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On the independence of the value function for stochastic differential games of the probability space

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Abstract

We show that the value function in a stochastic differential game does not change if we keep the same space $(\Omega, \mathcal{F})$ but introduce probability measures by means of Girsanov's transformation *depending* on the policies of the players. We also show that the value function does not change if we allow the driving Wiener processes to depend on the policies of the players. Finally, we show that the value function does not change if we perform a random time change with the rate depending on the policies of the players.

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1. Introduction

Let $\mathbb{R}^d = \{x = (x^1, \dots, x^d)\}$ be a d-dimensional Euclidean space and let $d_1 \geq d$ be an integer. Assume that we are given separable metric spaces A and B, and let, for each $\alpha \in A$, $\beta \in B$, the following functions on $\mathbb{R}^d$ be given:

(i) $d \times d_1$ matrix-valued $\sigma^{\alpha\beta}(x) = \sigma(\alpha, \beta, x) = (\sigma^{\alpha\beta}_{ij}(x))$,

(ii) $\mathbb{R}^d$-valued $b^{\alpha\beta}(x) = b(\alpha, \beta, x) = (b^{\alpha\beta}_i(x))$, and

(iii) real-valued functions $c^{\alpha\beta}(x) = c(\alpha, \beta, x) \geq 0$, $f^{\alpha\beta}(x) = f(\alpha, \beta, x)$, and $g(x)$.

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Under natural assumptions which will be specified later, on a probability space $(\Omega, \mathcal{F}, P)$ carrying a d_1-dimensional Wiener process w_t one associates with these objects and a bounded domain $G \subset \mathbb{R}^d$ a stochastic differential game with the diffusion term $\sigma^{\alpha\beta}(x)$, drift term $b^{\alpha\beta}(x)$, discount rate $c^{\alpha\beta}(x)$, running cost $f^{\alpha\beta}(x)$, and the final cost $g(x)$ payed when the underlying process first exits from G.

After the order of players is specified in a certain way it turns out (see our Remark 2.2) that the value function $v(x)$ of this differential game is a unique continuous in $\bar{G}$ viscosity solution of the Isaacs equation

$$H[v] = 0 \tag{1.1}$$

in G with boundary condition $v = g$ on ∂G, where for a sufficiently smooth function $u = u(x)$

$$H[u](x) = \sup_{\alpha\in A} \inf_{\beta\in B} [L^{\alpha\beta}u(x) + f^{\alpha\beta}(x)], \tag{1.2}$$

$$L^{\alpha\beta}u(x) := a^{\alpha\beta}_{ij}(x)D_{ij}u(x) + b^{\alpha\beta}_i(x)D_i u(x) - c^{\alpha\beta}(x)u(x),$$

$$a^{\alpha\beta}(x) := (1/2)\sigma^{\alpha\beta}(x)(\sigma^{\alpha\beta}(x))^*, \qquad D_i = \partial/\partial x^i, \qquad D_{ij} = D_i D_j.$$

We will assume that σ and b are uniformly Lipschitz with respect to x, $\sigma\sigma^*$ is uniformly nondegenerate, and c and f are uniformly bounded. In such a situation uniqueness of continuous viscosity solutions or even continuous L_p viscosity solutions of (1.2) is shown in [5] and therefore the fact of the independence of v of the probability space seems to be obvious.

Roughly speaking, the goal of this paper is to show that the value function does not change even if we keep the same space $(\Omega, \mathcal{F})$ but introduce probability measures by means of Girsanov's transformation *depending* on the policies of the players. We also show that the value function does not change if we allow the driving Wiener processes to depend on the policies of the players. Finally, we show that the value function does not change if we perform a random time change with the rate depending on the policies of the players.

These facts are well known in the theory of controlled diffusion processes and play there a very important role, in particular, while estimating the derivatives of the value function. A rather awkward substitute of them for stochastic differential games was used for the same purposes in [12]. Applying the results presented here one can make many constructions in [12] more natural and avoid introducing auxiliary "shadow" processes.

However, not all proofs in [12] can be simplified using our present methods. We deliberately avoided discussing the way to use the external parameters in contrast with [12] just to make the presentation more transparent.

Our proofs do not use anything from the theory of viscosity solutions and are based on a version of Święch's [14] idea as presented in [10] and a general solvability theorem in class $C^{1,1}$ of Isaacs equations from [9].

It is quite possible that Święch's approach from [14] based on inf sup convolutions can be further developed and used to prove our main results. We prefer using the above mentioned result from [9] for two reasons.

First, in previous articles we used this result to establish a W^2_p-solvability theorem for fully nonlinear equations with a "relaxed" convexity assumption, to showing that the L_p-viscosity solutions of Isaacs equations with VMO coefficients are in $C^{1+\kappa}$, and to establishing an algebraic rate of convergence of finite-difference approximations to solutions of Isaacs equations. Here we provide one more application of this result of [9].

The second reason is that we want to justify a claim made in Section 5 of [12] and in the continuation of this paper to prove a sharper theorem about how fast the approximate solutions from [9] converge to the value function.

The article is organized as follows. In Section 2 we present our main result, Theorem 2.1. We prove it in Section 3 under the additional assumption that the corresponding Isaacs equation has a smooth solution. Then in Section 4 we allow the solutions to belong to the Sobolev class W^2_d. Section 5 contains a general approximation result, which allows us in Section 6 to use a result from [9] (see Theorem 2.2) and conclude the proof of Theorem 2.1 in the general case.

2. Main result

We start with our assumptions.

Assumption 2.1. (i) The functions $\sigma^{\alpha\beta}(x), b^{\alpha\beta}(x), c^{\alpha\beta}(x)$, and $f^{\alpha\beta}(x)$ are continuous with respect to $\beta \in B$ for each (α, x) and continuous with respect to $\alpha \in A$ uniformly with respect to $\beta \in B$ for each x. The function $g(x)$ is bounded and continuous.

(ii) The functions $c^{\alpha\beta}(x)$ and $f^{\alpha\beta}(x)$ are uniformly continuous with respect to x uniformly with respect to $(\alpha, \beta) \in A \times B$ and for any $x \in \mathbb{R}^d$ and $(\alpha, \beta) \in A \times B$

$$\|\sigma^{\alpha\beta}(x)\|, |b^{\alpha\beta}(x)|, |c^{\alpha\beta}(x)|, |f^{\alpha\beta}(x)| \le K_0,$$

where K_0 is a fixed constant and for a matrix σ we denote $\|\sigma\|^2 = \operatorname{tr}\sigma\sigma^*$,

(iii) For any $(\alpha, \beta) \in A \times B$ and $x, y \in \mathbb{R}^d$ we have

$$\|\sigma^{\alpha\beta}(x) - \sigma^{\alpha\beta}(y)\| + |b^{\alpha\beta}(x) - b^{\alpha\beta}(y)| \le K_0|x - y|.$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $\{\mathcal{F}_t, t \ge 0\}$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ such that each $\mathcal{F}_t$ is complete with respect to $\mathcal{F}, P$.

The set of progressively measurable A-valued processes $\alpha_t = \alpha_t(\omega)$ is denoted by $\mathfrak{A}$. Similarly we define $\mathfrak{B}$ as the set of B-valued progressively measurable functions. By $\mathbb{B}$ we denote the set of $\mathfrak{B}$-valued functions $\boldsymbol{\beta}(\alpha_\cdot)$ on $\mathfrak{A}$ such that, for any $T \in (0, \infty)$ and any $\alpha^1_\cdot, \alpha^2_\cdot \in \mathfrak{A}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \le T) = 1, \tag{2.1}$$

we have

$$P(\boldsymbol{\beta}_t(\alpha^1_\cdot) = \boldsymbol{\beta}_t(\alpha^2_\cdot) \text{ for almost all } t \le T) = 1.$$

Definition 2.1. A function $p^{\alpha_\cdot\beta_\cdot}_t = p^{\alpha_\cdot\beta_\cdot}_t(\omega)$ given on $\mathfrak{A} \times \mathfrak{B} \times \Omega \times [0, \infty)$ with values in some measurable space is called *a control adapted process* if, for any $(\alpha_\cdot, \beta_\cdot) \in \mathfrak{A} \times \mathfrak{B}$, it is progressively measurable in (ω, t) and, for any $T \in (0, \infty)$, we have

$$P(p^{\alpha^1_\cdot\beta^1_\cdot}_t = p^{\alpha^2_\cdot\beta^2_\cdot}_t \text{ for almost all } t \le T) = 1$$

as long as

$$P(\alpha^1_t = \alpha^2_t, \beta^1_t = \beta^2_t \text{ for almost all } t \le T) = 1.$$

Assumption 2.2. For each $\alpha_\cdot \in \mathfrak{A}$ and $\beta_\cdot \in \mathfrak{B}$ we are given control adapted processes

(i) $w^{\alpha_\cdot\beta_\cdot}_t, t \ge 0$, which are standard d_1-dimensional Wiener process relative to the filtration $\{\mathcal{F}_t, t \ge 0\}$,

(ii) $r^{\alpha_\cdot\beta_\cdot}_t, t \ge 0$, and $\pi^{\alpha_\cdot\beta_\cdot}_t, t \ge 0$, which are real-valued and $\mathbb{R}^{d_1}$-valued, respectively,

(iii) for all values of the arguments

$$\delta_1^{-1} \geq r_t^{\alpha.\beta.} \geq \delta_1, \quad |\pi_t^{\alpha.\beta.}| \leq K_1,$$

where $\delta_1 > 0$ and $K_1 \in (0, \infty)$ are fixed constants.

