n+1}^2), \quad (2.7)$$

where the constant  $C$  depends only on  $K, \kappa$  and  $T$ .

**Proof.** The first inequality of condition (2.6) implies Assumption 2.3. In view of Theorem 2.2, Eq. (1.2) has a unique strong solution  $(p, q)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$  such that  $p \in C([0, T], L^2) \cap L^\infty([0, T], H^1)(a.s.)$ , and estimate (2.5) holds true.

Now we apply induction to prove this theorem.

Assume that the assertion of Theorem 2.3 holds true for  $n = m - 1$  ( $m \geq 1$ ), that is

$$p \in \mathbb{H}^{m+1}, \quad q \in \mathbb{H}^m, \quad p \in C([0, T], H^{m-1})(a.s.),$$

and inequality (2.7) holds for  $n = m - 1$ .

Note that Eq. (1.2) can be rewritten into divergence form like (1.1) since  $Da$  and  $D\sigma$  are bounded. Therefore, by the integration by parts, it is not hard to show that for any multi-index  $\alpha$  s.t.  $|\alpha| = m$ , the function pair  $(D^\alpha p, D^\alpha q) \in \mathbb{H}^1 \times \mathbb{H}^0$  satisfies the following equation (in the sense of Definition 2.1(i))

$$\begin{cases} du = -(a^{ij} D_{ij} u + \sigma^{ik} D_i v^k + \tilde{F}) dt + v^k dW_t^k, \\ u(T, x) = D^\alpha \phi(x), \quad x \in \mathbb{R}^d, \end{cases} \quad (2.8)$$

with the unknown functions  $u$  and  $v$ . Here ( $|\alpha| = m$ )

$$\begin{aligned} \tilde{F} &= D^\alpha F + \sum_{|\beta|+|\gamma|=|\alpha|, |\beta| \geq 1} [(D^\beta a^{ij})(D^\gamma p_{x^i x^j}) + (D^\beta \sigma^i)(D^\gamma q_{x^i})] \\ &\quad + \sum_{|\beta|+|\gamma|=|\alpha|} [(D^\beta b^i)(D^\gamma p_{x^i}) - (D^\beta c)(D^\gamma p) + (D^\beta v)(D^\gamma q)]. \end{aligned}$$

From our assumption for  $n = m - 1$  and condition (2.6), we see that  $\tilde{F} \in \mathbb{H}^0$ . Moreover, from estimate (2.7) for  $n = m - 1$ , we obtain that ( $|\alpha| = m$ )

$$\begin{aligned} \|\tilde{F}\|_0^2 &\leq C(\kappa, K, T)(\|D^\alpha F\|_0^2 + \|p\|_{m+1}^2 + \|q\|_m^2) \\ &\leq C(\kappa, K, T)(\|F\|_m^2 + \|\phi\|_m^2). \end{aligned}$$

Then applying Theorem 2.2 to Eq. (2.8), we obtain that  $(D^\alpha p, D^\alpha q) \in \mathbb{H}^2 \times \mathbb{H}^1$ , and  $D^\alpha p \in C([0, T], L^2) \cap L^\infty([0, T], H^1)$  (a.s.), and moreover (recall  $|\alpha| = m$ )

$$\|D^\alpha p\|_2^2 + \|D^\alpha q\|_1^2 + E \sup_{t \leq T} \|D^\alpha p(t, \cdot)\|_1^2 \leq C(\|F\|_m^2 + E\|\phi\|_{m+1}^2).$$

The proof is complete.  $\square$

**Remark 2.1.** In contrast to Theorem 3.1 in [23], Theorems 2.2 and 2.3 in this paper require less regularity of the coefficients to reach similar conclusions due to the different approach. To be specific, Zhou [23] applies duality analysis directly to Eq. (1.1), actually requiring high regularity of the coefficients (see p. 278 in [23]), while we start our discussion from simple equations, and then borrow some classical ideas from the theory of PDEs, such as perturbation and the continuation method, to study the general equations.

**Remark 2.2.** In the case of  $n - d/2 > 2$ , the function pair  $(p, q)$  satisfies Eq. (1.2) for all  $(t, x) \in [0, T] \times \mathbb{R}^d$  and  $\omega \in \Omega'$  s.t.  $\mathbb{P}(\Omega') = 1$ , which is a classical solution of Eq. (1.2) (see e.g. [12, Def. 2.1]).

**Remark 2.3.** In this paper, all constants denoted by  $C$  are independent of  $d_1$ , which allows us to extend our results (Theorems 2.1–2.3) to the more general case of Eq. (1.2) which is driven by a Hilbert-space-valued Wiener process.

### 3. Backward stochastic evolution equations in Hilbert spaces

In this section, we consider backward stochastic evolution equations in Hilbert spaces. The basic form of the main result (Proposition 3.2) in this section is first obtained by Hu and Peng [8]. However, they did not give any rigorous proof. In order to be self-contained, we provide here a proof of this result with details, and establish an estimate which did not appear in [8].

Let  $V$  and  $H$  be two separable (real) Hilbert spaces such that  $V$  is densely embedded in  $H$ . We identify  $H$  with its dual space, and denote by  $V^*$  the dual of  $V$ . Then we have  $V \subset H \subset V^*$ . Denote by  $\|\cdot\|_V$ ,  $\|\cdot\|_H$  and  $\|\cdot\|_{V^*}$  the norms of  $V$ ,  $H$  and  $V^*$  respectively, by  $(\cdot, \cdot)$  the inner product in  $H$ , and by  $\langle \cdot, \cdot \rangle$  the duality product between  $V$  and  $V^*$ .

Consider three processes  $v, m$  and  $v^*$  defined on  $\Omega \times [0, T]$  with values in  $V, H$  and  $V^*$ , respectively. Let  $v(\omega, t)$  be measurable with respect to  $(\omega, t)$  and be  $\mathcal{F}_t$ -measurable with respect to  $\omega$  for a.e.  $t$ ; for any  $\eta \in V$  the quantity  $\langle \eta, v^*(\omega, t) \rangle$  is  $\mathcal{F}_t$ -measurable in  $\omega$  for a.e.  $t$  and is measurable with respect to  $(\omega, t)$ . Assume that  $m(\omega, t)$  is strongly continuous in  $t$  and is  $\mathcal{F}_t$ -measurable with respect to  $\omega$  for any  $t$ , and is a local martingale. Let  $\langle m \rangle$  be the increasing process for  $\|m\|_H^2$  in the Doob–Meyer Decomposition (see e.g. [9, p. 1240]).

Proceeding identically to the proof of Theorem 3.2 in Krylov and Rozovskii [9], we have the following result concerning the Itô’s formula, which is the backward version of [9, Thm. 3.2].

**Lemma 3.1.** Let  $\varphi \in L^2(\Omega, \mathcal{F}_T, H)$ . Suppose that for every  $\eta \in V$  and almost every  $(\omega, t) \in \Omega \times [0, T]$ , it holds that

$$(\eta, v(t)) = (\eta, \varphi) + \int_t^T \langle \eta, v^*(s) \rangle ds + (\eta, m(T) - m(t)).$$

Then there exist a set  $\Omega' \subset \Omega$  s.t.  $P(\Omega') = 1$  and a function  $h(t)$  with values in  $H$  such that

- (a)  $h(t)$  is  $\mathcal{F}_t$ -measurable for any  $t \in [0, T]$  and strongly continuous with respect to  $t$  for any  $\omega$ , and  $h(t) = v(t)$  (in the space  $H$ ) for a.s.  $(\omega, t) \in \Omega \times [0, T]$ , and  $h(T) = \varphi$  for any  $\omega \in \Omega'$ ;  
 (b) for any  $\omega \in \Omega'$  and any  $t \in [0, T]$ ,

$$\|h(t)\|_H^2 = \|\varphi\|_H^2 + 2 \int_t^T \langle v(s), v^*(s) \rangle ds + 2 \int_t^T (h(s), dm(s)) - \langle m \rangle_T + \langle m \rangle_t.$$

Denote  $H^{\otimes d_1} = \{v = (v^1, v^2, \dots, v^{d_1}) : v^k \in H, k = 1, 2, \dots, d_1\}$ . The norm in  $H^{\otimes d_1}$  is defined by  $\|v\|_{H^{\otimes d_1}} = (\sum_k \|v^k\|_H^2)^{1/2}$ .

