

# Small noise asymptotic expansions for stochastic PDE's driven by dissipative nonlinearity and Lévy noise

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## Abstract

We study a reaction–diffusion evolution equation perturbed by a space–time Lévy noise. The associated Kolmogorov operator is the sum of the infinitesimal generator of a  $C_0$ -semigroup of strictly negative type acting on a Hilbert space and a nonlinear term which has at most polynomial growth, is non necessarily Lipschitz and is such that the whole system is dissipative.

The corresponding Itô stochastic equation describes a process on a Hilbert space with dissipative nonlinear, non globally Lipschitz drift and a Lévy noise. Under smoothness assumptions on the nonlinearity, asymptotics to all orders in a small parameter in front of the noise are given, with detailed estimates on the remainders.

Applications to nonlinear SPDEs with a linear term in the drift given by a Laplacian in a bounded domain are included. As a particular case we provide the small noise asymptotic expansions for the SPDE equations of FitzHugh–Nagumo type in neurobiology with external impulsive noise.

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## 1. Introduction

In many areas of investigations, in natural as well as technical and socio-economical sciences, a description of phenomena in terms of (partial) differential equations (PDEs) is quite natural and has received a lot of attention, also in recent years. However the necessity of taking care of stochastic (or random) influences on systems primarily described by (P)DEs in particular through stochastic (P)DEs, i.e., (S(P)DEs) has also came to the forefront of research; see, e.g. [75,49,33,32,63,61,38,42]. In the present paper we concentrate on PDE's perturbed by a space–time noise of the additive type. Such SPDEs have been studied extensively particularly in the case of Gaussian noises and they have found applications in several areas, from physics to biology and financial mathematics; see e.g. [75,33,32,36–38,42,13,27,17,64]. The extension to the treatment of additive Lévy type noises (which are more general in the sense that random variables with Lévy distributions extend Gaussian random variables) is relatively more recent; see e.g. [14,18,20,21,25,54,59,60,63,66]. A natural question which arises in such extensions from a deterministic description of phenomena to a stochastic one, is in which sense one can recover the deterministic description by “switching off” the noise and possibly obtain “small noise expansions” around the deterministic limit. In the case of SDE's (as opposite to SPDEs) this is a rather well studied problem, especially in the case of Gaussian noises and it has also relations with the study of the classical limit from quantum mechanics (see e.g. [6,34,35,67,76,72,43,15,19,12,57,58,56,68,44,45]). The infinite dimensional case and the case of SPDEs is less studied, even in the case of Gaussian noises, see however [29,13,16,65,58,57]. Concerning applications, the case of stochastic perturbations of the FitzHugh–Nagumo equations of neurobiology and its relations with the classical, deterministic FitzHugh–Nagumo equations is particularly interesting, due to the fact that those equations, and their extensions to the case where the underlying euclidean domain in space is replaced by a network, are extensively used in neurobiology, see e.g. [2,26,3,50,53,73–75]. In two recent papers [4,5] a systematic study of SPDEs with additive Gaussian noise which includes in particular the above stochastic FitzHugh–Nagumo equations, has been given, together with a detailed study of the corresponding diffusion expansion around the deterministic limit. One basic difficulty which had to be overcome (besides the infinite dimensionality of the stochastic process involved) consisted in the non global Lipschitz character of the nonlinear terms. A global Lipschitz condition would in fact exclude the interesting case of the FitzHugh–Nagumo equations; similarly other interesting equations like those arising in stochastic quantization [11,1,30,31,46,47,55,62], hydrodynamics [7,42] or solid state physics, [8] (e.g., Allen–Cahn equations), would be excluded. Despite the interest in modeling the noise in such systems through a Lévy-type one instead of a Gaussian one, which has motivations in all the areas which have been mentioned, apparently a corresponding study of asymptotic expansions around the underlying deterministic systems has yet to be performed. We shall here adopt the method used in [4] to cope with this case. The adaptation involves, in particular, using methods developed by [63] to handle stochastic convolutions in the case of Lévy noise. Let us note that our results seem to be new even in the finite dimensional case, where small stochastic perturbation expansions have also not been provided in details for equations of the type we consider. Before we go over to describe the contents of the present paper, let us mention that our study of SPDEs with Lévy noise can also be related to the study of certain pseudo-differential equations with such noises which occur in quantum field theory and statistical mechanics (see e.g. [9,10]). Also relations to certain problems in the study of statistics of processes described by Lévy noises should be mentioned [39,40].

## 2. Outline of the paper

Let us consider the following deterministic nonlinear evolution problem:

$$\begin{cases} d\phi(t) = [A\phi(t) + F(\phi(t))]dt, & t \in [0, +\infty) \\ \phi(0) = u^0, & u^0 \in H, \end{cases} \quad (2.1)$$

where  $A$  is a linear operator on a separable Hilbert space  $H$  which generates a  $C_0$ -semigroup of strict negative type. The term  $F$  is a *smooth* nonlinear, quasi- $m$ -dissipative mapping from the domain  $D(F) \subset H$  (dense in  $H$ ) with values in  $H$  (this means that there exists  $\omega \in \mathbb{R}$  such that  $(F - \omega I)$  is  $m$ -dissipative in the sense of [33, p. 73]), with (at most) polynomial growth at infinity and satisfying some further assumptions which will be specified in [Hypothesis 3.1](#) below. Existence and uniqueness of solutions for Eq. (2.1) is discussed in [Proposition 4.2](#) below.

Our aim is to study a stochastic (white noise) perturbation of (2.1) and to write its (unique) solution as an expansion in powers of a parameter  $\varepsilon > 0$ , which controls the strength of the noise, as  $\varepsilon$  goes to zero. More precisely, we are concerned with the following stochastic Cauchy problem on the Hilbert space  $H$ :

$$\begin{cases} du(t) = [Au(t) + F(u(t))]dt + \varepsilon\sqrt{Q}dL(t), & t \in [0, +\infty) \\ u(0) = u^0, & u^0 \in K, \end{cases} \quad (2.2)$$

where  $A$  and  $F$  are as described above,  $L$  is a mean square integrable Lévy process taking values in a Hilbert space  $U$ ,  $Q$  is a positive trace class linear operator from  $H$  to  $H$  and  $\varepsilon > 0$  is the parameter which determines the magnitude of the stochastic perturbation. The initial datum  $u^0$  takes values in a continuously embedded Banach space  $K$  of  $H$ . A unique solution of the problem (2.2) can be shown to exist exploiting as in [24] results on stochastic differential equations (contained, e.g., in [32,33]). Our purpose is to show that the solution of the Eq. (2.2), which will be denoted by  $u = u(t)$ ,  $t \in [0, +\infty)$ , can be written as

$$u(t) = \phi(t) + \varepsilon u_1(t) + \cdots + \varepsilon^n u_n(t) + R_n(t, \varepsilon),$$

where  $n$  depends on the differentiability order of  $F$ . The function  $\phi(t)$  solves the associated deterministic problem (2.1),  $u_1(t)$  is the stochastic process which solves the following linear stochastic (non-autonomous) equation

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dL(t), & t \in [0, +\infty) \\ u_1(0) = 0, \end{cases} \quad (2.3)$$

while for each  $k = 2, \dots, n$ ,  $u_k(t)$  solves the following non-homogeneous linear differential equation with stochastic coefficients

$$\begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt, \\ u_k(0) = 0. \end{cases} \quad (2.4)$$

$\Phi_k(t)$  is a stochastic process which depends on  $u_1(t), \dots, u_{k-1}(t)$  and the Fréchet derivatives of  $F$  up to order  $k$ , see [Section 5](#) for details.

Let us shortly describe the content of the different sections of the present paper. In [Section 3](#) we set the basic assumptions needed to perform the construction of solutions and their asymptotic expansion. In [Section 4](#) we describe the mild solutions to SDE's driven by Lévy processes on Hilbert spaces, basically following the setting of [63]. Since our expansion will be around solutions of the corresponding deterministic equations, we start by presenting results on the latter equations ([Section 4.1](#)). In the [Section 4.2](#) we present the setting for the stochastic perturbation, first describing the noise. In [Section 5](#) we describe the basic assumptions on the nonlinear term and provide its Taylor expansion. [Section 6](#) contains the main results, in

particular the construction of the expansion, the proof of its asymptotic character and of detailed estimates on remainders, to any order. We close with an application to the case of a stochastic FitzHugh–Nagumo equation on a network.

### 3. Assumptions and basic estimates

Before recalling some known results on problems of the types (2.1)–(2.4), we begin by presenting our notation and assumptions. We are concerned with a real separable Hilbert space, with the inner product  $\langle \cdot, \cdot \rangle$ . Moreover, in what follows,  $(B, |\cdot|)$  is a reflexive Banach space continuously embedded into  $H$  as a dense Borel subset and  $(K, |\cdot|)$  is a reflexive Banach space continuously embedded in  $B$ . On  $H$  we consider a linear operator  $A : D(A) \subset H \rightarrow H$ , a nonlinear operator  $F : D(F) \subset H \rightarrow H$  with dense domain in  $H$  and a bounded linear operator  $Q$  from  $H$  to  $H$ . Moreover, we are given a complete probability space  $(\Omega, \mathcal{F}, (F_t)_{t \geq 0}, \mathbb{P})$  which satisfies the usual conditions, i.e., the probability space is complete,  $\mathcal{F}$  contains all  $\mathbb{P}$ -null subsets of sets in  $\mathcal{F}$  and the filtration  $(F_t)_{t \geq 0}$  is right continuous. Further, for any trace-class linear operator  $Q$ , we will denote by  $\text{Tr } Q$  its trace; if  $f$  is any mapping on  $H$  which is Fréchet differentiable up to order  $n$ ,  $n \in \mathbb{N}$ , we will denote by  $f^{(i)}$ ,  $i = 1, \dots, n$  its  $i$ -th Fréchet derivative and by  $D(f^{(i)})$  the corresponding domain (for a short survey on Fréchet differentiable mappings we refer to Section 4). For any  $j \in \mathbb{N}$  and any vector space  $X$ ,  $L(X^j; X)$  denotes the space of  $j$ -linear bounded mappings from  $X^j$  into  $X$  while the space of linear bounded mappings from  $X$  into  $L(X^j; X)$  is denoted by  $L^j(X)$ . We denote by  $|\cdot|_X$  the norm on  $X$ , by  $\|\cdot\|_{L^j(X)}$  the norm of any  $j$ -linear operator on  $X$  and by  $\|\cdot\|_{HS}$  the Hilbert–Schmidt norm of any linear operator on  $X$ . Finally, for any  $p \geq 1$ , we will denote by  $\mathcal{C}_{\mathcal{F}}([0, T]; L^p(\Omega; X))$  the space of  $X$ -valued, adapted  $p$  integrable processes  $Y$  on the time interval  $[0, T]$  such that the following norm is finite

$$\|Y\| = \left( \sup_{t \in [0, T]} \mathbb{E} |Y(t)|_X^p \right)^{1/p} < \infty.$$

**Hypothesis 3.1.** i. The operator  $A : D(A) \subset H \rightarrow H$  generates an analytic semigroup  $(e^{tA})_{t \geq 0}$ , on  $H$  of strict negative type such that

$$\|e^{tA}\|_{L(H)} \leq e^{-\omega t}, \quad t \geq 0$$

with  $\omega$  a strictly positive, real constant.

