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Functional limit theorems for renewal shot noise processes with increasing response functions

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Abstract

We consider renewal shot noise processes with response functions which are eventually nondecreasing and regularly varying at infinity. We prove weak convergence of renewal shot noise processes, properly normalized and centered, in the space $D[0, \infty)$ under the J_1 or M_1 topology. The limiting processes are either spectrally nonpositive stable Lévy processes, including the Brownian motion, or inverse stable subordinators (when the response function is slowly varying), or fractionally integrated stable processes or fractionally integrated inverse stable subordinators (when the index of regular variation is positive). The proof exploits fine properties of renewal processes, distributional properties of stable Lévy processes and the continuous mapping theorem.

Keywords: Continuous mapping theorem, fractionally integrated (inverse) stable process; Functional limit theorem; M_1 topology; Renewal shot noise process; Spectrally negative stable process

1. Introduction

Let $(\xi_k)_{k \in \mathbb{N}}$ be independent copies of a positive random variable ξ. Denote

$$S_0 := 0, \qquad S_n := \xi_1 + \cdots + \xi_n, \quad n \in \mathbb{N}$$

and

$$N(t) := \#\{k \in \mathbb{N}_0 : S_k \le t\} = \inf\{k \in \mathbb{N} : S_k > t\}, \quad t \in \mathbb{R}.$$

It is clear that $N(t) = 0$ for $t < 0$.

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Let $D := D[0, \infty)$ denote the Skorohod space of right-continuous real-valued functions on $[0, \infty)$ with finite limits from the left. Elements of D are sometimes called *càdlàg functions*. For a càdlàg function h, we define

$$X(t) := \sum_{k\geq 0} h(t - S_k) 1_{\{S_k \leq t\}} = \int_{[0,\,t]} h(t - y)\mathrm{d}N(y), \quad t \geq 0, \tag{1}$$

and call $(X(t))_{t\geq 0}$ a renewal shot noise process. The function h is called an *impulse response function* or just *response function*. Note that the so defined $X(t)$ is a.s. finite, for each $t \geq 0$.

Processes (1) and more general shot noise processes have been used to model a lot of diverse phenomena; see, for instance, [19,36] and references therein. More recent contributions [23,30] have discussed applications in risk theory and finance, respectively. A non-exhaustive list of works concerning mathematical aspects of shot noise processes is given in [1].

Since h is càdlàg, for every $t \geq 0$, $(X(ut))_{u\geq 0}$ is a random element taking values in D. Our aim is to prove the weak convergence of, properly normalized and centered, $X(ut)$ in D under the J_1 or M_1 topology. In what follows the symbols $\overset{J_1}{\Rightarrow}$, $\overset{M_1}{\Rightarrow}$ and $\Rightarrow$ mean that the convergence takes place under the J_1 topology, under the M_1 topology or under either of these, respectively. The J_1 topology is the commonly used topology in D (see [8,37]). We recall that $\lim_{n\to\infty} x_n = x$ in $D[0, T]$, $T > 0$, under the M_1 topology if

$$\lim_{n\to\infty} \inf \max\left(\sup_{t\in[0,\,1]} |r_n(t) - r(t)|, \sup_{t\in[0,\,1]} |u_n(t) - u(t)|\right) = 0,$$

where the infimum is taken over all parametric representations (u, r) of x and (u_n, r_n) of $x_n, n \in \mathbb{N}$. We refer the reader to pp. 80–82 in [37] for further details and definitions. The M_1 topology which like the J_1 topology was introduced in Skorohod's seminal paper [33] is not that common. Its appearances in the probability literature are comparatively rare. An incomplete list of works which have effectively used the M_1 topology in diverse applied problems includes [4,5,25,28,29,35]. Remark 12.3.2 in [37] gives more references.

The J_1 convergence in $D[0, 1]$, as $n \to \infty$, of $\sum_{k\geq 0} h(t - n^{-1}S_k)1_{\{S_k\leq nt\}}$, properly normalized and centered, to a Gaussian process can be derived from more general results obtained in [20]. When $(N(t))$ is the Poisson process, a functional convergence to a Gaussian process and an infinite variance stable process was proved in [23] (see also [18] and references therein) and [24], respectively, for shot noise processes which are more general than ours. We are not aware of any papers which would prove functional limit theorems for the shot noise processes $X(ut)$ in the case of a *general* renewal process $(N(t))$. In particular, in this wider framework a new technique is needed intended to replace the characteristic function approach available in the Poisson case. To some extent, this has served as the first motivation for the present research. Second (and more importantly), based on the technique developed in [15,16] we expect that a particular case of Theorem 1.1 with h being the distribution function of a positive random variable will form a basis for obtaining functional limit theorems for the number of occupied boxes in the Bernoulli sieve (see [16] for the definition and further details).

While the weak convergence of the renewal shot noise processes with eventually nonincreasing response functions will be investigated in a forthcoming paper [21], here we only consider the renewal shot noise processes with eventually nondecreasing response functions. Theorem 1.1 which is our main result relies heavily upon known functional limit theorems for $N(t)$. To shorten the presentation the latter are not given as a separate statement. Rather they are included in

Theorem 1.1 as a particular case with $h(y) = 1_{[0,\infty)}(y)$. Note that all bounded eventually nondecreasing h with positive $\lim_{t\to\infty} h(t)$ satisfy (2) below with $\beta = 0$ and $\ell^*(x) \equiv \lim_{t\to\infty} h(t)$. Therefore these are covered by the theorem.

Theorem 1.1. *Let $h : \mathbb{R}^+ \to \mathbb{R}$ be a locally bounded, right-continuous and eventually nondecreasing function, and*

$$h(x) \sim x^\beta \ell^*(x), \quad x \to \infty, \tag{2}$$

for some $\beta \in [0,\infty)$ and some ℓ^* slowly varying at ∞.

(A1) *If $\sigma^2 := \operatorname{Var} \xi < \infty$ then*

$$\frac{X(ut) - \mu^{-1}\int_{[0,ut]} h(y)\mathrm{d}y}{h(t)\sqrt{\sigma^2\mu^{-3}t}} \overset{J_1}{\Rightarrow} \int_{[0,u]} (u-y)^\beta \mathrm{d}W_2(y), \quad t \to \infty,$$

where $\mu := \mathbb{E}\xi < \infty$ and $(W_2(u))_{u\ge 0}$ is a Brownian motion.

(A2) *If $\sigma^2 = \infty$ and*

$$\int_{[0,x]} y^2 \mathbb{P}\{\xi \in \mathrm{d}y\} \sim \ell(x), \quad x \to \infty,$$

for some ℓ slowly varying at ∞, then

$$\frac{X(ut) - \mu^{-1}\int_{[0,ut]} h(y)\mathrm{d}y}{h(t)\mu^{-3/2}c(t)} \overset{J_1}{\Rightarrow} \int_{[0,u]} (u-y)^\beta \mathrm{d}W_2(y), \quad t \to \infty,$$

where $c(t)$ is any positive continuous function such that $\lim_{t\to\infty} \frac{t\ell(c(t))}{c^2(t)} = 1$ and $(W_2(u))_{u\ge 0}$ is a Brownian motion.

(A3) *If*

$$\mathbb{P}\{\xi > x\} \sim x^{-\alpha}\ell(x), \quad x \to \infty, \tag{3}$$

for some $\alpha \in (1,2)$ and some ℓ slowly varying at ∞, then

$$\frac{X(ut) - \mu^{-1}\int_{[0,ut]} h(y)\mathrm{d}y}{h(t)\mu^{-1-1/\alpha}c(t)} \overset{M_1}{\Rightarrow} \int_{[0,u]} (u-y)^\beta \mathrm{d}W_\alpha(y), \quad t \to \infty,$$

where $c(t)$ is any positive continuous function such that $\lim_{t\to\infty} \frac{t\ell(c(t))}{c^\alpha(t)} = 1$ and $(W_\alpha(u))_{u\ge 0}$ is an α-stable Lévy process such that $W_\alpha(1)$ has the characteristic function

$$z \mapsto \exp\left\{-|z|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i\sin(\pi\alpha/2)\operatorname{sgn}(z))\right\}, \quad z \in \mathbb{R}. \tag{4}$$

(A4) *If condition (3) holds for some $\alpha \in (0,1)$ then*

$$\frac{\mathbb{P}\{\xi > t\}}{h(t)} X(ut) \overset{J_1}{\Rightarrow} \int_{[0,u]} (u-y)^\beta \mathrm{d}V_\alpha(y), \quad t \to \infty,$$

where $(V_\alpha(u))_{u\ge 0}$ is an inverse α-stable subordinator defined by

$$V_\alpha(u) := \inf\{s \ge 0 : D_\alpha(s) > u\},$$

where $(D_\alpha(t))_{t\ge 0}$ is an α-stable subordinator with $-\log \mathbb{E}e^{-sD_\alpha(1)} = \Gamma(1-\alpha)s^\alpha$, $s \ge 0$.

Remark 1.2. Theorem 1.1 does not cover one case for which we have the following *conjecture*:

(A5) If condition (3) holds with $\alpha = 1$ then

$$\frac{m(t)}{h(t)c(t/m(t))}\left(X(ut) - \frac{1}{m(c(t/m(t)))}\int_{[0,\,ut]} h(y)\mathrm{d}y\right) \overset{M_1}{\Rightarrow} \int_{[0,\,u]} (u-y)^{\beta}\mathrm{d}W_1(y),$$

where $c(t)$ is any positive continuous function such that $\lim_{t\to\infty} \frac{t\ell(c(t))}{c(t)} = 1, m(t) := \int_{[0,\,t]} \mathbb{P}\{\xi > y\}\mathrm{d}y, t > 0$, and $(W_1(u))_{u\geq 0}$ is a 1-stable Lévy process such that $W_1(1)$ has the characteristic function

$$z \mapsto \exp\{-|z|(\pi/2 - i\log|z|\,\mathrm{sgn}(z))\}, \quad z \in \mathbb{R}.$$

The rest of the paper is organized as follows. In Section 2 we recall a simplified definition of the stochastic integral in the case when the integrand is a deterministic function. In Section 3 we discuss properties of the limiting processes appearing in Theorem 1.1. The proof of Theorem 1.1 is given in Section 4. In Section 5 we discuss an extension of Theorem 1.1 to response functions h concentrated on the whole line. Finally Appendix collects all the needed auxiliary information.

