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# Convergence in total variation on Wiener chaos

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**Abstract**

Let $\{F_n\}$ be a sequence of random variables belonging to a finite sum of Wiener chaoses. Assume further that it converges in distribution towards $F_\infty$ satisfying $\mathrm{Var}(F_\infty) > 0$. Our first result is a sequential version of a theorem by Shigekawa (1980) [23]. More precisely, we prove, without additional assumptions, that the sequence $\{F_n\}$ actually converges in total variation and that the law of $F_\infty$ is absolutely continuous. We give an application to discrete non-Gaussian chaoses. In a second part, we assume that each $F_n$ has more specifically the form of a multiple Wiener–Itô integral (of a fixed order) and that it converges in $L^2(\Omega)$ towards $F_\infty$. We then give an upper bound for the distance in total variation between the laws of $F_n$ and $F_\infty$. As such, we recover an inequality due to Davydov and Martynova (1987) [5]; our rate is weaker compared to Davydov and Martynova (1987) [5] (by a power of 1/2), but the advantage is that our proof is not only sketched as in Davydov and Martynova (1987) [5]. Finally, in a third part we show that the convergence in the celebrated Peccati–Tudor theorem actually holds in the total variation topology.


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## 1. Introduction

In a seminal paper of 2005, Nualart and Peccati [19] discovered the surprising fact that convergence in distribution for sequences of multiple Wiener–Itô integrals to the Gaussian is equivalent to convergence of just the fourth moment. A new line of research was born. Indeed, since the publication of this important paper, many improvements and developments on this

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theme have been considered. (For an overview of the existing literature, we refer the reader to the book [14], to the survey [12] or to the constantly updated web page [13].)

Let us only state one of these results, whose proof relies on the combination of Malliavin calculus and Stein's method (see, e.g., [14, Theorem 5.2.6]). When $F, G$ are random variables, we write $d_{TV}(F, G)$ to indicate the total variation distance between the laws of $F$ and $G$, that is,

$$d_{TV}(F, G) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P(F \in A) - P(G \in A)| = \frac{1}{2} \sup_{\phi} |E[\phi(F)] - E[\phi(G)]| ,$$

where the first (resp. second) supremum is taken[1] over Borel sets $A$ of $\mathbb{R}$ (resp. over continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ which are bounded by 1).

**Theorem 1.1.** *If $k \geqslant 2$ is an integer, if $F$ is an element of the kth Wiener chaos $\mathcal{H}_k$ satisfying $E[F^2] = 1$ and if $N \sim \mathcal{N}(0, 1)$, then*

$$d_{TV}(F, N) \leqslant \sqrt{\frac{4k-4}{3k}} \sqrt{\left|E\left[F^4\right] - 3\right|}.$$

As an almost immediate corollary of Theorem 1.1, we get the surprising fact that if a sequence of multiple Wiener–Itô integrals with unit variance converges in distribution to the standard Gaussian law, then it automatically converges in total variation [14, Corollary 5.2.8]. The main thread of the present paper is the seek for other instances where such a phenomenon could occur. In particular, a pivotal role will be played by the sequences having the form of a (vector of) multiple Wiener–Itô integral(s) or, more generally, belonging to a *finite* sum of Wiener chaoses. As we said, the proof of Theorem 1.1 relies in a crucial way to the use of Stein's method. In a non-discrete framework (which is the case here), it is fairly understood that this method can give good results with respect to the total variation distance only in dimension one (see [3]) and when the target law is Gaussian (see [4]). Therefore, to reach our goal we need to introduce completely new ideas with respect to the existing literature. As anticipated, we will manage to exhibit three different situations where the convergence in distribution turns out to be equivalent to the convergence in total variation. In our new approach, an important role is played by the fact that the Wiener chaoses enjoy many nice properties, such as hypercontractivity (Theorem 2.1), product formula (2.7) or Hermite polynomial representation of multiple integrals (2.3).

Let us now describe our main results in more detail. Our first example focuses on sequences belonging to a finite sum of chaoses and may be seen as a sequential version of a theorem by Shigekawa [23]. More specifically, let $\{F_n\}$ be a sequence in $\bigoplus_{k=0}^{p} \mathcal{H}_k$ (where $\mathcal{H}_k$ stands for the $k$th Wiener chaos; by convention $\mathcal{H}_0 = \mathbb{R}$), and assume that it converges in distribution towards a random variable $F_\infty$. Assume moreover that the variance of $F_\infty$ is not zero. Let $d_{FM}$ denote the Fortet–Mourier distance, defined by

$$d_{FM}(F, G) = \sup_{\phi} |E[\phi(F)] - E[\phi(G)]| ,$$

where the supremum is taken over 1-Lipschitz functions $\phi : \mathbb{R} \to \mathbb{R}$ which are bounded by 1. We prove that there exists a constant $c > 0$ such that, for any $n \geqslant 1$,

$$d_{TV}(F_n, F_\infty) \leqslant c\, d_{FM}(F_n, F_\infty)^{\frac{1}{2p+1}}. \tag{1.1}$$

[1] One can actually restrict to *bounded* Borel sets without changing the value of the supremum; this easy remark is going to be used many times in the forthcoming proofs.

Since it is well-known that $d_{FM}$ metrizes the convergence in distribution (see, e.g., [6, Theorem 11.3.3]), our inequality (1.1) implies in particular that $F_n$ converges to $F_\infty$ not only in distribution, but also in total variation. Besides, one can further prove that the law of $F_\infty$ is absolutely continuous with respect to the Lebesgue measure. This fact is an interesting first step towards a full description of the closure in distribution[2] of the Wiener chaoses $\mathcal{H}_k$, which is still an open problem except when $k = 1$ (trivial) or $k = 2$ (see [16]). We believe that our method is robust enough to be applied to some more general situations, and here is a short list of possible extensions of (1.1) that we plan to study in some subsequent papers:

(i) extension to the multidimensional case;
(ii) improvement of the rate of convergence;
(iii) extension to other types of chaoses (in the spirit of [8]).

As a first step towards point (iii) and using some techniques of Mossel et al. [10], we establish in Theorem 3.2 that, if $\mu$ is the law of a sequence of multilinear polynomials with low influences, bounded degree and unit variance, then it necessarily admits a density with respect to Lebesgue measure.

Our second example is concerned with sequences belonging to a fixed order Wiener chaos $\mathcal{H}_k$ (with $k \geqslant 2$) and when we have convergence in $L^2(\Omega)$. More precisely, let $\{F_n\}$ be a sequence of the form $F_n = I_k(f_n)$ (with $I_k$ the $k$th multiple Wiener–Itô integral) and assume that it converges in $L^2(\Omega)$ towards a random variable $F_\infty = I_k(f_\infty)$. Assume moreover that $E[F_\infty^2] > 0$. Then, there exists a constant $c > 0$ such that, for any $n \geqslant 1$,

$$d_{TV}(F_n, F_\infty) \leqslant c\, \|f_n - f_\infty\|^{\frac{1}{2k}}. \tag{1.2}$$

Actually, the inequality (1.2) is not new. It was shown in 1987 by Davydov and Martynova in [5] (with the better factor $\frac{1}{k}$ instead of $\frac{1}{2k}$). However, it is a pity that [5] contains only a sketch of the proof of (1.2). Since it is not clear (at least for us!) how to complete the missing details, we believe that our proof may be of interest as it is fully self-contained. Moreover, we are hopeful that our approach could be used in the multivariate framework as well, which would solve an open problem (see indeed [1] and comments therein). Once again, we postpone this possible extension in a subsequent paper.

Finally, we develop a third example. It arises when one seeks for a multidimensional counterpart of Theorem 1.1, that is, when one wants to prove that one can replace for free the convergence in distribution in the statement of the Peccati–Tudor theorem [14, Theorem 6.2.3] by a convergence in total variation. We prove, without relying to Stein's method but in the same spirit as in the famous proof of the Hörmander theorem by Malliavin [9], that if a sequence of vectors of multiple Wiener–Itô integrals converges in law to a Gaussian vector having a non-degenerate covariance matrix, then it necessarily converges in total variation. This result solves, in the multidimensional framework, a problem left open after the discovery of Theorem 1.1.

Our paper contains results closely connected to those of the paper [7] by Hu et al.. The investigations were done independently and at about the same time. In [7], the authors focus on the convergence of random vectors $\{F_n\}$ which are functionals of Gaussian processes to a normal $\mathcal{N}(0, I_d)$. More specifically, they work under a negative moment condition (in the spirit of our Theorem 4.2 and whose validity may be sometimes difficult to check in concrete situations) which enables them to show that the density of $F_n$ (as well as its first derivatives) converges to

[2] It is worthwhile noting that the Wiener chaoses are closed for the convergence in *probability*, as shown by Schreiber [22] in 1969.

the Gaussian density. Applications to sequences of random variables in the second Wiener chaos is then discussed. It is worth mentioning that the philosophy of our paper is a bit different. We are indeed interested in exhibiting instances for which, *without* further assumptions, the convergence in law (to a random variable which is *not necessarily* Gaussian) turns out to be equivalent to the convergence in total variation.[3]

The rest of the paper is organized as follows. In Section 2, we first recall some useful facts about multiple Wiener–Itô integrals and Malliavin calculus. We then prove inequality (1.1) in Section 3. The proof of (1.2) is done in Section 4. Finally, our extension of the Peccati–Tudor Theorem is given in Section 5.

## 2. Preliminaries

This section contains the elements of Gaussian analysis and Malliavin calculus that are used throughout this paper. See the monographs [14,17] for further details.

### 2.1. Isonormal processes and multiple Wiener–Itô integrals

Let $\mathfrak{H}$ be a real separable Hilbert space. For any $k \geqslant 1$, we write $\mathfrak{H}^{\otimes k}$ and $\mathfrak{H}^{\odot k}$ to indicate, respectively, the $k$th tensor power and the $k$th symmetric tensor power of $\mathfrak{H}$; we also set by convention $\mathfrak{H}^{\otimes 0} = \mathfrak{H}^{\odot 0} = \mathbb{R}$. When $\mathfrak{H} = L^2(A, \mathcal{A}, \mu) =: L^2(\mu)$, where $\mu$ is a $\sigma$-finite and non-atomic measure on the measurable space $(A, \mathcal{A})$, then $\mathfrak{H}^{\otimes k} = L^2(A^k, \mathcal{A}^k, \mu^k) =: L^2(\mu^k)$, and $\mathfrak{H}^{\odot k} = L^2_s(A^k, \mathcal{A}^k, \mu^k) := L^2_s(\mu^k)$, where $L^2_s(\mu^k)$ stands for the subspace of $L^2(\mu^k)$ composed of those functions that are $\mu^k$-almost everywhere symmetric. We denote by $X = \{X(h) : h \in \mathfrak{H}\}$ an *isonormal Gaussian process* over $\mathfrak{H}$. This means that $X$ is a centered Gaussian family, defined on some probability space $(\Omega, \mathcal{F}, P)$, with a covariance structure given by the relation $E[X(h)X(g)] = \langle h, g\rangle_{\mathfrak{H}}$. We also assume that $\mathcal{F} = \sigma(X)$, that is, $\mathcal{F}$ is generated by $X$.