Moreover, if  $A_B$  denotes the part of  $A$  in the reflexive Banach space  $B$ , that is

$$D(A_B) := \{x \in D(A) \cap B; Ax \in B\}, \quad A_B x = Ax,$$

then  $A_B$  generates an analytic semigroup (of negative type)  $e^{tA_B}$ ,  $t \geq 0$  on  $B$ .

ii. The mapping  $F : D(F) \subset H \rightarrow H$  is continuous, nonlinear, Fréchet differentiable up to order  $n$  for some positive integer  $n$  and quasi- $m$ -dissipative, i.e., there exist  $\eta > 0$  such that

$$\langle F(u) - F(v) - \eta(u - v), u - v \rangle < 0, \quad \text{for all } u, v \in D(F).$$

iii. If  $F_B^{(j)}$ ,  $j = 1, \dots, n$  denotes the part of  $F^{(j)}$  in  $B$ , that is

$$D(F_B^{(j)}) := \left\{ x \in D(F^{(j)}) \cap K; F_B^{(j)}(x) \in B \right\}, \quad F_B^{(j)}(x) = F^{(j)}(x),$$

then there exists a reflexive Banach space  $K$  densely and continuously embedded in  $B$  which makes the following assumptions satisfied:

(a) there exists a positive real number  $\gamma$  and a positive natural number  $n$  such that:

$$|F_B(u)|_B \leq \gamma \left( 1 + |u|_K^{2n+1} \right), \quad u \in K,$$

- (b) for some  $n$  and any  $u \in D(F_K^{(i)})$ ,  $i = 1, \dots, n$ , there exist positive real constants  $\gamma_i$ ,  $i = 1, \dots, n$  such that

$$\|F_B^{(i)}(u)\|_{L^j(B)} \leq \gamma_i(1 + |u|_K^{2n+1-i}), \quad \text{with } n \text{ as in (iii), } u \in K.$$

- iv. The constants  $\omega, \eta$  satisfy the inequality  $\omega - \eta > 0$ ; this implies that the term  $A + F$  is  $m$ -dissipative in the sense of [32], [33, p. 73].  
v. The term  $L$  is a Lévy process (for example in the sense of [22,63]) on some Hilbert space  $U$ ; moreover we assume that

$$\int_U |y|^m \nu(dy) < \infty,$$

for all  $m \in \mathbb{N}$ , where  $\nu$  is the jump intensity measure introduced in Section 4.2.

- vi.  $Q$  is a positive linear bounded operator on  $H$  of trace class, that is  $\text{Tr } Q < \infty$ .

**Example 3.2.** Let us give an example of a mapping  $F$  satisfying the above hypothesis (in view of the application to stochastic neuronal models, which we will present in Section 6). Let  $H = L^2(\Lambda)$  with  $\Lambda \subset \mathbb{R}^n$ , bounded and open; set  $B := L^{2(2n+1)}(\Lambda)$ ,  $K := L^{2(2n+1)^2}(\Lambda)$  and let  $F$  be a multinomial of odd degree  $2n + 1$ ,  $n \in \mathbb{N}$ , i.e. a mapping of the form  $F(u) = g_{2n+1}(u)$ , where  $g_{2n+1}(u)$ ,  $u \in H$ , is a polynomial of degree  $2n + 1$ , that is,  $g_{2n+1}(u) = a_0 + a_1 u + \dots + a_{2n+1} u^{2n+1}$ , with  $a_i \in \mathbb{R}$ ,  $i = 0, \dots, 2n + 1$ . Then it is easy to prove that  $D(F) = L^{2(2n+1)}(\Lambda) \subsetneq L^2(\Lambda)$ ,  $n > 0$ ,  $D(F) = L^2(\Lambda) = H$ ,  $n = 0$  and (by using the Hölder inequality)  $D(F^{(i)}) = L^{2(2n+1-i)}(\Lambda)$ . Moreover, it turns out that, for any  $u \in K$ ,  $F^{(i)}(u)$  can be identified with the element  $g_m^{(i)}(u)$  (both in  $D(F)$  and  $K$ ). Consequently,

$$\begin{aligned} |F(u)|_B &= \left( \int_{\Lambda} |g_{2n+1}(u(\xi))|^{2(2n+1)} d\xi \right)^{1/(2(2n+1))} \\ &\leq C_{2n+1} \left( 1 + \int_{\Lambda} |u(\xi)|^{2(2n+1)^2} d\xi \right)^{1/(2(2n+1))} \\ &= C_{2n+1} (1 + |u|_K^{2n+1}) \end{aligned}$$

and, similarly,

$$\begin{aligned} |\nabla^{(j)} F(u)|_{L^j(K; B)} &\leq C_{2n+1-j} (1 + |u|_K^{2n+1-j}) \\ &= C_{2n-j} (1 + |u|_K^{2n+1-j}), \quad j = 0, 1, \dots, m, \end{aligned}$$

for some constants  $C_l > 0$ ,  $l = 0, 1, \dots, 2n$ . Hence  $F$  satisfies Hypothesis 3.1(ii), (iii). Further, in the case  $g_3(u) = -u(u - 1)(u - \xi)$ ,  $0 < \xi < 1$ , the corresponding mapping  $F$  coincides with the nonlinear term of the first equation in the FitzHugh–Nagumo system (see Example 6.4 below).

#### 4. Mild solutions to SDE's driven by Lévy on Hilbert spaces

In this section we basically use the setting of [63].

##### 4.1. The deterministic case

Let  $A_0$  be a densely defined linear operator on a Banach space  $B$ , with domain  $D(A_0)$ . Let assume that the differential equation

$$\begin{cases} \frac{dy}{dt} = A_0 y \\ y(0) = y_0 \in D(A_0) \end{cases} \quad (4.1)$$

has a unique solution  $y(t)$ ,  $t \geq 0$ ,  $y(t) \in B$ . The equation being linear, we have

$$y(t) = S(t)y_0, \quad t \geq 0,$$

with  $S(t)$  a linear operator from  $D(A_0)$  into  $B$ . If for each  $t \geq 0$ ,  $S(t)$  has a *continuous* extension to all of  $B$ , and for each  $z \in B$ ,  $t \rightarrow S(t)z$  is continuous, then one says that the Cauchy problem (4.1) is well posed.  $t \rightarrow S(t)z$ , defined then for all  $z \in B$ , is called a generalized solution to (4.1). One has that  $(S(t))_{t \geq 0}$  is a  $C_0$ -semigroup:

- i.  $S(0) = \mathbb{1}$ ,  $S(t)S(s) = S(t+s)$ ,  $t, s \geq 0$ ,
- ii.  $\|S(t)z - z\|_B \rightarrow 0$  as  $t \downarrow 0$ , for every  $z \in B$ .

Let  $D(A)$  be the definition domain of the generator  $A$  of  $S(t)$ , i.e.  $S(t) = e^{(tA)}$ . We have  $D(A) \supset D(A_0)$  and  $A$  is an extension of  $A_0$ . Moreover, see, e.g. [63, Theorem 9.2]:

- i.  $\|S(t)z\|_B \leq e^{\omega t} M \|z\|_B$ , for some  $\omega, M > 0 \forall z \in B, \forall t \geq 0$ ,
- ii.  $A$  is closed and for any  $z \in D(A)$ ,  $t > 0$  one has  $S(t)z \in D(A)$  and  $\frac{d}{dt}S(t)z = AS(t)z = S(t)Az$ . In particular for  $z = y_0 \in D(A)$ ,  $t \rightarrow S(t)y_0$  solves (4.1) with  $A$  replacing  $A_0$ .

Now, let  $H$  be a Hilbert space such that  $B \subset H$ , with a dense, continuous embedding,  $B$  being a Borel subset of  $H$ . Let  $\psi(t)$ ,  $t \geq 0$  be  $H$ -valued and continuously differentiable. Then the “variation of constants formula”

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\psi(s) ds, \quad t \geq 0 \quad (4.2)$$

solves

$$\begin{cases} \frac{dy}{dt}(t) = Ay(t) + \psi(t) \\ y(0) = y_0 \in H. \end{cases} \quad (4.3)$$

In general, whenever the integral in (4.2) has a meaning for a given  $y_0$  in  $H$ , one says that (4.2) is a *mild solution* of

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + \psi(t) \\ y(0) = y_0. \end{cases} \quad (4.4)$$

The formal definition of mild solution is given below.

**Definition 4.1.** Let  $y_0 \in H$ ; we say that the function  $\phi : [0, \infty) \rightarrow H$  is a mild solution of Eq. (2.1) if it is continuous (in  $t$ ), with values in  $H$  and it satisfies:

$$\phi(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}\psi(s) ds, \quad t \in [0, +\infty), \quad (4.5)$$

with the integral existing in the sense of Bochner integrals on Hilbert spaces.

In the case of  $\psi$  being substituted by a mapping  $F : D(F) \subset H \rightarrow H$  satisfying the assumptions given in Hypothesis 3.1 we have the following result.