Assume that linear operators

$$\mathcal{L}(\omega, t) : V \rightarrow V^*, \quad \mathcal{M}^k(\omega, t) : H \rightarrow V^*,$$

and functions  $\varphi(\omega)$ ,  $f(\omega, t)$  taking values in  $H$  and  $V^*$ , respectively, are given for  $t \in [0, T]$ ,  $\omega \in \Omega$ . Denote  $\mathcal{M} = (\mathcal{M}^1, \mathcal{M}^2, \dots, \mathcal{M}^{d_1})$ , then we define a linear operator  $\mathcal{M} : H^{\otimes d_1} \rightarrow V^*$  as follows:

$$\mathcal{M}v = \sum_k \mathcal{M}^k v^k, \quad \forall v \in H^{\otimes d_1}.$$

Consider the linear backward stochastic evolution equation (we use the summation convention)

$$u(t) = \varphi + \int_t^T [\mathcal{L}u(s) + \mathcal{M}v(s) + f(s)]ds - \int_t^T v^k(s) dW_s^k. \quad (3.1)$$

**Definition 3.1.** An  $\mathcal{F}_t$ -adapted process  $(u, v)$  valued in  $V \times H^{\otimes d_1}$  is called a solution of Eq. (2.1), if  $u \in L^2(\Omega \times (0, T), \mathcal{P}, V)$  and  $v \in L^2(\Omega \times (0, T), \mathcal{P}, H^{\otimes d_1})$ , such that for every  $\eta \in V$  and a.e.  $(\omega, t) \in \Omega \times [0, T]$ , it holds that

$$(\eta, u(t)) = (\eta, \varphi) + \int_t^T \langle \eta, \mathcal{L}u(s) + \mathcal{M}v(s) + f(s) \rangle ds - \int_t^T (\eta, v^k(s)) dW_s^k.$$

**Remark 3.1.** From Lemma 3.1, we know that a solution of Eq. (3.1), in the sense of Definition 3.1, always has a continuous version in  $H$ .

**Remark 3.2.** When  $\mathcal{L}$  is the infinitesimal generator of a  $C_0$ -semigroup (so independent of  $(\omega, t)$ ), another notion of the solution of Eq. (3.1), i.e. so-called the mild solution, is also studied in many literature, see e.g. [8,10,21] for details.

Now we study the existence and uniqueness of the solution of Eq. (3.1). We need the following

**Assumption 3.1.** There exist two constants  $\lambda, A > 0$  such that for any  $(\omega, t) \in \Omega \times [0, T]$ ,

$$\begin{aligned} 2\langle x, \mathcal{L}x \rangle + \|\mathcal{M}^*x\|_{H^{\otimes d_1}}^2 &\leq -\lambda\|x\|_V^2 + A\|x\|_H^2, \\ \|\mathcal{L}x\|_{V^*} &\leq A\|x\|_V, \quad \forall x \in V, \end{aligned} \quad (3.2)$$

where  $\mathcal{M}^* : V \rightarrow H^{\otimes d_1}$  is the adjoint operator of  $\mathcal{M}$ . The first inequality is called the coercivity condition (see e.g. [16, p. 87]).

The main result of this section is the following.

**Proposition 3.2.** *Let Assumption 3.1 be satisfied. Suppose that*

$$f \in L^2(\Omega \times (0, T), \mathcal{P}, V^*), \quad \varphi \in L^2(\Omega, \mathcal{F}_T, H). \quad (3.3)$$

Then Eq. (3.1) has a unique solution  $(u, v)$  in the space  $L^2(\Omega \times (0, T), \mathcal{P}, V \times H^{\otimes d_1})$  such that  $u \in C([0, T], H)$  (a.s.), and moreover,

$$\begin{aligned} E \sup_{t \leq T} \|u(t)\|_H^2 + E \int_0^T \left( \|u(t)\|_V^2 + \|v(t)\|_{H^{\otimes d_1}}^2 \right) dt \\ \leq C \left( E \int_0^T \|f(t)\|_{V^*}^2 dt + E \|\varphi\|_H^2 \right), \end{aligned}$$

where the constant  $C = C(\lambda, \Lambda, T)$ .

**Proof.** Step 1. Assume the existence of the solution of Eq. (3.1) in the sense of Definition 3.1. In view of Lemma 3.1, we have  $u \in C([0, T], H)$  (a.s.). Now we deduce estimate (3.3).

First we claim that  $E \sup_{t \leq T} \|u(t)\|_H^2 < \infty$ . Indeed, note that  $u(\omega, 0) \in H$  is  $\mathcal{F}_0$ -measurable, thus is deterministic. Define a sequence of stopping times as

$$\tau_n(\omega) = \inf\{t; \sup_{s \leq t} \|u(\omega, s)\|_H \geq n\} \wedge T.$$

It is clear that  $\tau_n \uparrow T$  a.s. Then applying Itô's formula to  $\|u(t)\|_H^2$  and from Assumption 3.1, we have

$$\begin{aligned} \|u(t \wedge \tau_n)\|_H^2 &= \|u(0)\|_H^2 - \int_0^{t \wedge \tau_n} \left[ 2\langle u(s), \mathcal{L}u(s) \rangle + 2((\mathcal{M}^k)^* u(s), v^k(s)) \right. \\ &\quad \left. + 2\langle u(s), f(s) \rangle - \|v(s)\|_{H^{\otimes d_1}}^2 \right] ds + \int_0^{t \wedge \tau_n} 2\langle u(s), v^k(s) \rangle dW_t^k \\ &\leq \|u(0)\|_H^2 + C(\Lambda) \int_0^T \left( \|u(s)\|_V^2 + \|v(s)\|_{H^{\otimes d_1}}^2 + \|f(s)\|_{V^*}^2 \right) ds \\ &\quad + 2 \int_0^{t \wedge \tau_n} \langle u(s), v^k(s) \rangle dW_t^k. \end{aligned}$$

On the other hand, from the Burkholder–Davis–Gundy (BDG) inequality, we have

$$\begin{aligned} E \left| \sup_{t \leq \tau_n} \int_0^t \langle u(s), v^k(s) \rangle dW_t^k \right| &\leq C \left[ E \int_0^{\tau_n} \|u(t)\|_H^2 \|v(t)\|_{H^{\otimes d_1}}^2 dt \right]^{1/2} \\ &\leq \frac{1}{4} E \sup_{t \leq \tau_n} \|u(t)\|_H^2 + C E \int_0^T \|v(t)\|_{H^{\otimes d_1}}^2 dt. \end{aligned} \quad (3.4)$$

Therefore, we have

$$E \sup_{t \leq \tau_n} \|u(t)\|_H^2 \leq 2\|u(0)\|_H^2 + C(\Lambda) E \int_0^T \left( \|u(t)\|_V^2 + \|v(t)\|_{H^{\otimes d_1}}^2 + \|f(t)\|_{V^*}^2 \right) ds.$$

Note that the constant  $C$  is independent of  $n$ . Passing  $n$  to infinity, we obtain that  $E \sup_{t \leq T} \|u(t)\|_H^2 < \infty$ .