For every $k \geqslant 1$, the symbol $\mathcal{H}_k$ stands for the $k$th *Wiener chaos* of $X$, defined as the closed linear subspace of $L^2(\Omega, \mathcal{F}, P) =: L^2(\Omega)$ generated by the family $\{H_k(X(h)) : h \in \mathfrak{H}, \|h\|_{\mathfrak{H}} = 1\}$, where $H_k$ is the $k$th Hermite polynomial given by

$$H_k(x) = (-1)^k e^{\frac{x^2}{2}} \frac{d^k}{dx^k}\left(e^{-\frac{x^2}{2}}\right). \tag{2.3}$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$. For any $k \geqslant 1$, the mapping $I_k(h^{\otimes k}) = H_k(X(h))$ can be extended to a linear isometry between the symmetric tensor product $\mathfrak{H}^{\odot k}$ (equipped with the modified norm $\sqrt{k!}\,\|\cdot\|_{\mathfrak{H}^{\otimes k}}$) and the $k$th Wiener chaos $\mathcal{H}_k$. For $k = 0$, we write $I_0(c) = c$, $c \in \mathbb{R}$. A crucial fact is that, when $\mathfrak{H} = L^2(\mu)$, for every $f \in \mathfrak{H}^{\odot k} = L^2_s(\mu^k)$ the random variable $I_k(f)$ coincides with the $k$-fold multiple Wiener–Itô stochastic integral of $f$ with respect to the centered Gaussian measure (with control $\mu$) canonically generated by $X$ (see [17, Section 1.1.2]).

It is well-known that $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_k$. It follows that any square-integrable random variable $F \in L^2(\Omega)$ admits the following *Wiener–Itô chaotic expansion*

$$F = \sum_{k=0}^{\infty} I_k(f_k), \tag{2.4}$$

[3] When we are dealing with sequences of random variables that have a law which is absolutely continuous with respect to the Lebesgue measure, which is going to be always the case in our paper, it is worthwhile noting that the convergence in total variation is actually equivalent to the $L^1$-convergence of densities.

where $f_0 = E[F]$, and the $f_k \in \mathfrak{H}^{\odot k}$, $k \geqslant 1$, are uniquely determined by $F$. For every $k \geqslant 0$, we denote by $J_k$ the orthogonal projection operator on the $k$th Wiener chaos. In particular, if $F \in L^2(\Omega)$ is as in (2.4), then $J_k F = I_k(f_k)$ for every $k \geqslant 0$.

Let $\{e_i, i \geqslant 1\}$ be a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot k}$ and $g \in \mathfrak{H}^{\odot l}$, for every $r = 0, \ldots, k \wedge l$, the *contraction* of $f$ and $g$ of order $r$ is the element of $\mathfrak{H}^{\otimes(k+l-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}. \tag{2.5}$$

Notice that the definition of $f \otimes_r g$ does not depend on the particular choice of $\{e_i, i \geqslant 1\}$, and that $f \otimes_r g$ is not necessarily symmetric; we denote its symmetrization by $f \widetilde{\otimes}_r g \in \mathfrak{H}^{\odot(k+l-2r)}$. Moreover, $f \otimes_0 g = f \otimes g$ equals the tensor product of $f$ and $g$ while, for $k = l$, $f \otimes_k g = \langle f, g \rangle_{\mathfrak{H}^{\otimes k}}$. When $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ and $r = 1, \ldots, k \wedge l$, the contraction $f \otimes_r g$ is the element of $L^2(\mu^{k+l-2r})$ given by

$$\begin{aligned} f \otimes_r g(x_1, \ldots, x_{k+l-2r}) = \int_{A^r} & f(x_1, \ldots, x_{k-r}, a_1, \ldots, a_r) g(x_{k-r+1}, \ldots, x_{k+l-2r}, \\ & a_1, \ldots, a_r) d\mu(a_1) \cdots d\mu(a_r). \end{aligned} \tag{2.6}$$

It can also be shown that the following *product formula* holds: if $f \in \mathfrak{H}^{\odot k}$ and $g \in \mathfrak{H}^{\odot l}$, then

$$I_k(f) I_l(g) = \sum_{r=0}^{k \wedge l} r! \binom{k}{r} \binom{l}{r} I_{k+l-2r}(f \widetilde{\otimes}_r g). \tag{2.7}$$

Finally, we state a very useful property of Wiener chaos (see [11] or [14, Corollary 2.8.14]), which is going to be used several times in the sequel (notably in the proofs of Lemmas 2.4 and 5.3).

**Theorem 2.1** (*Hypercontractivity*)**.** *Let $F \in \mathcal{H}_k$ with $k \geqslant 1$. Then, for all $r > 1$,*

$$E\left[|F|^r\right]^{1/r} \leqslant (r-1)^{k/2} E[F^2]^{1/2}.$$

## *2.2. Malliavin calculus*

We now introduce some basic elements of the Malliavin calculus with respect to the isonormal Gaussian process $X = \{X(h), h \in \mathfrak{H}\}$. Let $\mathcal{S}$ be the set of all cylindrical random variables of the form

$$F = g\left(X(\phi_1), \ldots, X(\phi_n)\right), \tag{2.8}$$

where $n \geqslant 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function such that its partial derivatives have polynomial growth, and $\phi_i \in \mathfrak{H}$, $i = 1, \ldots, n$. The *Malliavin derivative* of $F$ with respect to $X$ is the element of $L^2(\Omega, \mathfrak{H})$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \left(X(\phi_1), \ldots, X(\phi_n)\right) \phi_i.$$

In particular, $DX(h) = h$ for every $h \in \mathfrak{H}$. By iteration, one can define the $m$th derivative $D^m F$, which is an element of $L^2(\Omega, \mathfrak{H}^{\odot m})$ for every $m \geqslant 2$. For $m \geqslant 1$ and $p \geqslant 1$, $\mathbb{D}^{m,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,p}$, defined by the relation

$$\|F\|_{m,p}^p = E\left[|F|^p\right] + \sum_{i=1}^m E\left[\|D^i F\|_{\mathfrak{H}^{\otimes i}}^p\right].$$

We often use the notation $\mathbb{D}^\infty := \bigcap_{m\geqslant 1}\bigcap_{p\geqslant 1}\mathbb{D}^{m,p}$.

**Remark 2.2.** Any random variable $Y$ that is a finite linear combination of multiple Wiener–Itô integrals is an element of $\mathbb{D}^\infty$. Moreover, if $Y \neq 0$, then the law of $Y$ admits a density with respect to the Lebesgue measure—see [23] or [14, Theorem 2.10.1].

The Malliavin derivative $D$ obeys the following *chain rule*. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable with bounded partial derivatives and if $F = (F_1, \dots, F_n)$ is a vector of elements of $\mathbb{D}^{1,2}$, then $\varphi(F) \in \mathbb{D}^{1,2}$ and

$$D\,\varphi(F) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(F) DF_i. \tag{2.9}$$

**Remark 2.3.** By approximation, it is easily checked that Eq. (2.9) continues to hold in the following two cases: (i) $F_i \in \mathbb{D}^\infty$ and $\varphi$ has continuous partial derivatives with at most polynomial growth, and (ii) $F_i \in \mathbb{D}^{1,2}$ has an absolutely continuous distribution and $\varphi$ is Lipschitz continuous.

Note also that a random variable $F$ in $L^2(\Omega)$ is in $\mathbb{D}^{1,2}$ if and only if $\sum_{k=1}^\infty k\|J_k F\|_{L^2(\Omega)}^2 < \infty$ and, in this case, $E\left[\|DF\|_{\mathfrak{H}}^2\right] = \sum_{k=1}^\infty k\|J_k F\|_{L^2(\Omega)}^2$. If $\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$ (with $\mu$ non-atomic), then the derivative of a random variable $F$ in $L^2(\Omega)$ can be identified with the element of $L^2(A \times \Omega)$ given by

$$D_x F = \sum_{k=1}^\infty k I_{k-1}\left(f_k(\cdot, x)\right), \quad x \in A. \tag{2.10}$$

We denote by $\delta$ the adjoint of the operator $D$, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathfrak{H})$ belongs to the domain of $\delta$, noted $\operatorname{Dom}\delta$, if and only if it verifies $|E\langle DF, u\rangle_{\mathfrak{H}}| \leqslant c_u \|F\|_{L^2(\Omega)}$ for any $F \in \mathbb{D}^{1,2}$, where $c_u$ is a constant depending only on $u$. If $u \in \operatorname{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship (customarily called *integration by parts formula*)

$$E[F\delta(u)] = E[\langle DF, u\rangle_{\mathfrak{H}}], \tag{2.11}$$

which holds for every $F \in \mathbb{D}^{1,2}$. More generally, if $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta$ are such that the three expectations $E[F^2\|u\|_{\mathfrak{H}}^2]$, $E\left[F^2\delta(u)^2\right]$ and $E\left[\langle DF, u\rangle_{\mathfrak{H}}^2\right]$ are finite, then $Fu \in \operatorname{Dom}\delta$ and

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{\mathfrak{H}}. \tag{2.12}$$

The operator $L$, defined as $L = \sum_{k=0}^\infty -kJ_k$, is the *infinitesimal generator of the Ornstein–Uhlenbeck semigroup*. The domain of $L$ is

$$\operatorname{Dom}L = \left\{F \in L^2(\Omega) : \sum_{k=1}^\infty k^2\,\|J_k F\|_{L^2(\Omega)}^2 < \infty\right\} = \mathbb{D}^{2,2}.$$

There is an important relation between the operators $D$, $\delta$ and $L$. A random variable $F$ belongs to $\mathbb{D}^{2,2}$ if and only if $F \in \mathrm{Dom}\,(\delta D)$ (i.e. $F \in \mathbb{D}^{1,2}$ and $DF \in \mathrm{Dom}\delta$) and, in this case,

$$\delta DF = -LF. \tag{2.13}$$

In particular, if $F \in \mathbb{D}^{2,2}$ and $H, G \in \mathbb{D}^{1,2}$ are such that $HG \in \mathbb{D}^{1,2}$, then

$$\begin{aligned} -E[HG\,LF] = E[HG\,\delta DF] = E[\langle D(HG), DF\rangle_{\mathfrak{H}}] = E[H\langle DG, DF\rangle_{\mathfrak{H}}] \\ + E[G\langle DH, DF\rangle_{\mathfrak{H}}]. \end{aligned} \tag{2.14}$$

### 2.3. A useful result

In this section, we state and prove the following lemma, which will be used several times in the sequel.