**Proposition 4.2.** Under Hypothesis 3.1 there exists a unique mild solution  $\phi = \phi(t)$ ,  $t \in [0, \infty)$  of the deterministic problem

$$\begin{cases} \frac{dy}{dt} = A_0y + F(y) \\ y(0) = y_0 \in D(A_0) \end{cases} \quad (4.6)$$

such that

$$|\phi(t)|_H \leq e^{-2(\omega-\eta)t} |u^0|_H, \quad t \geq 0. \quad (4.7)$$

**Proof.** The proof of existence and uniqueness can be found, e.g. in [32, Theorem 7.13, p. 203], while estimate (4.7) is a direct consequence of the application of Gronwall's lemma to the following inequality

$$\begin{aligned} \frac{d}{dt} |\phi(t)|_H^2 &= 2\langle A\phi(t), \phi(t) \rangle + 2\langle F(\phi(t)), \phi(t) \rangle \\ &\leq -2(\omega - \eta) |\phi(t)|_H^2. \quad \square \end{aligned}$$

**Remark 4.3.** It can be shown that, under Hypothesis 3.1, there exists a  $K$ -continuous version of the unique solution of Eq. (4.5) such that, for any  $T > 0$ ,  $p \geq 1$

$$\sup_{t \in [0, T]} |\phi(t)|_K^p < \infty.$$

(See [33, Section 5.5.2, Proposition 5.5.6].) Hence, in the following, by  $\phi$  we will always understand this  $K$ -valued version of the solution of (2.1).

#### 4.2. The stochastically perturbed case

Let  $G$  be a linear operator from a Hilbert space  $U$  into a Hilbert space  $H$ . Let  $S(t)$  be a  $C_0$ -semigroup on the Hilbert space  $H$ . Assume the generator  $(A, D(A))$  of  $S(t)$  in  $H$  is almost  $m$ -dissipative (i.e.  $[(\lambda \mathbb{1} - A) + \eta]H = H$  for any  $\lambda > 0$  and some  $\eta \in \mathbb{R}$ : [63, p. 180]; this is equivalent to quasi  $m$ -dissipative in the sense of [33, p. 73]). Assume  $B \subset H$  as in 4.1 and that the restriction  $A_B$  of  $A$  to  $B$  is also almost  $m$ -dissipative. Let  $L$  be a square-integrable mean zero Lévy process taking values in a Hilbert space  $K$ . I.e.,  $L = (L(t))_{t \geq 0}$  takes values in a Hilbert space  $K$ , has independent, stationary, increments, one has  $L(0) = \vec{0}$ , and  $L(t)$  is stochastically continuous (see [63, Definition 4.1, p. 38]). Let  $Q$  be the covariance of  $L$ . Then  $Q^{\frac{1}{2}}(K)$  is the reproducing kernel Hilbert space (RKHS) of  $L$ , we assume that  $Q^{\frac{1}{2}}(K)$  is embedded into  $U$ .

We recall the following basic notions and results.

**Definition 4.4.** Let  $\nu$  be a finite measure on a Hilbert space  $U$  such that  $\nu(\{0\}) = 0$ . A compound Poisson process with Lévy measure (also called jump intensity measure)  $\nu$  is a càdlàg Lévy process  $L$  satisfying

$$P(L(t) \in \Gamma) = e^{-\nu(U)t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \nu^{*k}(\Gamma), \quad t \geq 0, \quad \Gamma \in \mathcal{B}(U).$$

$\mathcal{B}(U)$  being the  $\sigma$ -algebra of Borel subsets of  $U$ .

Given a Borel set  $I$  separated from  $\{0\}$ , write

$$\pi_I(t) = \sum_{s \leq t} \chi_I(\Delta L(s)), \quad t \geq 0,$$

where  $DL(s) = L(s) - L(s-)$  and  $\chi_I$  denotes the characteristic function on the set  $I$ .

The càdlàg property of  $L$  implies that  $\pi_I$  is  $\mathbb{Z}_+$ -valued. We notice that it is a Lévy process with jumps of size 1 and thus, a Poisson process (see [63, Proposition 4.9(iv)] for more details). We

also have that  $\mathbb{E}\pi_I(t) = t\mathbb{E}\pi_I(1) = t\nu(I)$ , where  $\nu$  is a measure that is finite on sets separated from  $\{0\}$ . We shall write

$$L_I(t) = \sum_{s \leq t} \chi_I(\Delta L(s)) \Delta L(s).$$

Then  $L_I$  is a well-defined Lévy process. The theorem below provides the corresponding Lévy–Khinchin decomposition:

**Theorem 4.5.** i. If  $\nu$  is a jump intensity measure corresponding to a Lévy process then

$$\int_U (|y|_U^2 \wedge 1) \nu(dy) < \infty.$$

ii. Every Lévy process has the following representation:

$$L(t) := at + \sqrt{Q}W(t) + \sum_{k=1}^{\infty} \left( L_{I_k}(t) - t \int_{I_k} y \nu(dy) \right) + L_{I_0}(t),$$

where  $I_0 := \{x : |x|_U \geq r_0\}$ ,  $I_k := \{x : r_k \leq |x|_U < r_{k-1}\}$ ,  $(r_k)$  is an arbitrary sequence decreasing to 0,  $W$  is a Wiener process, all members of the representation are independent processes and the series converges  $\mathbb{P}$ -a.s., uniformly on each bounded subinterval  $[0, \infty)$ .

In the following (see [Hypothesis 3.1](#)), with no loss of generality, we assume that

$$\sum_{k=1}^{\infty} \int_{I_k} y \nu(dy) = 0. \quad (4.8)$$

We also assume throughout that the Lévy process is a pure jump process, i.e.  $a = 0$  and  $Q = 0$  and that

$$\int_U |y|^m \nu(dy) < \infty, \quad \text{for all } m \in \mathbb{N}, \quad (4.9)$$

which leads to the representation

$$L(t) = \sum_{k=1}^{\infty} L_{I_k}(t) + L_{I_0}(t),$$

in view of assumptions (4.8) and (4.9).

Let  $L_A(t) = \int_0^t S(t-s) \sqrt{Q} dL(s)$ ,  $t \geq 0$ , be the Lévy Ornstein–Uhlenbeck process associated with  $S$ ,  $\sqrt{Q}$ ,  $L$ , assumed to exist and have a càdlàg version in  $B$  (the latter is satisfied if  $B$  is a Hilbert space  $K$  and  $S(t)$  is a contraction on  $K$ , see e.g. [63, p. 155], or  $S$  is analytic and  $L$  takes values in  $D((-A)^\alpha)$  for some  $\alpha > 0$ ; see, e.g. [63, p. 155]). Assume  $F$  is an operator on  $H$  (possibly nonlinear, nor everywhere defined) satisfying [Hypothesis 3.1](#).

An adapted  $B$ -valued process  $X$  is said to be a *càdlàg mild solution* to

$$\begin{cases} dX(t) = AX(t) dt + F(X(t)) dt + \sqrt{Q} dL(t) \\ X(0) = x \in D(F) \end{cases} \quad (4.10)$$

if it is càdlàg in  $B$  and satisfies,  $P$ -a.s., the equation  $X(t) = S(t)x + \int_0^t S(t-s)F(X(s)) ds + L_A(t)$ ,  $t \geq 0$ , with  $X(s) \in D(F)$  for  $s \geq 0$  [63, p. 182]. The formal definition of mild solution for the stochastic problem (4.10) is given below; next we recall the definition of stochastic convolution and we list some of its properties.



**Definition 4.6.** Let  $u^0 \in K$ . A predictable  $H$ -valued process  $u := (u(t))_{t \geq 0}$  is called a mild solution to the Cauchy problem (2.2) with initial condition  $u^0 \in D(F)$  if for arbitrary  $t \geq 0$  we have

$$u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon \int_0^t e^{(t-s)A}\sqrt{Q}dL(s), \quad \mathbb{P}\text{-a.s.}$$

$L_A(t) := \int_0^t e^{(t-s)A}\sqrt{Q}dL(s)$  is called a stochastic convolution and under our hypothesis it is a well defined mean square continuous  $\mathcal{F}_t$ -adapted process with values in  $B$  and càdlàg trajectories (see e.g., [63, Proposition 9.28, p. 163]).

The first integral on the right hand side is defined pathwise in the Bochner sense,  $\mathbb{P}$ -almost surely.

For further use, in the following we introduce some additional condition on the stochastic convolution:

**Hypothesis 4.7.** The stochastic convolution  $L_A(t)$ ,  $t \geq 0$  introduced in Definition 4.6, admits a  $K$ -valued version such that, for any  $T > 0$ , it satisfies the following estimate

$$\mathbb{E} \left( \sup_{t \in [0, T]} |L_A(t)|_K^m \right) \leq C_T \quad (4.11)$$

for every  $m \in \mathbb{N}$  and some positive constant  $C_T$  (possibly depending on  $T$ ).

**Example 4.8.** Let us give an example for the setting  $(H, B, K, L, A, Q)$  where  $L_A$  is well-defined and Hypothesis 4.7 is satisfied. This example is related to the application to the stochastic FitzHugh–Nagumo model which we discuss in Example 6.4. Let  $H, B, K$  be as in Example 3.2. Let  $A = \Delta$  be the Laplacian in  $L^2(\Lambda)$  with Neumann boundary conditions on the boundary  $\partial \Lambda$  of the bounded open subset  $\Lambda$  of  $\mathbb{R}^n$ . Let  $Q$  be a bounded trace class operator commuting with  $A$  and  $L$  be a Lévy process such that the corresponding measure  $\nu$  satisfies

$$\int_{L^2(\Lambda)} |x|_{W^{\beta, 2(2n+1)}}^m \nu(dx) < \infty \quad \text{for all } m \in \mathbb{N},$$

where  $W^{\beta, 2(2n+1)}$  is a fractional Sobolev space with given  $\beta > 0$ . Finally, let  $(A_{18}, D(A_{18}))$  denote the generator of the heat semigroup with Neumann boundary conditions on  $L^{18}(\Lambda)$ . By [32, Appendix]  $L_A(t) \in D(-A_{2(2n+1)}^\gamma)$ ,  $\gamma > 0$ ; in particular  $L_A(t) \in K$ ,  $L_A$  being in addition a Lévy process. This implies the bound in Hypothesis 4.7.

The next result concerns the existence and uniqueness of solutions for the stochastically perturbed problem. Moreover, we shall use Hypothesis 4.7 above concerning the Ornstein–Uhlenbeck process associated with  $e^{tA}$ ,  $\sqrt{Q}$  and  $L$  in order to prove a useful estimate on the solution.