Now using Itô's formula to  $\|u(t)\|_H^2$  once more and from [Assumption 3.1](#), we have

$$\begin{aligned}
 \|u(t)\|_H^2 &= \|\varphi\|_H^2 + \int_t^T \left[ 2\langle u(s), \mathcal{L}u(s) \rangle + 2((\mathcal{M}^k)^*u(s), v^k(s)) + 2\langle u(s), f(s) \rangle \right. \\
 &\quad \left. - \|v(s)\|_{H^{\otimes d_1}}^2 \right] ds - \int_t^T 2(u(s), v^k(s)) dW_t^k \\
 &\leq \|\varphi\|_H^2 + \int_t^T \left[ 2\langle u(s), \mathcal{L}u(s) \rangle + (1 + \varepsilon)\|\mathcal{M}^*u(s)\|_{H^{\otimes d_1}}^2 + \frac{1}{1 + \varepsilon}\|v(s)\|_{H^{\otimes d_1}}^2 \right. \\
 &\quad \left. - \|v(s)\|_{H^{\otimes d_1}}^2 + \varepsilon\|u(s)\|_V^2 + \frac{1}{\varepsilon}\|f(s)\|_{V^*}^2 \right] ds - \int_t^T 2(u(s), v^k(s)) dW_t^k \\
 &\leq \|\varphi\|_H^2 + \int_t^T \left[ -2\varepsilon\langle u(s), \mathcal{L}u(s) \rangle + (1 + \varepsilon)(-\lambda\|u(s)\|_V^2 + A\|u(s)\|_H^2) \right. \\
 &\quad \left. - \frac{\varepsilon}{1 + \varepsilon}\|v(s)\|_{H^{\otimes d_1}}^2 + \varepsilon\|u(s)\|_V^2 + \frac{1}{\varepsilon}\|f(s)\|_{V^*}^2 \right] ds - \int_t^T 2(u(s), v^k(s)) dW_t^k \\
 &\leq \|\varphi\|_H^2 + \int_t^T \left\{ [2\varepsilon A - \lambda(1 + \varepsilon) + \varepsilon]\|u(s)\|_V^2 + (1 + \varepsilon)A\|u(s)\|_H^2 \right. \\
 &\quad \left. - \frac{\varepsilon}{1 + \varepsilon}\|v(s)\|_{H^{\otimes d_1}}^2 + \frac{1}{\varepsilon}\|f(s)\|_{V^*}^2 \right\} ds - \int_t^T 2(u(s), v^k(s)) dW_t^k.
 \end{aligned}$$

Taking  $\varepsilon$  small enough such that  $2\varepsilon A - \lambda(1 + \varepsilon) + \varepsilon < 0$ , we have

$$\begin{aligned}
 \|u(t)\|_H^2 &+ \int_t^T \left( \|u(s)\|_V^2 + \|v(s)\|_{H^{\otimes d_1}}^2 \right) ds \\
 &\leq C(\lambda, A) \left[ \|\varphi\|_H^2 + \int_t^T \left( \|u(s)\|_H^2 + \|f(s)\|_{V^*}^2 \right) ds \right] - 2 \int_t^T (u(s), v^k(s)) dW_s^k. \quad (3.5)
 \end{aligned}$$

Since  $E \sup_{t \leq T} \|u(t)\|_H^2 < \infty$ , repeating (3.4), we know that  $\int_0^T (u(s), v^k(s)) dW_s^k$  is a uniformly integrable martingale. Then taking expectation on the both sides of (3.5) and from the Gronwall inequality, we have

$$\begin{aligned}
 \sup_{t \leq T} E \|u(t)\|_H^2 &+ E \int_0^T \left( \|u(t)\|_V^2 + \|v(t)\|_{H^{\otimes d_1}}^2 \right) dt \\
 &\leq C e^{CT} \left( E \|\varphi\|_H^2 + E \int_0^T \|f(t)\|_{V^*}^2 dt \right). \quad (3.6)
 \end{aligned}$$

Recalling (3.5) and from the BDG inequality, we get

$$\begin{aligned}
 E \sup_{t \leq T} \|u(t)\|_H^2 &\leq C(\lambda, A, T) \left[ \|\varphi\|_H^2 + \int_0^T \left( \|u(s)\|_H^2 + \|f(s)\|_{V^*}^2 \right) dt \right] \\
 &\quad + \frac{1}{2} E \sup_{t \leq T} \|u(t)\|_H^2 + C E \int_0^T \|v(t)\|_{H^{\otimes d_1}}^2 dt,
 \end{aligned}$$

and this along with (3.6) yields estimate (3.4).

*Step 2.* We use the Galerkin approximate to prove the existence.

Fix a standard complete orthogonal basis  $\{e_i : i = 1, 2, 3, \dots\}$  in the space  $H$  which is also an orthogonal basis in the space  $V$ .

Consider the following system of BSDEs in  $\mathbb{R}^n$

$$\begin{aligned} u_n^i(t) = & (e_i, \varphi) + \int_t^T [\langle e_i, \mathcal{L}(s)e_j \rangle u_n^j(s) + (e_i, \mathcal{M}^k(s)e_j)v_n^{jk}(s) + \langle e_i, f(s) \rangle] ds \\ & - \int_t^T v_n^{ik}(s) dW_s^k, \end{aligned} \quad (3.7)$$

with the unknown processes  $u_n^i$  and  $v_n^i = (v_n^{i1}, \dots, v_n^{id_1})$  ( $i = 1, \dots, n$ ) taken values in  $\mathbb{R}$  and  $\mathbb{R}^{d_1}$ , respectively. It is clear that

$$E(e_i, \varphi)^2 < \infty, \quad E \int_0^T \langle e_i, f(s) \rangle^2 ds < \infty.$$

Thus system (3.7) has a unique continuous solution (see e.g. [14]). Define

$$u_n(t) = \sum_{i=1}^n u_n^i(t)e_i, \quad v_n(t) = \sum_{i=1}^n v_n^i(t)e_i.$$

Applying Itô's formula to  $\|u_n\|_H^2$  and from similar arguments as in Step 1, we have

$$\begin{aligned} E \sup_{t \leq T} \|u_n(t)\|_H^2 + E \int_0^T (\|u_n(t)\|_V^2 + \|v_n(t)\|_{H^{\otimes d_1}}^2) dt \\ \leq C(\lambda, A) \left( E \int_0^T \|f(t)\|_{V^*}^2 dt + E \|\varphi\|_H^2 \right). \end{aligned} \quad (3.8)$$

This inequality implies that there exists a subsequence  $\{n'\}$  of  $\{n\}$  and a pair  $(u, v) \in L^2(\Omega \times (0, T), \mathcal{P}, V \times H^{\otimes d_1})$  such that

$$\begin{aligned} u_{n'} & \rightarrow u \quad \text{weakly in } L^2(\Omega \times (0, T), \mathcal{P}, V), \\ v_{n'} & \rightarrow v \quad \text{weakly in } L^2(\Omega \times (0, T), \mathcal{P}, H^{\otimes d_1}). \end{aligned}$$

Let  $\xi$  be an arbitrary bounded random variable on  $(\Omega, \mathcal{F})$  and  $\psi$  be an arbitrary bounded measurable function on  $[0, T]$ .