**Lemma 2.4.** *Fix $p \geqslant 2$, and let $\{F_n\}$ be a sequence of non-zero random variables belonging to the finite sum of chaoses $\bigoplus_{k=0}^{p} \mathcal{H}_k$. Assume that $F_n$ converges in distribution as $n \to \infty$. Then $\sup_{n\geqslant 1} E[|F_n|^r] < \infty$ for all $r \geqslant 1$.*

**Proof.** Let $Z$ be a positive random variable such that $E[Z] = 1$. Consider the decomposition $Z = Z\mathbf{1}_{\{Z\geqslant 1/2\}} + Z\mathbf{1}_{\{Z<1/2\}}$ and take the expectation. One deduces, using Cauchy–Schwarz, that

$$1 \leqslant \sqrt{E[Z^2]}\sqrt{P(Z \geqslant 1/2)} + \frac{1}{2},$$

that is,

$$E[Z^2]\,P(Z \geqslant 1/2) \geqslant \frac{1}{4}. \tag{2.15}$$

On the other hand, Theorem 2.1 implies the existence of $c_p > 0$ (a constant depending only on $p$) such that $E\left[F_n^4\right] \leqslant c_p\,E\left[F_n^2\right]^2$ for all $n \geqslant 1$. Combining this latter fact with (2.15) yields, with $Z = F_n^2/E[F_n^2]$,

$$P\left(F_n^2 \geqslant \frac{1}{2}E[F_n^2]\right) \geqslant \frac{1}{4c_p}. \tag{2.16}$$

The sequence $\{F_n\}_{n\geqslant 1}$ converging in distribution, it is tight and one can choose $M > 0$ large enough so that $P(F_n^2 > M) < \frac{1}{4c_p}$ for all $n \geqslant 1$. By applying (2.16), one obtains that

$$P\left(F_n^2 \geqslant M\right) < \frac{1}{4c_p} \leqslant P\left(F_n^2 \geqslant \frac{1}{2}\,E[F_n^2]\right),$$

from which one deduces immediately that $\sup_{n\geqslant 1} E[F_n^2] \leqslant 2M < \infty$. The desired conclusion follows from Theorem 2.1. $\square$

### 2.4. Carbery–Wright inequality

The proof of (1.2) shall rely on the following nice inequality due to Carbery and Wright [2]. We state it in the case of standard Gaussian random variables only. But its statement is actually more general, as it works under a log-concave density assumption.

**Theorem 2.5** (*Carbery–Wright*). *There exists an absolute constant $c > 0$ such that, for all polynomials $Q : \mathbb{R}^n \to \mathbb{R}$ of degree at most $d$, all independent random variables $X_1, \dots, X_n \sim \mathcal{N}(0,1)$ and all $\alpha > 0$,*

$$E[Q(X_1, \dots, X_n)^2]^{\frac{1}{2d}} \, P(|Q(X_1, \dots, X_n)| \leqslant \alpha) \leqslant c \, d \, \alpha^{\frac{1}{d}}. \tag{2.17}$$

**Proof.** See [2, Theorem 8]. $\square$

The power of $\alpha$ in the inequality (2.17) is sharp. To see it, it suffices to consider the case where $n = 1$ and $Q(x) = x^d$; we then have

$$P(|X_1|^d \leqslant \alpha) = P(|X_1| \leqslant \alpha^{1/d}) \sim_{\alpha \to 0^+} \sqrt{\frac{2}{\pi}} \, \alpha^{\frac{1}{d}}.$$

## 3. An asymptotic version of a theorem by Shigekawa

Our first result, which may be seen as an asymptotic version of Shigekawa [23], reads as follows.

**Theorem 3.1.** *Fix $p \geqslant 2$, and let $\{F_n\}$ be a sequence of random variables belonging to the finite sum of chaoses $\bigoplus_{k=0}^{p} \mathcal{H}_k$. Assume that $F_n$ converges in distribution towards $F_\infty$ satisfying $\mathrm{Var}(F_\infty) > 0$. Then, the following three assertions hold true:*

1. *the sequence $\{F_n\}$ is uniformly bounded in all the $L^r(\Omega)$: that is, $\sup_{n \geqslant 1} E[|F_n|^r] < \infty$ for all $r \geqslant 1$;*
2. *there exists $c > 0$ such that, for all $n \geqslant 1$,*

$$d_{TV}(F_n, F_\infty) \leqslant c \, d_{FM}(F_n, F_\infty)^{\frac{1}{2p+1}}. \tag{3.18}$$

   *In particular, $F_n$ converges in total variation towards $F_\infty$;*

3. *the law of $F_\infty$ is absolutely continuous with respect to the Lebesgue measure.*

**Proof.** The first point comes directly from Lemma 2.4. The rest of the proof is divided into four steps. Throughout the proof, the letter $c$ stands for a non-negative constant *independent* of $n$ (but which may depend on $p$, $\{F_n\}$ or $F_\infty$) and whose value may change from line to line.

*First step*. We claim that there exists $c > 0$ such that, for all $n \geqslant 1$:

$$P(\|DF_n\|_{\mathfrak{H}} \leqslant \lambda) \leqslant c \, \frac{\lambda^{\frac{1}{p-1}}}{\mathrm{Var}(F_n)^{\frac{1}{2p-2}}}. \tag{3.19}$$

Indeed, let $f_{k,n}$ be the elements of $\mathfrak{H}^{\odot k}$ such that $F_n = E[F_n] + \sum_{k=1}^{p} I_k(f_{k,n})$. Using the product formula (2.7), we can write:

$$\begin{aligned} \|DF_n\|_{\mathfrak{H}}^2 &= \sum_{k,l=1}^{p} kl \langle I_{k-1}(f_{k,n}), I_{l-1}(f_{l,n}) \rangle_{\mathfrak{H}} \\ &= \sum_{k,l=1}^{p} kl \sum_{r=1}^{k \wedge l} (r-1)! \binom{k-1}{r-1} \binom{l-1}{r-1} I_{k+l-2r}(f_{k,n} \tilde{\otimes}_r f_{l,n}). \end{aligned}$$

Now, let $\{e_i\}_{i\geqslant 1}$ be an orthonormal family of $\mathfrak{H}$ and decompose

$$f_{k,n}\tilde{\otimes}_r f_{l,n} = \sum_{m_1,m_2,\ldots,m_{k+l-2r}=1}^{\infty} \alpha_{m_1,\ldots,m_{k+l-2r},n}\, e_{m_1}\otimes\cdots\otimes e_{m_{k+l-2r}}.$$

Also, set

$$g_{k,l,r,n,s} = \sum_{m_1,m_2,\ldots,m_{k+l-2r}=1}^{s} \alpha_{m_1,\ldots,m_{k+l-2r},n}\, e_{m_1}\otimes\cdots\otimes e_{m_{k+l-2r}},$$

$$Y_{s,n} = \sum_{k,l=1}^{p} kl \sum_{r=1}^{k\wedge l}(r-1)!\binom{k-1}{r-1}\binom{l-1}{r-1} I_{k+l-2r}(g_{k,l,r,n,s}).$$

First, it is clear that $g_{k,l,r,n,s}\to f_{k,n}\tilde{\otimes}_r f_{l,n}$ as $s$ tends to infinity in $\mathfrak{H}^{\otimes(k+l-2r)}$. Hence, using the isometry property of Wiener–Itô integrals we conclude that

$$Y_{s,n}\overset{L^2}{\to}\|DF_n\|^2_{\mathfrak{H}}\quad\text{as } s\to\infty. \tag{3.20}$$

We deduce that there exists a strictly increasing sequence $\{s_l\}$ such that $Y_{s_l,n}\to\|DF_n\|^2_{\mathfrak{H}}$ as $l\to\infty$ almost surely. Second, we deduce from a well-known result by Itô that, with $k=k_1+\cdots+k_m$,

$$I_k(e_1^{\otimes k_1}\otimes\cdots\otimes e_m^{\otimes k_m}) = \prod_{i=1}^{m} H_{k_i}\left(X(e_i)\right).$$

Here, $H_k$ stands for the $k$th Hermite polynomial and has degree $k$, see (2.3). Also, one should note that the value of $I_k(e_1^{\otimes k_1}\otimes\cdots\otimes e_m^{\otimes k_m})$ is not modified when one permutes the order of the elements in the tensor product. Putting these two facts together, we can write

$$Y_{s,n} = Q_{s,n}\left(X(e_1),\ldots,X(e_s)\right),$$

for some polynomial $Q_{s,n}$ of degree at most $2p-2$. Consequently, we deduce from Theorem 2.5 that there exists a constant $c>0$ such that, for any $n\geqslant 0$ and any $\lambda>0$,

$$P\left(\left|Y_{s,n}\right|\leqslant\lambda^2\right)\leqslant c\,E[Y_{s,n}^2]^{-\frac{1}{4p-4}}\lambda^{1/(p-1)}.$$

Next, we can use Fatou's lemma to deduce that, for any $n\geqslant 0$ and any $\lambda>0$,

$$\begin{aligned}P(\|DF_n\|_{\mathfrak{H}}\leqslant\lambda) &\leqslant P\left(\liminf_{l\to\infty}\{|Y_{s_l,n}|\leqslant 2\lambda^2\}\right)\\ &\leqslant \liminf_{l\to\infty} P(|Y_{s_l,n}|\leqslant 2\lambda^2)\leqslant c\,E\left[\|DF_n\|^4_{\mathfrak{H}}\right]^{-1/(4p-4)}\lambda^{1/(p-1)}.\end{aligned}\tag{3.21}$$

Finally, by applying the Poincaré inequality (that is, $\mathrm{Var}(F_n)\leqslant E[\|DF_n\|^2_{\mathfrak{H}}]$), we get

$$E\left[\|DF_n\|^4_{\mathfrak{H}}\right]\geqslant E\left[\|DF_n\|^2_{\mathfrak{H}}\right]^2\geqslant\mathrm{Var}(F_n)^2,$$

which, together with (3.21), implies the desired conclusion (3.19).

*Second step*. We claim that: (i) the law of $F_n$ is absolutely continuous with respect to the Lebesgue measure when $n$ is large enough, and that (ii) there exists $c>0$ and $n_0\in\mathbb{N}$ such that, for all $\varepsilon > 0$, the following inequality holds:

$$\sup_{n\geqslant n_0} E\left[\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right] \leqslant c\,\varepsilon^{\frac{1}{2p-1}}. \tag{3.22}$$

The first point is a direct consequence of Shigekawa [23], but one can also give a direct proof by using (3.19). Indeed Point 1 together with the assumption that $F_n$ converges in distribution to $F_\infty$ implies that $\mathrm{Var}(F_n) \to \mathrm{Var}(F_\infty) > 0$. By letting $\lambda \to 0$ in (3.19), we deduce that, for $n$ large enough (so that $\mathrm{Var}(F_n) > 0$), we have $P(\|DF_n\|_{\mathfrak{H}} = 0) = 0$. Then, the Bouleau–Hirsch criterion (see, e.g., [17, Theorem 2.1.3]) ensures that the law of $F_n$ is absolutely continuous with respect to the Lebesgue measure.