**Theorem 4.9.** Assume that  $A$  and  $F$  satisfy Hypothesis 3.1. Assume that  $A$  and  $Q$  satisfy Hypothesis 4.7. Then there exists a unique càdlàg mild solution of (4.10) for any  $x \in B$ . For each  $x \in H$  there exists a unique generalized solution for (4.10) (in the sense that  $\exists (X_n)_{n \in \mathbb{N}}$ ,  $X_n \in B$ , unique càdlàg mild solutions of (4.10) with  $X_n(0) = x$  s.t.  $|X_n(t) - X(t)|_H \rightarrow 0$  uniformly on each bounded interval). Moreover (4.10) defines Feller families on  $B$  and on  $H$  (in the sense that the Markov semigroup  $P(t)$  associated with  $X(t)$  maps, for any  $t \geq 0$ ,  $C_b(H)$  into  $C_b(H)$  and  $C_b(B)$  into  $C_b(B)$ ).

Moreover, the solution  $X$  to (4.10) belongs to the space  $\mathcal{L}^p(\Omega; C([0, T]; H))$ , i.e., is such that

$$\mathbb{E} \left( \sup_{t \in [0, T]} |X(t)|_H^p \right) < +\infty, \quad (4.12)$$

for any  $p \in [2, \infty)$ .

**Proof.** The first part of the result is proven in [63, Theorem 10.14]. We only have to prove the estimate (4.12). Let  $z(t) := X(t) - L_A(t)$ ; then it is not difficult to show that  $z(t)$  is the unique solution of the following deterministic equation:

$$\begin{cases} z'(t) = Az(t) + F(z(t) + L_A(t)) \\ z(0) = u^0 \end{cases}$$

with  $z'(t) := \frac{d}{dt} z(t)$ .

With no loss of generality (because of inclusion results for  $L^p$ -spaces with respect to bounded measures) we can assume that  $p = 2a, a \in \mathbb{N}$ . Now combining condition (i) with (ii) in Hypothesis 3.1 and recalling Newton's binomial formula we have:

$$\begin{aligned} \frac{d}{dt} |z(t)|_H^{2a} &= 2a \langle z'(t), z(t) \rangle |z(t)|_H^{2a-2} \\ &= 2a \langle Az(t) + F(z(t) + L_A(t)), z(t) \rangle |z(t)|_H^{2a-2} \\ &\leq -2a\omega |z(t)|_H^{2a} + 2a \langle F(z(t) + L_A(t)), z(t) \rangle |z(t)|_H^{2a-2} \\ &\leq -2a(\omega - \eta) |z(t)|_H^{2a} + 2a |F(L_A(t))|_H |z(t)|_H^{2a-1} \\ &\leq -2a(\omega - \eta) |z(t)|_H^{2a} + 2a \frac{C_a}{\xi} |F(L_A(t))|_H^{2a} + C_a 2a\xi |z(t)|_H^{2a}, \end{aligned} \quad (4.13)$$

for some constant  $C_a > 0$  and a sufficiently small  $\xi > 0$  such that  $-2a(\omega - \eta) + 2a\xi C_a < 0$ . Applying the previous inequality and Gronwall's lemma we get:

$$|z(t)|_H^{2a} \leq e^{(-2a(\omega - \eta) + \xi C_a 2a)t} |u^0|_H^{2a} + \frac{2aC_a}{\xi} \int_0^t e^{-2a(\omega - \eta)(t-s)} |F(L_A(s))|_H^{2a} ds.$$

Then there exists a positive constant  $C$  such that:

$$\begin{aligned} |X(t)|_H^{2a} &\leq C \left( e^{(-2a(\omega - \eta) + \xi C_a 2a)t} |u^0|_H^{2a} + 2a \right. \\ &\quad \times \left. \int_0^t e^{-2a(\omega - \eta)(t-s)} |F(L_A(s))|_H^{2a} ds + |L_A(t)|_H^{2a} \right). \end{aligned} \quad (4.14)$$

Since by condition (iii) in Hypothesis 3.1, the restriction of  $F$  to  $K$  has (at most) polynomial growth at infinity in the  $K$ -norm and, by the assumption on  $L_A(t)$  made in Hypothesis 4.7,  $L_A$  takes value in  $K$ , for any  $a \in \mathbb{N}$  we have:

$$|F(L_A(t))|_H^{2a} \leq C_{a,m} (1 + |L_A(t)|_K^m)^{2a} \leq C_{a,m} (1 + |L_A(t)|_K^{2am}),$$

for some positive constant  $C_{a,m}$  depending on  $m$  and  $a$ . Moreover, we observe that, again by Hypothesis 4.7, it holds that

$$\mathbb{E} \left( \sup_{t \in [0, T]} |L_A(t)|_K^{2am} \right) \leq C'_{a,m,T},$$

where  $C'_{a,m,T}$  is again a positive constant depending on  $m, a$  and  $T$ ; hence

$$\begin{aligned} & \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t e^{-2a(\omega - \eta)(t-s)} |F(L_A(s))|_H^{2a} ds \right] \\ & \leq \tilde{C} \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t e^{-2a(\omega - \eta)(t-s)} (1 + |L_A(t)|_K^{2am}) ds \right] \\ & \leq \tilde{C} \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t e^{-2a(\omega - \eta)(t-s)} ds + C'_{a,m} \int_0^t e^{-2a(\omega - \eta)} ds \right] \leq \bar{C}, \end{aligned} \quad (4.15)$$

for some positive constants  $\tilde{C}, \bar{C}$  depending on  $a, m$  and  $T$ . Consequently, putting together inequalities (4.14), (4.15), we obtain

$$\mathbb{E} \left( \sup_{t \in [0, T]} |X(t)|_H^{2a} \right) \leq C |u^0|_H^{2a} + \bar{\bar{C}},$$

for some positive constant  $\bar{\bar{C}}$ , so that the proposition follows.  $\square$

## 5. Properties of the nonlinear term $F$ and Taylor expansions

In this section we study the nonlinear term  $F$  in order to write its Taylor expansion around the solution  $\phi(t)$  of (4.5) with respect to an increment given in terms of powers of  $\varepsilon$ . In order to do that we recall some basic properties of Fréchet differentiable functions.

Let  $U$  and  $V$  be two real Banach spaces. For a mapping  $F : U \rightarrow V$  the Gâteaux differential at  $u \in U$  in the direction  $h \in U$  is defined as

$$\nabla F(u)[h] = \lim_{s \rightarrow 0} \frac{F(u + sh) - F(u)}{s},$$

whenever the limit exists in the topology of  $V$  (see for example [52, p. 12]).

We notice that if  $\nabla F(u)[h]$  exists in a neighborhood of  $u_0 \in U$  and is continuous in  $u$  at  $u_0$  and also continuous in  $h$  at  $h = 0$ , then  $\nabla F(u)[h]$  is linear in  $h$  (see for instance [52, Problem 1.61, p. 15]). If  $\nabla F(u_0)[h]$  has this property for all  $u_0 \in U_0 \subseteq U$  and all  $h \in U$  we shall say that  $F$  belongs to the space  $G^1(U_0; V)$ . If  $F$  is continuous from  $U$  to  $V$  and  $F \in G^1(U_0; V)$  and one has  $F(u + h) = F(u) + \nabla F(u)[h] + R(u, h)$ , for any  $u \in U_0$  with:

$$\lim_{|h|_U \rightarrow 0} \frac{|R(u, h)|_V}{|h|_U} = 0 \quad (5.1)$$

with  $|\cdot|_V$  and  $|\cdot|_U$  denoting respectively the norm in  $V$  and  $U$ , then the map  $h \rightarrow \nabla F(u)[h]$  is a bounded linear operator from  $U_0$  to  $V$ , and  $\nabla F(u)[h]$  is, by definition, the unique Fréchet differential of  $F$  at  $u \in U_0$  with increment  $h \in U$ . The function  $R(u, h)$  is called the remainder of this Fréchet differential, while the operator sending  $h$  into  $\nabla F(u)[h]$  is then called the Fréchet derivative of  $F$  at  $u$  and is usually denoted by  $F'(u)$  (see for instance [52, pp. 15–16, Problem 1.6.2 and Lemma 1.6.3]). We have then  $\nabla F(u)[h] = F'(u) \cdot h$ , with the symbol  $\cdot$  denoting the action of the linear bounded operator  $F'(u)$  on  $h$ .

The mapping  $F'(u)$  is also called the gradient of  $F$  at  $u$  (see for example [52, p. 15]) and it coincides with the Gâteaux derivative of  $F$  at  $u$ . We shall denote by  $\mathcal{F}^{(1)}(U_0, V)$  the subset

of  $G^1(U_0, V)$  such that the Fréchet derivative exists at any point of  $U_0$ . Similarly we introduce the Fréchet derivative  $F''(u)$  of  $F'$  at  $u \in U$ . This is a bounded linear map from a subset  $D(F')$  of  $U$  into  $L(U, V)$  ( $L(U, V)$  being the space of bounded linear operators from  $U$  to  $V$ ). One has thus  $F'' \in L(U, L(U, V))$ . If we choose  $h, k \in U$  then  $F''(u) \cdot k \in L(U, V)$  and  $(F''(u) \cdot k) \cdot h \in V$ . The latter is also written  $F''(u) h k$  or  $F''(u)[h, k]$ . The mapping  $F''(u)[h, k]$  is bilinear in  $h, k$ , for any given  $u \in D(F'')$  and it can be identified with the Gâteaux differential  $\nabla^{(2)} F(u)[h, k]$  of  $\nabla F(u)[h]$  in the direction  $k$ , the latter looked upon as a map from  $U$  to  $L(U, V)$ . Similarly one defines the  $j$ -th Fréchet derivative  $F^{(j)}(u)$  and the  $j$ -th Gâteaux derivative  $\nabla F^{(j)}(u)[h_1, \dots, h_j]$ . The function  $F^{(j)}(u)$  acts  $j$ -linearly on  $h_1, \dots, h_j$  with  $h_i \in U$  for any  $i = 1, \dots, j$ . Let  $U_0$  be an open subset of  $U$  and consider the space  $\mathcal{F}^{(j)}(U_0, V)$  of maps  $F$  from  $U$  to  $V$  such that  $F^{(j)}(u)$  exists at all  $u \in U_0$  and is uniformly continuous on  $U_0$ . The following Taylor formula holds for any  $u, h \in U$  for which  $F(h)$  and  $F(u+h)$  are well defined (i.e.  $h$  and  $u+h$  are elements of  $D(F)$ ), and  $j = 1, \dots, n+1$  with  $u \in \cap_{j=1}^n \mathcal{F}^{(j)}(U_0, V)$ :

$$\begin{aligned} F(u+h) &= F(u) + \nabla F(u)[h] + \frac{1}{2} \nabla^{(2)} F(u)[h, h] + \dots \\ &\quad + \frac{1}{n!} \nabla^{(n)} F(u) \underbrace{[h, \dots, h]}_{n\text{-terms}} + R^{(n)}(u; h), \end{aligned} \quad (5.2)$$

where  $|R^{(n)}(u; h)|_U \leq C_{u,n} \cdot |h|_U^n$  for some constant  $C_{u,n}$  depending only on  $u$  and  $n$  (see for example [51, Theorem X.1.2]).