From Eq. (3.7), for  $n \in \mathbb{N}^*$  and  $e_i \in \{e_i\}$ , where  $i \leq n$ , we have

$$\begin{aligned} E \int_0^T \xi \psi(t) (e_i, u_{n'}(t)) dt = E \int_0^T \xi \psi(t) \left\{ (e_i, \varphi) + \int_t^T [\langle e_i, \mathcal{L}u_{n'}(s) \rangle + (e_i, \mathcal{M}^k v_{n'}^k(s)) \right. \\ \left. + \langle e_i, f(s) \rangle] ds - \int_t^T (e_i, v_{n'}^k(s)) dW_s^k \right\} dt. \end{aligned}$$

Evidently, we have

$$E \int_0^T \xi \psi(t) (e_i, u_{n'}(t)) dt \rightarrow E \int_0^T \xi \psi(t) (e_i, u(t)) dt.$$

In view of the second condition of Assumption 3.1 and estimate (3.8), we get

$$E \left| \int_t^T \xi \langle e_i, \mathcal{L}u_{n'}(s) \rangle ds \right| < C < \infty,$$

where the constant  $C$  is independent of  $n'$ . It is also clear that

$$E \int_t^T \xi \langle e_i, \mathcal{L}u_{n'}(s) \rangle ds \rightarrow E \int_t^T \xi \langle e_i, \mathcal{L}u(s) \rangle ds, \quad \forall t \in [0, T].$$

Hence from Fubini's Theorem and Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} E \int_0^T \xi \psi(t) \int_t^T \langle e_i, \mathcal{L}u_{n'}(s) \rangle ds dt &= \int_0^T \psi(t) E \int_t^T \xi \langle e_i, \mathcal{L}u_{n'}(s) \rangle ds dt, \\ &\rightarrow \int_0^T \psi(t) E \int_t^T \xi \langle e_i, \mathcal{L}u(s) \rangle ds dt. \end{aligned}$$

Similarly, we have

$$E \int_0^T \xi \psi(t) \int_t^T \langle e_i, \mathcal{M}^k v_{n'}^k(s) \rangle ds dt \rightarrow E \int_0^T \xi \psi(t) \int_t^T \langle e_i, \mathcal{M}^k v^k(s) \rangle ds dt.$$

From the second condition of [Assumption 3.1](#) and estimate (3.8), we have

$$E \left| \xi \int_t^T \langle e_i, v_{n'}^k(s) \rangle dW_s^k \right| < C < \infty,$$

where the constant  $C$  is independent of  $n'$ . Since

$$(e_i, v_{n'}^k(\cdot)) \rightarrow (e_i, v^k(\cdot)) \quad \text{weakly in } L^2(0, T),$$

from a known result (see [16, p. 63, Thm. 4]), we have that for every  $t \in [0, T]$ ,

$$\int_t^T \langle e_i, v_{n'}^k(s) \rangle dW_s^k \rightarrow \int_t^T \langle e_i, v^k(s) \rangle dW_s^k \quad \text{weakly in } L^2(\Omega, \mathcal{F}_T, \mathbb{R}).$$

Hence, using Lebesgue's Dominated Convergence Theorem, we have

$$E \int_0^T \xi \psi(t) \int_t^T \langle e_i, v_{n'}^k(s) \rangle dW_s^k dt \rightarrow E \int_0^T \xi \psi(t) \int_t^T \langle e_i, v^k(s) \rangle dW_s^k dt.$$

To sum up, we obtain that for a.e.  $(\omega, t) \in \Omega \times [0, T]$ ,

$$(e_i, u(t)) = (e_i, \varphi) + \int_t^T \langle e_i, \mathcal{L}u(s) + \mathcal{M}v(s) + f(s) \rangle ds - \int_t^T \langle e_i, v^k(s) \rangle dW_s^k.$$

Thus the existence is proved and our proof is complete.  $\square$

**Remark 3.3.** As for [Proposition 3.2](#), we mention two similar results in [21, Thm. 3.6] and Al-Hussein [1, Thm. 4.2]. However, in the former reference, the critical technique is Yosida's approximation which requires that the operator  $\mathcal{L}$  is the infinitesimal generator of a  $C_0$  semi-group, i.e., the operator  $\mathcal{L}$  must be independent of  $(\omega, t)$ , while in our situation the operator  $\mathcal{L}$  could be dependent on  $(\omega, t)$ . The latter reference focuses on the problem of the semi-linear BSPDEs driven by infinite-dimensional martingales, moreover, not like ours, the BSPDEs he studies do not involve the unbounded operator  $\mathcal{M}$ .

**Proof of Theorem 2.1.** In order to apply [Proposition 3.2](#), we set

$$H = L^2(\mathbb{R}^d) = H^0, \quad V = H^1, \quad V^* = H^{-1},$$

and for any  $u \in H^1, v \in H^0$ , define

$$\begin{aligned} \mathcal{L}u &= D_i(a^{ij}D_j u) + b^i D_i u - cu, \\ \mathcal{M}^k v &= D_i(\sigma^{ik}v) + v^k v. \end{aligned} \tag{3.9}$$

The inner product in  $H$  (and the duality product between  $V$  and  $V^*$ ) is defined by

$$(u, v) = \int_{\mathbb{R}^d} u(x)v(x)dx.$$

It is clear that  $(\mathcal{M}^k)^*(t)u = \sigma^{ik}D_i u + v^k u$  for  $u \in H^1$ . From [Assumption 2.2](#) and Green's formula, we have that for any  $u \in H^1$ ,

$$\begin{aligned} 2\langle u, \mathcal{L}u \rangle + \|\mathcal{M}^*u\|_H^2 &= 2 \int_{\mathbb{R}^d} [-a^{ij}D_i u D_j u + b^i u D_i u - c|u|^2]dx \\ &\quad + \sum_{k=1}^{d_1} \int_{\mathbb{R}^d} |\sigma^{ik}D_i u + v^k u|^2 dx \\ &\leq - \int_{\mathbb{R}^d} (2a^{ij} - \sigma^{ik}\sigma^{kj})D_i u D_j u dx + \frac{\kappa}{2}\|u\|_1^2 + C(\kappa, K)\|u\|_0^2 \\ &\leq -\frac{\kappa}{2}\|u\|_1^2 + C(\kappa, K)\|u\|_0^2. \end{aligned}$$

Moreover, for any  $u, v \in H^1$ , we have

$$\begin{aligned} \langle \mathcal{L}u, v \rangle &= \int_{\mathbb{R}^d} (-a^{ij}D_i u D_j v + b^i v D_i u - cuv)dx \\ &\leq C(K)\|u\|_1\|v\|_1, \end{aligned}$$

which implies that  $\|\mathcal{L}u\|_{-1} \leq C(K)\|u\|_1$ . Then [Theorem 2.1](#) follows from [Proposition 3.2](#). The proof is complete.  $\square$

#### 4. Proof of [Theorem 2.2](#)

First we study the equations with the coefficients  $a$  and  $\sigma$  independent of the variable  $x$ .