Finally we introduce

$$a^{\alpha\beta}(x) := (1/2)\sigma^{\alpha\beta}(x)(\sigma^{\alpha\beta}(x))^*,$$

fix a domain $G \subset \mathbb{R}^d$, and impose the following.

Assumption 2.3. G is a bounded domain of class C^2 and there exists a constant $\delta \in (0, 1)$ such that for any $\alpha \in A$, $\beta \in B$, and $x, \lambda \in \mathbb{R}^d$

$$\delta|\lambda|^2 \leq a_{ij}^{\alpha\beta}(x)\lambda^i\lambda^j \leq \delta^{-1}|\lambda|^2.$$

Remark 2.1. As is well known, if Assumption 2.3 is satisfied, then there exists a bounded from above $\Psi \in C_{loc}^2(\mathbb{R}^d)$ such that $\Psi > 0$ in G, $\Psi = 0$ on ∂G, and for all $\alpha \in A$, $\beta \in B$, and $x \in G$

$$L^{\alpha\beta}\Psi(x) + c^{\alpha\beta}\Psi(x) \leq -1. \tag{2.2}$$

For $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ consider the following Itô equation

$$x_t = x + \int_0^t r_s^{\alpha.\beta.}\sigma^{\alpha_s\beta_s}(x_s)\, dw_s^{\alpha.\beta.} + \int_0^t [r_s^{\alpha.\beta.}]^2\big[b^{\alpha_s\beta_s}(x_s) + \sigma^{\alpha_s\beta_s}(x_s)\pi_s^{\alpha.\beta.}\big]\, ds. \tag{2.3}$$

Observe that Eq. (2.3) satisfies the usual hypothesis, that is for any $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, $x \in \mathbb{R}^d$, and $T \in (0, \infty)$ it has a unique solution on $[0, T]$ denoted by $x_t^{\alpha.\beta.x}$ and $x_t^{\alpha.\beta.x}$ is a control adapted process for each x.

Set

$$\phi_t^{\alpha.\beta.x} = \int_0^t [r_s^{\alpha.\beta.}]^2 c^{\alpha_s\beta_s}(x_s^{\alpha.\beta.x})\, ds,$$

$$\psi_t^{\alpha.\beta.x} = -(1/2)\int_0^t [r_s^{\alpha.\beta.}]^2|\pi_s^{\alpha.\beta.}|^2\, ds - \int_0^t r_s^{\alpha.\beta.}\pi_s^{\alpha.\beta.}\, dw_s^{\alpha.\beta.},$$

define $\tau^{\alpha.\beta.x}$ as the first exit time of $x_t^{\alpha.\beta.x}$ from G, and introduce

$$v(x) = \inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\boldsymbol{\beta}(\alpha.)}\Big[\int_0^\tau r_t^2 f(x_t)e^{-\phi_t-\psi_t}\, dt + g(x_\tau)e^{-\phi_\tau-\psi_\tau}\Big], \tag{2.4}$$

where the indices $\alpha.$, $\boldsymbol{\beta}$, and x at the expectation sign are written to mean that they should be placed inside the expectation sign wherever and as appropriate, that is

$$E_x^{\alpha.\beta.}\Big[\int_0^\tau r_t^2 f(x_t)e^{-\phi_t-\psi_t}\, dt + g(x_\tau)e^{-\phi_\tau-\psi_\tau}\Big]$$
$$:= E\Big[g(x_{\tau^{\alpha.\beta.x}}^{\alpha.\beta.x})e^{-\phi_{\tau^{\alpha.\beta.x}}^{\alpha.\beta.x}-\psi_{\tau^{\alpha.\beta.x}}^{\alpha.\beta.x}} + \int_0^{\tau^{\alpha.\beta.x}} [r_t^{\alpha.\beta.x}]^2 f^{\alpha_t\beta_t}(x_t^{\alpha.\beta.x})e^{-\phi_t^{\alpha.\beta.x}-\psi_t^{\alpha.\beta.x}}\, dt\Big].$$

Observe that, formally, the value x_τ may not be defined if $\tau = \infty$. In that case we set the corresponding terms to equal zero. The above definitions make perfect sense due to our Remark 2.3.

Here is our main result.

Theorem 2.1. *Under the above assumptions the function $v(x)$ is independent of the choice of the probability space, filtration and control adapted process $(r,\pi,w)_t^{\alpha.\beta.}$, it is bounded and continuous in $\bar{G}$.*

Remark 2.2. Once we know that $v(x)$ is independent of the choice of the probability space, filtration and control adapted process $(r,\pi,w)_t^{\alpha.\beta.}$, we can take any probability space carrying a d_1-dimensional Wiener process w_t and construct $v(x)$ by setting $w_t^{\alpha.\beta.}=w_t$, $r\equiv 1$, $\pi\equiv 0$. In that case we are in the position to apply the results of [1,6,11] according to which v is continuous in $\bar{G}$ and satisfies the dynamic programming principle. Then it is a standard fact that v is a viscosity solution of (1.1) (see, for instance, [1,6,14]). Indeed, if a smooth function $\psi(x)$ is such that $\psi(x)\geq v(x)$ in a neighborhood of $x_0\in G$ and $\psi(x_0)=v(x_0)$, then by defining $\gamma_\varepsilon^{\alpha.\beta.}$, $\varepsilon>0$ as the first exit time of $x_t^{\alpha.\beta.x_0}$ from an ε-neighborhood of x_0 for all small ε we have

$$\begin{aligned}\psi(x_0)=v(x_0)&=\inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}}E_{x_0}^{\alpha.\boldsymbol{\beta}(\alpha.)}\Big[\int_0^{\gamma_\varepsilon}f(x_t)e^{-\phi_t}\,dt+v(x_{\gamma_\varepsilon})e^{-\phi_{\gamma_\varepsilon}}\Big]\\&\leq\inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}}E_{x_0}^{\alpha.\boldsymbol{\beta}(\alpha.)}\Big[\int_0^{\gamma_\varepsilon}f(x_t)e^{-\phi_t}\,dt+\psi(x_{\gamma_\varepsilon})e^{-\phi_{\gamma_\varepsilon}}\Big].\end{aligned}$$

On the other hand set $H[\psi]=-h$ and observe that by Theorem 4.1 of [10]

$$\psi(x_0)=\inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}}E_{x_0}^{\alpha.\boldsymbol{\beta}(\alpha.)}\Big[\int_0^{\gamma_\varepsilon}(f+h)(x_t)e^{-\phi_t}\,dt+\psi(x_{\gamma_\varepsilon})e^{-\phi_{\gamma_\varepsilon}}\Big].$$

It follows that

$$\inf_{\alpha.\in\mathfrak{A}}\inf_{\beta.\in\mathfrak{B}}E_{x_0}^{\alpha.\beta.}\int_0^{\gamma_\varepsilon}h(x_t)e^{-\phi_t}\,dt\leq 0,\tag{2.5}$$

and if we assume that $H[\psi](x_0)<0$, then $h>0$ in an ε-neighborhood of x_0 and (2.5) is impossible, since c is bounded and σ and b are bounded so that $E_{x_0}^{\alpha.\beta.}\gamma_\varepsilon$ is bounded away from zero. Hence $H[\psi](x_0)\geq 0$ and v is a viscosity subsolution by definition. Similarly one shows that it is a viscosity supersolution.

Provided that we know that continuous viscosity solutions are unique the above argument proves the fact that the value function is independent of the probability space (if we drop out r and π and take w independent of the policies). Jensen [2] proved uniqueness for *Lipschitz* continuous viscosity solutions to the fully nonlinear second order elliptic PDE not explicitly depending on x in a bounded domain. Related results in the same year with H depending on x were published in Jensen–Lions–Souganidis [4].

In what concerns uniformly nondegenerate Isaacs equations, Trudinger in [15] proves the existence and uniqueness of continuous viscosity solutions for Isaacs equations if the coefficients are continuous and a is $1/2$ Hölder continuous uniformly with respect to α,β (see Corollary 3.4 there). Uniqueness is also stated for Isaacs equations with Lipschitz continuous a as Corollary 5.11 in [3]. Jensen and Święch in [5] further relaxed the requirement on a and proved uniqueness of continuous even L_p-viscosity solutions.