2. Defining a stochastic integral via integration by parts

There is a general definition of a stochastic integral with integrand being a locally bounded predictable process and integrator being a semimartingale, in particular, a Lévy process (see, for instance, Theorem 23.4 in [22]). However, when the integrand is a deterministic function of bounded variation there is an equivalent definition which is much simpler. It turns out that the latter stochastic integral can be defined in terms of usual Lebesgue–Stieltjes integral and integration by parts.

Let $f, g \in D[a,b], b > a \geq 0$ and f has bounded variation. Using Lemma A.7 we define the integral $\int_{(a,b]} f(b-y)\mathrm{d}g(y)$ by formal integration by parts

$$\int_{(a,b]} f(b-y)\mathrm{d}g(y) = f(0-)g(b) - f((b-a)-)g(a) - \int_{(a,b]} g(y)\mathrm{d}f(b-y).$$

Now if $(W(y))_{y\geq 0} = (W(y,\omega))_{y\geq 0}$ is a Lévy process (it has paths in D) the definition above with $g(y) := W(y,\omega)$, for each ω, provides a pathwise construction of the stochastic integral for all ω:

$$\int_{(a,b]} f(b-y)\mathrm{d}W(y) = f(0-)W(b) - f((b-a)-)W(a) - \int_{(a,b]} W(y)\mathrm{d}f(b-y). \tag{5}$$

From this definition and continuity theorem for characteristic functions we conclude that

$$\log\mathbb{E}\exp\left(\mathrm{i}t\int_{(a,\,b]} f(b-y)\mathrm{d}W(y)\right) = \int_{(a,\,b]} \log\mathbb{E}\exp\left(\mathrm{i}tf(b-y)W(1)\right)\mathrm{d}y,$$
$$t \in \mathbb{R} \tag{6}$$

(see Lemma 5.1 in [15] for a similar argument). Let $f^* \in L_2[a,b]$ and $(W_2(y))_{y\geq 0}$ be a Brownian motion. Then

$$-\log\mathbb{E}\exp\left(\mathrm{i}t\int_{[a,\,b]} f^*(y)\mathrm{d}W_2(y)\right) = 2^{-1}t^2\int_{[a,\,b]} (f^*)^2(y)\mathrm{d}y, \quad t \in \mathbb{R}.$$

Hence the random variable $\int_{[a,b]} f^*(y)\mathrm{d}W_2(y)$ has the same law as $W_2(1)\sqrt{\int_{[a,b]}(f^*)^2(y)\mathrm{d}y}$ which implies the moment formulas to be used in the sequel:

$$
\begin{aligned}
&\mathbb{E}\left(\int_{[a,b]} f^*(y)\mathrm{d}W_2(y)\right)^2 = \int_{[a,\,b]} (f^*)^2(y)\mathrm{d}y,\\
&\mathbb{E}\left(\int_{[a,b]} f^*(y)\mathrm{d}W_2(y)\right)^4 = 3\left(\int_{[a,\,b]} (f^*)^2(y)\mathrm{d}y\right)^2.
\end{aligned}
\tag{7}
$$

Of course, all moments of odd orders equal zero.

3. Properties of the limit processes in Theorem 1.1

Recall that $(W_2(u))_{u\geq 0}$ denotes a Brownian motion and, for $\alpha \in (1,2)$, $(W_\alpha(u))_{u\geq 0}$ denotes an α-stable Lévy process such that $W_\alpha(1)$ has the characteristic function given in (4).

Let $\beta > 0$. The limit processes $\big(Y_{\alpha,\,\beta}(u)\big)_{u\geq 0}$ defined by

$$
Y_{\alpha,\,\beta}(u) := \int_{[0,\,u]} (u-y)^\beta \mathrm{d}W_\alpha(y) = \beta \int_{[0,\,u]} (u-y)^{\beta-1} W_\alpha(y)\mathrm{d}y \tag{8}
$$

are called the *α-stable Riemann–Liouville processes* or *fractionally integrated α-stable processes* (see, for instance, [2]).

We now establish some properties of the processes $\big(Y_{\alpha,\,\beta}(u)\big)$.

(P1) Their paths are continuous a.s.

This follows from the second equality in (8) and Lemma A.8(a).

(P2) They are self-similar with Hurst parameter $\beta+\alpha^{-1}$, i.e., for every $c>0$

$$
\big(Y_{\alpha,\,\beta}(cu)\big)_{u\geq 0} \overset{\text{f.d.}}{=} \left(c^{\beta+\alpha^{-1}} Y_{\alpha,\,\beta}(u)\right)_{u\geq 0},
$$

where $\overset{\text{f.d.}}{=}$ denotes the equality of finite-dimensional distributions (see [13] for an accessible introduction to the theory of self-similar processes).

We only prove this property for two-dimensional distributions. For any $0<u_1<u_2$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$
\begin{aligned}
\beta^{-1}\big(\alpha_1 Y_{\alpha,\,\beta}(cu_1) + \alpha_2 Y_{\alpha,\,\beta}(cu_2)\big) &= \alpha_1 \int_{[0,\,cu_1]} (cu_1-y)^{\beta-1} W_\alpha(y)\mathrm{d}y\\
&\quad + \alpha_2 \int_{[0,\,cu_2]} (cu_2-y)^{\beta-1} W_\alpha(y)\mathrm{d}y\\
&= c^\beta \int_{[0,\,u_2]} (\alpha_1(u_1-y)^{\beta-1} 1_{[0,u_1]}(y)\\
&\quad + \alpha_2(u_2-y)^{\beta-1}) W_\alpha(cy)\mathrm{d}y\\
&\overset{d}{=} c^{\beta+\alpha^{-1}} \int_{[0,\,u_2]} (\alpha_1(u_1-y)^{\beta-1} 1_{[0,u_1]}(y)\\
&\quad + \alpha_2(u_2-y)^{\beta-1}) W_\alpha(y)\mathrm{d}y\\
&= \beta^{-1} c^{\beta+\alpha^{-1}} (\alpha_1 Y_{\alpha,\,\beta}(u_1) + \alpha_2 Y_{\alpha,\,\beta}(u_2)),
\end{aligned}
$$

where the second equality follows by the change of variable, and the third is a consequence of the self-similarity with parameter α^{-1} of $(W_\alpha(u))$.

(P3) For fixed $u>0$,

$$
Y_{\alpha,\,\beta}(u) \overset{d}{=} \int_{[0,\,u]} y^\beta \mathrm{d}W_\alpha(y) \overset{d}{=} \left(\frac{u^{\alpha\beta+1}}{\alpha\beta+1}\right)^{1/\alpha} W_\alpha(1).
$$

While the first distributional equality follows from the fact that, for fixed u,

$$(W_\alpha(u) - W_\alpha(u-y))_{y\in[0,\,u]} \overset{d}{=} (W_\alpha(y))_{y\in[0,\,u]}\,,$$

the second is implied by the equality

$$\log \mathbb{E}\exp\left(\mathrm{i}tY_{\alpha,\,\beta}(u)\right) = \int_{[0,\,u]} \log \mathbb{E}\exp\left(\mathrm{i}ty^\beta W_\alpha(1)\right)\mathrm{d}y, \quad t\in\mathbb{R}.$$

(see (6)).

(P4) The increments of $\left(Y_{\alpha,\,\beta}(u)\right)$ are neither independent, nor stationary.

Let $0 < v < u$ and $\alpha = 2$. Since $(W_2(u))$ has independent increments $X_{2,\,\beta}(v)$ and $\int_{[v,\,u]}(u-y)^\beta \mathrm{d}W_2(y)$ are independent. Set

$$r_\beta(u,v) := \left(\int_{[0,\,v]} \left((u-y)^\beta - (v-y)^\beta\right)^2 \mathrm{d}y\right)^{1/2}.$$

It seems that the integral cannot be evaluated in terms of elementary functions. Fortunately we only have to check that $r_\beta(u,v) \neq 0$, for *some* $v < u$. Using the inequality $(x-y)^2 \geq 2^{-1}x^2 - y^2$, $x, y \in \mathbb{R}$ we conclude that

$$\begin{aligned} r_\beta^2(u,v) &\geq \int_{[0,\,v]} \left(2^{-1}(u-y)^{2\beta} - (v-y)^{2\beta}\right)\mathrm{d}y \\ &= \frac{1}{2\beta+1}\left(2^{-1}(u^{2\beta+1} - (u-v)^{2\beta+1}) - v^{2\beta+1}\right). \end{aligned}$$

There is a unique solution x^* to the equation

$$x^{2\beta+1} - (x-1)^{2\beta+1} = 2.$$

Taking any $x > x^* \vee 1$ and any $v > 0$ we have $r_\beta^2(xv, v) > 0$. In view of the distributional equality

$$\begin{aligned} &\left(\int_{[0,\,v]}(v-y)^\beta \mathrm{d}W_2(y), \int_{[0,\,xv]}\left((xv-y)^\beta - (v-y)^\beta\right)\mathrm{d}W_2(y)\right) \\ &\quad\overset{d}{=} \left(\left(\frac{v^{2\beta+1}}{2\beta+1}\right)^{1/2} W_2(1), r_\beta(xv,v)W_2(1)\right), \end{aligned}$$

$X_{2,\,\beta}(v)$ and $\int_{[0,\,xv]}\left((xv-y)^\beta - (v-y)^\beta\right)\mathrm{d}W_2(y)$ are strongly dependent. Therefore, $X_{2,\,\beta}(v)$ and $X_{2,\,\beta}(xv) - X_{2,\,\beta}(v)$ are not independent.