**Proposition 4.1.** *Let [Assumptions 2.1](#) and [2.2](#) be satisfied with the functions  $a, \sigma$  independent of  $x$ . Suppose  $F \in \mathbb{H}^0$ ,  $\phi \in L^2(\Omega, \mathcal{F}_T, H^1)$ . Then Eq. (1.2) has a unique strong solution  $(p, q)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$  such that  $p \in C([0, T], H^1)(a.s.)$ , and moreover,*

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C(\kappa, K, T)(\|F\|_0^2 + E\|\phi\|_1^2). \quad (4.1)$$

**Proof.** *Step 1.* In this step we assume, in addition, that  $b = 0, c = 0, v = 0$ . Then Eq. (1.2) has the following simple form

$$dp = -[a^{ij}(t)D_{ij}p + \sigma^{ik}(t)D_i q^k + F]dt + q^k dW_t^k, \quad p|_{t=T} = \phi. \quad (4.2)$$

In order to apply [Proposition 3.2](#), we set

$$\begin{aligned} H &= H^1, & V &= H^2, & V^* &= H^0, \\ \mathcal{L}(t) &= a^{ij}(t)D_{ij}, & \mathcal{M}^k(t) &= \sigma^{ik}(t)D_i. \end{aligned}$$

The inner product in  $H$  (and the duality product between  $V$  and  $V^*$ ) is defined by

$$(u, v) = \int_{\mathbb{R}^d} u(x)v(x)dx + \sum_{l=1}^d \int_{\mathbb{R}^d} D_l u(x)D_l v(x)dx.$$

It is clear that  $\mathcal{M}^*(t) = \sigma^i(t)D_i$ . From Green's formula and the super-parabolicity condition (see [Assumption 2.2](#)), for any  $u \in H^2$ , we have

$$\begin{aligned} 2\langle u, \mathcal{L}u \rangle + \|\mathcal{M}^*u\|_H^2 &= -2 \int_{\mathbb{R}^d} a^{ij}(t) D_i u(x) D_j u(x) dx \\ &\quad - 2 \sum_{l=1}^d \int_{\mathbb{R}^d} a^{ij}(t) D_{il} u(x) D_{jl} u(x) dx + \int_{\mathbb{R}^d} |\sigma(t) Du(x)|^2 dx \\ &\quad + \sum_{l=1}^d \int_{\mathbb{R}^d} |\sigma(t) Du_{x^l}(x)|^2 dx \\ &\leq -\kappa \left[ \int_{\mathbb{R}^d} |Du(x)|^2 dx + \sum_{l=1}^d \int_{\mathbb{R}^d} |Du_{x^l}(x)|^2 dx \right] \\ &\leq -\kappa \|u\|_2^2 + \kappa \|u\|_1^2. \end{aligned}$$

Moreover, it is clear that  $\|\mathcal{L}u\|_0 \leq C(K)\|u\|_2$ . Thus condition (3.2) is satisfied. Then from [Proposition 3.2](#), there exists a unique function pair  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$  s.t.  $p \in C([0, T], H^1)$  (a.s.), satisfying the equation

$$p(t, \cdot) = \phi(\cdot) + \int_t^T [\mathcal{L}p(s, \cdot) + \mathcal{M}q(s, \cdot) + F(s, \cdot)] dt - \int_0^T q^k(s, \cdot) dW_t^k,$$

in the sense of [Definition 3.1](#), which means that the above equation holds in the space  $L^2$  for any  $t \in [0, T]$  and a.e.  $\omega \in \Omega$ , and furthermore, the pair  $(p, q)$  is the strong solution of Eq. (4.2).

It is clear that a strong solution of Eq. (4.2) is actually a weak solution of Eq. (4.2) (in the sense of [Definition 2.1\(i\)](#)). Therefore, the uniqueness of the strong solution is implied by the uniqueness of the weak solution.

**Step 2.** Now we remove the additional assumption made in Step 1.

Since the functions  $a$  and  $\sigma$  are independent of  $x$ , we can rewrite Eq. (1.2) into divergence form like (1.1). In view of [Theorem 2.1](#), Eq. (1.2) has a unique weak solution  $(p, q)$  in the space  $\mathbb{H}^1 \times \mathbb{H}^0$ . Consider the following

$$du = -(a^{ij} D_{ij} u + \sigma^{ik} D_i v^k + \tilde{F}) dt + v^k dW_t^k,$$

where  $\tilde{F} = b^i D_i p - cp + v^k q^k + F$  belongs to  $\mathbb{H}^0$ . From the result in Step 1, the above equation has a unique solution  $(u, v)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$  such that  $u \in C([0, T], H^1)$  (a.s.) and  $u, v$  satisfy estimate (4.1). By the uniqueness of the weak solution, we have that  $p = u$  and  $q = v$ . The proof is complete.  $\square$

Next, we prove a perturbation result.

**Lemma 4.2.** *Let Assumptions 2.1 and 2.2 be satisfied with  $b = 0, c = 0, v = 0$ . Assume that for a constant  $\delta > 0$  and for any  $(\omega, t, x)$  we have*

$$|a(t, x) - a_0(t)| \leq \delta, \quad |\sigma(t, x) - \sigma_0(t)| \leq \delta, \quad (4.3)$$

where  $a_0(t)$  and  $\sigma_0(t)$  are some functions of  $(t, \omega)$  satisfying [Assumptions 2.1 and 2.2](#). Suppose  $F \in \mathbb{H}^0, \phi \in L^2(\Omega, \mathcal{F}_T, H^1)$ .

Under the above assumptions, we assert that there exists a constant  $\delta(\kappa, K, T) > 0$  such that if  $\delta \leq \delta(\kappa, K, T)$ , then Eq. (1.2) has a unique strong solution  $(p, q)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$  such

that  $p \in C([0, T], H^1)(a.s.)$  and moreover,

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C(\kappa, K, T)(\|F\|_0^2 + E\|\phi\|_1^2). \quad (4.4)$$

**Proof.** In view of Proposition 4.1, we know that for any  $(u, v) \in \mathbb{H}^2 \times \mathbb{H}^1$ , the equation

$$\begin{cases} dp = -[a_0^{ij} D_{ij} p + \sigma_0^{ik} D_i q^k + (a^{ij} - a_0^{ij}) D_{ij} u \\ \quad + (\sigma^{ik} - \sigma_0^{ik}) D_i v^k + F] dt + q^k dW_t^k, \\ p(T, x) = \phi(x), \quad x \in \mathbb{R}^d \end{cases} \quad (4.5)$$

has a unique solution  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$  such that  $p \in C([0, T], H^1)$  (a.s.). By denoting  $(p, q) = T(u, v)$ , we define a linear operator

$$T : \mathbb{H}^2 \times \mathbb{H}^1 \rightarrow \mathbb{H}^2 \times \mathbb{H}^1.$$

Then from estimate (4.1), we can easily obtain that for any  $(u_i, v_i) \in \mathbb{H}^2 \times \mathbb{H}^1$ ,  $i = 1, 2$ ,

$$\|T(u_1 - u_2, v_1 - v_2)\|_{2,1}^2 \leq C\delta \|u_1 - u_2, v_1 - v_2\|_{2,1}^2, \quad (4.6)$$

where we denote

$$\|(u, v)\|_{2,1}^2 = \|u\|_2^2 + \|v\|_1^2.$$

Taking  $\delta = (2C)^{-1} = (2C(\kappa, K, T))^{-1}$ , we have that the operator  $T$  is a contraction in  $\mathbb{H}^2 \times \mathbb{H}^1$ , which implies the existence of the solution of Eq. (1.2) in the space  $\mathbb{H}^2 \times \mathbb{H}^1$ .

Next, applying estimate (4.1) to Eq. (4.5), we have

$$\|(p, q)\|_{2,1}^2 \leq C\delta \|(p, q)\|_{2,1}^2 + C(\|F\|_{0,\mathbb{R}_+^d}^2 + E\|\phi\|_{1,\mathbb{R}_+^d}^2).$$

Taking  $\delta = (2C)^{-1}$ , we obtain the required estimate, which also implies the uniqueness.  $\square$

Now we prove a priori estimate for the strong solution of equation (1.2).