Now, let us prove (3.22). We deduce from (3.19) that, for any $\lambda, \varepsilon > 0$,

$$\begin{aligned} E\left[\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right] &\leqslant E\left[\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\mathbf{1}_{\{\|DF_n\|_{\mathfrak{H}}>\lambda\}}\right] + P(\|DF_n\|_{\mathfrak{H}}\leqslant\lambda) \\ &\leqslant \frac{\varepsilon}{\lambda^2} + c\,\mathrm{Var}(F_n)^{-1/(2p-2)}\lambda^{1/(p-1)}. \end{aligned}$$

As we said, we have that $\mathrm{Var}(F_n) \to \mathrm{Var}(F_\infty) > 0$ as $n \to \infty$. Therefore, there exists $a > 0$ such that $\mathrm{Var}(F_n) > a$ for $n$ large enough (say $n \geqslant n_0$). We deduce that there exists $c > 0$ such that, for any $\lambda, \varepsilon > 0$,

$$\sup_{n\geqslant n_0} E\left[\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right] \leqslant c\left(\frac{\varepsilon}{\lambda^2}+\lambda^{1/(p-1)}\right). \tag{3.23}$$

Choosing $\lambda = \varepsilon^{\frac{p-1}{2p-1}}$ concludes the proof of (3.22).

*Third step*. We claim that there exists $c > 0$ such that, for all $n, m$ large enough,

$$d_{TV}(F_n, F_m) \leqslant c\, d_{FM}(F_n, F_m)^{\frac{1}{2p+1}}. \tag{3.24}$$

Set $p_\alpha(x) = \frac{1}{\alpha\sqrt{2\pi}}e^{-\frac{x^2}{2\alpha^2}}$, $x \in \mathbb{R}$, $0 < \alpha \leqslant 1$. Let $A$ be a bounded Borel set. It is easily checked that

$$\|\mathbf{1}_A * p_\alpha\|_\infty \leqslant \|\mathbf{1}_A\|_\infty \|p_\alpha\|_1 = 1 \leqslant \frac{1}{\alpha} \tag{3.25}$$

and, since $p'_\alpha(x) = -\frac{x}{\alpha^2}p_\alpha(x)$, that

$$\begin{aligned} \|(\mathbf{1}_A * p_\alpha)'\|_\infty = \|\mathbf{1}_A * p'_\alpha\|_\infty &= \frac{1}{\alpha^2}\sup_{x\in\mathbb{R}}\left|\int_{\mathbb{R}}\mathbf{1}_A(x-y)\,y\,p_\alpha(y)dy\right| \\ &\leqslant \frac{1}{\alpha^2}\int_{\mathbb{R}}|y|p_\alpha(y)dy = \frac{1}{\alpha}\sqrt{\frac{2}{\pi}} \leqslant \frac{1}{\alpha}. \end{aligned} \tag{3.26}$$

Let $n, m$ be large integers. Using (3.22), (3.25), (3.26) and that $F_n$ has a density when $n$ is large enough (Step 2(i)), we can write

$$\begin{aligned} |P(F_n \in A) - P(F_m \in A)| &\leqslant |E[\mathbf{1}_A * p_\alpha(F_n) - \mathbf{1}_A * p_\alpha(F_m)]| \\ &+ \left|E\left[(\mathbf{1}_A(F_n) - \mathbf{1}_A * p_\alpha(F_n))\left(\frac{\|DF_n\|_{\mathfrak{H}}^2}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}+\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \end{aligned}$$

$$
\begin{aligned}
&+\left|E\left[(\mathbf{1}_A(F_m)-\mathbf{1}_A*p_\alpha(F_m))\left(\frac{\|DF_m\|_{\mathfrak{H}}^2}{\|DF_m\|_{\mathfrak{H}}^2+\varepsilon}+\frac{\varepsilon}{\|DF_m\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \\
&\leqslant \frac{1}{\alpha}d_{FM}(F_n,F_m)+2E\left[\frac{\varepsilon}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right]+2E\left[\frac{\varepsilon}{\|DF_m\|_{\mathfrak{H}}^2+\varepsilon}\right] \\
&+\left|E\left[(\mathbf{1}_A(F_n)-\mathbf{1}_A*p_\alpha(F_n))\frac{\|DF_n\|_{\mathfrak{H}}^2}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \\
&+\left|E\left[(\mathbf{1}_A(F_m)-\mathbf{1}_A*p_\alpha(F_m))\frac{\|DF_m\|_{\mathfrak{H}}^2}{\|DF_m\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \\
&\leqslant \frac{1}{\alpha}d_{FM}(F_n,F_m)+c\,\varepsilon^{1/(2p-1)} \\
&+2\sup_{n\geqslant n_0}\left|E\left[(\mathbf{1}_A(F_n)-\mathbf{1}_A*p_\alpha(F_n))\frac{\|DF_n\|_{\mathfrak{H}}^2}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right]\right|.
\end{aligned}
$$

Now, set $\Psi(x)=\int_{-\infty}^x \mathbf{1}_A(s)ds$ and let us integrate by parts through (2.14). We get

$$
\begin{aligned}
&\left|E\left[(\mathbf{1}_A(F_n)-\mathbf{1}_A*p_\alpha(F_n))\frac{\|DF_n\|_{\mathfrak{H}}^2}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \\
&=\left|E\left[\frac{1}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\langle D(\Psi(F_n)-\Psi*p_\alpha(F_n)),DF_n\rangle_{\mathfrak{H}}\right]\right| \\
&=\left|E\left[(\Psi(F_n)-\Psi*p_\alpha(F_n))\left(\left\langle DF_n,D\left(\frac{1}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right)\right\rangle_{\mathfrak{H}}\right.\right.\right. \\
&\qquad\left.\left.\left.+\frac{LF_n}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \\
&=\left|E\left[(\Psi(F_n)-\Psi*p_\alpha(F_n))\left(-\frac{2\langle D^2F_n,DF_n\otimes DF_n\rangle_{\mathfrak{H}^{\otimes 2}}}{(\|DF_n\|_{\mathfrak{H}}^2+\varepsilon)^2}+\frac{LF_n}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \\
&\leqslant \frac{1}{\varepsilon}E\left[|\Psi(F_n)-\Psi*p_\alpha(F_n)|\left(2\|D^2F_n\|_{\mathfrak{H}^{\otimes 2}}+|LF_n|\right)\right].
\end{aligned}
\tag{3.27}
$$

On the other hand, we have

$$
\begin{aligned}
|\Psi(x)-\Psi*p_\alpha(x)| &= \left|\int_{\mathbb{R}}p_\alpha(y)\left(\int_{-\infty}^x(\mathbf{1}_A(z)-\mathbf{1}_A(z-y))\,dz\right)dy\right| \\
&\leqslant \int_{\mathbb{R}}p_\alpha(y)\left|\int_{-\infty}^x\mathbf{1}_A(z)dz-\int_{-\infty}^x\mathbf{1}_A(z-y)dz\right|dy \\
&\leqslant \int_{\mathbb{R}}p_\alpha(y)\left|\int_{x-y}^x\mathbf{1}_A(z)dz\right|dy\leqslant\int_{\mathbb{R}}p_\alpha(y)\,|y|\,dy\leqslant\sqrt{\frac{2}{\pi}}\alpha.
\end{aligned}
\tag{3.28}
$$

Moreover, $F_n$ is bounded in $L^2(\Omega)$ (see indeed Point 1) and $D^2F_n = \sum_{k=2}^{p} k(k-1)I_{k-2}(f_{k,n})$. We deduce that $\sup_{n\geqslant 1} E[|LF_n|] < \infty$ (since $F_n \in \bigoplus_{k=0}^{p} \mathcal{H}_k$) and

$$\sup_n E\left[\|D^2F_n\|_{\mathfrak{H}^{\otimes 2}}\right] < \infty,$$

implying in turn, thanks to (3.28), that

$$\sup_{n\geqslant n_0}\left|E\left[(\mathbf{1}_A(F_n) - \mathbf{1}_A * p_\alpha(F_n))\,\frac{\|DF_n\|_{\mathfrak{H}}^2}{\|DF_n\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \leqslant c\,\frac{\alpha}{\varepsilon}.$$

Thus, there exists $c > 0$ and $n_0 \in \mathbb{N}$ such that, for any $n, m \geqslant n_0$, any $0 < \alpha \leqslant 1$ and any $\varepsilon > 0$,

$$d_{TV}(F_n, F_m) \leqslant c\left(\frac{1}{\alpha}d_{FM}(F_n, F_m) + \frac{\alpha}{\varepsilon} + \varepsilon^{\frac{1}{2p-1}}\right).$$

Choosing $\alpha = \left(\frac{1}{2}d_{FM}(F_n, F_m)\right)^{\frac{2p}{2p+1}}$ (observe that $\alpha \leqslant 1$) and $\varepsilon = d_{FM}(F_n, F_m)^{\frac{2p-1}{2p+1}}$ leads to our claim (3.24).

*Fourth and final step*. Since the Fortet–Mourier distance $d_{FM}$ metrizes the convergence in distribution (see, e.g., [6, Theorem 11.3.3]), our assumption ensures that $d_{FM}(F_n, F_m) \to 0$ as $n, m \to \infty$. Thanks to (3.24), we conclude that $P_{F_n}$ is a Cauchy sequence for the total variation distance. But the space of bounded measures is complete for the total variation distance, so $P_{F_n}$ must converge towards $P_{F_\infty}$ in the total variation distance. Letting $m \to \infty$ in (3.24) yields the desired inequality (3.18). The proof of point 2 is done.

Let $A$ be a Borel set of Lebesgue measure zero. By Step 2(i), we have $P(F_n \in A) = 0$ when $n$ is large enough. Since $d_{TV}(F_n, F_\infty) \to 0$ as $n \to \infty$, we deduce that $P(F_\infty \in A) = 0$, proving that the law of $F_\infty$ is absolutely continuous with respect to the Lebesgue measure by the Radon–Nikodym theorem. The proof of point 3 is done. □

Let us give an application of Theorem 3.1 to the study of the absolute continuity of laws which are limits of multilinear polynomials with low influences and bounded degrees. We use techniques from Mossel et al. [10].

**Theorem 3.2.** *Let $p \geqslant 1$ be an integer and let $X_1, X_2, \ldots$ be independent random variables. Assume further that $E[X_k] = 0$ and $E[X_k^2] = 1$ for all k and that there exists $\varepsilon > 0$ such that $\sup_k E|X_k|^{2+\varepsilon} < \infty$. For any $m \geqslant 1$, let $n = n(m)$ and let $Q_m \in \mathbb{R}[x_1, \ldots, x_{n(m)}]$ be a real polynomial of the form*

$$Q_m(x_1, \ldots, x_{n(m)}) = \sum_{S\subset\{1,\ldots,n(m)\}} c_{S,m}\prod_{i\in S} x_i,$$

*with $\sum_{|S|>0} c_{S,m}^2 = 1$. Suppose moreover that the contribution of each $x_i$ to $Q_m(x_1, \ldots, x_{n(m)})$ is uniformly negligible, that is,*

$$\lim_{m\to\infty}\sup_{1\leqslant k\leqslant n(m)}\sum_{S:k\in S} c_{S,m}^2 = 0, \tag{3.29}$$

*and that the degree of $Q_m$ is at most $p$, that is,*

$$\max_{S:\,c_{s,m}\neq 0}|S| \leqslant p. \tag{3.30}$$

*Finally, let $F$ be a limit in law of $Q_m(X_1, \dots, X_{n(m)})$ (possibly through a subsequence only) as $m \to \infty$. Then the law of $F$ has a density with respect to the Lebesgue measure.*

**Proof.** Using [10, Theorem 2.2] and because of (3.29) and (3.30), we deduce that $F$ is also a limit in law of $Q_m(G_1, \dots, G_{n(m)})$, where the $G_i$'s are independent $N(0, 1)$ random variables. Moreover, because of (3.30), it is straightforward that $Q_m(G_1, \dots, G_{n(m)})$ may be realized as an element belonging to the sum of the $p$ first Wiener chaoses. Also, due to $\sum_{|S|>0} c_{S,m}^2 = 1$, we have that the variance of $Q_m(G_1, \dots, G_{n(m)})$ is 1. Therefore, the desired conclusion is now a direct consequence of Theorem 3.1. □

**Remark 3.3.** One cannot remove the assumption (3.29) in the previous theorem. Indeed, without this assumption, it is straightforward to construct easy counterexamples to the conclusion of Theorem 3.1. For instance, it is clear that the conclusion is not reached if one considers $Q_m(x_1, \dots, x_{n(m)}) = x_1$ together with a *discrete* random variable $X_1$.