Let $\alpha \in (1,2)$. If the increments were independent the continuous process $\left(Y_{\alpha,\,\beta}(u)\right)$ would be Gaussian (see Theorem 5 on p. 189 in [14]) which is not the case.

If the increments were stationary the characteristic function of $Y_{\alpha,\,\beta}(u) - Y_{\alpha,\,\beta}(v)$ for $0 < v < u$ would be a function of $u - v$. This is however not the case as is seen from formula

$$\begin{aligned} &\log \mathbb{E}\exp\left(\mathrm{i}t\left(Y_{\alpha,\,\beta}(u) - Y_{\alpha,\,\beta}(v)\right)\right) \\ &\quad= \int_{[0,\,u]} \log \mathbb{E}\exp\left(\mathrm{i}t\left((u-y)^\beta - (v-y)^\beta 1_{[0,\,v]}(y)\right)W_\alpha(1)\right)\mathrm{d}y, \quad t\in\mathbb{R}. \end{aligned}$$

Recall that, for $\alpha \in (0, 1)$, $(V_\alpha(u))_{u\ge 0}$ denotes an inverse α-stable subordinator. Let $\beta > 0$. The limit processes $\big(Z_{\alpha,\beta}(u)\big)_{u\ge 0}$ defined by

$$Z_{\alpha,\beta}(u) := \int_{[0,u]} (u-y)^\beta \mathrm{d}V_\alpha(y) = \beta \int_{[0,u]} (u-y)^{\beta-1} V_\alpha(y)\mathrm{d}y,$$

where the integral is a pathwise Lebesgue–Stieltjes integral, will be called the *fractionally integrated inverse α-stable subordinators*.

We now establish some properties of these processes.

(Q1) Their paths are continuous a.s.

Obvious.

(Q2) They are self-similar with Hurst parameter $\beta + \alpha$.

This is implied by the self-similarity with index α of $(V_\alpha(u))$.

(Q3) The law of $Z_{\alpha,\beta}(u)$ is uniquely determined by its moments

$$\mathbb{E}\big(Z_{\alpha,\beta}(u)\big)^k = u^{k(\alpha+\beta)} \frac{k!}{\Gamma^k(1-\alpha)} \prod_{j=1}^{k} \frac{\Gamma(\beta+1+(j-1)(\alpha+\beta))}{\Gamma(j(\alpha+\beta)+1)}, \quad k \in \mathbb{N}, \tag{9}$$

where $\Gamma(\cdot)$ is the gamma function. In particular,

$$Z_{\alpha,\beta}(1) \overset{d}{=} \int_0^R e^{-cZ_\alpha(t)}\mathrm{d}t, \tag{10}$$

where R is a random variable with the standard exponential law which is independent of $(Z_\alpha(u))_{u\ge 0}$ a drift-free subordinator with no killing and the Lévy measure

$$\nu_\alpha(\mathrm{d}t) = \frac{e^{-t/\alpha}}{(1-e^{-t/\alpha})^{\alpha+1}} 1_{(0,\infty)}(t)\mathrm{d}t,$$

and $c := (\alpha+\beta)/\alpha$.

From the results obtained in [27] it follows that $(V_\alpha(u))$ is a local time at level 0 for the $2(1-\alpha)$-dimensional Bessel process. Therefore, (9) is nothing else but a specialization of formula (4.3) in [17].

One can check that

$$\Phi_\alpha(x) := -\log \mathbb{E}e^{-xZ_\alpha(1)} = \frac{\Gamma(1-\alpha)\Gamma(\alpha x+1)}{\Gamma(\alpha(x-1)+1)} - 1, \quad x \ge 0.$$

Formula (9) with $u = 1$ can be rewritten in an equivalent form

$$\begin{aligned}\mathbb{E}Z^k_{\alpha,\beta}(1) &= \frac{k!}{(\Phi_\alpha(c)+1)\cdots(\Phi_\alpha(ck)+1)} \\ &= \frac{k!}{\prod_{j=1}^{k}(1-\alpha+j(\alpha+\beta))\mathrm{B}(1-\alpha, 1+k(\alpha+\beta))}, \quad k \in \mathbb{N},\end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function, which, by Theorem 2(i) in [7], entails distributional equality (10). From the inequality

$$\int_0^R e^{-cZ_\alpha(t)}\mathrm{d}t \le R,$$

and the fact that $\mathbb{E}e^{aR} < \infty$, for $a \in (0, 1)$, we conclude that the law of $Z_{\alpha,\,\beta}(1)$ has some finite exponential moments and thereby is uniquely determined by its moments.

(Q4) Their increments are not stationary.

When $\alpha + \beta \neq 1$ this follows from the fact that $\mathbb{E}Z_{\alpha,\beta}(u)$ is a function of $u^{\alpha+\beta}$ rather than u. The case $\alpha + \beta = 1$ follows by continuity.

In [26] it was shown that $(V_\alpha(u))$ does not have independent increments. Although we believe it is also the case for $\big(Z_{\alpha,\,\beta}(u)\big)$, we refrain from investigating this.

4. Proof of Theorem 1.1

Cases (A1)–(A3). The functional limit theorems

$$W^{(t)}(u) := \frac{N(ut) - ut}{b(t)} \Rightarrow W_\alpha(u), \quad t \to \infty,$$

with case dependent $b(t)$ and $W_\alpha(u)$, can be found, for instance, in Theorem 1b(i) [10].

For $t > 0$ set

$$X_t(u) := \frac{X(ut) - \int_{[0,\,ut]} h(y)\mathrm{d}y}{b(t)h(t)}, \quad u \geq 0.$$

Also recall the notation

$$Y_{\alpha,\,\beta}(u) := \beta \int_{[0,\,u]} W_\alpha(y)(u-y)^{\beta-1}\mathrm{d}y = \int_{[0,\,u]} (u-y)^\beta \mathrm{d}W_\alpha(y), \quad u \geq 0,$$

if $\beta > 0$, and set $Y_{\alpha,\,0}(u) := W_\alpha(u)$, $u \geq 0$, if $\beta = 0$.

We proceed by showing that, in the subsequent analysis, we can replace h by a nondecreasing and continuous on $\mathbb{R}^+$ function h^* with $h^*(0) = 0$ such that $h^*(t) \sim h(t)$, $t \to \infty$. To this end, we will use the two step reduction.

Suppose we have already proved that

$$X_t^*(u) := \frac{\int_{[0,\,ut]} h^*(ut-y)\mathrm{d}N(y) - \int_{[0,\,ut]} h^*(y)\mathrm{d}y}{b(t)h^*(t)} \Rightarrow Y_{\alpha,\,\beta}(u), \quad t \to \infty.$$

Now to ensure the convergence $X_t(u) \Rightarrow Y_{\alpha,\,\beta}(u)$, $t \to \infty$, it suffices to check that, for any $T > 0$,

$$\frac{\sup_{u\in[0,\,T]} \left|\int_{[0,\,ut]} \left(h(ut-y) - h^*(ut-y)\right)\mathrm{d}N(y)\right|}{b(t)h(t)} \xrightarrow{P} 0, \quad t \to \infty, \tag{11}$$

and

$$\frac{\sup_{u\in[0,\,T]} \left|\int_{[0,\,ut]} \left(h(y) - h^*(y)\right)\mathrm{d}y\right|}{b(t)h(t)} \to 0, \quad t \to \infty. \tag{12}$$

Step 1. We first prove an intuitively clear fact that the behavior of h near zero does not influence the asymptotics of X_t. In particular, if, given $a > 0$, we replace h by any càdlàg function $\widehat{h}$ such that $\widehat{h}(t) = h(t)$ for $t \geq a$ the asymptotics of X_t will not change. Indeed,

$$\begin{aligned}
&\left| \int_{[0,\, u]} \big(h(t(u-y)) - \widehat{h}(t(u-y)) \big) \, \mathrm{d}_y N(ty) \right| \\
&\quad = \left| \int_{(u-a/t,\, u]} \big(h(t(u-y)) - \widehat{h}(t(u-y)) \big) \, \mathrm{d}_y N(ty) \right| \\
&\quad \leq \sup_{y \in [0,\, a]} |h(y) - \widehat{h}(y)| \, (N(ut) - N(ut-a)) \, .
\end{aligned}$$

Since h and $\widehat{h}$ are càdlàg, they are locally bounded. After noting that the local boundedness entails the finiteness of the last supremum, and that in all cases b is regularly varying with positive index, an appeal to Lemma A.1 allows us to conclude that, for any $T > 0$,

$$\begin{aligned}
&\frac{\sup_{u \in [0,\, T]} \left| \int_{[0,\, u]} \big(h(t(u-y)) - \widehat{h}(t(u-y)) \big) \, \mathrm{d}_y N(ty) \right|}{b(t)} \\
&\quad \leq \sup_{y \in [0,\, a]} |h(y) - \widehat{h}(y)| \frac{\sup_{u \in [0,\, T]} (N(ut) - N(ut-a))}{b(t)} \overset{P}{\to} 0, \quad t \to \infty. \qquad (13)
\end{aligned}$$

Arguing in a similar but simpler way we conclude that, for any $T > 0$,

$$\frac{\sup_{u \in [0,\, T]} \left| \int_{[0,\, ut]} \big(h(y) - \widehat{h}(y) \big) \, \mathrm{d}y \right|}{b(t)} \to 0, \quad t \to \infty. \qquad (14)$$

This justifies the claim. In particular, choosing a large enough we can make $\widehat{h}$ nondecreasing on $\mathbb{R}^+$. Besides that, we will take $\widehat{h}$ such that $\widehat{h}(t) = 0$ for $t \in [0, b]$ for some $b > 0$ to be specified later.