**Lemma 4.3.** *Let the conditions of Theorem 2.2 be satisfied. In addition, assume that the function pair  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$  is a strong solution of Eq. (1.2). Then there exists a constant  $C$  depends only on  $K, \kappa, T$  and the function  $\gamma$  such that*

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C(\|F\|_0^2 + E\|\phi\|_1^2). \quad (4.7)$$

**Proof.** Step 1. In view of the definition of the strong solution (Definition 2.1), we know that the process  $p(t, \cdot)$  is an  $L^2$ -valued semimartingale. Then applying Itô's formula for Hilbert-valued semimartingales (see e.g. [4, p. 105]), we have

$$\begin{aligned} \|p(t, \cdot)\|_0^2 &= \|\phi\|_0^2 + 2 \int_t^T \int_{\mathbb{R}^d} p[a^{ij} D_{ij} p + b^i D_i p - cp + \sigma^{ik} D_i q^k + v^k q^k + F] dx ds \\ &\quad - \int_t^T \|q(s, \cdot)\|_0^2 ds - 2 \int_t^T \int_{\mathbb{R}^d} p q^k dx dW_s^k. \end{aligned}$$

Taking expectations and from the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|q\|_0^2 &\leq E\|\phi\|_0^2 + 2E \int_0^T \int_{\mathbb{R}^d} p[a^{ij} D_{ij} p + b^i D_i p - cp + \sigma^{ik} D_i q^k + v^k q^k + F] dx dt \\ &\leq E\|\phi\|_0^2 + \varepsilon(\|p\|_2^2 + \|q\|_1^2) + C(\varepsilon, K)\|p\|_1^2 + \|F\|_0^2, \end{aligned} \quad (4.8)$$

where  $\varepsilon$  is a small positive number to be specified later.

Step 2. In view of [Assumption 2.3](#), we can take a small  $\rho$  such that for any  $(\omega, t)$  and  $x, y \in \mathbb{R}^d$ ,

$$|a(t, x) - a(t, y)| \leq \delta, \quad |\sigma(t, x) - \sigma(t, y)| \leq \delta \quad (4.9)$$

if  $|x - y| \leq 4\rho$ , where  $\delta = \delta(\kappa, K, T)$  is taken from [Lemma 4.2](#).

Denote  $B_r(z) = \{x \in \mathbb{R}^d : |x - z| < r\}$ . Then take a nonnegative function  $\zeta \in C_0^\infty(\mathbb{R}^d)$  such that  $\text{supp}(\zeta) \subset B_{2\rho}(0)$ ,  $\zeta(x) = 1$  for  $|x| \leq \rho$ . For any  $z \in \mathbb{R}^d$ , define

$$\zeta^z(x) = \zeta(x - z), \quad p^z(t, x) = p(t, x)\zeta^z(x), \quad q^z(t, x) = q(t, x)\zeta^z(x). \quad (4.10)$$

In addition, define  $\eta^z(x) = \zeta(\frac{x-z}{2})$ . It is not hard to check that the functions  $p^z, q^z$  satisfy the equation (in the sense of [Definition 2.1\(ii\)](#))

$$dp^z = -(\tilde{a}^{ij} D_{ij} p^z + \tilde{\sigma}^{ik} D_i q^{z,k} + \tilde{F})dt + q^{z,k} dW_t^k, \quad (4.11)$$

where (observe that  $p^z = 0, q^z = 0$  whenever  $\eta^z \neq 1$ )

$$\begin{aligned} \tilde{a}^{ij}(t, x) &= a^{ij}(t, x)\eta^z(x) + a^{ij}(t, z)(1 - \eta^z(x)), \\ \tilde{\sigma}^{ik}(t, x) &= \sigma^{ik}(t, x)\eta^z(x) + \sigma^{ik}(t, z)(1 - \eta^z(x)), \\ \tilde{F}(t, x) &= (F\zeta^z)(t, x) + (b^i \zeta^z - 2a^{ij} D_j \zeta^z) D_i p(t, x) \\ &\quad - (c\zeta^z + a^{ij} D_{ij} \zeta^z) p(t, x) + (v^k \zeta^z - \sigma^{ik} D_i \zeta^z) q^k(t, x). \end{aligned}$$

The choice of  $\rho$  shows that  $\tilde{a}$  and  $\tilde{\sigma}$  satisfy condition [\(4.3\)](#) with  $a_0(t) = a(t, z)$  and  $\sigma_0(t) = \sigma(t, z)$ . Since  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$ , it is easy to see that  $\tilde{F} \in \mathbb{H}^0$ . Therefore, from [Lemma 4.2](#), [Eq. \(4.11\)](#) has a unique solution  $(u, v)$  in the space  $\mathbb{H}^2 \times \mathbb{H}^1$ . From the uniqueness of the solution, we know that  $p^z = u$  and  $q^z = v$ . Note that  $\text{supp}(p^z), \text{supp}(q^z) \subset B_{2\rho}(z)$ . From estimate [\(4.4\)](#), we get

$$\begin{aligned} &E \int_0^T [\|p(t, \cdot)\|_{2, B_\rho(z)}^2 + \|q(t, \cdot)\|_{1, B_\rho(z)}^2] dt + E \sup_{t \leq T} \|p(t, \cdot)\|_{1, B_\rho(z)}^2 \\ &\leq C \left\{ E \|\phi\|_{1, B_{2\rho}(z)}^2 + E \int_0^T [\|F(t, \cdot)\|_{0, B_{2\rho}(z)}^2 + \|p(t, \cdot)\|_{1, B_{2\rho}(z)}^2 \right. \\ &\quad \left. + \|q(t, \cdot)\|_{0, B_{2\rho}(z)}^2] dt \right\}, \end{aligned}$$

where we denote by  $\|\cdot\|_{n, B_r(z)}$  the norm of  $H^n(B_r(z))$ . Integrating this inequality with respect to all  $z \in \mathbb{R}^d$ , we obtain that

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C (\|F\|_0^2 + E \|\phi\|_1^2 + \|p\|_1^2 + \|q\|_1^2),$$

where the constant  $C$  depends only on  $K, \kappa, T$  and  $\gamma$ . Recalling inequality [\(4.8\)](#) and taking  $\varepsilon$  small enough (for instant,  $\varepsilon = (2C)^{-1}$ ), we have

$$\|p\|_2^2 + \|q\|_1^2 + E \sup_{t \leq T} \|p(t, \cdot)\|_1^2 \leq C (\|F\|_0^2 + E \|\phi\|_1^2 + \|p\|_1^2). \quad (4.12)$$

Observe that the above estimate also holds if we replace the initial time zero by any  $s \in [0, T)$ , which means

$$E \|p(s, \cdot)\|_1^2 \leq C \left( \|F\|_0^2 + E \|\phi\|_1^2 + E \int_s^T \|p(t, \cdot)\|_1^2 dt \right),$$

and this along with the Gronwall inequality yields that

$$\|p\|_1^2 = \int_0^T E \|p(s, \cdot)\|_1^2 \leq C e^{CT} (\|F\|_0^2 + E \|\phi\|_1^2).$$

Recalling (4.12), the proof is complete.  $\square$

**Proof of Theorem 2.2.** The uniqueness of the strong solution of Eq. (1.2) is implied by estimate (4.7). We shall use the continuation method to prove the existence.