## 4. Continuity of the law of $I_k(f)$ with respect to $f$

In this section, we are mainly interested in the continuity of the law of $I_k(f)$ with respect to its kernel $f$. Our first theorem is a result going in the same direction. It exhibits a sufficient condition that allows one to pass from a convergence in law to a convergence in total variation.

**Theorem 4.1.** *Let $\{F_n\}_{n\geqslant 1}$ be a sequence of $\mathbb{D}^{1,2}$ satisfying*

(i) $\frac{DF_n}{\|DF_n\|_{\mathfrak{H}}^2} \in \mathrm{dom}\delta$ *for any $n \geqslant 1$;*

(ii) $C := \sup_{n\geqslant 1} E\left|\delta\left(\frac{DF_n}{\|DF_n\|_{\mathfrak{H}}^2}\right)\right| < \infty$;

(iii) $F_n \overset{\text{law}}{\to} F_\infty$ *as $n \to \infty$.*

*Then*

$$d_{TV}(F_n, F_\infty) \leqslant \left(\sqrt{2} + \frac{4C}{\sqrt{\pi}}\right)\sqrt{d_{FM}(F_n, F_\infty)}.$$

*In particular, $F_n$ tends to $F_\infty$ in total variation.*

**Proof.** Let $A$ be a bounded Borel set and set $p_\alpha(x) = \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{x^2}{2\alpha^2}}$, $x \in \mathbb{R}$, $0 < \alpha \leqslant 1$. Since $\int_0^\cdot (\mathbf{1}_A(x) - \mathbf{1}_A * p_\alpha)\,dx$ is Lipschitz and $F_m, F_n$ admit a density, we have using (2.11) that, for any $n, m \geqslant 1$,

$$\begin{aligned} P(F_n \in A) - P(F_m \in A) = {} & E[\mathbf{1}_A * p_\alpha(F_n)] - E[\mathbf{1}_A * p_\alpha(F_m)] \\ & + E\left[\delta\left(\frac{DF_n}{\|DF_n\|_{\mathfrak{H}}^2}\right)\int_0^{F_n} (\mathbf{1}_A - \mathbf{1}_A * p_\alpha)(x)dx\right] \\ & - E\left[\delta\left(\frac{DF_m}{\|DF_m\|_{\mathfrak{H}}^2}\right)\int_0^{F_m} (\mathbf{1}_A - \mathbf{1}_A * p_\alpha)(x)dx\right]. \end{aligned}$$

Using (3.25) and (3.26), we can write

$$|E[\mathbf{1}_A * p_\alpha(F_n)] - E[\mathbf{1}_A * p_\alpha(F_m)]| \leqslant \frac{1}{\alpha} d_{FM}(F_n, F_m).$$

On the other hand, we have, for any $x \in \mathbb{R}$,

$$\begin{aligned}\left|\int_0^x (\mathbf{1}_A - \mathbf{1}_A * p_\alpha)(v)dv\right| &= \left|\int_0^x du \int_{\mathbb{R}} dv p_\alpha(v) (\mathbf{1}_A(u) - \mathbf{1}_A(u-v))\right| \\ &= \left|\int_{\mathbb{R}} dv\, p_\alpha(v) \int_0^x (\mathbf{1}_A(u) - \mathbf{1}_A(u-v))\, du\right| \\ &= \left|\int_{\mathbb{R}} dv\, p_\alpha(v) \left(\int_{x-v}^x \mathbf{1}_A(u)du - \int_0^{-v} \mathbf{1}_A(u)du\right)\right| \\ &\leqslant 2\int_{\mathbb{R}} |v| p_\alpha(v)dv = 2\sqrt{\frac{2}{\pi}}\,\alpha.\end{aligned}$$

By putting all these facts together, we get that

$$|P(F_n \in A) - P(F_m \in A)| \leqslant \frac{1}{\alpha} d_{FM}(F_n, F_m) + 4C\sqrt{\frac{2}{\pi}}\alpha.$$

To conclude, it remains to choose $\alpha = \sqrt{\frac{1}{2} d_{FM}(F_n, F_m)}$ and then to let $m \to \infty$ as in the fourth step of the proof of Theorem 3.1. □

In [21], Poly and Malicet prove that, if $F_n \stackrel{\mathbb{D}^{1,2}}{\to} F_\infty$ and $P(\|DF_\infty\|_{\mathfrak{H}} > 0) = 1$, then $d_{TV}(F_n, F_\infty) \to 0$. Nevertheless, their proof does not give any idea on the rate of convergence. The following result is a kind of quantitative version of the aforementioned result in [21].

**Theorem 4.2.** *Let $\{F_n\}_{n\geqslant 1}$ be a sequence in $\mathbb{D}^{1,2}$ such that each $F_n$ admits a density. Let $F_\infty \in \mathbb{D}^{2,4}$ and let $0 < \alpha \leqslant 2$ be such that $E\left[\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^\alpha}\right] < \infty$. If $F_n \stackrel{\mathbb{D}^{1,2}}{\to} F_\infty$ then there exists a constant $c > 0$ depending only of $F_\infty$ such that, for any $n \geqslant 1$,*

$$d_{TV}(F_n, F_\infty) \leqslant c\|F_n - F_\infty\|_{\mathbb{D}^{1,2}}^{\frac{\alpha}{\alpha+2}}.$$

**Proof.** Throughout the proof, the letter $c$ stands for a non-negative constant *independent* of $n$ and whose value may change from line to line. Let $A$ be a bounded Borel set of $\mathbb{R}$. For all $0 < \varepsilon \leqslant 1$, one has (using that $F_n$ has a density to perform the integration by parts, see Remark 2.3)

$$\begin{aligned}P(F_n \in A) - P(F_\infty \in A) &= E\left[\frac{\left\langle D\int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx, DF_\infty\right\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2 + \varepsilon}\right] \\ &\quad + E\left[(\mathbf{1}_A(F_n) - \mathbf{1}_A(F_\infty)) \frac{\varepsilon}{\|DF_\infty\|_{\mathfrak{H}}^2 + \varepsilon}\right] \\ &\quad - E\left[\mathbf{1}_A(F_n) \frac{\langle D(F_n - F_\infty), DF_\infty\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2 + \varepsilon}\right]. \qquad (4.31)\end{aligned}$$

But, see (2.12),

$$\left\langle D\int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx, DF_\infty\right\rangle_{\mathfrak{H}} = -\delta\left(DF_\infty \int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx\right) + LF_\infty \int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx.$$

Therefore

$$\begin{aligned}
&\left| E\left[ \frac{\left\langle D\int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx,\, DF_\infty\right\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \\
&\quad = \left| E\left[ \int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx \left( -\left\langle DF_\infty, D\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right\rangle_{\mathfrak{H}} + \frac{LF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \\
&\quad = \left| E\left[ \int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx \left( \frac{2\langle D^2F_\infty, DF_\infty\otimes DF_\infty\rangle_{\mathfrak{H}^{\otimes 2}}}{\left(\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon\right)^2} + \frac{LF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right)\right]\right| \\
&\quad \leqslant \frac{c}{\varepsilon}\|F_n-F_\infty\|_2,
\end{aligned}$$

the last inequality following from Cauchy–Schwarz and the fact that $F_\infty \in \mathbb{D}^{2,4}$. On the other hand,

$$\left| E\left[ \mathbf{1}_A(F_n)\frac{\langle D(F_n-F_\infty), DF_\infty\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| \leqslant \frac{c}{\varepsilon}\|F_n-F_\infty\|_{\mathbb{D}^{1,2}}.$$

Finally, let us observe that:

$$\begin{aligned}
\left| E\left[ (\mathbf{1}_A(F_n)-\mathbf{1}_A(F_\infty))\frac{\varepsilon}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| &\leqslant E\left[\frac{\varepsilon}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right] \\
&\leqslant \varepsilon^{\frac{\alpha}{2}} E\left[\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^{\alpha}}\right].
\end{aligned}$$

Therefore, putting all these facts together and with $\varepsilon = \|F_n-F_\infty\|_{\mathbb{D}^{1,2}}^{\frac{2}{\alpha+2}}$, we get

$$|P(F_n\in A)-P(F_\infty\in A)| \leqslant c\left(\varepsilon^{\frac{\alpha}{2}}+\frac{1}{\varepsilon}\|F_n-F_\infty\|_{\mathbb{D}^{1,2}}\right) \leqslant c\|F_n-F_\infty\|_{\mathbb{D}^{1,2}}^{\frac{\alpha}{\alpha+2}},$$

which is the desired conclusion. □

Let us now study the continuity of the law of $I_k(f)$ with respect to its kernel $f$. Before offering another proof of the main result in Davydov and Martynova [5] (see our comments about this in the introduction), we start with a preliminary lemma.

**Lemma 4.3.** *Let $F = I_k(f)$ with $k \geqslant 2$ and $f \in \mathfrak{H}^{\odot k}$ non identically zero. There exists $c > 0$ such that, for all $\varepsilon > 0$,*

$$P(\|DF\|_{\mathfrak{H}}^2 \leqslant \varepsilon) \leqslant c\,\varepsilon^{\frac{1}{2k-2}}.$$

**Proof.** Throughout the proof, the letter $c$ stands for a non-negative constant *independent* of $n$ and whose value may change from line to line. The proof is very close to that of Step 1 in Theorem 3.1. Let $\{e_i\}_{i\geqslant 1}$ be an orthonormal basis of $\mathfrak{H}$. One can decompose $f$ as

$$f = \sum_{i_1,\ldots,i_k=1}^{\infty} c_{i_1,\ldots,i_k}\, e_{i_1}\otimes\cdots\otimes e_{i_k}. \tag{4.32}$$

For each $n \geqslant 1$, set

$$f_n = \sum_{i_1,\ldots,i_k=1}^{n} c_{i_1,\ldots,i_k}\, e_{i_1} \otimes \cdots \otimes e_{i_k}.$$

As $n \to \infty$, one has $f_n \to f$ in $\mathfrak{H}^{\otimes k}$ or, equivalently, $I_k(f_n) \to I_k(f)$ in $L^2(\Omega)$. We deduce that there exists a strictly increasing sequence $\{n_l\}$ such that $I_k(f_{n_l}) \to I_k(f)$ almost surely as $l \to \infty$.