Now let us consider the case  $U = H$ , with  $H$  being the same Hilbert space appearing in problem (2.1). Let  $F$  be as in Hypothesis 3.1 and set  $U_0 = D(F)$ . Let us define for  $0 < \varepsilon \leq 1$  the function  $h(t)$ ,  $t \geq 0$ :

$$h(t) = \sum_{k=1}^n \varepsilon^k u_k(t) + r^{(n)}(t; \varepsilon),$$

where the functions  $u_k(t)$ ,  $k = 1, \dots, n$  and  $r^{(n)}(t; \varepsilon)$  are  $p$ -mean integrable continuous stochastic processes with values in  $H$ , defined on the whole interval  $[0, T]$  for  $p \in [2, \infty)$ . Moreover we suppose  $r^{(n)}(\cdot; \varepsilon) = o(\varepsilon^n)$ , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{t \in [0, T]} \frac{|r^{(n)}(t; \varepsilon)|^p}{\varepsilon^n} \right] = 0, \quad \text{for any } T > 0.$$

Let  $\phi$  be a  $p$ -mean integrable continuous stochastic process with values in the Banach space  $K$ . Then using the above Taylor formula we have

$$\begin{aligned} F(\phi(t) + h(t)) &= F(\phi(t)) + \nabla F(\phi(t))[h(t)] + \frac{1}{2} \nabla^{(2)} F[h(t), h(t)] + \dots \\ &\quad + \frac{1}{n!} \nabla^{(n)} F(u) \underbrace{[h(t), \dots, h(t)]}_{n\text{-terms}} + R^{(n)}(\phi(t); h(t)), \end{aligned} \quad (5.3)$$

and, recalling that for any  $j = 1, \dots, n$ ,  $\nabla^{(j)} F(\phi(t))$  is multilinear, we have

$$\frac{1}{j!} \nabla^{(j)} F(\phi(t)) \underbrace{[h(t), \dots, h(t)]}_{j\text{-terms}}$$

$$= \frac{1}{j!} \sum_{k_1+\dots+k_j=j}^{nj} \varepsilon^{k_1+\dots+k_j} \nabla^{(j)} F(\phi(t)) [u_{k_1}(t), \dots, u_{k_j}(t)] + \mathbf{o}_j(\varepsilon^{nj}) \quad (5.4)$$

where  $\mathbf{o}_j(\varepsilon^{nj})$  is the contribution to the right member of the above equality coming from the term  $r^{(n)}(t; \varepsilon)$  and satisfies the estimate

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_{t \in [0, T]} \frac{|\mathbf{o}_j(\varepsilon^{nj})|^p}{\varepsilon^{nj}} \right] = 0, \quad \text{for any } T > 0.$$

We notice that any derivative appearing in the member on the right hand side of (5.4) is multiplied by the parameter  $\varepsilon$  raised to a power between  $j$  and  $nj$ .

Taking into account the above equality we can rewrite (5.3) as

$$\begin{aligned} F(\phi(t) + h(t)) &= F(\phi(t)) + \sum_{k=1}^n \varepsilon^k \nabla F(\phi(t)) [u_k(t)] + \sum_{j_1+j_2=2}^n \frac{\varepsilon^{j_1+j_2}}{2!} \nabla^{(2)} F(\phi(t)) \\ &\quad \times [u_{j_1}(t), u_{j_2}(t)] + \dots + \sum_{j_1+\dots+j_k=k}^n \frac{\varepsilon^{j_1+\dots+j_k}}{k!} \nabla^{(k)} \\ &\quad \times F(\phi(t)) [u_{j_1}(t), \dots, u_{j_k}(t)] \\ &\quad + \dots + \frac{\varepsilon^n}{n!} \nabla^{(n)} F(\phi(t)) [u_1(t), \dots, u_1(t)] + R_1^{(n)}(\phi(t); h(t), \varepsilon), \end{aligned} \quad (5.5)$$

where the quantity  $R_1^{(n)}(\phi(t); h(t), \varepsilon)$  is given in terms of the derivatives of  $F$  with the parameter  $\varepsilon$  raised to powers greater than  $n$ , in terms of the  $n$ -th remainder  $R^{(n)}(\phi(t); h(t))$  in the Taylor expansion of the map  $F$  (as stated in Eq. (5.2)) and in terms of the remainders  $\mathbf{o}_j(\varepsilon^{nj})$ ,  $j = 2, \dots, n$  introduced in (5.4). Namely, we have:

$$\begin{aligned} R_1^{(n)}(\phi(t); h(t), \varepsilon) &= \sum_{j=2}^n \sum_{i_1+\dots+i_j=n+1}^{nj} \varepsilon^{i_1+\dots+i_j} \frac{1}{j!} \nabla^{(j)} F(\phi(t)) [u_{i_1}(t), \dots, u_{i_j}(t)] \\ &\quad + \sum_{j=2}^n \mathbf{o}_j(\varepsilon^{nj}) + R^{(n)}(\phi(t); h(t)), \end{aligned} \quad (5.6)$$

$R^{(n)}(\phi(t); h(t))$  being as in (5.2) (with  $u$  replaced by  $\phi$ ). In this way Eq. (5.5) can be rearranged as

$$\begin{aligned} F(\phi(t) + h(t)) &= F(\phi(t)) + \sum_{j=2}^n \varepsilon^j \left( \sum_{i_1+\dots+i_j=j}^n \frac{1}{j!} \nabla^{(j)} F(\phi(t)) [u_{i_1}(t), \dots, u_{i_j}(t)] \right) \\ &\quad + R_1^{(n)}(\phi(t); h(t), \varepsilon). \end{aligned} \quad (5.7)$$

**Lemma 5.1.** *Let  $R_1^{(n)}$  be as in formula (5.6). Then for all  $p \in [2, \infty)$  and  $T > 0$  there exists a constant  $C > 0$ , depending on  $|\phi|_K, \dots, |u_n|_H, \nabla^{(1)} F, \dots, \nabla^{(n)} F, p, n$ , such that:*

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \right] \leq C \varepsilon^{p(n+1)}$$

for all  $0 < \varepsilon \leq 1$ .

**Proof.** First of all we notice that

$$\sum_{j=2}^n \mathbf{o}_j(\varepsilon^{nj}) = \mathbf{O}(\varepsilon^{2n}),$$

meaning that

$$\left| \sum_{j=2}^n \mathbf{o}(\varepsilon^{nj}) \right| \leq C_n \varepsilon^{2n}, \quad \varepsilon \rightarrow 0, \quad (5.8)$$

for some constant  $C_n > 0$ . Now since:

$$\begin{aligned} R_1^{(n)}(\phi(t); h(t), \varepsilon) &= \sum_{j=2}^n \sum_{i_1+\dots+i_j=n+1}^{nj} \varepsilon^{i_1+\dots+i_j} \frac{1}{j!} \nabla^{(j)} F(\phi(t)) [u_{i_1}(t), \dots, u_{i_j}(t)] \\ &\quad + \sum_{j=2}^n \mathbf{o}_j(\varepsilon^{nj}) + R^{(n)}(\phi(t); h(t)), \end{aligned}$$

using the estimate given in condition (iii.b) in [Hypothesis 3.1](#) and (5.8), for  $\varepsilon \in (0, 1]$  we have

$$\begin{aligned} &|R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \\ &\leq C_{n,p}^1 \varepsilon^{(n+1)p} \left[ \left( \max_{j=1,\dots,n} \|\nabla^{(j)} F(\phi(t))\|_{L^j(K)} \right)^p \left( \sum_{i=1}^n |u_i(t)|_H^p \right) \right] \\ &\quad + (\mathbf{O}(\varepsilon^{2n}))^p + C_{n,p}^2 |R^{(n)}(\phi(t); h(t))|_H^p \\ &\leq C_{n,p}^{(1)} \varepsilon^{(n+1)p} \max_{j=1,\dots,n} \left[ \gamma_j^p (1 + |\phi(t)|_K^{m-j})^p \right] \left( \sum_{i=1}^n |u_i(t)|_H^p \right) \\ &\quad + C_n \varepsilon^{2np} + C_{n,p}^{(2)} |R^{(n)}(\phi(t); h(t))|_H^p \\ &\leq \tilde{C}_n \varepsilon^{(n+1)p} + C_{n,p}^{(2)} |R^{(n)}(\phi(t); h(t))|_H^p, \end{aligned} \quad (5.9)$$

where  $C_{n,p}^1, C_{n,p}^{(1)}, C_{n,p}^{(2)}$  are constants depending only on  $n, p$  and the constant  $C_n$  in (5.8) while  $\tilde{C}_n$  is a suitable positive constant depending on  $p, n, \max_{j=1,\dots,n} \left[ \gamma_j^p (1 + |\phi(t)|_K^{m-j})^p \right]$  ( $\gamma_i$  being the constants appearing in [Hypothesis 3.1](#), condition (iii)) and  $|u_i(t)|_H^p, i = 1, \dots, n$ . We notice that the above inequality follows by recalling that the deterministic function  $\phi(t)$  is bounded (in the  $H$ -norm) (see [Proposition 4.2](#)).