Step 2. Set $h^*(t) := \mathbb{E}\widehat{h}((t-\theta)^+)$, where θ is a random variable with the standard exponential distribution. It is clear that $\widehat{h}(t) \geq h^*(t), t \geq 0$. By Lemma A.4, h^* is continuous on $\mathbb{R}^+$ with $h^*(0) = 0$ and $h^*(t) \sim \widehat{h}(t) \sim h(t), t \to \infty$. Furthermore,

$$\int_{[0,\, t]} \big(\widehat{h}(y) - h^*(y) \big) \, \mathrm{d}y \sim h(t), \quad t \to \infty,$$

which immediately implies

$$\begin{aligned}
\frac{\sup_{u \in [0,\, T]} \left| \int_{[0,\, ut]} \big(\widehat{h}(y) - h^*(y) \big) \, \mathrm{d}y \right|}{b(t) h(t)} &= \frac{\int_{[0,\, Tt]} \big(\widehat{h}(y) - h^*(y) \big) \, \mathrm{d}y}{b(t) h(t)} \\
&\sim \frac{T^\beta}{b(t)} \to 0, \quad t \to \infty.
\end{aligned}$$

In combination with (14) the latter proves (12).

Now we intend to apply Lemma A.3 with $K_1 = \widehat{h}$ and $K_2 = h^*$. Since

$$\lim_{t \to \infty} \frac{\widehat{h}(t) + h^*(t)}{\int_{[0,\, t]} \big(\widehat{h}(y) - h^*(y) \big) \, \mathrm{d}y} = 2$$

and

$$\int_{[0,\,Tt]} \left(\widehat{h}(y) - h^*(y)\right) \mathrm{d}y \sim\ T^{\beta} h(t), \quad t \to \infty,$$

and in all cases $b(t)$ is regularly varying with positive index, we have

$$\frac{\sup_{u\in[0,\,T]} \left|\int_{[0,\,ut]} \left(\widehat{h}(ut-y) - h^*(ut-y)\right) \mathrm{d}N(y)\right|}{b(t)h(t)} = \frac{\sup_{u\in[0,\,T]} \int_{[0,\,ut]} \left(\widehat{h}(ut-y) - h^*(ut-y)\right) \mathrm{d}N(y)}{b(t)h(t)} \xrightarrow{P} 0, \quad t \to \infty.$$

This together with (13) leads to (11).

By Potter's bound (Theorem 1.5.6(iii) in [11]) for any chosen $A > 1$, $\delta \in (0, \alpha\beta)$ if $\beta > 0$ and $\delta \in (0, 1/2)$ if $\beta = 0$ (we take $\alpha = 2$ in cases (A1) and (A2)) there exists t_0 such that

$$\frac{h^{*\alpha}(ty)}{h^{*\alpha}(t)} \le A y^{\alpha\beta-\delta},$$

whenever $y \le 1$ and $ty \ge t_0$. Choosing $b = t_0$ in the definition of $\widehat{h}$, i.e., $\widehat{h}(t) = h^*(t) = 0$ for $t \in [0, t_0]$ we can and do assume that

$$\frac{h^{*\alpha}(Tty)}{h^{*\alpha}(Tt)} \le A y^{\alpha\beta-\delta} \quad \text{and} \quad \frac{h^*(Tt)}{h^*(t)} \le A\left(T^{\beta+\delta} \vee T^{\beta-\delta}\right) =: C(T), \tag{15}$$

whenever $T > 0$, $y \le 1$, $Tt \ge t_0$ and $t \ge t_0$. The second inequality in (15) is just Potter's bound.

Setting

$$h_t(x) := h^*(tx)/h^*(t),$$

using the fact that $N(0) = 1$ a.s. and integrating by parts, we have, for $t > 0$ and $u > 0$

$$\begin{aligned} X_t^*(u) &= \int_{[0,\,u]} h_t(u-y)\mathrm{d}_y W^{(t)}(y) \\ &= \frac{h^*(ut)}{b(t)h^*(t)} + \int_{(0,\,u]} h_t(u-y)\mathrm{d}_y W^{(t)}(y) \\ &= \int_{(0,\,u]} W^{(t)}(y)\mathrm{d}_y\left(-h_t(u-y)\right). \end{aligned}$$

It suffices to show that,[1]

$$\int_{(0,\,u]} \left(W^{(t)}(y) - W_\alpha(y)\right) \mathrm{d}_y\left(-h_t(u-y)\right) \xrightarrow{P} 0, \quad t \to \infty, \tag{16}$$

in D under the J_1 topology in cases (A1) and (A2) and under the M_1 topology in case (A3), and

$$\begin{aligned} \int_{(0,\,u]} W_\alpha(y)\mathrm{d}_y\left(-h_t(u-y)\right) &\Rightarrow \int_{(0,\,u]} W_\alpha(y)\mathrm{d}_y\left(-(u-y)^\beta\right) \\ &= Y_{\alpha,\,\beta}(u), \quad t \to \infty. \end{aligned} \tag{17}$$

[1] Although $W^{(t)}$ and W_α are not necessarily defined on a common probability space we can assume that by virtue of Skorohod's representation theorem.

The convergence of finite dimensional distributions in (17) holds by Lemma A.5 and the continuous mapping theorem (see the proof for case (A4) for more details). Therefore, as far as relation (17) is concerned we only have to prove the tightness.

Cases (A1) *and* (A2). If $\lim_{t\to\infty} x_t = x$ in D under the J_1 topology and x is continuous then, for any $T > 0$, $\lim_{t\to\infty} \sup_{u\in[0,\,T]} |x_t(u) - x(u)| = 0$. Hence, using the monotonicity of h_t we obtain

$$\sup_{u\in[0,\,T]} \left| \int_{(0,\,u]} (x_t(y) - x(y))\, \mathrm{d}_y\, (-h_t(u-y)) \right| \\ \leq \sup_{u\in[0,\,T]} \sup |x_t(u) - x(u)| h_t(T) \to 0, \quad t \to \infty.$$

Since $(W_2(u))$ is a Brownian motion which has a.s. continuous paths (16) follows by the continuous mapping theorem.

By Lemma A.8(b), for each $t > 0$, the process on the left-hand side of (17) has a.s. continuous paths. Therefore we will prove that the convergence in (17) takes place under the uniform topology in $C[0, \infty)$ which is more than was claimed in (17). To this end, it suffices to show that the mentioned convergence holds in $C[0,\ T]$, for any $T > 0$. We can write, for any $T > 0$,

$$\begin{aligned} \int_{(0,\,Tu]} W_2(y)\mathrm{d}_y\,(-h_t(Tu-y)) &= \frac{h^*(Tt)}{h^*(t)} \int_{(0,\,u]} W_2(Ty)\mathrm{d}_y\,(-h_{Tt}(u-y)) \\ &\overset{(5)}{=} \frac{h^*(Tt)}{h^*(t)} \int_{(0,\,u]} h_{Tt}(u-y)\mathrm{d}W_2(Ty) \\ &=: \widehat{X}_t(u), \quad u \in [0, 1]. \end{aligned} \tag{18}$$

Hence it remains to check the tightness of $\big(\widehat{X}_t(u)\big)$ in $C[0, 1]$. With $u, v \in [0, 1]$, $u > v$

$$\begin{aligned} &\left(\frac{h^*(t)}{h^*(Tt)}\right)^4 \mathbb{E}\left(\widehat{X}_t(u) - \widehat{X}_t(v)\right)^4 \\ &= \mathbb{E}\left(\int_{[0,\,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))\,\mathrm{d}_y W_2(Ty) + \int_{[v,\,u]} h_{Tt}(u-y)\mathrm{d}_y W_2(Ty)\right)^4 \\ &= \mathbb{E}\left(\int_{[0,\,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))\,\mathrm{d}_y W_2(Ty)\right)^4 \\ &\quad + 6\mathbb{E}\left(\int_{[0,\,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))\,\mathrm{d}_y W_2(Ty)\right)^2 \\ &\quad \times \mathbb{E}\left(\int_{[v,\,u]} h_{Tt}(u-y)\mathrm{d}_y W_2(Ty)\right)^2 + \mathbb{E}\left(\int_{[v,\,u]} h_{Tt}(u-y)\mathrm{d}_y W_2(Ty)\right)^4 \\ &= 3T^2\Bigg(\left(\int_{[0,\,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))^2\,\mathrm{d}y\right)^2 \\ &\quad + 2\int_{[0,\,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))^2\,\mathrm{d}y \\ &\quad \times \int_{[v,\,u]} h_{Tt}^2(u-y)\mathrm{d}y + \left(\int_{[v,\,u]} h_{Tt}^2(u-y)\mathrm{d}y\right)^2\Bigg) \end{aligned}$$

$$= 3T^2 \left(\int_{[0,v]} (h_{Tt}(u-y) - h_{Tt}(v-y))^2 \,\mathrm{d}y + \int_{[v,u]} h_{Tt}^2(u-y)\mathrm{d}y \right)^2$$
$$= 3T^2 \left(\int_{[v,u]} h_{Tt}^2(y)\mathrm{d}y - 2\int_{[0,v]} h_{Tt}(v-y)\,(h_{Tt}(u-y) - h_{Tt}(v-y)) \right)^2$$
$$\le 3T^2 \left(\int_{[v,u]} h_{Tt}^2(y)\mathrm{d}y \right)^2.$$

Here the second equality follows since $(W_2(u))$ has independent increments, and the moments of odd orders of the integrals involved equal zero. The third equality is a consequence of (7). The last inequality is explained by the fact that the functions h_{Tt} are nonnegative and nondecreasing.

Hence when $Tt \ge t_0$ and $T \ge t_0$ we have

$$\begin{aligned}\mathbb{E}\left(\widehat{X}_t(u) - \widehat{X}_t(v)\right)^4 &\le 3T^2 \frac{h^{*4}(Tt)}{h^{*4}(t)} \left(\int_{[v,u]} h_{Tt}^2(y)\mathrm{d}y \right)^2 \\ &\overset{(15)}{\le} 3T^2 C^4(T) A \left(\int_{[v,u]} y^{2\beta-\delta}\mathrm{d}y \right)^2 \\ &= \frac{3T^2 C^4(T) A}{(2\beta-\delta+1)^2} \left(u^{2\beta-\delta+1} - v^{2\beta-\delta+1} \right)^2.\end{aligned}$$

If $u < v$ the same inequality holds. Hence, the required tightness follows by formula (12.51) and Theorem 12.3 in [8].