We will use Theorem 2.1 to prove in a subsequent article a result to state which we need a few new objects. In the end of Section 1 of [9] a function $P(u_{ij},u_i,u)$ is constructed defined for all symmetric $d\times d$ matrices (u_{ij}), $\mathbb{R}^d$-vectors (u_i), and $u\in\mathbb{R}$ such that it is convex, positive-homogeneous of degree one, is Lipschitz continuous, and at all points of differentiability of P

for all values of arguments we have $P_u \leq 0$ and

$$\hat{\delta}|\lambda|^2 \leq P_{u_{ij}}\lambda^i\lambda^j \leq \hat{\delta}^{-1}|\lambda|^2,$$

where $\hat{\delta}$ is a constant in $(0,1)$ depending only on d, K_0, and δ. For smooth enough functions $u(x)$ introduce

$$P[u](x) = P(D_{ij}u(x), D_iu(x), u(x)).$$

We now state part of Theorem 1.1 of [9] which we need even in the present article.

Theorem 2.2. *Let $g \in C^{1,1}(\mathbb{R}^d)$. Then for any $K \geq 0$ the equation*

$$\max(H[u], P[u] - K) = 0 \tag{2.6}$$

in G (a.e.) with boundary condition $u = g$ on ∂G has a unique solution $u \in C^{0,1}(\bar{G}) \cap C^{1,1}_{loc}(G)$.

The result we are aiming at in a subsequent article consists of proving the conjecture stated in [9]:

Theorem 2.3. *Denote by u_K the function from Theorem 2.2 and assume that G and g are of class C^3. Then there exists a constant N such that $|v - u_K| \leq N/K$ on G for $K \geq 1$.*

A very week version of this theorem was already used in [13] for establishing a rate of convergence of finite-difference approximations for solutions of Isaacs equations.

We finish this section with a useful technical result.

Lemma 2.4. *For any $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ the process*

$$\exp(-\psi^{\alpha.\beta.x}_{t\wedge\tau^{\alpha.\beta.x}})$$

is a uniformly integrable martingale on $[0,\infty)$. Furthermore, there exists a constant N independent of $\alpha. \in \mathfrak{A}$, $\beta. \in \mathfrak{B}$, and $x \in \mathbb{R}^d$ such that

$$E^{\alpha.\beta.}_x \int_0^\tau e^{-\psi_s}\,ds \leq N. \tag{2.7}$$

Finally,

$$E^{\alpha.\beta.}_x e^{-\psi_\tau} = 1. \tag{2.8}$$

Proof. Owing to (2.2) by Itô's formula

$$\begin{aligned} E^{\alpha.\beta.}_x G(x_{t\wedge\tau})e^{-\psi_{t\wedge\tau}} &= G(x) + E^{\alpha.\beta.}_x \int_0^{t\wedge\tau} r_s^2[LG + cG](x_s)e^{-\psi_s}\,ds \\ &\leq G(x) - \delta_1^2 E^{\alpha.\beta.}_x \int_0^{t\wedge\tau} e^{-\psi_s}\,ds, \end{aligned}$$

and (2.7) follows. To prove (2.8) use that

$$1 = E^{\alpha.\beta.}_x e^{-\psi_{t\wedge\tau}} = E^{\alpha.\beta.}_x e^{-\psi_\tau} I_{\tau\leq t} + E^{\alpha.\beta.}_x e^{-\psi_t} I_{\tau>t},$$

where the last term decreases as t increases, which is seen from the formula, and tends to zero as $t \to \infty$ since its integral with respect to t over $[0,\infty)$ is finite being equal to the left-hand side of (2.7).

Finally, the first assertion of the lemma follows from (2.8) due to the well-known properties of martingales. The lemma is proved.

Remark 2.3. In light of the proof of Lemma 2.4

$$0=\lim_{t\to\infty}E_x^{\alpha.\beta.}e^{-\psi_t}I_{\tau>t}=\lim_{t\to\infty}E_x^{\alpha.\beta.}e^{-\psi_\tau}I_{\tau>t}=E_x^{\alpha.\beta.}e^{-\psi_\tau}I_{\tau=\infty}.$$

Hence defining the terms containing x_τ as zero on the set where $\tau=\infty$ is indeed natural. Lemma 2.4 shows that the function v is well defined and one can rewrite its definition as

$$v(x)=\inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}}E_x^{\alpha.\boldsymbol{\beta}(\alpha.)}\Big[\int_0^\tau r_t^2 f(x_t)e^{-\phi_t}\,dt+g(x_\tau)\Big]e^{-\psi_\tau},$$

which calls for changes of probability measure by using Girsanov's theorem.

3. Proof of Theorem 2.1 in case that the Isaacs equation has a smooth solution

In this section we replace Assumption 2.1(iii) with a weaker one.

Assumption 3.1. The functions $\sigma^{\alpha\beta}(x)$ and $b^{\alpha\beta}(x)$ are uniformly continuous with respect to x uniformly with respect to $(\alpha,\beta)\in A\times B$.

However, this time there is no guarantee that Eq. (2.3) has a unique solution and we impose the following.

Assumption 3.2. Eq. (2.3) satisfies the usual hypothesis, that is for any $\alpha.\in\mathfrak{A}$, $\beta.\in\mathfrak{B}$, $x\in\mathbb{R}^d$, and $T\in(0,\infty)$ it has a unique solution on $[0,T]$ denoted by $x_t^{\alpha.\beta.x}$ and $x_t^{\alpha.\beta.x}$ is a control adapted process for each x.

We also assume that we are given two functions $\hat{u},\check{u}\in C^2(\bar{G})$.

Theorem 3.1. (i) *If $H[\hat{u}]\le 0$ in G and $\hat{u}\ge g$ on ∂G, then $v\le\hat{u}$ in $\bar{G}$.*
(ii) *If $H[\check{u}]\ge 0$ in G and $\check{u}\le g$ on ∂G, then $v\ge\check{u}$ in $\bar{G}$.*
(iii) *If $\hat{u}$ and $\check{u}$ are as in* (i) *and* (ii) *and $\hat{u}=\check{u}$, then v is independent of the choice of the probability space, filtration, r,π, and w.*

We need three lemmas.

Lemma 3.2. *Set $\kappa_n(t)=[nt]/n$. Then there exists a constant N such that for all $n\ge 1$, $x\in\mathbb{R}^d$, $\alpha.\in\mathfrak{A}$, $\beta.\in\mathfrak{B}$ we have*

$$E_x^{\alpha.\beta.}\int_0^\tau e^{-\phi_t-\psi_t}|x_t-x_{\kappa_n(t)}|^2\,dt\le N/n. \tag{3.1}$$

Proof. For each fixed t while estimating

$$E_x^{\alpha.\beta.}e^{-\phi_t-\psi_t}|x_t-x_{\kappa_n(t)}|^2 I_{\tau>t}$$

Girsanov's theorem allows us to assume that $\pi\equiv 0$. In that case for simplicity of notation we will drop the indices $x,\alpha.,\beta.$ and observe that

$$|x_t-x_{\kappa_n(t)}|^2\le 2\Big|\int_{k/n}^t\sigma(x_s)\,dw_s\Big|^2+2\Big|\int_{k/n}^t b(x_s)\,ds\Big|^2$$

so that for $t\in[k/n,(k+1)/n]$

$$E_x^{\alpha.\beta.}\{e^{-\phi_t}|x_t-x_{k/n}|^2\mid\mathcal{F}_{k/n}\}\le E_x^{\alpha.\beta.}\{|x_t-x_{k/n}|^2\mid\mathcal{F}_{k/n}\}\le N/n,$$

where N depends only on d and K_0. Hence, owing also to (2.7) the left-hand side of (3.1) is dominated by

$$\int_0^\infty E_x^{\alpha.\beta.} e^{-\phi_t}|x_t - x_{\kappa_n(t)}|^2 I_{\tau>t}\,dt \le \int_0^\infty E_x^{\alpha.\beta.}|x_t - x_{\kappa_n(t)}|^2 I_{\tau>\kappa_n(t)}\,dt$$
$$\le Nn^{-1}\int_0^\infty E_x^{\alpha.\beta.} I_{\tau>\kappa_n(t)}\,dt = Nn^{-1}E_x^{\alpha.\beta.}(\tau + 1/n) \le Nn^{-1}.$$

The lemma is proved.

For a stopping time γ we say that a process ξ_t is a submartingale on $[0,\gamma]$ if $\xi_{t\wedge\gamma}$ is a submartingale. Similar definition applies to supermartingales.

The proof of the following lemma and Lemma 3.4 follows a version of Święch's [14] idea as it is presented in [10].