Define

$$\begin{aligned} \mathcal{L}_0 &= a^{ij}(t, 0) D_{ij} + b^i(t, x) D_i - c(t, x), & \mathcal{M}_0^k &= \sigma^{ik}(t, 0) D_i + v^k(t, x), \\ \mathcal{L}_1 &= a^{ij}(t, x) D_{ij} + b^i(t, x) D_i - c(t, x), & \mathcal{M}_1^k &= \sigma^{ik}(t, x) D_i + v^k(t, x). \end{aligned}$$

For each  $\lambda \in [0, 1]$ , set

$$\mathcal{L}_\lambda = (1 - \lambda) \mathcal{L}_0 + \lambda \mathcal{L}_1, \quad \mathcal{M}_\lambda^k = (1 - \lambda) \mathcal{M}_0^k + \lambda \mathcal{M}_1^k.$$

Consider the following equation

$$dp = -(\mathcal{L}_\lambda p + \mathcal{M}_\lambda^k q^k + F) dt + q^k dW_t^k, \quad p|_{t=T} = \phi. \quad (4.13)$$

Observe that the coefficients of Eq. (4.13) satisfy the conditions of Theorem 2.2 with the same  $K, \kappa$  and  $\gamma$ . Hence a priori estimate (4.7) holds for Eq. (4.13) for each  $\lambda \in [0, 1]$  with the same constant  $C$  (i.e., independent of  $\lambda$ ).

Assume that for a  $\lambda = \lambda_0 \in [0, 1]$ , Eq. (4.13) is solvable, i.e., it has a unique solution  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$  for any  $F \in \mathbb{H}^0$  and any  $\phi \in L^2(\Omega, \mathcal{F}_T, H^1)$ . For other  $\lambda \in [0, 1]$ , we can rewrite (4.13) as

$$dp = -\{\mathcal{L}_{\lambda_0} p + \mathcal{M}_{\lambda_0}^k q^k + (\lambda - \lambda_0)[(\mathcal{L}_1 - \mathcal{L}_0)p + (\mathcal{M}_1^k - \mathcal{M}_0^k)q^k] + F\} dt + q^k dW_t^k.$$

Thus for any  $(u, v) \in \mathbb{H}^2 \times \mathbb{H}^1$ , the equation

$$dp = -\{\mathcal{L}_{\lambda_0} p + \mathcal{M}_{\lambda_0}^k q^k + (\lambda - \lambda_0)[(\mathcal{L}_1 - \mathcal{L}_0)u + (\mathcal{M}_1^k - \mathcal{M}_0^k)v^k] + F\} dt + v^k dW_t^k$$

with the terminal condition  $p|_{t=T} = \phi$  has a unique solution  $(p, q) \in \mathbb{H}^2 \times \mathbb{H}^1$ . By denoting  $T(u, v) = (p, q)$ , we define a linear operator

$$T : \mathbb{H}^2 \times \mathbb{H}^1 \rightarrow \mathbb{H}^2 \times \mathbb{H}^1.$$

Then from estimate (4.7), we can easily obtain that for any  $(u_i, v_i) \in \mathbb{H}^2 \times \mathbb{H}^1$ ,  $i = 1, 2$ ,

$$\|T(u_1 - u_2, v_1 - v_2)\|_{2,1}^2 \leq C |\lambda - \lambda_0| \| (u_1 - u_2, v_1 - v_2) \|_{2,1}^2, \quad (4.14)$$

where we denote

$$\|(u, v)\|_{2,1}^2 = \|u\|_2^2 + \|v\|_1^2.$$

Recall that the constant  $C$  in (4.14) is independent of  $\lambda$ . Set  $\theta = (2C)^{-1}$ . Then the operator is a contraction in  $\mathbb{H}^2 \times \mathbb{H}^1$  as long as  $|\lambda - \lambda_0| \leq \theta$ , which implies that Eq. (4.13) is solvable if  $|\lambda - \lambda_0| \leq \theta$ .

The solvability of equation (4.13) for  $\lambda = 0$  has been given by Proposition 4.1. Starting from  $\lambda = 0$ , one can reach  $\lambda = 1$  in finite steps, and this finishes the proof of solvability of Eq. (1.2).

The assertion that  $p \in C([0, T], L^2) \cap L^\infty([0, T], H^1)$  (a.s.) easily follows from Lemma 3.1 and estimate (4.7). The proof of Theorem 2.2 is complete.  $\square$

## 5. An application: a comparison theorem

It is well known that the comparison theorem plays an important role in the theory of PDEs and BSDEs. Thus a comparison theorem for BSPDEs is reasonably supposed to be equally important in the research of BSPDEs. Ma and Yong [12] obtains some comparison theorems for strong solutions of BSPDEs by Itô's formula, and discuss some potential applications. In this section, we deduce a comparison theorem for the strong solution of Eq. (1.2) based on the results in [12] under weaker conditions.

Our main result in this section is the following

**Theorem 5.1.** *Let the conditions of Theorem 2.2 be satisfied. Suppose for any  $(\omega, t)$ ,  $F(t, \cdot) \geq 0$  and  $\phi \geq 0$ . Then  $p(t, \cdot) \geq 0$  a.s., for every  $t \in \times[0, T]$ .*

The proof of the above theorem needs the following lemma. In what follows, we denote  $a^- = -(a \wedge 0)$  for  $a \in \mathbb{R}$ .

**Lemma 5.2.** *Let the conditions of Theorem 2.2 be satisfied. In addition, assume that the functions  $Da^{ij}$  and  $D\sigma^{ik}$  are bounded (by a constant  $L$ ). Let  $(p, q)$  be the strong solutions of Eq. (1.2). Then for some constant  $C$ ,*

$$E \int_{\mathbb{R}^d} [p(t, x)^-]^2 dx \leq e^{C(T-t)} \left\{ E \int_{\mathbb{R}^d} [\phi(x)^-]^2 dx + E \int_t^T \int_{\mathbb{R}^d} [F(s, x)^-]^2 dx ds \right\}. \quad (5.1)$$

**Proof.** Define a function  $h(r) : \mathbb{R} \rightarrow [0, \infty)$  as follows:

$$h(r) = \begin{cases} r^2, & r \leq -1, \\ (6r^3 + 8r^4 + 3r^5)^2, & -1 \leq r \leq 0, \\ 0, & r \geq 0. \end{cases} \quad (5.2)$$

One can directly check that  $h$  is  $C^2$  and

$$h(0) = h'(0) = h''(0) = 0, \quad h(-1) = 1, \quad h'(-1) = -2, \quad h''(-1) = 2.$$

For any  $\varepsilon > 0$ , let  $h_\varepsilon(r) = \varepsilon^2 h(r/\varepsilon)$ . The function  $h$  has the following properties:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} h_\varepsilon(r) &= (r^-)^2, & \lim_{\varepsilon \rightarrow 0} h'_\varepsilon(r) &= -2r^-, & \text{uniformly;} \\ |h''_\varepsilon(r)| &\leq C, \quad \forall \varepsilon > 0, r \in \mathbb{R}; & \lim_{\varepsilon \rightarrow 0} h''_\varepsilon(r) &= \begin{cases} 2, & r < 0, \\ 0, & r > 0. \end{cases} \end{aligned}$$