Proof of (16) for case (A3). We first note that the functions h_t are absolutely continuous with densities

$$h_t'(y) = \frac{t\left(h^*(ty) - e^{-ty} \int_{[0,ty]} h^*(x) e^x \mathrm{d}x \right)}{h^*(t)}.$$

The renewal process N has only unit jumps. Hence, $\lim_{t\to\infty} J(W^{(t)}) = 0$ a.s., where $J(\cdot)$ denotes the maximum-jump functional defined in (28). Since $W^{(t)} \overset{M_1}{\Rightarrow} W_\alpha$, $t \to \infty$, an appeal to Lemma A.6 and the continuous mapping theorem complete the proof. □

Before turning to the proof of (17) in case (A3) let us recall the following. The process $(W_\alpha(u))$ has no positive jumps, equivalently, the Lévy measure of $W_\alpha(1)$ is concentrated on the negative halfline. Therefore, it follows from Theorem 25.3 in [31] and the fact that the function $x \to (x \vee 1)^\gamma$, $\gamma > 0$ is submultiplicative, that the power moments of all positive orders of $W_\alpha^+(1)$ are finite.[2] Also it is well-known that

$$\mathbb{P}\{W_\alpha(1) < -x\} \sim \mathrm{const}\, x^{-\alpha}, \quad x \to \infty. \tag{19}$$

For a formal proof one can use the explicit form of characteristic function of $W_\alpha(1)$, Theorem 8.1.10 in [11] and the fact that the right tail of the law of $W_\alpha(1)$ is very light (in particular, it is clearly dominated by the left tail).

Proof of (17) for case (A3). We will prove that the convergence in (17) takes place in D under the M_1 topology. To this end, it suffices to show that the mentioned convergence holds in

[2] Moreover, the exponential moments of all positive orders of $W_\alpha(1)$ are finite.

$D[0, T]$, for any $T > 0$. Define $\widehat{X}_t(u)$ as in (18) but using W_α instead of W_2. Then the task reduces to proving the tightness of the so defined $\big(\widehat{X}_t(u)\big)$ in $D[0, 1]$. By Theorem 1 in [3], the required tightness will follow once we have proved that

$$\mathbb{P}\{M\left(\widehat{X}_t(u_1), \widehat{X}_t(u), \widehat{X}_t(u_2)\right) > \varepsilon\} \le L\varepsilon^{-\nu}(u_2 - u_1)^{1+\rho}, \quad 0 \le u_1 \le u \le u_2 \le 1, \quad (20)$$

for large enough t and some positive constants L, ν and ρ, where for $x_1, x_2, x_3 \in \mathbb{R} M(x_1, x_2, x_3) := 0$ if $x_2 \in [x_1 \wedge x_3, x_1 \vee x_3]$, and $:= |x_2 - x_1| \wedge |x_3 - x_2|$, otherwise, for x_1, x_2, x_3 in $\mathbb{R}$.

We have

$$\begin{aligned}
&\mathbb{P}\left\{M\left(\widehat{X}_t(u_1), \widehat{X}_t(u), \widehat{X}_t(u_2)\right) > \varepsilon\right\} \\
&\quad = \mathbb{P}\left\{|\widehat{X}_t(u_1) - \widehat{X}_t(u)| > \varepsilon, |\widehat{X}_t(u_2) - \widehat{X}_t(u)| > \varepsilon, \widehat{X}_t(u) < \widehat{X}_t(u_1) \wedge \widehat{X}_t(u_2)\right\} \\
&\qquad + \mathbb{P}\left\{|\widehat{X}_t(u_1) - \widehat{X}_t(u)| > \varepsilon, |\widehat{X}_t(u_2) - \widehat{X}_t(u)| > \varepsilon, \widehat{X}_t(u) > \widehat{X}_t(u_1) \vee \widehat{X}_t(u_2)\right\} \\
&\qquad + \mathbb{P}\left\{\widehat{X}_t(u_1) \wedge \widehat{X}_t(u_2) > \widehat{X}_t(u) + \varepsilon\right\} + \mathbb{P}\left\{\widehat{X}_t(u_1) \vee \widehat{X}_t(u_2) < \widehat{X}_t(u) - \varepsilon\right\} \\
&\quad = \mathbb{P}\left\{\widehat{X}_t(u) - \widehat{X}_t(u_1) > \varepsilon, \widehat{X}_t(u_2) - \widehat{X}_t(u) < -\varepsilon\right\} \\
&\qquad + \mathbb{P}\left\{\widehat{X}_t(u) - \widehat{X}_t(u_1) < -\varepsilon, \widehat{X}_t(u_2) - \widehat{X}_t(u) > \varepsilon\right\} \\
&\quad =: I_t(u_1, u, u_2) + J_t(u_1, u, u_2).
\end{aligned}$$

Using (18) and formula (6) with characteristic function of $W_\alpha(1)$ given by (4) we arrive at the distributional equality

$$\begin{aligned}
\frac{h^*(t)}{h^*(Tt)}\left(\widehat{X}_t(u) - \widehat{X}_t(u_1)\right) &= \int_{(0,\,u_1]} (h_{Tt}(u-y) - h_{Tt}(u_1-y))\,\mathrm{d}W_\alpha(Ty) \\
&\quad + \int_{(u_1,\,u]} h_{Tt}(u-y)\mathrm{d}W_\alpha(Ty) \\
&\overset{d}{=} T^\alpha\left(W_\alpha(1)\left(\int_{(0,\,u_1]} (h_{Tt}(u-y) - h_{Tt}(u_1-y))^\alpha\,\mathrm{d}y\right)^{1/\alpha}\right. \\
&\quad \left. + W'_\alpha(1)\left(\int_{(u_1,\,u]} h^\alpha_{Tt}(u-y)\mathrm{d}y\right)^{1/\alpha}\right) \\
&\overset{d}{=} T^\alpha W_\alpha(1)\left(\int_{(0,\,u_1]} (h_{Tt}(u-y) - h_{Tt}(u_1-y))^\alpha\,\mathrm{d}y\right. \\
&\quad \left. + \int_{(u_1,\,u]} h^\alpha_{Tt}(u-y)\mathrm{d}y\right)^{1/\alpha} \\
&=: T^\alpha W_\alpha(1) a_t(u_1, u),
\end{aligned}$$

where $W'_\alpha(1)$ and $W_\alpha(1)$ are i.i.d. Similarly

$$\begin{aligned}
\frac{h^*(t)}{h^*(Tt)}\left(\widehat{X}_t(u_2) - \widehat{X}_t(u)\right) &= \int_{(0,\,u]} (h_{Tt}(u_2-y) - h_{Tt}(u-y))\,\mathrm{d}W_\alpha(Ty) \\
&\quad + \int_{(u,\,u_2]} h_{Tt}(u_2-y)\mathrm{d}W_\alpha(Ty)
\end{aligned}$$

$$\stackrel{d}{=} T^\alpha \left(W_\alpha(1) \left(\int_{(0,\,u]} (h_{Tt}(u_2 - y) - h_{Tt}(u - y))^\alpha \,\mathrm{d}y \right)^{1/\alpha} \right.$$
$$\left. + W^*_\alpha(1) \left(\int_{(u,\,u_2]} h^\alpha_{Tt}(u_2 - y)\mathrm{d}y \right)^{1/\alpha} \right)$$
$$=: T^\alpha \left(W_\alpha(1) b_t(u, u_2) + W^*_\alpha(1) c_t(u, u_2) \right),$$

where $W_\alpha(1)$ and $W^*_\alpha(1)$ are i.i.d. and $W_\alpha(1)$ is the same as in the previous display.

Using the second inequality in (15) and setting $D(T) := T^\alpha C(T)$ we obtain, for large enough t,

$$I_t(u_1, u, u_2) = \mathbb{P}\left\{ T^\alpha \frac{h^*(Tt)}{h^*(t)} W_\alpha(1) a_t(u_1, u) > \varepsilon, T^\alpha \frac{h^*(Tt)}{h^*(t)} \right.$$
$$\left. \times \left(W_\alpha(1) b_t(u, u_2) + W^*_\alpha(1) c_t(u, u_2) \right) < -\varepsilon \right\}$$
$$\le \mathbb{P}\left\{ W_\alpha(1) > (a_t(u_1, u) D(T))^{-1} \varepsilon, W_\alpha(1) b_t(u, u_2) \right.$$
$$\left. + W^*_\alpha(1) c_t(u, u_2) < -D^{-1}(T)\varepsilon \right\}$$
$$\le \mathbb{P}\left\{ W_\alpha(1) > (a_t(u_1, u) D(T))^{-1} \varepsilon \right\} \mathbb{P}$$
$$\times \left\{ W^*_\alpha(1) c_t(u, u_2) < -D^{-1}(T)\varepsilon \left(1 + \frac{b_t(u, u_2)}{a_t(u_1, u)} \right) \right\}$$
$$\le \mathbb{P}\left\{ W_\alpha(1) > (a_t(u_1, u) D(T))^{-1} \varepsilon \right\} \mathbb{P}\left\{ W^*_\alpha(1) < -(c_t(u, u_2) D(T))^{-1} \varepsilon \right\}.$$

In view of (19), there exists a positive constant $q = q(\varepsilon)$ such that

$$\mathbb{P}\{W^*_\alpha(1) < -x\} \le q x^{-\alpha}, \tag{21}$$

whenever $x \ge \varepsilon(\alpha\beta - \delta + 1)^{1/\alpha}(A^{1/\alpha} D(T))^{-1}$ (the constants A and δ were defined in the paragraph that contains formula (15)). Further, for large enough t,

$$c^\alpha_t(u, u_2) = \int_{[0,\,u_2 - u)} h^\alpha_{Tt}(y)\mathrm{d}y \stackrel{(15)}{\le} A \int_{[0,\,u_2 - u]} y^{\alpha\beta - \delta}\mathrm{d}y = \frac{A}{\alpha\beta - \delta + 1}(u_2 - u)^{\alpha\beta - \delta + 1}$$
$$\le \frac{A}{\alpha\beta - \delta + 1}.$$