Lemma 3.3. *Let $H[\hat{u}] \le 0$ in G. Then for any $x \in \mathbb{R}^d$, $\alpha. \in \mathfrak{A}$, and $\varepsilon > 0$, there exist a sequence $\beta^n_\cdot(\alpha.) = \beta^n_\cdot(\alpha., x, \varepsilon) \in \mathfrak{B}$, $n = 1, 2, \ldots$, and a sequence of increasing continuous $\{\mathcal{F}_t\}$-adapted processes $\eta^{n\varepsilon}_t(\alpha.) = \eta^{n\varepsilon}_t(\alpha., x)$ with $\eta^{n\varepsilon}_0(\alpha.) = 0$ such that*

$$\sup_n E\eta^{n\varepsilon}_\infty(\alpha.) < \infty, \tag{3.2}$$

the processes

$$\kappa^{n\varepsilon}_t(\alpha.) := \hat{u}(x^n_t)e^{-\phi^n_t-\psi^n_t} - \eta^{n\varepsilon}_t(\alpha.) + \int_0^t [r^n_t]^2 f^n_s(x^n_s)e^{-\phi^n_s-\psi^n_s}\,ds,$$

where

$$(x^n_t, \phi^n_t, \psi^n_t) = (x_t, \phi_t, \psi_t)^{\alpha.\beta^n_\cdot(\alpha.)x}, \qquad f^n_t(x) = f^{\alpha_t\beta^n_t(\alpha.)}(x), \qquad r^n_t = r_t^{\alpha.\beta^n_\cdot(\alpha.)}, \tag{3.3}$$

are supermartingales on $[0, \tau^{\alpha.\beta^n_\cdot(\alpha.)x}]$, and

$$\varlimsup_{n\to\infty}\sup_{\alpha.\in\mathfrak{A}} E\eta^{n\varepsilon}_\tau(\alpha.) \le N\varepsilon, \tag{3.4}$$

where N is independent of x and ε. Finally,

$$\sup_{\alpha.\in\mathfrak{A}}\sup_n E\sup_{t\ge0}|\kappa^{n\varepsilon}_{t\wedge\tau}(\alpha.)| < \infty. \tag{3.5}$$

Proof. Since B is separable and $a^{\alpha\beta}$, $b^{\alpha\beta}$, $c^{\alpha\beta}$, and $f^{\alpha\beta}$ are continuous with respect to β one can replace B in (1.2) with an appropriate countable subset $B_0 = \{\beta_1, \beta_2, \ldots\}$. Then for each $\alpha \in A$ and $x \in G$ define $\beta(\alpha, x)$ as $\beta_i \in B_0$ with the least i such that

$$0 \ge L^{\alpha\beta_i}\hat{u}(x) + f^{\alpha\beta_i}(x) - \varepsilon. \tag{3.6}$$

For each i the right-hand side of (3.6) is Borel in x and continuous in α. Therefore, it is a Borel function of (α, x), implying that $\beta(\alpha, x)$ also is a Borel function of (α, x). For $x \notin G$ set $\beta(\alpha, x) = \beta^*$, where β^* is a fixed element of B. Then we have that in G

$$0 \ge L^{\alpha\beta(\alpha,x)}\hat{u}(x) + f^{\alpha\beta(\alpha,x)}(x) - \varepsilon. \tag{3.7}$$

After that fix x, define $\beta^{n0}_t(\alpha.) = \beta(\alpha_t, x)$, $t \ge 0$, and for $k \ge 1$ introduce $\beta^{nk}_t(\alpha.)$ recursively so that

$$\begin{aligned}\beta^{nk}_t(\alpha.) &= \beta^{n(k-1)}_t(\alpha.) \quad \text{for } t < k/n,\\ \beta^{nk}_t(\alpha.) &= \beta(\alpha_t, x^{nk}_{k/n}) \quad \text{for } t \ge k/n,\end{aligned} \tag{3.8}$$

where x_t^{nk}, $k = 1, 2, \dots$, is a unique solution of

$$x_t = x + \int_0^t r_s^{\alpha.\beta.^{n(k-1)}(\alpha.)} \sigma(\alpha_s, \beta_s^{n(k-1)}(\alpha.), x_s)\, dw_s^{\alpha.\beta.^{n(k-1)}(\alpha.)}$$

$$+ \int_0^t [r_s^{\alpha.\beta.^{n(k-1)}}]^2 \big[b(\alpha_s, \beta_s^{n(k-1)}(\alpha.), x_s) + \sigma(\alpha_s, \beta_s^{n(k-1)}(\alpha.), x_s)\pi_s^{\alpha.\beta.^{n(k-1)}}\big]\, ds. \tag{3.9}$$

To show that the above definitions make sense, observe that, by Assumption 3.2, x_t^{n1} is well defined for all t. Therefore, $\beta_t^{n1}(\alpha.)$ is also well defined, and by induction we conclude that x_t^{nk} and $\beta_t^{nk}(\alpha.)$ are well defined for all $k \geq 1$.

Furthermore, owing to (3.8) it makes sense to define

$$\beta_t^n(\alpha.) = \beta_t^{nk}(\alpha.) \quad \text{for } t < k/n.$$

Notice that by definition $x_t^n := x_t^{\alpha.\beta.^n(\alpha.)x}$ satisfies the equation

$$x_t = x + \int_0^t r_s^{\alpha.\beta.^n(\alpha.)} \sigma(\alpha_s, \beta_s^n(\alpha.), x_s)\, dw_s^{\alpha.\beta.^n(\alpha.)}$$

$$+ \int_0^t [r_s^{\alpha.\beta.^n}]^2 \big[b(\alpha_s, \beta_s^n(\alpha.), x_s) + \sigma(\alpha_s, \beta_s^n(\alpha.), x_s)\pi_s^{\alpha.\beta.^n(\alpha.)}\big]\, ds. \tag{3.10}$$

For $t < k/n$ we have $\beta_t^n(\alpha.) = \beta_t^{n(k-1)}(\alpha.)$, so that for $t \leq k/n$ Eq. (3.10) coincides with (3.9) owing to the fact that $r_t^{\alpha.\beta.}$, $\pi_t^{\alpha.\beta.}$, and $w_t^{\alpha.\beta.}$ are control adapted. It follows that (a.s.)

$$x_t^n = x_t^n(\alpha.) = x_t^{nk} \quad \text{for all } t \leq k/n,$$

so that (a.s.)

$$\beta_t^{nk}(\alpha.) = \beta(\alpha_t, x_{k/n}^n)$$

for all $t \geq k/n$. Therefore, if $(k-1)/n \leq t < k/n$, then

$$\begin{aligned} &\beta_t^n(\alpha.) = \beta_t^{n(k-1)}(\alpha.) = \beta(\alpha_t, x_{(k-1)/n}^n), \\ &\beta_s^n := \beta_s^n(\alpha.) = \beta(\alpha_s, x_{\kappa_n(s)}^n), \end{aligned} \tag{3.11}$$

and x_t^n satisfies

$$x_t^n = x + \int_0^t r_s^n \sigma(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_s^n)\, dw_s^n$$

$$+ \int_0^t [r_s^n]^2 \big[b(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_s^n) + \sigma(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_s^n)\pi_s^n\big]\, ds \tag{3.12}$$

with $(r, \pi, w)_s^n = (r, \pi, w)_s^{\alpha.\beta.^n}$.

Introduce $\tau^n = \tau^n(\alpha.)$ as the first exit time of $x_t^n = x_t^n(\alpha.)$ from G and set

$$\phi_t^n = \phi_t^{\alpha.\beta.^n x}, \qquad \psi_t^n = \psi_t^{\alpha.\beta.^n x}.$$

Observe that by Itô's formula

$$\hat{u}(x_{t\wedge\tau^n}^n) e^{-\phi_{t\wedge\tau^n}^n - \psi_{t\wedge\tau^n}^n} = \hat{u}(x) + \int_0^{t\wedge\tau^n} [r_s^n]^2 e^{-\phi_s^n - \psi_s^n} L^{\alpha_s\beta_s^n} \hat{u}(x_s^n)\, ds + m_t^n, \tag{3.13}$$

where m_s^n is a martingale. Here according to our assumptions on the uniform continuity in x of the data and $D_{ij}\hat{u}(x)$ we have that for $s < \tau^n$ (notice the change of x_s^n to $x_{\kappa_n(s)}^n$)

$$\begin{aligned}L^{\alpha_s\beta_s^n}\hat{u}(x_s^n) \le\ & a_{ij}(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_{\kappa_n(s)}^n)D_{ij}\hat{u}(x_{\kappa_n(s)}^n)\\ &+ b_i(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_{\kappa_n(s)}^n)D_i\hat{u}(x_{\kappa_n(s)}^n)\\ &- c(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_{\kappa_n(s)}^n)\hat{u}(x_{\kappa_n(s)}^n) + \chi(x_s^n - x_{\kappa_n(s)}^n)\end{aligned}$$

where $\chi(y)$ is a (nonrandom) bounded function on $\mathbb{R}^d$ such that $\chi(y) \to 0$ as $y \to 0$. All such functions will be denoted by χ even if they may change from one occurrence to another.