On the other hand, this is a well-known result from Itô that, with $k = k_1 + \cdots + k_m$, one has

$$I_k(e_1^{\otimes k_1} \otimes \cdots \otimes e_m^{\otimes k_m}) = \prod_{i=1}^{m} H_{k_i}\left(X(e_i)\right),$$

with $H_k$ the $k$th Hermite polynomial given by (2.3). Also, one should note that the value of $I_k(e_1^{\otimes k_1} \otimes \cdots \otimes e_m^{\otimes k_m})$ is not modified when one permutes the order of the elements in the tensor product. It is deduced from these two facts that

$$I_k(f_n) = Q_{n,k}\left(X(e_1), \ldots, X(e_n)\right),$$

where $Q_{n,k}$ is a polynomial of degree at most $k$. Theorem 2.5 ensures the existence of a constant $c > 0$ such that, for all $n \geqslant 1$ and $\varepsilon > 0$,

$$P\left(|I_k(f_n)| \leqslant \varepsilon\, \|f_n\|_{\mathfrak{H}^{\otimes k}}\right) \leqslant c\, \varepsilon^{1/k}.$$

Next, we can use Fatou's lemma to deduce that, for any $\varepsilon > 0$,

$$\begin{aligned} P\left(|I_k(f)| \leqslant \varepsilon\, \|f\|_{\mathfrak{H}^{\otimes k}}\right) &\leqslant P\left(\liminf_{l\to\infty}\{|I_k(f_{n_l})| \leqslant 2\varepsilon \|f_{n_l}\|_{\mathfrak{H}^{\otimes k}}\}\right) \\ &\leqslant \liminf_{l\to\infty} P(|I_k(f_{n_l})| \leqslant 2\varepsilon \|f_{n_l}\|_{\mathfrak{H}^{\otimes k}}) \leqslant c\, \varepsilon^{1/k}. \end{aligned}$$

Equivalently,

$$P\left(|I_k(f)| \leqslant \varepsilon\right) \leqslant c\, \|f\|_{\mathfrak{H}^{\otimes k}}^{-1/k}\, \varepsilon^{1/k}.$$

Now, assume for a while that $\langle f, h\rangle_{\mathfrak{H}} = 0$ for all $h \in \mathfrak{H}$. By (4.32), we have

$$\langle f, h\rangle_{\mathfrak{H}} = \sum_{i_1,\ldots,i_k=1}^{\infty} c(i_1,\ldots,i_k)\langle e_{i_1}, h\rangle_{\mathfrak{H}}\; e_{i_2} \otimes \cdots \otimes e_{i_k},$$

implying in turn, because $\langle f, h\rangle_{\mathfrak{H}} = 0$ for all $h \in \mathfrak{H}$, that

$$\sum_{i_2,\ldots,i_k=1}^{\infty} \left(\sum_{i_1=1}^{\infty} c(i_1,\ldots,i_k)\langle e_{i_1}, h\rangle_{\mathfrak{H}}\right)^2 = 0 \quad \text{for all } h \in \mathfrak{H}.$$

By choosing $h = e_i$, $i = 1, 2, \ldots$, we get that $c(i_1,\ldots,i_k) = 0$ for any $i_1,\ldots,i_k \geqslant 1$, that is, $f = 0$. This latter fact being in contradiction with our assumption, one deduces that there exists $h \in \mathfrak{H}$ so that $\langle f, h\rangle_{\mathfrak{H}} \neq 0$. Consequently,

$$\begin{aligned} P\left(\|DF\|_{\mathfrak{H}}^2 \leqslant \varepsilon\right) &\leqslant P\left(|\langle DF, h\rangle_{\mathfrak{H}}| \leqslant \sqrt{\varepsilon}\|h\|_{\mathfrak{H}}\right) \\ &= P\left(|I_{k-1}(\langle f, h\rangle)_{\mathfrak{H}}| \leqslant \frac{1}{k}\sqrt{\varepsilon}\|h\|_{\mathfrak{H}}\right) \leqslant c\, \varepsilon^{\frac{1}{2k-2}}, \end{aligned}$$

which is the desired conclusion. $\square$

Finally, we state and prove the following result, which gives a precise estimate for the continuity of $I_k(f)$ with respect to $f$. This is almost the main result of Davydov and Martynova [5], see our comments in the introduction. Moreover, with respect to what we would have obtained by applying (3.18), here the rate is $\frac{1}{2k}$ (which is better than $\frac{1}{2k+1}$, immediate consequence of (3.18)).

**Theorem 4.4.** *Fix $k \geqslant 2$, and let $\{f_n\}_{n\geqslant 1}$ be a sequence of elements of $\mathfrak{H}^{\odot k}$. Assume that $f_\infty = \lim_{n\to\infty} f_n$ exists in $\mathfrak{H}^{\otimes k}$ and that each $f_n$ as well as $f_\infty$ are not identically zero. Then there exists a constant c, depending only on k and $f_\infty$, such that, for all $n \geqslant 1$,*

$$d_{TV}(I_k(f_n), I_k(f_\infty)) \leqslant c\,\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes k}}^{\frac{1}{2k}}$$

*for any $n \geqslant 1$.*

**Proof.** Set $F_n = I_k(f_n)$ and $F_\infty = I_k(f_\infty)$. Let $A$ be a bounded Borel set of $\mathbb{R}$, and fix $0 < \varepsilon \leqslant 1$. Since $f_n, f_\infty \not\equiv 0$, Shigekawa theorem (see [23], or [14, Theorem 2.10.1], or Theorem 3.1) ensures that $F_n$ and $F_\infty$ both have a density. We deduce that

$$\left\langle D\int_{F_\infty}^{F_n} \mathbf{1}_A(x)dx, DF_\infty\right\rangle_{\mathfrak{H}} = (\mathbf{1}_A(F_n) - \mathbf{1}_A(F_\infty))\,\|DF_\infty\|_{\mathfrak{H}}^2 + \mathbf{1}_A(F_n)\langle D(F_n - F_\infty), DF_\infty\rangle_{\mathfrak{H}},$$

implying in turn that

$$\begin{aligned} P(F_n\in A) - P(F_\infty \in A) &= E\left[\frac{\left\langle D\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx, DF_\infty\right\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right] \\ &\quad - E\left[\frac{\mathbf{1}_A(F_n)\langle D(F_n-F_\infty), DF_\infty\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right] \\ &\quad + E\left[(\mathbf{1}_A(F_n)-\mathbf{1}_A(F_\infty))\,\frac{\varepsilon}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right]. \end{aligned}$$

First, using (2.12) and next $\delta(DF_\infty) = -LF_\infty = kF_\infty$ we can write

$$\begin{aligned} &\left|E\left[\frac{\left\langle D\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx, DF_\infty\right\rangle_{\mathfrak{H}}}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right]\right| = \left|E\left[\delta\left(\frac{DF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right)\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx\right]\right| \\ &= \left|E\left[\frac{kF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx\right]\right. \\ &\quad \left. - E\left[\left\langle DF_\infty, D\left(\frac{1}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\right)\right\rangle_{\mathfrak{H}}\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx\right]\right| \\ &= \left|E\left[\frac{kF_\infty}{\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon}\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx\right]\right. \\ &\quad \left. + E\left[\frac{2\langle D^2F_\infty, DF_\infty\otimes DF_\infty\rangle_{\mathfrak{H}^{\otimes 2}}}{(\|DF_\infty\|_{\mathfrak{H}}^2+\varepsilon)^2}\int_{F_\infty}^{F_n}\mathbf{1}_A(x)dx\right]\right| \end{aligned}$$

$$\leqslant \frac{1}{\varepsilon}\,E\left[\left(k|F_\infty| + 2\|D^2F_\infty\|_{\mathfrak{H}^{\otimes 2}}\right)|F_n - F_\infty|\right]$$
$$\leqslant \frac{1}{\varepsilon}\,\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes k}}\sqrt{k!\,E\left[\left(k|F_\infty| + 2\|D^2F_\infty\|_{\mathfrak{H}^{\otimes 2}}\right)^2\right]},$$

where the last inequality comes from Cauchy–Schwarz and the isometry property of multiple integrals. Second, using $\frac{1}{k}E\left[\|DF_\infty\|^2_{\mathfrak{H}}\right] = E\left[F_\infty \times \frac{1}{k}\delta DF_\infty\right] = E[F_\infty^2]$ and

$$\frac{1}{k}E\left[\|D(F_n - F_\infty)\|^2\right] = E[(F_n - F_\infty)^2] = k!\|f_n - f_\infty\|^2_{\mathfrak{H}^{\otimes k}},$$

we have

$$\left|E\left[\frac{\mathbf{1}_A(F_n)\langle D(F_n - F_\infty), DF_\infty\rangle_{\mathfrak{H}}}{\|DF_\infty\|^2_{\mathfrak{H}} + \varepsilon}\right]\right| \leqslant \frac{1}{\varepsilon}\,\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes k}}\sqrt{k^2k!E[F_\infty^2]}.$$

Third,

$$\begin{aligned}\left|E\left[(\mathbf{1}_A(F_n) - \mathbf{1}_A(F_\infty))\,\frac{\varepsilon}{\|DF_\infty\|^2_{\mathfrak{H}} + \varepsilon}\right]\right| &\leqslant E\left[\frac{\varepsilon}{\|DF_\infty\|^2_{\mathfrak{H}} + \varepsilon}\right]\\ &\leqslant E\left[\frac{\varepsilon}{\|DF_\infty\|^2_{\mathfrak{H}} + \varepsilon}\,\mathbf{1}_{\left\{\|DF_\infty\|^2 > \varepsilon^{\frac{2k-2}{2k-1}}\right\}}\right]\\ &\quad + P\left(\|DF_\infty\|^2_{\mathfrak{H}} \leqslant \varepsilon^{\frac{2k-2}{2k-1}}\right)\\ &\leqslant \varepsilon^{\frac{1}{2k-1}} + c\,\varepsilon^{\frac{1}{2k-1}} = c\,\varepsilon^{\frac{1}{2k-1}},\end{aligned}$$

where the last inequality comes from Lemma 4.3.

By summarizing, we get

$$|P(F_n \in A) - P(F_\infty \in A)| \leqslant \frac{c}{\varepsilon}\,\|f_n - f_\infty\|_{\mathfrak{H}^{\otimes k}} + c\,\varepsilon^{\frac{1}{2k-1}}.$$

The desired conclusion follows by choosing $\varepsilon = \|f_n - f_\infty\|^{\frac{2k-1}{2k}}_{\mathfrak{H}^{\otimes k}}$. □

## 5. The Peccati–Tudor theorem holds in total variation

Let us first recall the Peccati–Tudor theorem [20].