In view of this inequality (21) can be applied to estimate

$$I_t(u_1, u, u_2) \le \mathbb{P}\left\{ W^*_\alpha(1) < -(c_t(u, u_2) D(T))^{-1} \varepsilon \right\}$$
$$\le q D^\alpha(T) \varepsilon^{-\alpha} c^\alpha_t(u, u_2) \le \frac{A q D^\alpha(T)}{\alpha\beta - \delta + 1} \varepsilon^{-\alpha} (u_2 - u_1)^{\alpha\beta - \delta + 1}.$$

When $\beta > 0$ this crude bound suffices. When $\beta = 0$ we need a more refined estimate for $\mathbb{P}\left\{ W_\alpha(1) > (a_t(u_1, u) D(T))^{-1} \varepsilon \right\}$. To this end, we first work towards estimating $a_t(u_1, u)$. Since $\alpha > 1$ and $1 - \delta \in (0, 1)$,

$$(x + y)^\alpha \ge x^\alpha + y^\alpha \quad \text{and} \quad (x + y)^{1-\delta} \le x^{1-\delta} + y^{1-\delta} \quad \text{for all } x, y \ge 0.$$

Hence

$$\begin{aligned} a_t^\alpha(u_1, u) &\le \int_{(0,\,u_1]} \left(h_{Tt}^\alpha(u-y) - h_{Tt}^\alpha(u_1-y)\right) \mathrm{d}y + \int_{(u_1,\,u]} h_{Tt}^\alpha(u-y)\mathrm{d}y \\ &= \int_{(u_1,\,u]} h_{Tt}^\alpha(y)\mathrm{d}y \overset{(15)}{\le} A \int_{(u_1,u]} y^{-\delta}\mathrm{d}y \le \frac{A}{1-\delta}\left(u^{1-\delta} - u_1^{1-\delta}\right) \\ &\le \frac{A}{1-\delta}(u-u_1)^{1-\delta} \le \frac{A}{1-\delta}(u_2-u_1)^{1-\delta}. \end{aligned} \tag{22}$$

Using (22) and the Markov inequality we conclude that

$$\begin{aligned} \mathbb{P}\left\{W_\alpha(1) > (a_t(u_1,u)D(T))^{-1}\varepsilon\right\} &\le \mathbb{E}(W_\alpha^+(1))^\alpha D^\alpha(T)\varepsilon^{-\alpha} a_t^\alpha(u_1,u) \\ &\le \frac{\mathbb{E}(W_\alpha^+(1))^\alpha A D^\alpha(T)}{1-\delta}\varepsilon^{-\alpha}(u_2-u_1)^{1-\delta}. \end{aligned}$$

Combining pieces together leads to the inequality

$$I_t(u_1,u,u_2) \le \frac{\mathbb{E}(W_\alpha^+(1))^\alpha q A^2 D^{2\alpha}(T)}{(1-\delta)^2}\varepsilon^{-2\alpha}(u_2-u_1)^{2(1-\delta)},$$

which holds for large enough t in the case $\beta = 0$ and serves our needs as $1-\delta > 1/2$. Starting with a trivial estimate

$$\begin{aligned} J_t(u_1,u,u_2) &\le \mathbb{P}\left\{W_\alpha^*(1) > (a_t(u_1,u)D(T))^{-1}\varepsilon\right\} \\ &\quad \times \mathbb{P}\left\{W_\alpha(1) < -(c_t(u,u_2)D(T))^{-1}\varepsilon\right\}, \end{aligned}$$

we observe that $J_t(u_1,u,u_2)$ is bounded from above by the same quantity as $I_t(u_1,u,u_2)$. Summarizing we have proved that (20) holds with $L = \frac{2\mathbb{E}(W_\alpha^+(1))^\alpha q A^2 D^{2\alpha}(T)}{(1-\delta)^2}$, $\nu = 2\alpha$ and $\rho = 1-2\delta$ when $\beta = 0$ and with $L = \frac{2AqD^\alpha(T)}{\alpha\beta-\delta+1}$, $\nu = \alpha$ and $\rho = \alpha\beta - \delta$ when $\beta > 0$.

Case (A4). We first note that the functional limit theorem

$$V^{(t)}(u) := \mathbb{P}\{\xi > t\}N(ut) \overset{J_1}{\Rightarrow} V_\alpha(u), \quad t \to \infty,$$

was proved in Corollary 3.7 in [26].

We could have proceeded as above, by checking (16) and (17). However, in the present case the situation is much easier. Indeed, for each $t > 0$, the process $\left(X_t^*(u)\right)$ defined by

$$X_t^*(u) = \int_{(0,\,u]} V^{(t)}(y)\mathrm{d}_y\left(-h_t(u-y)\right), \quad u \ge 0,$$

has nondecreasing paths (as the convolution of two nondecreasing functions). Recall further that $(V_\alpha(u))$ is a generalized inverse function of a stable subordinator. Since the paths of the latter are right-continuous and strictly increasing, $\left(Z_{\alpha,0}(u)\right) := (V_\alpha(u))$ has continuous and nondecreasing sample paths. If $\beta > 0$, $\left(Z_{\alpha,\beta}(u)\right)$ has continuous paths by Lemma A.8. By Theorem 3 in [9] the desired functional limit theorem will follow once we have established the convergence of finite dimensional distributions.

We will only investigate the two-dimensional convergence. The other cases can be treated similarly. Since $X_t^*(0) = 0$ a.s., we only have to prove that, for fixed $0 < u < v < \infty$ and any

$\alpha_1, \alpha_2 \in \mathbb{R}$,

$$\alpha_1 X_t^*(u) + \alpha_2 X_t^*(v) \xrightarrow{d} \alpha_1 Z_{\alpha,\beta}(u) + \alpha_2 Z_{\alpha,\beta}(v), \quad t \to \infty. \tag{23}$$

For fixed $w > 0$ and each $t > 0$, define measures $\nu_{t,w}$ and ν_w on $[0, w]$ by

$$\nu_{t,w}(c, d] := \frac{h^*(t(w-c)) - h^*(t(w-d))}{h^*(t)}, \quad 0 \le c < d \le w$$

and

$$\nu_w(c, d] := (w-c)^\beta - (w-d)^\beta, \quad 0 \le c < d \le w,$$

where β is assumed positive.

Case $\beta > 0$. As $t \to \infty$, the measures $\nu_{t,w}$ weakly converge on $[0, w]$ to ν_w. Now relation (23) follows immediately from the equality

$$\alpha_1 X_t^*(u) + \alpha_2 X_t^*(v) = \int_{(0,u]} V^{(t)}(y) \left(\alpha_1 \nu_{t,u}(\mathrm{d}y) + \alpha_2 \nu_{t,v}(\mathrm{d}y)\right) + \alpha_2 \int_{(u,v]} V^{(t)}(y) \nu_{t,v}(\mathrm{d}y), \tag{24}$$

Lemma A.5 and the continuous mapping theorem.

Case $\beta = 0$. Now, as $t \to \infty$, the measures $\nu_{t,w}$ weakly converge on $[0, w]$ to δ_w (delta measure). Using (24) and arguing as before we arrive at (23).

5. Extension to *h*'s defined on $\mathbb{R}$

Let $h : \mathbb{R} \to \mathbb{R}$ be a right-continuous function with finite limits from the left. Unlike the situation considered in the previous sections the corresponding shot noise process is not necessarily well-defined. However we do not investigate the a.s. finiteness of $X(t)$ in the most general situation. Rather we prove that under appropriate assumptions on h which cover most of practically interesting cases the result of Theorem 1.1 continues to hold.

Theorem 5.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a right-continuous function with finite limits from the left such that*

$$h(x) \sim x^\beta \ell^*(x), \quad x \to \infty,$$

for some $\beta \in [0, \infty)$ and some ℓ^* slowly varying at ∞. Assume also that h is nondecreasing in the neighborhood of $+\infty$, and nondecreasing and integrable in the neighborhood of $-\infty$. Then the result of Theorem 1.1 is valid.

Proof. It suffices to prove that, for any $T > 0$ and any $c > 0$,

$$\frac{\sup_{u \in [0,T]} \sum_{k \ge 0} h(ut - S_k) 1_{\{S_k > ut\}}}{t^c} = \frac{\sup_{u \in [0,Tt]} \sum_{k \ge 0} h(u - S_k) 1_{\{S_k > u\}}}{t^c}$$
$$= \frac{\sup_{u \in [0,Tt]} \sum_{k \ge 0} h(u - S_{N(u)+k})}{t^c} \xrightarrow{P} 0, \quad t \to \infty.$$

As it was shown at the beginning of the proof of Theorem 1.1, without loss of generality, we can modify h on any finite interval in any way that would lead to a right-continuous resulting function with finite limits from the left. In particular, we will assume that $\widetilde{h} : [0, \infty) \to [0, \infty)$ defined by $\widetilde{h}(t) = h(-t)$, $t \geq 0$, is nonincreasing and $\widetilde{h}(0+) = 1$. Note that the integrability and monotonicity of h in the vicinity of $-\infty$ entail $\lim_{t\to\infty} \widetilde{h}(t) = 0$.

The random function $u \to S_{N(u)+k} - u$ attains a.s. its local minima $S_{k+1+j} - S_j$ at points S_j, $j \in \mathbb{N}$. Hence

$$\sup_{u\in[0,\,Tt]} \sum_{k\geq 0} \widetilde{h}(S_{N(u)+k} - u) = \sup_{1\leq j\leq N(Tt)-1} \sum_{k\geq 0} \widetilde{h}(S_{k+1+j} - S_j) =: \sup_{1\leq j\leq N(Tt)-1} \tau_j.$$

Note that the sequence $(\tau_j)_{j\in\mathbb{N}_0}$ is stationary. Since $\int_{[0,\,\infty)} \widetilde{h}(y)\mathrm{d}y < \infty$ implies $\int_{[0,\,\infty)} \widetilde{h}^p(y)\mathrm{d}y < \infty$ for any $p > 1$, we conclude that $\mathbb{E}\tau_0^p < \infty$ for any $p > 0$, by Theorem 3.7 in [1].