Then (3.7) shows that, for $s < \tau^n$,

$$\begin{aligned}L^{\alpha_s\beta_s^n}\hat{u}(x_s^n) &\le \varepsilon + \chi(x_s^n - x_{\kappa_n(s)}^n) - f(\alpha_s, \beta(\alpha_s, x_{\kappa_n(s)}^n), x_{\kappa_n(s)}^n)\\ &\le \varepsilon + \chi(x_s^n - x_{\kappa_n(s)}^n) - f^{\alpha_s\beta_s^n}(x_s^n),\end{aligned}$$

which along with (3.13) implies that, for

$$\begin{aligned}&\eta_t^{n\varepsilon} = \eta_t^{n\varepsilon}(\alpha.) = \delta_1^{-2}\int_0^{t\wedge\tau^n} e^{-\phi_s^n-\psi_s^n}[\varepsilon + \chi(x_s^n - x_{\kappa_n(s)}^n)]\,ds,\\ &\kappa_{t\wedge\tau^n}^{n\varepsilon} = \zeta_t^{n\varepsilon} + m_t^n,\end{aligned} \tag{3.14}$$

where $\zeta_t^{n\varepsilon}$ is a decreasing process.

Hence $\kappa_{t\wedge\tau^n}^{n\varepsilon}$ is at least a local supermartingale. Owing to Lemmas 2.4 and 3.2, (3.2) and (3.5) hold. It follows that the local supermartingale $\kappa_{t\wedge\tau^n}^{n\varepsilon}$ is, actually, a supermartingale.

Furthermore, Lemmas 2.4 and 3.2, the boundedness of χ, its continuity, and the fact that $\chi(0) = 0$ easily yield that

$$\sup_{\alpha.\in\mathfrak{A}} E\int_0^{\tau^n(\alpha.)} e^{-\phi_s^n-\psi_s^n}\chi(x_s^n(\alpha.) - x_{\kappa_n(s)}^n(\alpha.))\,ds \to 0 \tag{3.15}$$

as $n \to \infty$, which proves (3.4). The lemma is proved.

For treating $\check{u}$ we use the following result.

Lemma 3.4. *Let $H[\check{u}] \ge 0$ in G. Then for any $x \in \mathbb{R}^d$, $\boldsymbol{\beta} \in \mathbb{B}$, and $\varepsilon > 0$, there exist a sequence $\alpha_\cdot^n \in \mathfrak{A}$, $n = 1, 2, \dots$, and a sequence of increasing continuous $\{\mathcal{F}_t\}$-adapted processes $\eta_t^{n\varepsilon}(\boldsymbol{\beta})$ with $\eta_0^{n\varepsilon}(\boldsymbol{\beta}) = 0$ such that the processes*

$$\kappa_t^{n\varepsilon} := \check{u}(x_t^n)e^{-\phi_t^n-\psi_t^n} + \eta_t^{n\varepsilon}(\boldsymbol{\beta}) + \int_0^t [r_s^n]^2 f_s^n(x_s^n)e^{-\phi_s^n-\psi_s^n}\,ds,$$

where

$$\begin{aligned}&(x_t^n, \phi_t^n, \psi_t^n) = (x_t, \phi_t, \psi_t)^{\alpha_\cdot^n\boldsymbol{\beta}(\alpha_\cdot^n)x}, \qquad f_t^n(x) = f^{\alpha_t^n\boldsymbol{\beta}_t(\alpha_\cdot^n)}(x),\\ &r_t^n = r_t^{\alpha_\cdot^n\boldsymbol{\beta}(\alpha_\cdot^n)},\end{aligned} \tag{3.16}$$

are submartingales on $[0, \tau^{\alpha_\cdot^n\boldsymbol{\beta}(\alpha_\cdot^n)x}]$ and

$$\sup_n E\eta_\infty^{n\varepsilon}(\boldsymbol{\beta}) < \infty, \tag{3.17}$$

$$\overline{\lim_{n\to\infty}}\, E\eta_\tau^{n\varepsilon}(\boldsymbol{\beta}) \le N\varepsilon, \tag{3.18}$$

where N is independent of x, $\boldsymbol{\beta}$, and ε.

Finally,

$$\sup_n E \sup_{t\geq 0} |\kappa^{n\varepsilon}_{t\wedge\tau}| < \infty.$$

Proof. Owing to Assumption 2.1 the function

$$h(\alpha, x) := \inf_{\beta\in B}\big[L^{\alpha\beta}\check{u}(x) + f^{\alpha\beta}(x)\big]$$

is a finite Borel function of x and is continuous with respect to α. Its sup over A can be replaced with the sup over an appropriate countable subset of A and since

$$\sup_{\alpha\in A} h(\alpha, x) \geq 0,$$

similarly to how $\beta(\alpha, x)$ was defined in the proof of Lemma 3.3, one can find a Borel function $\bar{\alpha}(x)$ in such a way that

$$\inf_{\beta\in B}\big[L^{\bar{\alpha}(x)\beta}\check{u}(x) + f^{\bar{\alpha}(x)\beta}(x)\big] \geq -\varepsilon \tag{3.19}$$

in G. If $x \notin G$ we set $\bar{\alpha}(x) = \alpha^*$, where α^* is a fixed element of A.

After that we need some processes which we introduce recursively. Fix x and set $\alpha^{n0}_t \equiv \bar{\alpha}(x)$. Then define x^{n0}_t, $t \geq 0$, as a unique solution of the equation

$$\begin{aligned} x_t &= x + \int_0^t r_s^{\alpha^{n0}_\cdot \boldsymbol{\beta}(\alpha^{n0}_\cdot)} \sigma(\alpha^{n0}_s, \boldsymbol{\beta}_s(\alpha^{n0}_\cdot), x_s)\, dw_s^{\alpha^{n0}_\cdot \boldsymbol{\beta}(\alpha^{n0}_\cdot)} \\ &\quad + \int_0^t [r_s^{\alpha^{n0}_\cdot \boldsymbol{\beta}(\alpha^{n0}_\cdot)}]^2 \big[b(\alpha^{n0}_s, \boldsymbol{\beta}_s(\alpha^{n0}_\cdot), x_s) + \sigma(\alpha^{n0}_s, \boldsymbol{\beta}_s(\alpha^{n0}_\cdot), x_s)\pi_s^{\alpha^{n0}_\cdot \boldsymbol{\beta}(\alpha^{n0}_\cdot)}\big]\, ds. \end{aligned}$$

For $k \geq 1$ introduce α^{nk}_t so that

$$\begin{aligned} \alpha^{nk}_t &= \alpha^{n(k-1)}_t \quad \text{for } t < k/n, \\ \alpha^{nk}_t &= \bar{\alpha}(x^{n(k-1)}_{k/n}) \quad \text{for } t \geq k/n, \end{aligned}$$

where $x^{n(k-1)}_t$ is a unique solution of

$$\begin{aligned} x_t &= x + \int_0^t r_s^{\alpha^{n(k-1)}_\cdot \boldsymbol{\beta}(\alpha^{n(k-1)}_\cdot)} \sigma(\alpha^{n(k-1)}_s, \boldsymbol{\beta}_s(\alpha^{n(k-1)}_\cdot), x_s)\, dw_s^{\alpha^{n(k-1)}_\cdot \boldsymbol{\beta}(\alpha^{n(k-1)}_\cdot)} \\ &\quad + \int_0^t [r_s^{\alpha^{n(k-1)}_\cdot \boldsymbol{\beta}(\alpha^{n(k-1)}_\cdot)}]^2 \big[b(\alpha^{n(k-1)}_s, \boldsymbol{\beta}_s(\alpha^{n(k-1)}_\cdot), x_s) \\ &\quad + \sigma(\alpha^{n(k-1)}_s, \boldsymbol{\beta}_s(\alpha^{n(k-1)}_\cdot), x_s)\pi_s^{\alpha^{n(k-1)}_\cdot \boldsymbol{\beta}(\alpha^{n(k-1)}_\cdot)}\big]\, ds. \end{aligned} \tag{3.20}$$

As in the proof of Lemma 3.3 one can show that the above definitions make sense as well as the definition

$$\alpha^n_t = \alpha^{n(k-1)}_t \quad \text{for } t < k/n. \tag{3.21}$$

Next, by definition $x^n_t = x_t^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)x}$ satisfies

$$\begin{aligned} x_t &= x + \int_0^t r_s^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)} \sigma(\alpha^n_s, \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s)\, dw_s^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)} \\ &\quad + \int_0^t [r_s^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)}]^2 \big[b(\alpha^n_s, \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s) + \sigma(\alpha^n_s, \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s)\pi_s^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)}\big]\, ds. \end{aligned}$$

Eq. (3.21) and the definitions of $\mathbb{B}$ and of control adapted processes show that x_t^n satisfies (3.20) for $t \leq k/n$. Hence, (a.s.) $x_t^n = x_t^{n(k-1)}$ for all $t \leq k/n$ and (a.s.) for all $t \geq 0$, $\alpha_t^n = \bar{\alpha}(x^n_{\kappa_n(t)})$ and

$$x_t^n = x + \int_0^t r_s^n \sigma(\bar{\alpha}(x^n_{\kappa_n(s)}), \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s^n)\, dw_s^n$$
$$+ \int_0^t [r_s^n]^2 \big[b(\bar{\alpha}(x^n_{\kappa_n(s)}), \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s^n) + \sigma(\bar{\alpha}(x^n_{\kappa_n(s)}), \boldsymbol{\beta}_s(\alpha^n_\cdot), x_s^n)\pi_s^n\big]\, ds,$$

where $(r, \pi, w)_s^n = (r, \pi, w)_s^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)}$.