Since  $Da^{ij}$  and  $D\sigma^{ik}$  are bounded, Eq. (1.2) can be written into divergence form. Then applying Itô's formula for Hilbert-valued semimartingales (see e.g. [4, p. 105]) to  $h_\varepsilon(p(t, \cdot))$ , and from Green's formula, we obtain that

$$\begin{aligned} & E \int_{\mathbb{R}^d} h_\varepsilon(\phi(x)) dx - E \int_{\mathbb{R}^d} h_\varepsilon(p(t, x)) dx \\ &= E \int_t^T \int_{\mathbb{R}^d} \left\{ -h'_\varepsilon(p) D_i (a^{ij} D_j p + \sigma^{ik} q^k) - h'_\varepsilon(p) [(b^i - D_j a^{ij}) D_i p \right. \\ &\quad \left. - cp + (v^k - D_i \sigma^{ik}) q^k + F] + \frac{1}{2} h''_\varepsilon(p) |q|^2 \right\} dx dt \end{aligned}$$

$$= E \int_t^T \int_{\mathbb{R}^d} \left\{ \frac{1}{2} h''_\varepsilon(p) (2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2) - h'_\varepsilon(p) [(b^i - D_j a^{ij}) D_i p - cp + (v^k - D_i \sigma^{ik}) q^k + F] \right\} dx dt.$$

Let  $\varepsilon \rightarrow 0$  and from Lebesgue's Dominated Convergence Theorem, we have

$$\begin{aligned} & E \int_{\mathbb{R}^d} [\phi(x)^-]^2 dx - E \int_{\mathbb{R}^d} [(p(t, x))^-]^2 dx \\ &= E \int_t^T \int_{\mathbb{R}^d} \chi_{\{p \leq 0\}} \left\{ (2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2) - 2p[(b^i - D_j a^{ij}) D_i p - cp + (v^k - D_i \sigma^{ik}) q^k + F] \right\} dx dt. \end{aligned}$$

Note that (for  $\delta, \delta_1 > 0$ )

$$\begin{aligned} 2a^{ij} D_i p D_j p + 2\sigma^{ik} q^k D_i p + |q|^2 &\geq 2a^{ij} D_i p D_j p - (1 + \delta) |\sigma^i D_i p|^2 + \frac{\delta}{1 + \delta} |q|^2 \\ &\geq [-2\delta K + (1 + \delta)\kappa] |Dp|^2 + \frac{\delta}{1 + \delta} |q|^2, \\ -p[(b^i - D_j a^{ij}) D_i p - cp + (v^k - D_i \sigma^{ik}) q^k] \\ &\geq -\delta_1 (|Dp|^2 + |q|^2) - C(K, L) \delta_1^{-1} |p|^2. \end{aligned}$$

Then by taking  $\delta$  and  $\delta_1$  small enough (such that  $\delta_1 = \min\{-2\delta K + (1 + \delta)\kappa, \frac{\delta}{1 + \delta}\} > 0$ ), we have

$$\begin{aligned} & E \int_{\mathbb{R}^d} [\phi(x)^-]^2 dx - E \int_{\mathbb{R}^d} [(p(t, x))^-]^2 dx \\ &\geq E \int_t^T \int_{\mathbb{R}^d} \chi_{\{p \leq 0\}} [-C(\kappa, K, L) |p|^2 - 2pF] dx dt \\ &\geq E \int_t^T \int_{\mathbb{R}^d} [-C(\kappa, K, L) |p^-|^2 - 2p^- F^-] dx dt \\ &\geq E \int_t^T \int_{\mathbb{R}^d} [-C(\kappa, K, L) |p^-|^2 - |F^-|^2] dx dt, \end{aligned}$$

and this along with the Gronwall inequality implies inequality (5.1).  $\square$

**Proof of Theorem 5.1.** Fix a nonnegative function  $\zeta \in C_0^\infty(\mathbb{R}^d)$  such that  $\text{supp } \zeta \subset B_1(0)$ ,  $\int_{\mathbb{R}^d} \zeta = 1$ . Define  $\zeta_n(x) = n^d \zeta(nx)$ . For  $\varphi = a^{ij}, \sigma^{ik}$ , we define

$$\varphi_n(\omega, t, x) = (\varphi * \zeta_n)(\omega, t, x) = \int_{\mathbb{R}^d} \varphi(\omega, t, x - y) \zeta_n(y) dy.$$

It is clear that  $\varphi_n(\omega, t, \cdot) \in C^\infty(\mathbb{R}^d)$  for any  $(\omega, t, x)$ . Moreover, we have that  $|\varphi_n| \leq K$  and for any  $(\omega, t, x)$ ,

$$|D\varphi_n(\omega, t, x)| = \left| \int_{\mathbb{R}^d} \varphi(\omega, t, x - y) D\zeta_n(y) dy \right| \leq C(n)K.$$

It is not hard to check that  $a_n$  and  $\sigma_n$  satisfy [Assumption 2.3](#). Indeed, for any  $(\omega, t)$  and  $x, y \in \mathbb{R}^d$ , we have

$$|\varphi_n(\omega, t, x) - \varphi_n(\omega, t, y)| \leq \int_{\mathbb{R}^d} |\varphi(\omega, t, x - z) - \varphi(\omega, t, y - z)| \zeta_n(z) dz \leq \gamma(|x - y|).$$

We also claim that as  $n \rightarrow \infty$ ,

$$\varphi_n(\omega, t, x) \rightarrow \varphi(\omega, t, x), \quad \text{uniformly w.r.t. } (\omega, t, x). \quad (5.3)$$

Indeed, for any  $(\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d$ , we have

$$\begin{aligned} |\varphi_n(\omega, t, x) - \varphi(\omega, t, x)| &= \int_{\mathbb{R}^d} |\varphi(\omega, t, x - y) - \varphi(\omega, t, x)| \zeta_n(y) dy \\ &\leq \int_{|y| \leq 1/n} \gamma(|y|) \zeta_n(y) dy \leq \gamma(1/n) \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ , and this proves our claim.

Therefore, in view [Theorem 2.2](#), the following equation (for each  $n$ )

$$\begin{aligned} dp_n &= -(a_n^{ij} D_{ij} p_n + b^i D_i p_n - c p_n + \sigma_n^{ik} D_i q_n^k + v^k q_n^k + F) dt + q_n^k dW_t^k, \\ p_n|_{t=T} &= \psi \end{aligned}$$

has a unique strong solution  $(p_n, q_n) \in \mathbb{H}^2 \times \mathbb{H}^1$ , such that

$$\|p_n\|_2^2 + \|q_n\|_1^2 + E \sup_{t \leq T} \|p_n(t, \cdot)\|_1^2 \leq C(\|F\|_0^2 + E\|\phi\|_1^2), \quad (5.4)$$

where the constant  $C$  depends only on  $K, \kappa, T$  and the function  $\gamma$ , but is independent of  $n$ . It is easy to check that the function pair  $(p - p_n, q - q_n)$  satisfies the following equation

$$\begin{aligned} du &= -(a^{ij} D_{ij} u + b^i D_i u - cu + \sigma^{ik} D_i v^k + v^k v^k + F_n) dt + v^k dW_t^k, \\ u|_{t=T} &= 0 \end{aligned} \quad (5.5)$$

with the unknown functions  $u$  and  $v$ , where

$$F_n = (a^{ij} - a_n^{ij}) D_{ij} p_n + (\sigma^{ik} - \sigma_n^{ik}) D_i q_n^k.$$

In view of [\(5.3\)](#) and [\(5.4\)](#), we have

$$\|F_n\|_0 \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

and this along with estimate [\(2.5\)](#) implies that

$$p_n \rightarrow p, \quad \text{strongly in } \mathbb{H}^0.$$

On the other hand, it follows from [Lemma 5.2](#) that  $p_n(t, \cdot) \geq 0$  a.s. for every  $t \in [0, T]$ . Hence we get  $p(t, \cdot) \geq 0$  a.s. for every  $t \in [0, T]$ . The proof is complete.  $\square$

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