Now by the bound on  $R^{(n)}$  in the Eq. (5.2) we have that

$$|R^{(n)}(\phi(t); h(t))|_H^p \leq \hat{C}_n |h(t)|_H^{(n+1)p}$$

with  $\hat{C}_n$  depending on  $\phi(t)$  and  $n$  but independent of  $h(t)$ . Since  $h(t) = \sum_{k=1}^n \varepsilon^k u_k(t) + r^{(n)}(t; \varepsilon)$  with  $|r^{(n)}(t; \varepsilon)| \leq \tilde{C}_n \varepsilon^{n+1}$  for some  $\tilde{C}_n$ , then:

$$|R^{(n)}(\phi(t); h(t))|_H^p \leq \varepsilon^{(n+1)p} \hat{C}_{n,p} (|u_1(t)|_H, \dots, |u_n(t)|_H) \quad (5.10)$$

with  $\hat{C}_{n,p} = \hat{C}_{n,p}(|u_1(t)|_H, \dots, |u_n(t)|_H)$  independent of  $\varepsilon$ .

Hence by (5.9) and (5.10) we have that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \right] \leq C'_n \varepsilon^{n+1},$$

where  $C'_n := C'_n(p, \nabla^{(1)}F, \dots, \nabla^{(n)}F, |\phi|_H, \dots, |u_n|_H)$  is independent of  $\varepsilon$ . This gives the lemma, with  $C = C'_n$ .  $\square$

As we said before, we want to expand the solution of the Eq. (2.2) around  $\phi(t)$ , that is we want to write  $u(t)$  as:

$$u(t) = \phi(t) + \varepsilon u_1(t) + \dots + \varepsilon^n u_n(t) + R_n(t, \varepsilon), \quad (5.11)$$

(with the term  $R_n(t, \varepsilon) = \mathbf{O}(\varepsilon^{n+1})$ , for any  $t \geq 0$ ), where the processes  $(u_i(t))_{t \geq 0}$ ,  $i = 1, \dots, n$  can be found by using the Taylor expansion of  $F$  around  $\phi(t)$  and *matching terms* in the Eq. (2.2) for  $u$ . Given predictable  $H$ -valued stochastic processes  $w(t), v_1(t), \dots, v_n(t)$  let us use the notation:

$$\Phi_k(w(t)) [v_1(t), \dots, v_k(t)] := \sum_{j=2}^k \sum_{i_1+\dots+i_j=k} \nabla^{(j)} F(w(t)) [v_{i_1}(t), \dots, v_{i_j}(t)], \quad (5.12)$$

with  $i_1, \dots, i_j$ , running from 0 to  $k$  and the given restriction  $i_1 + \dots + i_n = k$ . With the above notation the processes  $u_1(t), \dots, u_n(t)$  occurring in (5.11) satisfy the following equations:

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dL(t), \\ u_1(0) = 0, \end{cases}$$

and

$$\begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt, \\ u_k(0) = 0, \end{cases} \quad (5.13)$$

with

$$\Phi_k(t) := \Phi_k(\phi(t)) [u_1(t), \dots, u_{k-1}(t)], \quad k \in \mathbb{N}, \quad n \geq k \geq 2. \quad (5.14)$$

Notice that while  $u_1(t)$  is the solution of a linear stochastic differential equation (with time dependent drift operator  $A + \nabla F(\phi(t))$ ), the processes  $u_2, \dots, u_n$  are solutions of non-homogenous differential equations with random coefficients whose meaning is given below.

**Definition 5.2.** Let  $2 \leq k \leq n$ . Then a predictable  $H$ -valued stochastic process  $u_k = u_k(t)$ ,  $t \geq 0$  is a solution of the problem (2.4) (i.e. (5.13)) if almost surely it satisfies the following integral equation

$$u_k(t) = \int_0^t e^{(t-s)A} \nabla F(\phi(s)) [u_k(s)] ds + \int_0^t \Phi_k(s) ds, \quad t \geq 0, \quad 2 \leq k \leq n,$$

with  $\phi$  as in Proposition 4.2 and  $\Phi_k$  as in (5.12) and (5.14).

In the following result we estimate the norm of  $\Phi_k$  in  $H$  by means of the norms of the Gâteaux derivatives of  $F$  and the norms of  $v_j(t)$ ,  $j = 1, \dots, k-1$ , where  $v_j(t)$  are  $H$ -valued stochastic processes.

**Lemma 5.3.** Let us fix  $2 \leq k \leq n$ ; let  $w(t)$  and  $v_1(t), \dots, v_{k-1}(t)$  be respectively a  $K$ -valued process and  $H$ -valued stochastic processes. Then  $\Phi_k(w(t)) [v_1(t), \dots, v_{k-1}(t)]$  as in (5.12) satisfies the following inequality

$$\|\Phi_k(w(t)) [v_1(t), \dots, v_{k-1}(t)]\|_H \leq C(1 + |w(t)|_K^{m-2})k^2 \left( k + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right),$$

where  $C$  is some positive constants depending on  $k$  and  $\gamma_j, j = 2, \dots, k$  are the constants introduced in Hypothesis 3.1.

**Proof.** We have

$$\begin{aligned} & \|\Phi_k(w(t)) [v_1(t), \dots, v_{k-1}(t)]\|_H \\ &= \left\| \sum_{j=2}^k \sum_{i_1+\dots+i_j=k} \frac{\nabla^{(j)} F(w(t)) [v_{i_1}(t), \dots, v_{i_j}(t)]}{j!} \right\|_H \\ &\leq \sum_{j=2}^k \sum_{i_1+\dots+i_j=k} \left\| \frac{\nabla^{(j)} F(w(t)) [v_{i_1}(t), \dots, v_{i_j}(t)]}{j!} \right\|_H \end{aligned} \quad (5.15)$$

and using the assumption (iii. b) in Hypothesis 3.1, we get

$$\begin{aligned} \|\Phi_k(t)\|_H &\leq \sum_{j=2}^k \sum_{i_1+\dots+i_j=k} \frac{1}{j!} \|\nabla F^{(j)}(w(t))\|_{L^j(H)} \prod_{l=1}^j |v_{i_l}(t)|_H \\ &\leq \sum_{j=2}^k \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} \sum_{i_1+\dots+i_j=k} \sum_{l=1}^j |v_{i_l}(t)|_H^j \\ &\leq \sum_{j=2}^k \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} \sum_{i_1+\dots+i_j=k} \left( j + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right) \\ &\leq \sum_{j=2}^k \frac{1}{j!} \gamma_j (1 + |w(t)|_K)^{m-j} k^2 \left( k + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right) \\ &\leq C(1 + |w(t)|_K^{m-2})k^2 \left( k + \sum_{l=1}^{k-1} |v_l(t)|_H^{k-1} \right), \end{aligned} \quad (5.16)$$

for some positive constant  $C$ , from which the assertion in Lemma 5.3 follows.  $\square$

**Remark 5.4.** Notice that by Lemma 5.3, if  $v_1, \dots, v_{k-1}$  are  $p$ -mean ( $p \in [2, \infty)$ ) integrable continuous stochastic processes then the same holds for  $\Phi_k$ .

## 6. Main results

**Proposition 6.1.** Under Hypothesis 3.1 the following stochastic differential equation:

$$\begin{cases} du_1(t) = [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dL(t), & t \in [0, +\infty) \\ u_1(0) = 0, \end{cases} \quad (6.1)$$



has, with  $\phi$  as in [Proposition 4.2](#), a unique mild solution satisfying, for any  $p \geq 2$ , the following estimate:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u_1(t)|_H^p \right] < +\infty, \quad \text{for any } T > 0. \quad (6.2)$$

**Proof.** First we show the uniqueness. Let us suppose that  $w_1(t)$  and  $w_2(t)$  are two solutions of (6.1). Then by Itô's formula we have:

$$\begin{aligned} d|w_1(t) - w_2(t)|_H^2 &= \langle A(w_1(t) - w_2(t)), w_1(t) - w_2(t) \rangle dt \\ &\quad + \langle \nabla F(\phi(t))[w_1(t) - w_2(t)], w_1(t) - w_2(t) \rangle dt, \end{aligned}$$

so that, by the dissipativity condition on  $A$  and the estimate on  $\nabla F$  in [Hypothesis 3.1](#), (iii), we have

$$d|w_1(t) - w_2(t)|_H^2 \leq -\omega |w_1(t) - w_2(t)|_H^2 + \gamma_1(1 + |\phi|_K^{m-1}) |w_1(t) - w_2(t)|_H^2.$$

Now uniqueness follows by applying Gronwall's lemma.

As far as the existence is concerned, we proceed by a fixed point argument. We introduce the mapping  $\Gamma$  from  $\mathcal{L}^p(\Omega; C([0, T]; H))$  into itself defined by

$$\Gamma(w(t)) := \int_0^t e^{(t-s)A} \nabla F(\phi(s))[w(s)] ds + L_A(t).$$

We are going to prove that there exists  $\tilde{T} > 0$  such that  $\Gamma$  is a contraction on  $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$ . In fact, for any  $v, w \in \mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$  we have, for any  $0 \leq t \leq \tilde{T}$ :

$$\begin{aligned} \|\Gamma(v(t)) - \Gamma(w(t))\|^p &= \mathbb{E} \left[ \sup_{t \in [0, \tilde{T}]} \left| \int_0^t e^{(t-s)A} \nabla F(\phi(s))[v(s) - w(s)] ds \right|_H^p \right] \\ &\leq \mathbb{E} \left[ \sup_{t \in [0, \tilde{T}]} \int_0^t \|e^{(t-s)A}\|_{L(H)^p} \|\nabla F(\phi(s))[v(s) - w(s)]\|_H^p ds \right] \\ &\leq \mathbb{E} \left[ \sup_{s \in [0, \tilde{T}]} \|\nabla F(\phi(s))[v(s) - w(s)]\|_H^p \right] \int_0^{\tilde{T}} \|e^{(\tilde{T}-s')A}\|_{L(H)^p} ds' \\ &\leq \mathbb{E} \left[ \sup_{s \in [0, \tilde{T}]} |v(s) - w(s)|_H^p \right] \gamma_1^p \left( 1 + |\phi(s)|_K^{m-1} \right)^p \frac{1}{\omega p} \left( 1 - e^{-\omega p \tilde{T}} \right) \\ &\leq \gamma_1^p (1 + |u^0|_K^{m-1})^p \|v - w\|^p \frac{1}{\omega p} \left( 1 - e^{-\omega p \tilde{T}} \right), \end{aligned}$$

where we used condition (iii) in [Hypothesis 3.1](#) for the third inequality and [Proposition 4.2](#) for the last inequality. Then if  $\tilde{T}$  is sufficiently small (depending on  $\omega, p, \gamma_1, \phi$ ), we see that  $\Gamma$  is a contraction on  $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$ .