Now, introduce τ^n as the first exit time of x_t^n from G, set

$$\beta_s^n = \boldsymbol{\beta}_s(\alpha^n_\cdot), \qquad \phi_t^n = \phi_t^{\alpha^n_\cdot \beta^n_\cdot x}, \qquad \psi_t^n = \psi_t^{\alpha^n_\cdot \beta^n_\cdot x}, \qquad r_s^n = r^{\alpha^n_\cdot \beta^n_\cdot},$$

and observe that by Itô's formula

$$\check{u}(x^n_{t\wedge\tau^n}) e^{-\phi^n_{t\wedge\tau^n} - \psi^n_{t\wedge\tau^n}} = \check{u}(x) + \int_0^{t\wedge\tau} [r_s^n]^2 e^{-\phi_s^n - \psi_s^n} L^{\alpha_s^n \beta_s^n} \check{u}(x_s^n)\, ds + m_t^n,$$

where m_s^n is a martingale and, for $s < \tau^n$,

$$L^{\alpha_s^n \beta_s^n} \check{u}(x_s^n) = a_{ij}(\bar{\alpha}(x^n_{\kappa_n(s)}), \beta_s^n, x_s^n) D_{ij} \check{u}(x_s^n)$$
$$+ b_i(\bar{\alpha}(x^n_{\kappa_n(s)}), \beta_s^n, x_s^n) D_i \check{u}(x_s^n) - c(\bar{\alpha}(x^n_{\kappa_n(s)}), \beta_s^n, x_s^n) \check{u}(x_s^n).$$

Similarly to the proof of Lemma 3.3 we derive from (3.19) that, for $s < \tau^n$,

$$\bar{L}^{\alpha_s^n \beta_s^n} \check{u}(x_s^n) \geq -\varepsilon - \chi(x_s^n - x^n_{\kappa_n(s)}) - f(\bar{\alpha}(x^n_{\kappa_n(s)}), \beta_s^n, x^n_{\kappa_n(s)})$$
$$= -\varepsilon - \chi(x_s^n - x^n_{\kappa_n(s)}) - f^{\alpha_s^n \beta_s^n}(x_s^n),$$

where $\chi(y)$ are (nonrandom) bounded functions on $\mathbb{R}^d$ such that $\chi(y) \to 0$ as $y \to 0$. It follows that

$$\check{u}(x^n_{t\wedge\tau^n}) e^{-\phi^n_{t\wedge\tau^n} - \psi^n_{t\wedge\tau^n}} + \int_0^{t\wedge\tau^n} [r_s^n]^2 f^{\alpha_s^n \beta_s^n}(x_s^n) e^{-\phi_s^n - \psi_s^n}\, ds + \eta_t^n = \zeta_t + m_t^n, \tag{3.22}$$

where ζ_t is an increasing process and

$$\eta_t^n = \eta_t^n(\boldsymbol{\beta}) = \delta_1^{-2} \int_0^{t\wedge\tau^n} e^{-\phi_s^n - \psi_s^n} [\varepsilon + \chi_\varepsilon(x_s^n - x^n_{\kappa_n(s)})]\, ds.$$

Hence the left-hand side of (3.22) is a local submartingale and we finish the proof in the same way as the proof of Lemma 3.3. The lemma is proved.

Proof of Theorem 3.1. (i) First we fix $x \in \mathbb{R}^d$, $\alpha_\cdot \in \mathfrak{A}$, and $\varepsilon > 0$, take $\beta^n_\cdot(\alpha_\cdot)$ form Lemma 3.3 and prove that the $\mathfrak{B}$-valued functions defined on $\mathfrak{A}$ by $\boldsymbol{\beta}^n(\alpha_\cdot) = \beta^n_\cdot(\alpha_\cdot)$ belong to $\mathbb{B}$. To do that observe that if (2.1) holds and $T \leq 1/n$, then (a.s.) $\beta_t^{n0}(\alpha^1_\cdot) = \beta_t^{n0}(\alpha^2_\cdot)$ for almost all $t \leq T$. By definition also (a.s.)

$$(r, \pi, w)_s^{\alpha^1_\cdot \beta^{n0}_\cdot(\alpha^1_\cdot)} = (r, \pi, w)_s^{\alpha^2_\cdot \beta^{n0}_\cdot(\alpha^2_\cdot)} \quad \text{for almost all } s \leq T.$$

By uniqueness of solutions of (2.3) (see Assumption 3.2), the processes x_t^{n1} found from (3.9) for $\alpha_\cdot = \alpha^1_\cdot$ and for $\alpha_\cdot = \alpha^2_\cdot$ coincide (a.s.) for all $t \leq T$.

If (2.1) holds and $1/n < T \leq 2/n$, then by the above solutions of (3.9) for $\alpha. = \alpha^1_\cdot$ and for $\alpha. = \alpha^2_\cdot$ coincide (a.s.) for $t = 1/n$ and then (a.s.) $\beta^{n1}_t(\alpha^1_\cdot) = \beta^{n1}_t(\alpha^2_\cdot)$ not only for all $t < 1/n$ but also for all $t \geq 1/n$, which implies that (a.s.)

$$(r, \pi, w)^{\alpha^1_\cdot \beta^{n1}(\alpha^1_\cdot)}_s = (r, \pi, w)^{\alpha^2_\cdot \beta^{n1}(\alpha^2_\cdot)}_s \quad \text{for almost all } s \leq T$$

and again the processes x^n_t found from (3.9) for $\alpha. = \alpha^1_\cdot$ and for $\alpha. = \alpha^2_\cdot$ coincide (a.s.) for all $t \leq T$.

By induction we get that if (2.1) holds for a $T \in (0, \infty)$ and we define k as the integer such that $k/n < T \leq (k+1)/n$, then (a.s.)

$$\beta^n_t(\alpha^1_\cdot) = \beta^{nk}_t(\alpha^1_\cdot) = \beta^{nk}_t(\alpha^2_\cdot) = \beta^n_t(\alpha^2_\cdot) \quad \text{for almost all } t < (k+1)/n, \tag{3.23}$$
$$(r, \pi, w)^{\alpha^1_\cdot \beta^{nk}(\alpha^1_\cdot)}_s = (r, \pi, w)^{\alpha^2_\cdot \beta^{nk}(\alpha^2_\cdot)}_s \quad \text{for almost all } s \leq T$$

and the processes x^n_t found from (3.9) for $\alpha. = \alpha^1_\cdot$ and for $\alpha. = \alpha^2_\cdot$ coincide (a.s.) for all $t \leq T$. This means that $\boldsymbol{\beta}^n \in \mathbb{B}$ indeed.

Furthermore, by the supermartingale property of $\kappa^{n\varepsilon}_t(\alpha.)$, we have

$$\hat{u}(x) \geq E^{\alpha.\boldsymbol{\beta}^n(\alpha.)}_x \Big[g(x_\tau) e^{-\phi_\tau - \psi_\tau} + \int_0^\tau r_t^2 f(x_t) e^{-\phi_t - \psi_t}\, dt \Big] - E\eta^{n\varepsilon}_\tau(\alpha.),$$

which owing to (3.4) yields

$$\hat{u}(x) \geq \varliminf_{n\to\infty} \sup_{\alpha.\in\mathfrak{A}} E^{\alpha.\boldsymbol{\beta}^n(\alpha.)}_x \Big[\int_0^\tau r_t^2 f(x_t) e^{-\phi_t - \psi_t}\, dt + g(x_\tau) e^{-\phi_\tau - \psi_\tau} \Big] - N\varepsilon.$$

In light of the arbitrariness of ε we conclude $\hat{u} \geq v$ and assertion (i) is proved.

(ii) Similarly to the above argument, for any $\boldsymbol{\beta} \in \mathbb{B}$,

$$\check{u}(x) \leq E^{\alpha^n_\cdot \boldsymbol{\beta}(\alpha^n_\cdot)}_x \Big[\int_0^\tau r_t^2 f(x_t) e^{-\phi_t - \psi_t}\, dt + g(x_\tau) e^{-\phi_\tau - \psi_\tau} \Big] + E\eta^{n\varepsilon}_\tau(\boldsymbol{\beta}).$$

It follows that

$$\check{u}(x) \leq \sup_{\alpha.\in\mathfrak{A}} E^{\alpha.\boldsymbol{\beta}(\alpha.)}_x \Big[\int_0^\tau r_t^2 f(x_t) e^{-\phi_t - \psi_t}\, dt + g(x_\tau) e^{-\phi_\tau - \psi_\tau} \Big] + \varlimsup_{n\to\infty} E\eta^{n\varepsilon}_\tau(\boldsymbol{\beta})$$
$$\leq \sup_{\alpha.\in\mathfrak{A}} E^{\alpha.\boldsymbol{\beta}(\alpha.)}_x \Big[\int_0^\tau r_t^2 f(x_t) e^{-\phi_t - \psi_t}\, dt + g(x_\tau) e^{-\phi_\tau - \psi_\tau} \Big] + N\varepsilon,$$

which in light of the arbitrariness of ε and $\boldsymbol{\beta} \in \mathbb{B}$ finally yields that $\check{u} \leq v$.