**Theorem 5.1.** *Let $d \geqslant 2$ and $k_d, \ldots, k_1 \geqslant 1$ be some fixed integers. Consider vectors*

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = (I_{k_1}(f_{1,n}), \ldots, I_{k_d}(f_{d,n})), \quad n \geqslant 1,$$

*with $f_{i,n} \in \mathfrak{H}^{\odot k_i}$. Let $N \sim \mathcal{N}_d(0, C)$ with* $\det(C) > 0$ *and assume that*

$$\lim_{n\to\infty} E[F_{i,n}F_{j,n}] = C(i, j), \quad 1 \leqslant i, j \leqslant d. \tag{5.33}$$

*Then, as $n \to \infty$, the following two conditions are equivalent:*

(a) *$F_n$ converges in law to $N$;*
(b) *for every $1 \leqslant i \leqslant d$, $F_{i,n}$ converges in law to $\mathcal{N}(0, C(i, i))$.*

The following result shows that the assertion (a) in the previous theorem may be replaced for free by an a priori stronger assertion, namely:

(a′) $F_n$ converges in total variation to $N$.

**Theorem 5.2.** *Let $d \geqslant 2$ and $k_d, \dots, k_1 \geqslant 1$ be some fixed integers. Consider vectors*

$$F_n = (F_{1,n}, \dots, F_{d,n}) = (I_{k_1}(f_{1,n}), \dots, I_{k_d}(f_{d,n})), \quad n \geqslant 1,$$

*with $f_{i,n} \in \mathfrak{H}^{\odot k_i}$. As $n \to \infty$, assume that $F_n \overset{\text{law}}{\to} N \sim \mathcal{N}_d(0, C)$ with $\det(C) > 0$. Then, $d_{TV}(F_n, N) \to 0$ as $n \to \infty$.*

During the proof of Theorem 5.2, we shall need the following auxiliary lemma. (Recall that $\mathcal{H}_k$ denotes the $k$th Wiener chaos of $X$.)

**Lemma 5.3.** *Let $\mathcal{A}$ be the class of sequences $\{Y_n\}_{n\geqslant 1}$ satisfying that:* (i) *there exists $p \in \mathbb{N}^*$ such that $Y_n \in \bigoplus_{k=0}^p \mathcal{H}_k$ for all $n$; and* (ii) *$\sup_{n\geqslant 1} E[Y_n^2] < \infty$. We have the following stability property for $\mathcal{A}$: if $\{Y_n\}_{n\geqslant 1}$ and $\{Z_n\}_{n\geqslant 1}$ both belong to $\mathcal{A}$, then $\{\langle DY_n, DZ_n\rangle_{\mathfrak{H}}\}_{n\geqslant 1}$ belongs to $\mathcal{A}$ too.*

**Proof.** Let $\{Y_n\}_{n\geqslant 1}$ and $\{Z_n\}_{n\geqslant 1}$ be two sequences of $\mathcal{A}$. We then have: (i) $Y_n = E[Y_n] + \sum_{k=1}^p I_k(g_{k,n})$ and $Z_n = E[Z_n] + \sum_{k=1}^p I_k(h_{k,n})$ for some integer $p$ and some elements $g_{k,n}$ and $h_{k,n}$ of $\mathfrak{H}^{\odot k}$; (ii) $\sup_{n\geqslant 1} \|g_{k,n}\|^2_{\mathfrak{H}^{\otimes k}} < \infty$ and $\sup_{n\geqslant 1} \|h_{k,n}\|^2_{\mathfrak{H}^{\otimes k}} < \infty$ for all $k = 1, \dots, p$. Using the product formula for multiple Wiener–Itô integrals, it is straightforward to check that

$$\langle DY_n, DZ_n\rangle_{\mathfrak{H}} = \sum_{k,l=1}^{p} kl \sum_{r=1}^{k\wedge l} (r-1)! \binom{k-1}{r-1} \binom{l-1}{r-1} I_{k+l-2r}(g_{k,n} \widetilde{\otimes}_r h_{l,n}).$$

We deduce in particular that $\langle DY_n, DZ_n\rangle_{\mathfrak{H}} \in \bigoplus_{k=0}^{2p} \mathcal{H}_k$. Moreover, since

$$\begin{aligned}\|g_{k,n} \widetilde{\otimes}_r h_{l,n}\|_{\mathfrak{H}^{\otimes k+l-2r}} &\leqslant \|g_{k,n} \otimes_r h_{l,n}\|_{\mathfrak{H}^{\otimes k+l-2r}} \leqslant \|g_{k,n}\|_{\mathfrak{H}^{\otimes k}} \|h_{l,n}\|_{\mathfrak{H}^{\otimes l}} \\ &\leqslant \frac{1}{2}\left(\|g_{k,n}\|^2_{\mathfrak{H}^{\otimes k}} + \|h_{l,n}\|^2_{\mathfrak{H}^{\otimes l}}\right),\end{aligned}$$

we have that $\sup_{n\geqslant 1} E\left[\langle DY_n, DZ_n\rangle^2_{\mathfrak{H}}\right] < \infty$. That is, the sequence $\{\langle DY_n, DZ_n\rangle_{\mathfrak{H}}\}_{n\geqslant 1}$ belongs to $\mathcal{A}$. $\square$

We are now in a position to prove Theorem 5.2.

**Proof of Theorem 5.2.** First, using Lemma 2.4 and because $F_n \overset{\text{law}}{\to} \mathcal{N}_d(0, C)$, it is straightforward to show that $E[F_{i,n}F_{j,n}] \to C(i, j)$ as $n \to \infty$ for all $i, j = 1, \dots, d$. Now, fix $M \geqslant 1$ and let $\phi \in \mathcal{C}_c^\infty([-M, M]^d)$. For any $i = 1, \dots, d$, define

$$T_i[\phi](x) = \int_0^{x_i} \phi(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d)dt, \quad x \in \mathbb{R}^d.$$

Also, set $T_{i_1,\dots,i_a} = T_{i_1} \circ \cdots \circ T_{i_a}$, so that $\partial_{i_1,\dots,i_a} T_{i_1,\dots,i_a}[\phi] = \phi$.

The following lemma, which exhibits mere regularizing properties for the operators $T_i$, is going to play a crucial role in the proof.

**Lemma 5.4.** *The function $T_{d,\dots,2,1}[\phi]$ satisfies the following two properties:*

- *for all $k = 1, \dots, d$, $\|T_{k,\dots,2,1}[\phi]\|_\infty \leqslant M^k \|\phi\|_\infty$;*
- *for all $x, y \in \mathbb{R}^d$, $|T_{d,\dots,2,1}[\phi](x) - T_{d,\dots,2,1}[\phi](y)| \leqslant M^{d-1}\|\phi\|_\infty \|x - y\|_1$.*

**Proof.** For any $x, y \in \mathbb{R}^d$ and $k = 1, \dots, d$, we have

$$\begin{aligned}
|T_{k,\dots,2,1}[\phi](x)| &\leqslant \int_0^{|x_1|}\int_0^{|x_2|}\cdots\int_0^{|x_k|} |\phi(t_1,\dots,t_k,x_{k+1},\dots,x_d)|dt_1dt_2\cdots dt_k \\
&\leqslant \int_0^{M}\int_0^{M}\cdots\int_0^{M} |\phi(t_1,\dots,t_k,x_{k+1},\dots,x_d)|dt_1dt_2\cdots dt_k \\
&\leqslant M^k\|\phi\|_\infty,
\end{aligned}$$

whereas

$$\begin{aligned}
&|T_{1,2,\dots,d}[\phi](x) - T_{1,2,\dots,d}[\phi](y)| \\
&\quad = \left|\int_0^{x_1}\cdots\int_0^{x_d}\phi(t_1,\dots,t_d)dt_1dt_2\cdots dt_d \right.\\
&\qquad \left. - \int_0^{y_1}\cdots\int_0^{y_d}\phi(t_1,\dots,t_d)dt_1dt_2\cdots dt_d\right| \\
&\quad = \left|\sum_{i=1}^d \int_0^{x_1}\cdots\int_0^{x_{i-1}}\int_{x_i}^{y_i}\int_0^{y_{i+1}}\cdots\int_0^{y_d}\phi(t_1,\dots,t_d)dt_1dt_2\cdots dt_d\right| \\
&\quad \leqslant M^{d-1}\|\phi\|_\infty \sum_{i=1}^d |x_i - y_i|. \quad \square
\end{aligned}$$

Let us go back to the proof of Theorem 5.2. Set

$$\Gamma_n = \begin{pmatrix} \langle DF_{1,n}, DF_{1,n}\rangle_{\mathfrak{H}} & \cdots & \langle DF_{d,n}, DF_{1,n}\rangle_{\mathfrak{H}} \\ \vdots & \dots & \vdots \\ \langle DF_{1,n}, DF_{d,n}\rangle_{\mathfrak{H}} & \cdots & \langle DF_{d,n}, DF_{d,n}\rangle_{\mathfrak{H}} \end{pmatrix},$$

the Malliavin matrix associated with $F_n$. Using the chain rule (2.9), we have

$$\begin{pmatrix} \langle D\phi(F_n), DF_{1,n}\rangle_{\mathfrak{H}} \\ \vdots \\ \langle D\phi(F_n), DF_{d,n}\rangle_{\mathfrak{H}} \end{pmatrix} = \Gamma_n \begin{pmatrix} \partial_1\phi(F_n) \\ \vdots \\ \partial_d\phi(F_n) \end{pmatrix}. \tag{5.34}$$

Solving (5.34) yields:

$$\partial_i\phi(F_n)\det(\Gamma_n) = \sum_{a=1}^d (\mathrm{adj}\,\Gamma_n)_{a,i}\langle D\phi(F_n), DF_{a,n}\rangle_{\mathfrak{H}}, \tag{5.35}$$

where $\mathrm{adj}(\cdot)$ stands for the usual adjugate matrix operator. By first multiplying (5.35) by $W \in \mathbb{D}^{1,2}$ and then taking the expectation, we get, using (2.14) as well,

$$\begin{aligned}
E[\partial_i\phi(F_n)\det(\Gamma_n)W] = -\sum_{a=1}^d E\left[\phi(F_n)\left(\langle D(W(\mathrm{adj}\,\Gamma_n)_{a,i}), DF_{a,n}\rangle_{\mathfrak{H}}\right.\right.& \\
\left.\left. +\ (\mathrm{adj}\,\Gamma_n)_{a,i}WLF_{a,n}\right)\right] = E[\phi(F_n)R_{i,n}(W)],&
\end{aligned} \tag{5.36}$$

where

$$R_{i,n}(W) = -\sum_{a=1}^{d} \left( \langle D(W(\mathrm{adj}\Gamma_n)_{a,i}), DF_{a,n} \rangle_{\mathfrak{H}} + (\mathrm{adj}\Gamma_n)_{a,i} WLF_{a,n} \right).$$

Thanks to [18, Lemma 6], we know that, for any $i, j = 1, \dots, d$,

$$\langle DF_{i,n}, DF_{j,n} \rangle_{\mathfrak{H}} \xrightarrow{L^2} \sqrt{k_i k_j}\, C(i,j) \quad \text{as } n \to \infty. \tag{5.37}$$

Also, Lemma 5.3 implies that $\langle DF_{i,n}, DF_{j,n} \rangle_{\mathfrak{H}}$ is in a finite sum of chaoses and is bounded in $L^2(\Omega)$. By hypercontractivity, we deduce that $\langle DF_{i,n}, DF_{j,n} \rangle_{\mathfrak{H}}$ is actually bounded in all the $L^p(\Omega)$, $p \geqslant 1$, and that the convergence in (5.37) extends in all the $L^p(\Omega)$. As a consequence,

$$\det(\Gamma_n) \xrightarrow{L^2} \det(C) \prod_{i=1}^{d} k_i =: \gamma > 0. \tag{5.38}$$