By the weak law of large numbers, for any $\delta > 0$, $\lim_{t\to\infty} \mathbb{P}\{N(Tt) > Tt(\mu^{-1} + \delta)\} = 0$, where μ^{-1} is interpreted as 0 when $\mu = \infty$. Choose $p > 0$ such that $pc > 1$. Then, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left\{\sup_{1\leq j\leq[Tt(\mu^{-1}+\delta)]} \tau_j > \varepsilon t^c\right\} &\leq \sum_{j=1}^{[Tt(\mu^{-1}+\delta)]} \mathbb{P}\{\tau_0 > \varepsilon t^c\}\\ &\leq [Tt(\mu^{-1} + \delta)]\varepsilon^{-p} t^{-pc} \mathbb{E}\tau_0^p \to 0, \quad t \to \infty,\end{aligned}$$

by the Markov inequality. Therefore, as $t \to \infty$,

$$\begin{aligned}\mathbb{P}\left\{\sup_{1\leq j\leq N(Tt)-1} \tau_j > \varepsilon t^c\right\} \leq{}& \mathbb{P}\left\{\sup_{1\leq j\leq[Tt(\mu^{-1}+\delta)]} \tau_j > \varepsilon t^c\right\}\\ &+ \mathbb{P}\{N(Tt) > Tt(\mu^{-1} + \delta)\} \to 0.\end{aligned}$$

The proof is complete. □

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Appendix

A.1. Probabilistic tools

Lemma A.1. *For any* $0 \leq a < b$*, any* $T > 0$ *and any* $c > 0$

$$\frac{\sup_{u\in[0,\,T]} (N(ut - a) - N(ut - b))}{t^c} \xrightarrow{P} 0, \quad t \to \infty. \tag{25}$$

Remark A.2. A perusal of the proof reveals that the rate of convergence to zero in (25) is not optimal. However, the present form of (25) serves our needs. In general, it seems very likely that the actual rate of a.s. convergence in (25) is the same as in Theorem 2 in [34]. Note however that the cited result assumed that $K_T \to \infty$ as $T \to \infty$ whereas we need $K_T = \text{const}$.

Proof. We start by writing

$$\begin{aligned}
\sup_{u\in[0,T]} (N(ut-a)-N(ut-b)) &= \sup_{u\in[0,T]} (N(ut-a)-N(ut-b))\, 1_{[b^{-1}t,\infty)}(u) \\
&\quad + \sup_{u\in[0,T]} (N(ut-a)-N(ut-b))\, 1_{[0,b^{-1}t)}(u) \\
&= \sup_{u\in[0,Tt-b]} (N(u+b-a)-N(u)) \\
&\quad + \sup_{u\in[0,T]} N(ut-a) 1_{[0,b^{-1}t)}(u) \\
&\le \sup_{u\in[0,Tt-b]} (N(u+b-a)-N(u)) + N(b-a).
\end{aligned}$$

To prove the equality

$$\sup_{u\in[0,S_{N(Tt-b)-1}]} (N(u+b-a)-N(u)) = \sup_{0\le k\le N(Tt-b)-1} (N(S_k+b-a)-N(S_k))$$

just note that obviously the right-hand side does not exceed the left-hand side, and that while u is traveling from S_k to $S_{k-1}-$ the numbers of S_j's falling into the interval $(u, u+b-a]$ can only decrease. In general, the following estimate holds true:

$$\begin{aligned}
\sup_{0\le k\le N(Tt-b)-1} (N(S_k+b-a)-N(S_k)) &\le \sup_{u\in[0,Tt-b]} (N(u+b-a)-N(u)) \\
&\le \sup_{0\le k\le N(Tt-b)} (N(S_k+b-a)-N(S_k)) \\
&=: Z(t).
\end{aligned}$$

A possible overestimate here is due to taking into account the extra interval $(Tt-a, S_{N(Tt-b)}+b-a]$.

By the weak law of large numbers, for any $\delta > 0$,

$$\lim_{t\to\infty} \mathbb{P}\{N(Tt-b) > Tt(\mu^{-1}+\delta)\} = 0, \tag{26}$$

where we set μ^{-1} to equal zero if $\mu = \mathbb{E}\xi = \infty$. It is known that $N(b-a)$ has exponential moments of all orders (see, for instance, Theorem 2 in [6]). Hence, for any $\gamma > 0$

$$\mathbb{P}\{N(b-a) > x\} = O(e^{-\gamma x}), \quad x \to \infty.$$

Now we conclude that, for any $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{0\le k\le [Tt(\mu^{-1}+\delta)]} (N(S_k+b-a)-N(S_k)) > \varepsilon t^c\right\} \\
&\quad\le \sum_{k=0}^{[Tt(\mu^{-1}+\delta)]} \mathbb{P}\{N(S_k+b-a)-N(S_k) > \varepsilon t^c\} \\
&\quad= ([Tt(\mu^{-1}+\delta)]+1)\mathbb{P}\{N(b-a)-1 > \varepsilon t^c\} \\
&\quad= O\left(t\exp(-\gamma\varepsilon t^c)\right) = o(1), \quad t\to\infty.
\end{aligned}\tag{27}$$

Therefore, in view of (26) and (27),

$$\begin{aligned}\mathbb{P}\{Z(t) > \varepsilon t^c\} &= \mathbb{P}\{Z(t) > \varepsilon t^c,\ N(Tt-b) > Tt(\mu^{-1}+\delta)\}\\ &\quad + \mathbb{P}\{Z(t) > \varepsilon t^c,\ N(Tt-b) \le Tt(\mu^{-1}+\delta)\}\\ &\le \mathbb{P}\{N(Tt-b) > Tt(\mu^{-1}+\delta)\}\\ &\quad + \mathbb{P}\left\{\max_{0\le k\le [Tt(\mu^{-1}+\delta)]} (N(S_k+b-a)-N(S_k)) > \varepsilon t^c\right\}\\ &= o(1), \quad t\to\infty. \quad \square\end{aligned}$$

Lemma A.3. *Let $K_1, K_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing functions such that $K_1(t) \ge K_2(t), t \in \mathbb{R}^+$. Assume that*

$$\limsup_{t\to\infty} \frac{K_1(t)+K_2(t)}{\int_{[0,\,t]} (K_1(y)-K_2(y))\,\mathrm{d}y} \le \text{const}.$$

Then, for any $c > 0$ and any $T > 0$,

$$\frac{\sup_{u\in[0,\,T]} \int_{[0,\,ut]} (K_1(ut-y)-K_2(ut-y))\,\mathrm{d}N(y)}{t^c \int_{[0,\,Tt]} (K_1(y)-K_2(y))\,\mathrm{d}y} \xrightarrow{P} 0, \quad t\to\infty.$$

Proof. We use the decomposition

$$\int_{[0,\,t]} (K_1(t-y)-K_2(t-y))\,\mathrm{d}N(y) = \int_{[0,\,[t]]} + \int_{[[t],\,t]} =: I_1(t)+I_2(t).$$

For $I_2(t)$ we have

$$\begin{aligned}I_2(t) &\le \int_{[[t],\,t]} K_1(t-y)\mathrm{d}N(y) \le K_1(t-[t])\,(N(t)-N([t]))\\ &\le K_1(1)\,(N(t)-N(t-1)).\end{aligned}$$

Hence, by Lemma A.1, for any $T > 0$,

$$t^{-c} \sup_{u\in[0,\,T]} I_2(ut) \xrightarrow{P} 0, \quad t\to\infty.$$

It remains to consider $I_1(t)$:

$$\begin{aligned}I_1(t) &= K_1(t)-K_2(t) + \sum_{j=0}^{[t]-1} \int_{(j,\,j+1]} (K_1(t-y)-K_2(t-y))\,\mathrm{d}N(y)\\ &\le K_1(t)-K_2(t) + \sum_{j=0}^{[t]-1} (K_1(t-j)-K_2(t-j-1))\,(N(j+1)-N(j))\\ &\le K_1(t) + \sup_{s\in[0,\,[t]]} (N(s+1)-N(s)) \sum_{j=0}^{[t]-1} (K_1(t-j)-K_2(t-j-1))\\ &\le K_1(t) + \sup_{s\in[0,\,[t]]} (N(s+1)-N(s)) \sum_{j=0}^{[t]-1} (K_1([t]+1-j)-K_2([t]-1-j))\end{aligned}$$

$$= \sup_{s\in[0,\,[t]]} (N(s+1)-N(s)) \times \left(\int_{[2,\,[t]]} (K_1(y)-K_2(y))\,\mathrm{d}y + O\left(K_1(t)+K_2(t)\right)\right).$$

Hence, for any $T > 0$,

$$\sup_{u\in[0,\,T]} I_1(ut) \le \sup_{u\in[0,\,T]} (N(ut+1)-N(ut)) \times \left(\int_{[2,\,[Tt]]} (K_1(y)-K_2(y))\,\mathrm{d}y + O\left(K_1(Tt)+K_2(Tt)\right)\right),$$

and, by Lemma A.1,

$$\frac{\sup_{u\in[0,\,T]} I_1(ut)}{t^c\int_{[0,\,Tt]}(K_1(y)-K_2(y))\,\mathrm{d}y} \xrightarrow{P} 0, \quad t\to\infty.$$

The proof is complete. □

A.2. Analytic tools

Lemma A.4. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a nondecreasing function which varies regularly at ∞ with index $\gamma\ge 0$, and $f(0)=0$. Let θ be a random variable with finite power moments of all positive orders whose absolutely continuous law is concentrated on $\mathbb{R}^+$. Then $f^*:\mathbb{R}^+\to\mathbb{R}^+$ defined by $f^*(t):=\mathbb{E}f((t-\theta)^+)$ is a continuous function with $f^*(0)=0$ such that $f^*(t)\sim f(t)$, $t\to\infty$. In particular, f^* varies regularly at ∞ with index γ. Furthermore,*

$$\int_{[0,\,t]} \left(f(y)-f^*(y)\right)\mathrm{d}y \sim \mathbb{E}\theta f(t), \quad t\to\infty.$$