This proves assertion (ii). Assertion (iii) is an obvious consequence of (i) and (ii). The theorem is proved.

4. The case of uniformly nondegenerate processes

As in Section 3 we replace Assumption 2.1(iii) with Assumptions 3.1 and 3.2, and we assume that we are given two functions $\hat{u}, \check{u} \in W^2_{d,loc}(G) \cap C(\bar{G})$.

In that case we have the following.

Theorem 4.1. (i) *If $H[\hat{u}] \leq 0$ (a.e.) in G and $\hat{u} \geq g$ on ∂G, then $v \leq \hat{u}$ in $\bar{G}$.*
(ii) *If $H[\check{u}] \geq 0$ (a.e.) in G and $\check{u} \leq g$ on ∂G, then $v \geq \check{u}$ in $\bar{G}$.*
(iii) *If $\hat{u}$ and $\check{u}$ are as in* (i) *and* (ii) *and $\hat{u} = \check{u}$, then $v = \hat{u}$ and v is independent of the choice of the probability space, filtration, r, π, and w.*

Proof. (i) We basically repeat the proof of Theorem 4.1 of [10] with considerable simplifications made possible due to our assumptions. It is well known that there exists a sequence $\hat{u}_n \in C^2(\bar{G})$ such that $\hat{u}_n \to \hat{u}$ in $C(\bar{D})$ and in $W^2_d(G')$ for any subdomain $G' \subset \bar{G}' \subset G$. Introduce $\hat{h}_n = H[\hat{u}_n]$,

$$f_n^{\alpha\beta}(x) = f^{\alpha\beta}(x) - \hat{h}_n(x),$$

and observe that owing to our continuity assumptions on σ, b, c, f, the functions $\hat{h}_n$ and $f_n^{\alpha\beta}(x)$ are continuous in x uniformly with respect to α, β and

$$\sup_{\alpha\in A}\inf_{\beta\in B}\left[L_n^{\alpha\beta}\hat{u}_n(x) + f_n^{\alpha\beta}(x)\right] = 0$$

in G. By Theorem 3.1, for any subdomain $G_1 \subset \bar{G}_1 \subset G$ we have in G_1 that

$$\hat{u}_n(x) \geq \inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\boldsymbol{\beta}(\alpha.)}\left[\hat{u}_n(x_{\tau_1})e^{-\phi_{\tau_1}-\psi_{\tau_1}} + \int_0^{\tau_1} r_t^2 f_n(x_t)e^{-\phi_t-\psi_t}\,dt\right], \tag{4.1}$$

where

$$\tau_1^{\alpha.\beta.x} = \inf\{t \geq 0 : x_t^{\alpha.\beta.x} \notin G_1\}.$$

Notice that

$$E_x^{\alpha.\beta.}|\hat{u}_n(x_{\tau_1}) - \hat{u}(x_{\tau_1})|e^{-\phi_{\tau_1}-\psi_{\tau_1}} \leq \sup_G |\hat{u}_n - \hat{u}|.$$

While estimating

$$I_n(x) := E_x^{\alpha.\beta.}\int_0^{\tau_1}|f_n - f|(x_t)e^{-\phi_t-\psi_t}\,dt = E_x^{\alpha.\beta.}e^{-\psi_{\tau_1}}\int_0^{\tau_1}|f_n - f|(x_t)e^{-\phi_t}\,dt,$$

Girsanov's theorem allows us to concentrate on $\pi \equiv 0$ and then the Alexandrov estimate guarantees that

$$I_n(x) \leq N\|\hat{h}_n\|_{\mathcal{L}_d(G_1)} = N\|H[\hat{u}_n] - H[\hat{u}]\|_{\mathcal{L}_d(G_1)} \leq N\|\hat{u}_n - \hat{u}\|_{W^2_d(G_1)},$$

where the constants N are independent of n (and x).

Hence by letting $n \to \infty$ in (4.1) we obtain that for $k = 1$

$$\hat{u}(x) \geq \inf_{\boldsymbol{\beta}\in\mathbb{B}}\sup_{\alpha.\in\mathfrak{A}} E_x^{\alpha.\boldsymbol{\beta}(\alpha.)}\left[\hat{u}(x_{\tau_k})e^{-\phi_{\tau_k}-\psi_{\tau_k}} + \int_0^{\tau_k} f(x_t)e^{-\phi_t-\psi_t}\,dt\right], \tag{4.2}$$

where $\tau_k^{\alpha.\beta.x}$ are defined as the first exit times of the processes $x_t^{\alpha.\beta.x}$ from an expanding sequence of subdomains $G_k \subset \bar{G}_k \subset G$ such that $\cup_k G_k = G$.

By letting $k \to \infty$ in (4.2) and repeating the proof of Theorem 2.2 of [10] given there in Section 6 we get that $\hat{u} \geq v$ in G as stated. Observe that in our situation in the proof of Theorem 2.2 of [10] we need not mollify $f^{\alpha\beta}(x)$ because by assumption it is uniformly continuous in x.

The proof of assertion (ii) is quite similar and as usual assertion (iii) is obtained by simply combining assertions (i) and (ii). The theorem is proved.

5. A general approximation result from above

In this section we suppose that all assumptions in Section 2 are satisfied. Set

$$A_1 = A$$

and let A_2 be a separable metric space having no common points with A_1.

Assumption 5.1. The functions $\sigma^{\alpha\beta}(x)$, $b^{\alpha\beta}(x)$, $c^{\alpha\beta}(x)$, and $f^{\alpha\beta}(x)$ are also defined on $A_2 \times B \times \mathbb{R}^d$ in such a way that they are *independent* of β (on $A_2 \times B \times \mathbb{R}^d$) and the assumptions in Section 2 are satisfied, of course, with A_2 in place of A.

Define

$$\hat{A} = A_1 \cup A_2.$$

Then we introduce $\hat{\mathfrak{A}}$ as the set of progressively measurable $\hat{A}$-valued processes and $\hat{\mathbb{B}}$ as the set of $\mathfrak{B}$-valued functions $\boldsymbol{\beta}(\alpha.)$ on $\hat{\mathfrak{A}}$ such that, for any $T \in [0,\infty)$ and any $\alpha^1_\cdot, \alpha^2_\cdot \in \hat{\mathfrak{A}}$ satisfying

$$P(\alpha^1_t = \alpha^2_t \text{ for almost all } t \le T) = 1,$$

we have

$$P(\boldsymbol{\beta}_t(\alpha^1_\cdot) = \boldsymbol{\beta}_t(\alpha^2_\cdot) \text{ for almost all } t \le T) = 1.$$

We fix an element $\alpha^* \in A_1$ and for $\alpha. \in \hat{\mathfrak{A}}$ define

$$(p\alpha)_t = \alpha_t \quad \text{if } \alpha_t \in A_1, \qquad (p\alpha)_t = \alpha^* \quad \text{if } \alpha_t \in A_2.$$

By using this projection operator we extend $(w,r,\pi)^{\alpha.\beta.}_t$ originally defined for $\alpha. \in \mathfrak{A}$ and $\beta. \in \mathfrak{B}$ as

$$(w,r,\pi)^{\alpha.\beta.}_t = (w,r,\pi)^{p\alpha.\beta.}_t \tag{5.1}$$

thereby now defined for $\alpha. \in \hat{\mathfrak{A}}$ and $\beta. \in \mathfrak{B}$.

Next, take a constant $K \ge 0$ and set

$$v_K(x) = \inf_{\boldsymbol{\beta}\in\hat{\mathbb{B}}} \sup_{\alpha.\in\hat{\mathfrak{A}}} v_K^{\alpha.\boldsymbol{\beta}(\alpha.)}(x),$$

where

$$\begin{aligned} v_K^{\alpha.\beta.}(x) &= E_x^{\alpha.\beta.}\Big[\int_0^\tau r_t^2 f_K(x_t)e^{-\phi_t-\psi_t}\,dt + g(x_\tau)e^{-\phi_\tau-\psi_\tau}\Big] \\ &=: v^{\alpha.\beta.}(x) - K E_x^{\alpha.\beta.}\int_0^\tau r_t^2 I_{\alpha_t\in A_2} e^{-\phi_t-\psi_t}\,dt, \end{aligned}$$

$$f_K^{\alpha\beta}(x) = f^{\alpha\beta}(x) - K I_{\alpha\in A_2}.$$

Notice that, obviously,

$$v(x) = \inf_{\boldsymbol{\beta}\in\hat{\mathbb{B}}} \sup_{\alpha.\in\hat{\mathfrak{A}}} v^{p\alpha.\boldsymbol{\beta}(p\alpha.)}(x).$$

These definitions make sense owing to Remark 2.3, which also implies that $v_K^{\alpha.\beta.}$ and $v^{\alpha.\beta.}$ and bounded in $\bar{G}$.