By considering the map  $\Gamma$  on intervals  $[0, \tilde{T}]$ ,  $[\tilde{T}, 2\tilde{T}]$ ,  $\dots$ ,  $[(N-1)\tilde{T}, T]$ ,  $\tilde{T} \equiv T/N$ ,  $N \in \mathbb{N}$ , we have that  $\Gamma$  is a contraction on  $\mathcal{L}^p(\Omega; C([0, T]; H))$  and hence we have the existence and uniqueness of the solution for the Eq. (6.1) in the space  $\mathcal{L}^p(\Omega; C([0, T]; H))$  for any  $p \in [2, \infty)$ .

Let us now consider the estimate (6.2). We write the Itô formula for the function  $|\cdot|_H^{2a}$ , applied to the process  $X$ . To this end, we recall the expressions for the first and second derivatives of the function  $H(x) := |x|^{2a}$ ,  $a \in \mathbb{N}$ .

We have

$$\begin{aligned}\nabla H(x) &= 2a|x|^{2(a-1)}x \\ \frac{1}{2}\text{Tr}(Q\nabla H^2(x)) &= a\text{Tr}(Q)|x|^{2(a-1)} + (a-1)a|x|^{2(a-2)}|\sqrt{Q}x|^2.\end{aligned}$$

Moreover (see, [23,36]), we recall that Itô formula implies:

$$dH(u(t)) = \nabla H(u(t-))du(t) + \frac{1}{2}\text{Tr}(Q\nabla H^2(u(t-)))du + d[u](t).$$

Although our computations are only formal, they can be justified using an approximation argument. By condition (iii) in Hypothesis 3.1 we have for all points in the probability space and  $p = 2a$  with  $a \in \mathbb{N}$ :

$$\begin{aligned}d|u_1(t)|^{2a} &= 2a\langle u_1(t-), du_1(t) \rangle_H |u_1(t)|^{2a-2} + a\text{Tr}(Q)|u_1(t-)|^{2(a-1)}dt \\ &\quad + (a-1)a|u_1(t-)|^{2(a-2)}|\sqrt{Q}u_1(t)|^2dt + d[u_1(t)](t).\end{aligned}\quad (6.3)$$

By the dissipativity of  $A + F$ , the first term in the above inequality is estimated by

$$\begin{aligned}\langle u_1(t-), du_1(t) \rangle_H |u_1(t)|^{2a-2} &= \langle Au_1(t), u_1(t) \rangle |u_1(t)|_H^{2a-2} + \langle \nabla F(\phi(t))[u_1(t)], \\ &\quad u_1(t) \rangle |u_1(t)|_H^{2a-2} + \langle \sqrt{Q}dL(t-), u_1(t) \rangle |u_1(t)|_H^{2a-2} \\ &\leq -\omega|u_1(t)|_H^{2a} + 2a\gamma(1 + |u^0|_K^{m-1})|u_1(t)|_H^{2a} \\ &\quad + 2a\langle \sqrt{Q}dL(t), u_1(t) \rangle |u_1(t)|_H^{2a-1} \\ &\leq -\tilde{\omega}|u_1(t)|_H^{2a} + \langle \sqrt{Q}L(t), u_1(t) \rangle |u_1(t)|_H^{2a-1},\end{aligned}\quad (6.4)$$

where  $\tilde{\omega} := \omega - \gamma(1 + |u^0|_H)$ .

Moreover, the second and third term in (6.3) can be estimated in the following way:

$$\begin{aligned}a\text{Tr}(Q)|u_1(t)|^{2(a-1)} + (a-1)a|u_1(t-)|^{2(a-2)}|\sqrt{Q}u_1(t)|^2 \\ \leq C_a \left( \epsilon \text{Tr}^{2a}(Q) + \frac{1}{\epsilon}|u_1(t)|^{2a} \right),\end{aligned}\quad (6.5)$$

for any  $\epsilon > 0$ , where we used the elementary inequality  $ab^{2(a-1)} \leq C_a(\epsilon a^{2a} + \frac{1}{\epsilon}b^{2a})$ , with  $C_a$  being a suitable positive constant. Therefore

$$\begin{aligned}|u_1(t)|_H^{2a} &\leq -2a \left( \tilde{\omega} - \frac{C_a}{\epsilon} \right) \int_0^t |u_1(s)|_H^{2a} ds + 2a \int_0^t \langle \sqrt{Q}dL(s), u_1(s) \rangle |u_1(s)|_H^{2a-1} \\ &\quad + C_a \epsilon \text{Tr}(Q)^{2a} T + \int_0^t \text{Tr } Q d|L|(s)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} &\leq - \left( 2a\tilde{\omega}T - \frac{C_a}{\epsilon} \right) \mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} \\ &\quad + 2a \mathbb{E} \sup_{t \leq T} \left| \int_0^t \langle \sqrt{Q}dL(s), u_1(s) \rangle |u_1(s)|_H^{2a-1} ds \right| \\ &\quad + C_a \epsilon T + T \int_H \text{Tr } Q |x|^2 \nu(dx),\end{aligned}$$

where we used the relation

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} [u_1](t) &\leq \mathbb{E} \int_0^T \text{Tr}(Q) d[L](t) = \mathbb{E} \int_0^T \text{Tr}(Q) d\langle L \rangle(t) \\ &= T \int_H \text{Tr}(Q) |x|^2 \nu(dx).\end{aligned}\quad (6.6)$$

By the Burkholder–Davis–Gundy inequality, see.e.g., ([63, p. 37], [41,48]) applied to

$$M(t) := \int_0^t \langle \sqrt{Q} dL(s), u_1(s) \rangle |u_1(s)|_H^{2a-1},$$

there exists a constant  $c_1$  such that

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} \left| \int_0^t \langle \sqrt{Q} dL(s), u_1(s-) \rangle |u_1(s)|_H^{2a-1} \right| \\ \leq c_1 \mathbb{E} \left( \left[ \int_0^t \langle \sqrt{Q} dL(s), u_1(s-) \rangle |u_1(s)|_H^{2a-1} \right]^2 (T) \right)^{1/2} \\ \leq c_1 \mathbb{E} \left( \sup_{t \leq T} |u_1(t)|^{2a} \int_0^T \text{Tr}(Q) d[L](s) \right)^{1/2} \\ \leq c_1 \mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} + \frac{c_1 T}{4\epsilon} \int_H \text{Tr} Q |x|^2 \nu(dx),\end{aligned}$$

where we used the elementary inequality  $ab \leq \epsilon a^2 + (1/4\epsilon)b^2$ ,  $\epsilon > 0$ . Collecting the above estimates we obtain

$$\begin{aligned}\mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} &\leq - \left( 2a\tilde{\omega} - \frac{c_a}{\epsilon} \right) T \mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} + 2c_1 \epsilon \mathbb{E} \sup_{t \leq T} |u_1(t)|_H^{2a} \\ &\quad + \left( \frac{c_1}{2\epsilon} + 1 \right) T \int_H \text{Tr} Q |x|^2 \mu(dx) + C_a \epsilon T,\end{aligned}$$

Hence

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u_1(t)|_H^{2a} \right] \leq C'_{a,T} e^{-2a(\tilde{\omega} - c_a/\epsilon)T} < C_{a,T},$$

where  $C_{a,T}$  is a positive constant and (6.2) follows.  $\square$

**Theorem 6.2.** Let us fix  $2 \leq k \leq n$ , assume that *Hypothesis 3.1* holds, and let  $u_1$  be the solution of the problem (2.3). Suppose moreover that  $u_j$  is the unique mild solution of the following Abstract Cauchy Problem (ACP):

$$\begin{cases} du_j(t) = [Au_j(t) + \nabla F(\phi(t))[u_j(t)]]dt, + \Phi_j(t)dt \\ u_j(0) = 0 \end{cases} \quad (\text{ACP}_j)$$

for  $j = 2, \dots, k-1$  satisfying:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u_j(t)|_H^p \right] < +\infty, \quad T > 0, \text{ for any } p \in [2, \infty); \quad (6.7)$$

then there exists a unique mild solution  $u_k(t)$  of the following non-homogeneous linear differential equation with stochastic coefficients (in the sense of Definition 5.2):

$$\begin{cases} du_k(t) = [Au_k(t) + \nabla F(\phi(t))[u_k(t)]]dt + \Phi_k(t)dt, & t \in [0, +\infty), \\ u_k(0) = 0 \end{cases} \quad (\text{ACP}_k)$$

and it satisfies the following estimate, for any  $T > 0$ :

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u_k(t)|_H^p \right] < +\infty. \quad (6.8)$$

**Proof.** We proceed by a fixed point argument, where the contraction is given by

$$\Gamma(y(t)) := \int_0^t e^{(t-s)A} \nabla F(\phi(s))[y(s)]ds + \int_0^t e^{(t-s)A} \Phi_k(s)ds$$

on  $\mathcal{L}^p(\Omega; C([0, T]; H))$ . In fact, arguing as in Proposition 6.1, we see that for  $\tilde{T} \in [0, T]$  sufficiently small,  $\Gamma$  is a contraction on  $\mathcal{L}^p(\Omega; C([0, \tilde{T}]; H))$ ,  $p \in [2, \infty)$ , so that the existence and the uniqueness of the solution for (ACP<sub>k</sub>) follows.