Using first (5.36) with $W = 1$ and $T_1[\phi]$ instead of $\phi$, and then iterating, yields

$$\begin{aligned} E\left[\phi(F_n)\det(\Gamma_n)\right] &= E\left[T_1[\phi](F_n) R_{1,n}(1)\right] \\ &= E\left[T_1[\phi](F_n)\frac{\gamma - \det(\Gamma_n)}{\gamma} R_{1,n}(1)\right] \\ &\quad + E\left[T_{2,1}[\phi](F_n) R_{2,n}\left(\frac{1}{\gamma} R_{1,n}(1)\right)\right] \\ &= \cdots \\ &= \sum_{k=1}^{d-1} E\left[T_{k,\dots,1}[\phi](F_n)\frac{\gamma - \det(\Gamma_n)}{\gamma} P_{k,n}\right] \\ &\quad + E\left[T_{d,\dots,1}[\phi](F_n) P_{d,n}\right], \end{aligned}$$

with $P_{1,n} = R_{1,n}(1)$ and $P_{k+1,n} = R_{k+1,n}(\frac{1}{\gamma} P_{k,n})$. As a consequence of Lemma 5.4, we get the following inequality:

$$\begin{aligned} |E[\phi(F_n)]| &\leqslant \frac{1}{\gamma}\|\phi\|_\infty \|\det(\Gamma_n) - \gamma\|_{L^2} + \frac{\sum_{k=1}^{d-1} M^k \|\phi\|_\infty}{\gamma} \|\det(\Gamma_n) - \gamma\|_{L^2} \\ &\quad \times \sup_{1 \leqslant k \leqslant d-1} \|P_{k,n}\|_{L^2} + \|P_{d,n}\|_{L^2} \|T_{d,\dots,1}[\phi]\|_\infty. \end{aligned} \tag{5.39}$$

Using Lemma 5.3, we have that $\{P_{k,n}\}_{n \geqslant 1} \in \mathcal{A}$ for all $k = 1, \dots, d$. Hence, we arrive at the following inequality:

$$|E[\phi(F_n)]| \leqslant c\left(\|\phi\|_\infty \|\det(\Gamma_n) - \gamma\|_{L^2} + \|T_{d,\dots,1}[\phi]\|_\infty\right), \quad n \geqslant 1,$$

where $c > 0$ denote a constant independent of $n$, and whose value can freely change from line to line in what follows. Similarly (more easily actually!), one also shows that

$$|E[\phi(N)]| \leqslant c\|T_{d,\dots,1}[\phi]\|_\infty.$$

Thus, if $\phi, \psi \in \mathcal{C}_c^\infty([-M, M]^d)$ are such that $\|\phi\|_\infty \leqslant 1$ and $\|\psi\|_\infty \leqslant 1$, we have, for all $n \geqslant 1$,

$$\begin{aligned} &|(E[\phi(F_n)] - E[\phi(N)]) - (E[\psi(F_n)] - E[\psi(N)])| \\ &\quad \leqslant c\|\det\Gamma_n - \gamma\|_{L^2} + c\|T_{d,\dots,1}[\phi - \psi]\|_\infty. \end{aligned} \tag{5.40}$$

Now, let $\rho : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be in $\mathcal{C}_c^\infty$ and satisfy $\int_{\mathbb{R}^d} \rho(x)dx = 1$. As usual, set $\rho_\alpha(x) = \frac{1}{\alpha^d}\rho(\frac{x}{\alpha})$ whenever $\alpha > 0$.

**Lemma 5.5.** *For all* $\alpha > 0$, *we have*

$$\|T_{d,\dots,1}[\phi - \phi * \rho_\alpha]\|_\infty \leqslant 2\alpha M^{d-1} \int_{\mathbb{R}^d} \|u\|_1 \rho(u)du.$$

**Proof.** We can write

$$\begin{aligned}
&T_{d,\dots,1}[\phi - \phi * \rho_\alpha](x) \\
&\quad = \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_d}\left[\int_{\mathbb{R}^d} (\phi(s_1,\dots,s_d) - \phi(s_1 - y_1,\dots,s_d - y_d))\right. \\
&\qquad \left. \times\ \rho_\alpha(y_1,\dots,y_d)dy\right] ds \\
&\quad = \int_{\mathbb{R}^d} dy\, \rho_\alpha(y_1,\dots,y_d) \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_d} (\phi(s_1,\dots,s_d) \\
&\qquad - \phi(s_1 - y_1,\dots,s_d - y_d))\, ds \\
&\quad = \int_{\mathbb{R}^d} dy\, \rho_\alpha(y_1,\dots,y_d)\left(T_{d,\dots,1}[\phi](x) - T_{d,\dots,1}[\phi](x-y)\right. \\
&\qquad \left. - \int_{-y_1}^0 \cdots \int_{-y_d}^0 \phi(s)ds\right).
\end{aligned}$$

According to Lemma 5.4, we have

$$\begin{aligned}
&|T_{d,\dots,1}[\phi](x) - T_{d,\dots,1}[\phi](x-y)| \leqslant M^{d-1}\|y\|_1 \quad \text{and} \\
&\left|\int_{-y_1}^0 \cdots \int_{-y_d}^0 \phi(s)ds\right| \leqslant M^{d-1}\|y\|_1.
\end{aligned}$$

By combining these two bounds with the above equality, we get

$$|T_{d,\dots,1}[\phi - \phi * \rho_\alpha](x)| \leqslant 2M^{d-1}\int_{\mathbb{R}^d} \rho_\alpha(y)\|y\|_1 dy = 2\alpha M^{d-1}\int_{\mathbb{R}^d}\|u\|_1\rho(u)du,$$

which is the announced result. □

Using the previous lemma and applying (5.40) with $\psi = \phi * \rho_\alpha$, we deduce that, for some constant $c$ independent of $\alpha > 0$ and $n \geqslant 1$,

$$\begin{aligned}
&|(E[\phi(F_n)] - E[\phi(N)]) - (E[\phi * \rho_\alpha(F_n)] - E[\phi * \rho_\alpha(N)])| \\
&\quad \leqslant c\left(\|\det(\Gamma_n) - \gamma\|_{L^2} + \alpha\right).
\end{aligned} \tag{5.41}$$

But

$$\begin{aligned}
|\phi * \rho_\alpha(x) - \phi * \rho_\alpha(x')| &\leqslant \frac{1}{\alpha}\int_{\mathbb{R}^d} |\phi(y)| \left|\rho\left(\frac{x-y}{\alpha}\right) - \rho\left(\frac{x'-y}{\alpha}\right)\right| dy \\
&\leqslant \frac{\|\rho'\|_\infty}{\alpha^2}\int_{\mathbb{R}^d}|\phi(y)|dy\, \|x - x'\|_1 \\
&\leqslant \frac{\|\rho'\|_\infty (2M)^d}{\alpha^2}\|x - x'\|_1,
\end{aligned}$$

that is, $\phi * \rho_\alpha$ is Lipschitz continuous with a constant of the form $c/\alpha^2$. We deduce that

$$|E[\phi * \rho_\alpha(F_n)] - E[\phi * \rho_\alpha(N)]| \leqslant \frac{c}{\alpha^2} d_W(F_n, N), \tag{5.42}$$

where $d_W(F_n, N)$ stands for the Wasserstein distance between $F_n$ and $N$, that is,

$$d_W(F_n, N) = \sup_{\phi \in \mathrm{Lip}(1)} |E[\phi(F_n)] - E[\phi(N)]|.$$

By plugging inequality (5.42) into (5.41), we deduce that, for all $\alpha > 0$ and all $n \geqslant 1$,

$$\sup_\phi |E[\phi(F_n)] - E[\phi(N)]| \leqslant \frac{c}{\alpha^2} d_W(F_n, N) + c\left(\| \det(\Gamma_n) - \gamma \|_{L^2} + \alpha\right),$$

where the supremum runs over the functions $\phi \in \mathcal{C}_c^\infty([-M, M]^d)$ with $\|\phi\|_\infty \leqslant 1$ and where $c$ is a constant independent of $n$ and $\alpha > 0$. By letting $n \to \infty$ (recall that $d_W(F_n, N) \to 0$ by Nourdin et al. [15, Proposition 3.10] and $\det(\Gamma_n) \overset{L^2}{\to} \gamma$ by (5.38)) and then $\alpha \to 0$, we get:

$$\lim_{n\to\infty} \sup_\phi |E[\phi(F_n)] - E[\phi(N)]| = 0,$$

so that the forthcoming Lemma 5.6 applies and allows to conclude. □

**Lemma 5.6.** *Let $F_\infty$ and $F_n$ be random vectors of $\mathbb{R}^d$, $d \geqslant 1$. As $n \to \infty$, assume that $F_n \overset{\text{law}}{\to} F_\infty$ and that, for all $M \geqslant 1$,*

$$A_M(n) := \sup_\phi |E[\phi(F_n)] - E[\phi(F_\infty)]| \to 0,$$

*where the supremum is taken over functions $\phi \in \mathcal{C}_c^\infty([-M, M]^d)$ which are bounded by 1. Then $d_{TV}(F_n, F_\infty) \to 0$ as $n \to \infty$.*

**Proof.** Let $\varepsilon > 0$. Using the tightness of $F_n$, we get that there exists $M_\varepsilon$ large enough such that

$$\sup_n P\left(\max_{1\leqslant i\leqslant d} |F_{i,n}| \geqslant M_\varepsilon\right) \leqslant \varepsilon \quad \text{and} \quad P\left(\max_{1\leqslant i\leqslant d} |F_{i,\infty}| \geqslant M_\varepsilon\right) \leqslant \varepsilon.$$

Let $\phi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R})$ with $\|\phi\|_\infty \leqslant 1$ and $M \geqslant M_\varepsilon + 1$. We have

$$\begin{aligned}|E[\phi(F_n) - \phi(F_\infty)]| &\leqslant \left|E\left[\mathbf{1}_{[-M_\varepsilon, M_\varepsilon]^d}(F_n)\phi(F_n) - \mathbf{1}_{[-M_\varepsilon, M_\varepsilon]^d}(F_\infty)\phi(F_\infty)\right]\right| + 2\varepsilon \\ &\leqslant \sup_{\psi \in E_M} |E[\psi(F_n) - \psi(F_\infty)]| + 2\varepsilon \leqslant A_M(n) + 2\varepsilon.\end{aligned}$$

Here, $E_M$ is the set of smooth functions $\psi$ with compact support in $[-M, M]^d$ which are bounded by 1. Hence, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} d_{TV}(F_n, F_\infty) = \frac{1}{2} \limsup_{n\to\infty} \sup_{\substack{\phi \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}): \\ \|\phi\|_\infty \leqslant 1}} |E[\phi(F_n) - \phi(F_\infty)]| \leqslant \varepsilon$$

and the desired conclusion follows. □

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