Proof. The fact $f^*(0)=0$ is trivial. The continuity (even differentiability) of f^* follows from the representation

$$f^*(t) = f(0)e^{-t} + e^{-t}\int_{[0,\,t]} f(y)e^{y}\mathrm{d}y.$$

By dominated convergence, $\lim_{t\to\infty} f^*(t)/f(t)=1$. This entails the regular variation of f^*. Further

$$\int_{[0,\,t]} \left(f(y)-f^*(y)\right)\mathrm{d}y = \mathbb{E}\int_{[(t-\theta)^+,\,t]} f(y)\mathrm{d}y = \int_{[0,\,t]} f(y)\mathrm{d}y\,\mathbb{P}\{\theta>t\} + \mathbb{E}1_{\{\theta\le t\}}\int_{[t-\theta,\,t]} f(y)\mathrm{d}y.$$

As $t\to\infty$, the first term on the right-hand side tends to 0, by the Markov inequality. The second term can be estimated as follows

$$\frac{\mathbb{E}f(t-\theta)\theta 1_{\{\theta\le t\}}}{f(t)} \le \frac{\mathbb{E}1_{\{\theta\le t\}}\int_{[t-\theta,\,t]} f(y)\mathrm{d}y}{f(t)} \le \mathbb{E}\theta.$$

Since, as $t\to\infty$, the term on the left-hand side converges to $\mathbb{E}\theta$, by dominated convergence, the proof is complete. □

Lemma A.5. *Let $0 \le a < b < \infty$. Assume that $\lim_{n\to\infty} x_n = x$ in D in the J_1 or M_1 topology. Assume also that, as $n \to \infty$, finite measures ν_n converge weakly on $[a, b]$ to a finite measure ν, and that the limiting measure ν is continuous (nonatomic). Then*

$$\lim_{t\to\infty} \int_{[a,\, b]} x_n(y)\nu_n(\mathrm{d}y) = \int_{[a,\, b]} x(y)\nu(\mathrm{d}y).$$

If x is continuous at point $c \in [a, b]$, and $\nu = \delta_c$ is the Dirac measure at point c then

$$\lim_{n\to\infty} \int_{[a,\, b]} x_n(y)\nu_n(\mathrm{d}y) = x(c).$$

Proof. Since the convergence in the J_1 topology entails the convergence in the M_1 topology, it suffices to investigate the case when $\lim_{n\to\infty} x_n = x$ in the M_1 topology.

Since $x \in D[a, b]$ the set D_x of its discontinuities is at most countable. By Lemma 12.5.1 in [37], convergence in the M_1 topology implies local uniform convergence at all continuity points of the limit. Hence $E := \{y : \text{there exists } y_n \text{ such that } \lim_{n\to\infty} y_n = y, \text{but } \lim_{n\to\infty} x_n(y_n) \ne x(y)\} \subseteq D_x$, and, if ν is continuous, we conclude that $\nu(E) = 0$. If x is continuous at c and $\nu = \delta_c$ then $c \notin E$, hence $\nu(E) = 0$. Now the statement follows from Lemma 2.1 in [12]. □

For $x \in D[0,\ T]$, $T > 0$, define the maximum-jump functional

$$J(x) := \sup_{t\in[0,\, T]} |x(t) - x(t-)|. \tag{28}$$

Lemma A.6. *Let $\lim_{n\to\infty} x_n = x$ in the M_1 topology in $D[0,\ T]$, and $\lim_{n\to\infty} J(x_n) = 0$. For $n \in \mathbb{N}$ let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing and absolutely continuous functions with $f_n(0) = 0$. Define*

$$y_n(u) := \int_{[0,\, u]} (x_n(y) - x(y))\, \mathrm{d}\left(-f_n(u - y)\right), \qquad y(u) := 0, \quad u \in [0,\ T].$$

Then $\lim_{n\to\infty} y_n = y$ in the M_1 topology in $D[0,\ T]$.

Proof. For $z \in D[0,\ T]$ denote by $\Pi(z)$ the set of all parametric representations of z (see pp. 80–82 in [37] for the definition). Since $\lim_{n\to\infty} x_n = x$, Theorem 12.5.1(i) in [37] implies that we can choose parametric representations $(u, r) \in \Pi(x)$ and $(u_n, r_n) \in \Pi(x_n)$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{t\in[0,\, 1]} |u_n(t) - u(t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\in[0,\, 1]} |r_n(t) - r(t)| = 0.$$

Furthermore, according to the proof of Lemma 4.3 in [28], we can assume that $r(t)$ is absolutely continuous with respect to the Lebesgue measure and that

$$x(r(t))r'(t) = u(t)r'(t) \quad \text{a.e. on } [0, 1], \tag{29}$$

where r' is the derivative of r.

Clearly, $(y, r) \in \Pi(y)$. By Lemma A.8(b), the functions y_n, $n \in \mathbb{N}$, are continuous. Hence, $(y_n(r), r) \in \Pi(y_n)$, $n \in \mathbb{N}$, and it suffices to prove that

$$\lim_{n\to\infty} \sup_{t\in[0,\, 1]} |y_n(r(t))| = 0.$$

We have

$$\begin{aligned}
y_n(r(t)) &= \int_{[0,\,r(t)]} (x_n(y) - x(y))\,\mathrm{d}\,(-f_n(r(t)-y)) \\
&= \int_{[0,\,t]} (x_n(r(y)) - x(r(y)))\,\mathrm{d}\,(-f_n(r(t)-r(y))) \\
&= \int_{[0,\,t]} (x_n(r(y)) - u(y))\,\mathrm{d}\,(-f_n(r(t)-r(y))) \\
&\quad + \int_{[0,\,t]} (u(y) - x(r(y)))\,r'(y) f_n'(r(t)-r(y))\mathrm{d}y \\
&\overset{(29)}{=} \int_{[0,\,t]} (x_n(r(y)) - u(y))\,\mathrm{d}\,(-f_n(r(t)-r(y)))\,.
\end{aligned}$$

From the proof of Lemma 4.2 in [28] it follows that

$$\lim_{n\to\infty} \sup_{t\in[0,\,1]} |x_n(r(t)) - u(t)| = 0,$$

whenever $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} J(x_n) = 0$. Hence, as $n \to \infty$,

$$\begin{aligned}
\sup_{t\in[0,\,1]} |y_n(r(t))| &\le \sup_{t\in[0,\,1]} \sup_{y\in[0,\,t]} |x_n(r(y)) - u(y)| f_n(r(t)) \\
&= \sup_{t\in[0,\,1]} |x_n(r(t)) - u(t)| f_n(T) \to 0. \quad \square
\end{aligned}$$

Lemma A.7. *Let F and G be left- and right-continuous functions of locally bounded variation, respectively. Then, for any real $a < b$,*

$$\int_{(a,\,b]} F(y)\mathrm{d}G(y) = F(b+)G(b) - F(a+)G(a) - \int_{(a,\,b]} G(y)\mathrm{d}F(y).$$

Proof. This follows along the lines of the proof of Theorem 11 on p. 222 in [32] which treats right-continuous functions F and G. $\square$

Lemma A.8. (a) *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and monotone function and $g : \mathbb{R}^+ \to \mathbb{R}$ be any locally bounded function such that the convolution $f \star g(x) := \int_{[0,\,x]} f(x-y)g(y)\mathrm{d}y$ is well-defined and finite. Then $f \star g$ is continuous on $\mathbb{R}^+$.*

(b) *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and nondecreasing function and $f : \mathbb{R}^+ \to \mathbb{R}$ be any locally bounded function. Then the Riemann–Stieltjes convolution $f \star g(x) := \int_{[0,\,x]} f(x-y)\mathrm{d}g(y)$ is continuous on $\mathbb{R}^+$.*

Proof. (a) With $\varepsilon > 0$ write for any $x \ge 0$

$$\begin{aligned}
|f \star g(x+\varepsilon) - f \star g(x)| &\le \int_{[0,\,x]} (f(x+\varepsilon-y) - f(x-y))\,|g(y)|\mathrm{d}y \\
&\quad + \int_{[x,\,x+\varepsilon]} f(x+\varepsilon-y)|g(y)|\mathrm{d}y.
\end{aligned}$$

As $\varepsilon \to 0$ the first integral goes to zero by monotone convergence. The function f must be integrable in the neighborhood of zero. With this at hand it remains to note that the second integral does not exceed

$$\sup_{y\in[x,\,x+\varepsilon]} |g(y)| \int_{[0,\,\varepsilon]} f(y)\mathrm{d}y \to |g(x+)| \times 0 = 0, \quad \varepsilon \to 0.$$

The case $\varepsilon < 0$ can be treated similarly.

(b) With $\varepsilon \in (0, 1)$ write for any $x \ge 0$

$$|f \star g(x+\varepsilon) - f \star g(x)| = \int_{[0,\,x]} f(y)\mathrm{d}\left(-g(x+\varepsilon-y) + g(x-y)\right) + \int_{[x,\,x+\varepsilon]} f(y)\mathrm{d}\left(-g(x+\varepsilon-y)\right).$$

The total variations of the integrators of the first integral are uniformly bounded. Furthermore, in view of the continuity of g, as $\varepsilon \to 0$, these integrators converge (pointwise) to zero. Hence, as $\varepsilon \to 0$ the first integral goes to zero by Helly's theorem for Lebesgue–Stieltjes integrals. The second integral does not exceed

$$\sup_{y\in[x,\,x+\varepsilon]} |f(y)| \left(g(\varepsilon) - g(0)\right) \to |f(x+)| \times 0 = 0, \quad \varepsilon \to 0.$$

The case $\varepsilon \in (-1, 0)$ can be treated similarly. □

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