Let us consider the estimate (6.8). By the condition (iv) in Hypothesis 3.1 we have, for  $p = 2a$  with  $a \in \mathbb{N}$  (and all points in the probability space):

$$\begin{aligned} \frac{d}{dt} |u_k(t)|_H^{2a} &= 2a \langle Au_k(t), u_k(t) \rangle |u_k(t)|_H^{2a-2} + 2a \langle \nabla F(\phi(t))[u_k(t)], u_k(t) \rangle |u_k(t)|_H^{2a-2} \\ &\quad + 2a \langle \Phi_k(t), u_k(t) \rangle |u_k(t)|_H^{2a-2} \\ &\leq -2a\omega |u_k(t)|_H^{2a} + 2a\gamma(1 + |u^0|_K) |u_k(t)|_H^{2a} + 2a |\Phi_k(t)|_H |u_k(t)|_H^{2a-1} \\ &\leq -2a\tilde{\omega} |u_k(t)|_H^{2a} + C_a |\Phi_k(t)|_H^{2a}, \end{aligned} \quad (6.9)$$

where  $\tilde{\omega} := \omega - \gamma(1 + |u^0|_K)$  as in the proof of Proposition 6.1. By the assumption (6.7) made on  $u_j(t)$ ,  $j = 1, \dots, k-1$  and Lemma 5.3 we have that:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |\Phi_k(t)|_H^{2a} \right] \leq C'_a, \quad T > 0,$$

so that taking the expectation of inequality (6.9) and applying Gronwall's lemma (similarly as in the proof of Proposition 6.1) we obtain:

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |u_k(t)|_H^{2a} \right] \leq C'_a e^{-2a\tilde{\omega}T} < C_a,$$

where  $C_a$  is a positive constant, and the theorem follows.  $\square$

We are now able to state the main result of this section:

**Theorem 6.3.** Under Hypothesis 3.1 the mild solution  $u(t)$  of (2.2) (in the sense of Definition 4.6) can be expanded in powers of  $\varepsilon > 0$  in the following form

$$u(t) = \phi(t) + \varepsilon u_1(t) + \dots + \varepsilon^n u_n(t) + R_n(t, \varepsilon), \quad n \in \mathbb{N},$$

where  $u_1$  is the solution of

$$\begin{aligned} du_1(t) &= [Au_1(t) + \nabla F(\phi(t))[u_1(t)]]dt + \sqrt{Q}dL(t) \\ u_1(0) &= 0, \end{aligned}$$

while  $u_k$ ,  $k = 2, \dots, n$  is the solution of

$$\begin{cases} du_k(t) = Au_k(t) + \nabla F(\phi(t))[u_k(t)]dt + \Phi_k(t)dt, \\ u_k(0) = 0. \end{cases} \quad (\text{ACP}_k)$$

The remainder  $R_n(t, \varepsilon)$  is defined by

$$\begin{aligned} R_n(t, \varepsilon) &:= u(t) - \phi(t) - \sum_{k=1}^n \varepsilon^k u_k(t) \\ &= \int_0^t e^{(t-s)A} \left( F(u(s)) - F(\phi(s)) \right. \\ &\quad \left. - \sum_{k=1}^n \varepsilon^k \nabla F(\phi(s))[u_k(s)] - \sum_{k=2}^n \varepsilon^k \Phi_k(s) \right) ds, \end{aligned} \quad (6.10)$$

and verifies the following inequality

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |R_n(t, \varepsilon)|_H^p \right] \leq C_p \varepsilon^{n+1},$$

with a constant  $C_p > 0$ .

**Proof.** Let us define  $R_n(t, \varepsilon)$ ,  $n \in \mathbb{N}$ , as stated in the theorem. Since by construction

- $\phi(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(\phi(s))ds$  (cf. Definition 4.1);
- $u(t) = e^{tA}u^0 + \int_0^t e^{(t-s)A}F(u(s))ds + \varepsilon L_A(t)$  (cf. Definition 4.6);
- $u_1(t) = \int_0^t e^{(t-s)A}\nabla F(\phi(s))[u_1(s)]ds + L_A(t)$  (cf. Proposition 6.1 and Definition 4.6);
- $u_k(t) = \int_0^t e^{(t-s)A}\nabla F(\phi(s))[u_k(s)]ds + \int_0^t e^{(t-s)A}\Phi_k(s)ds$  for  $k = 2, \dots, n$ , with  $\Phi_k(s) := \Phi_k(\phi(s)) [u_1(s), \dots, u_{k-1}(s)]$  defined in (5.14) (cf. Theorem 6.2 and Definition 4.6);

we have

$$\begin{aligned} R_n(t, \varepsilon) &= \int_0^t e^{(t-s)A} \left( F(u(s)) - F(\phi(s)) \right. \\ &\quad \left. - \sum_{k=1}^n \varepsilon^k \nabla F(\phi(s))[u_k(s)] - \sum_{k=2}^n \varepsilon^k \Phi_k(s) \right) ds. \end{aligned}$$

Recalling that  $R_1^{(n)}(\phi(s); h(s), \varepsilon) = F(u(s)) - F(\phi(s)) - \sum_{k=1}^n \varepsilon^k \nabla F(\phi(s))[u_k(s)] - \sum_{k=2}^n \varepsilon^k \Phi_k(s)$  we get:

$$\begin{aligned} \mathbb{E} \left[ \sup_{t \in [0, T]} |R_n(t, \varepsilon)|_H^p \right] &\leq \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} R_1^{(n)}(\phi(s); h(s), \varepsilon) ds \right|_H^p \right] \\ &\leq \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \|e^{(t-s)A}\|_{L(H)}^p |R_1^{(n)}(\phi(s); h(s), \varepsilon)|_H^p ds \right] \\ &\leq \mathbb{E} \left[ \sup_{t \in [0, T]} |R_1^{(n)}(\phi(t); h(t), \varepsilon)|_H^p \int_0^t e^{-\omega(t-s)p} ds \right] \\ &\leq C_{n,p} \varepsilon^{p(n+1)}, \end{aligned} \quad (6.11)$$

for some positive constant  $C_{n,p}$  (depending on  $n, p$ , but not on  $\varepsilon$ ), where in the second and third inequality we have used the contraction property of the semigroup generated by  $A$ . Now recalling Lemma 5.1 the inequality in Theorem 6.3 follows.  $\square$

**Example 6.4.** Our results apply in particular to stochastic PDEs describing the FitzHugh–Nagumo equation with a Lévy noise perturbation (related to those studied with a Gaussian noise, for example, in [69–71,24]).

The reference equation is given by (see [24, Eq. (1.1)])

$$\begin{cases} \partial_t v(t, x) = \partial_x(c(x)\partial_x v(t, x)) - p(x)v(t, x) - w(t, x) + f(v(t, x)) + \varepsilon \dot{L}_1(t, x), \\ \partial_t w(t, x) = \gamma v(t, x) - \alpha w(t, x) + \varepsilon \dot{L}_2(t, x), \\ \partial_x v(t, 0) = \partial_x v(t, 1) = 0, \\ v(0, x) = v_0(x), \quad w(0, x) = w_0(x), \end{cases} \quad (6.12)$$

with the parameter  $\varepsilon > 0$  in front of the noise, where  $u, w$  are real valued random variables,  $\alpha, \gamma$  are strictly positive real phenomenological constants and  $c, p$  are strictly positive smooth functions on  $[0, 1]$ . Moreover, the initial values  $v_0, w_0$  are in  $C([0, 1])$ . The nonlinear term is of the form  $f(v) = -v(v - 1)(v - \xi)$ , where  $\xi \in (0, 1)$ . Finally  $L_1, L_2$  are independent  $Q_i$ -Lévy processes with values in  $L^2(0, 1)$ , with  $Q_i$  positive trace class commuting operators, commuting also with  $A_0$ ,  $A_0$  being defined below. The above equation can be rewritten in the form of an infinite dimensional stochastic evolution equation on the space

$$H := L^2(0, 1) \times L^2(0, 1) \quad (6.13)$$

by introducing the following operators:

$$\begin{aligned} A_0 &:= \partial_x c(x) \partial_x, \\ D(A_0) &:= \{u \in H^2(0, 1); v_x(0) = v_x(1)\}, \quad \text{acting in } L^2(0, 1) \end{aligned}$$

and

$$A = \begin{pmatrix} A_0 - p & -I \\ \gamma I & \alpha I \end{pmatrix},$$

with domain  $D(A) := D(A_0) \times L^2(0, 1)$ , and

$$F \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -v(v - 1)(v - \xi) \\ 0 \end{pmatrix}, \quad \text{with } D(F) := L^6(0, 1) \times L^2(0, 1).$$

Further, we introduce the Banach space  $K := L^{18}(0, 1) \times L^2(0, 1)$ , endowed with the norm  $|\cdot|_K := |\cdot|_6 + |\cdot|_2$  and consider  $u^0 \in K$ . In this way, the Eq. (6.12) can be rewritten as

$$\begin{cases} du(t) = Au(t) + F(u(t))dt + \sqrt{Q}dL(t) \\ u(0) = u^0 := (v^0, w^0) \in K, \end{cases}$$

with  $A$  and  $F$  satisfying Hypothesis 3.1 when  $\xi^2 - \xi + 1 \leq 3 \min_{x \in [0, 1]} p(x)$ . In fact, the properties of the two operators  $A$  and  $F$  can be determined starting from the problems considered in [24,28]. In particular from [28, Section 2.2] the estimates on the nonlinear term  $F$  and its derivatives can easily be deduced. Moreover we claim that the stochastic convolution

$$L_A(t) := \int_0^t e^{(t-s)A} dL(s),$$

(where  $e^{tA}$ ,  $t \geq 0$  denotes the semigroup generated by  $A$ ) is well-defined and admits a continuous version with values into the space  $K$ . This fact can be proved by an application of [63] and its proof, taking into account that the domain of fractional powers of  $A$  are contained in  $K$  (cf. Appendix A – in particular Example A.5.2 – in [32]) and moreover we are assuming  $\text{Tr } Q < \infty$ .

Then by Theorem 6.3 we get an asymptotic expansion in powers of  $\varepsilon > 0$  of the solution, in terms of solutions of the corresponding deterministic FitzHugh–Nagumo equation and the solution of a system of (explicit) linear (non homogeneous) stochastic equations. The expansion holds for all orders in  $\varepsilon > 0$ . The remainders are estimated according to Theorem 6.3. These results should allow to obtain rigorously results similarly to those obtained numerically up to second order in  $\varepsilon$  in [72,71] in which the noise was of Gaussian type. Tuckwell, in particular, has made heuristic expansions up to second order in  $\varepsilon$  for the mean and the variance of the solution process  $u = (u(t))_{t \geq 0}$  (in the case of Gaussian noise, see [72,71]), proving in particular that one has enhancement (respectively reduction) of the mean according to whether the expansion is around which stable point of the stationary deterministic equation.

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