Theorem 5.1. *We have $v_K \to v$ uniformly on $\bar{G}$ as $K \to \infty$.*

Lemma 5.2. *Assume that $\pi \equiv 0$. Then there exists a constant N such that for any $\alpha. \in \hat{\mathfrak{A}}$, $\beta. \in \mathfrak{B}$, $x \in \mathbb{R}^d$, $T \in [0,\infty)$, and stopping time γ*

$$E_x^{\alpha.\beta.} \sup_{t\le T\wedge\gamma} |x_t - y_t| \le N e^{NT}\big(E_x^{\alpha.\beta.}\int_0^{T\wedge\gamma} I_{\alpha_t\in A_2}\,dt\big)^{1/2},$$

where

$$y_t^{\alpha.\beta.x} = x_t^{p\alpha.\beta.x}.$$

Proof. For simplicity of notation we drop the superscripts $\alpha.$, $\beta.$, x. Observe that x_t and y_t satisfy

$$x_t = x + \int_0^t r_s \sigma^{\alpha_s \beta_s}(x_s)\,dw_s + \int_0^t r_s^2 b^{\alpha_s\beta_s}(x_s)\,ds,$$

$$y_t = x + \int_0^t r_s \sigma^{\alpha_s \beta_s}(y_s)\,dw_s + \int_0^t r_s^2 b^{\alpha_s\beta_s}(y_s)\,ds + \eta_t,$$

where $\eta_t = I_t + J_t$,

$$I_t = \int_0^t r_s[\sigma^{p\alpha_s\beta_s}(y_s) - \sigma^{\alpha_s\beta_s}(y_s)]\,dw_s,$$

$$J_t = \int_0^t r_s^2[b^{p\alpha_s\beta_s}(y_s) - b^{\alpha_s\beta_s}(y_s)]\,ds.$$

By Theorem II.5.9 of [7] (where we replace the processes x_t and $\tilde{x}_t$ with appropriately stopped ones) for any $T \in [0,\infty)$ and any stopping time γ

$$E \sup_{t\le T\wedge\gamma} |x_t - y_t|^2 \le N e^{NT} E \sup_{t\le T\wedge\gamma} |\eta_t|^2, \tag{5.2}$$

where N depends only on K_1 and d, which by Theorem III.6.8 of [8] leads to

$$E \sup_{t\le T\wedge\gamma} |x_t - y_t| \le N e^{NT} E \sup_{t\le T\wedge\gamma} |\eta_t| \tag{5.3}$$

with the constant N being three times the one from (5.2).

By using Davis's inequality we see that for any $T \in [0,\infty)$

$$E \sup_{t\le T\wedge\gamma} |I_t| \le NE\Big(\int_0^{T\wedge\gamma} I_{\alpha_s\in A_2}\,ds\Big)^{1/2} \le N\Big(E\int_0^{T\wedge\gamma} I_{\alpha_s\in A_2}\,ds\Big)^{1/2}.$$

Furthermore, almost obviously

$$E \sup_{t\le T\wedge\gamma} |J_t| \le NE\int_0^{T\wedge\gamma} I_{\alpha_s\in A_2}\,ds \le NT^{1/2}\Big(E\int_0^{T\wedge\gamma} I_{\alpha_s\in A_2}\,ds\Big)^{1/2}$$

and this in combination with (5.3) proves the lemma.

Proof of Theorem 5.1. Without losing generality we may assume that $g \in C^3(\mathbb{R}^d)$ since the functions of this class uniformly approximate in $\bar{G}$ any g which is continuous in $\mathbb{R}^d$. Then notice that by Itô's formula for $g \in C^3(\mathbb{R}^d)$ we have

$$E_x^{\alpha.\beta.}\Big[\int_0^\tau r_t^2 f_K(x_t) e^{-\phi_t-\psi_t}\,dt + g(x_\tau)e^{-\phi_\tau-\psi_\tau}\Big]$$
$$= g(x) + E_x^{\alpha.\beta.}\int_0^\tau r_t^2[\hat{f}(x_t) - KI_{\alpha_t\in A_2}]e^{-\phi_t-\psi_t}\,dt,$$

where

$$\hat{f}^{\alpha\beta}(x) := f^{\alpha\beta}(x) + L^{\alpha\beta}g(x),$$

which is bounded and, for $(\alpha,\beta)\in\hat{A}\times B$, is uniformly continuous in x uniformly with respect to α,β. This argument shows that without losing generality we may (and will) also assume that $g=0$.

Next, since $\mathfrak{A}\subset\hat{\mathfrak{A}}$ and for $\alpha.\in\hat{\mathfrak{A}}$ and $\boldsymbol{\beta}\in\hat{\mathbb{B}}$ we have $\boldsymbol{\beta}(\alpha.)\in\mathfrak{B}$, it holds that

$$v_K\geq v.$$

To estimate v_K from above, take $\boldsymbol{\beta}\in\mathbb{B}$ and define $\hat{\boldsymbol{\beta}}\in\hat{\mathbb{B}}$ by

$$\hat{\boldsymbol{\beta}}_t(\alpha.)=\boldsymbol{\beta}_t(p\alpha.). \tag{5.4}$$

Also take any sequence $x^n\in\bar{G}$, $n=1,2,\dots$, recall that $r_t^{\alpha.\beta.}\geq\delta_1$, and find a sequence $\alpha^n_\cdot\in\hat{\mathfrak{A}}$ such that

$$\begin{aligned}v_K(x^n)&\leq\sup_{\alpha\in\hat{\mathfrak{A}}}E_{x^n}^{\alpha.\hat{\boldsymbol{\beta}}(\alpha.)}\int_0^\tau r_t^2f_K(x_t)e^{-\phi_t-\psi_t}\,dt\\&\leq 1/n+v^{\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)}(x^n)-K\delta_1^2E\int_0^{\tau^n}I_{\alpha^n_t\in A_2}e^{-\phi^n_t-\psi^n_t}\,dt,\end{aligned} \tag{5.5}$$

where

$$(\tau^n,\phi^n_t,\psi^n_t)=(\tau,\phi_t,\psi_t)^{\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)x^n}.$$

It follows that there is a constant N independent of n and K such that

$$E\int_0^{\tau^n}I_{\alpha^n_t\in A_2}e^{-\phi^n_t-\psi^n_t}\,dt\leq N/K. \tag{5.6}$$

Below by N we denote generic constants independent of n and K (and T once it appears).

We want to estimate the difference

$$v^{\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)}(x^n)-v^{p\alpha^n_\cdot\boldsymbol{\beta}(p\alpha^n_\cdot)}(x^n). \tag{5.7}$$

Observe that in the expression of this difference by the definition through the mathematical expectations of certain quantities the processes ψ_t involved are just the same, thanks to (5.1) and (5.4). This allows us to rewrite the mathematical expectations similarly to how it is done in Remark 2.3 and then by using Girsanov's theorem allows us to assume that $\pi_t\equiv 0$, at the expense that the underlying probability measures will now depend on n. However, for simplicity of notation we keep the symbol E for expectations with respect to the new probability measures depending on n. Thus, while estimating (5.7) we assume that $\pi_t\equiv 0$.

Introduce

$$\begin{aligned}&x^n_t=x_t^{\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)x^n}, &&y^n_t=x_t^{p\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)x^n},\\&c^n_t=c^{\alpha^n_t\hat{\beta}_t(\alpha^n_\cdot)}(x^n_t), &&pc^n_t=c^{p\alpha^n_t\hat{\beta}_t(\alpha^n_\cdot)}(y^n_t)\\&f^n_t=f^{\alpha^n_t\hat{\beta}_t(\alpha^n_\cdot)}(x^n_t), &&pf^n_t=f^{p\alpha^n_t\hat{\beta}_t(\alpha^n_\cdot)}(y^n_t)\\&r^n_t=r_t^{\alpha^n_\cdot\hat{\boldsymbol{\beta}}(\alpha^n_\cdot)}, &&p\phi^n_t=\int_0^t[r^n_s]^2pc^n_s\,ds,\end{aligned}$$

and define γ^n as the first exit time of y^n_t from G. Notice that, for any $T\in[0,\infty)$, (5.7) equals

$$I_{1n}(T)+I_{2n}(T)-I_{3n}(T),$$

where

$$I_{1n}(T)=E\int_0^{\tau^n\wedge\gamma^n\wedge T}[r^n_t]^2\big[f^n_t\exp(-\phi^n_t)-pf^n_t\exp(-p\phi^n_t)\big]\,dt,$$
$$I_{2n}(T)=E\int_{\tau^n\wedge\gamma^n\wedge T}^{\tau^n}[r^n_t]^2f^n_t\exp(-\phi^n_t)\,dt,$$
$$I_{3n}(T)=E\int_{\tau^n\wedge\gamma^n\wedge T}^{\gamma^n}[r^n_t]^2pf^n_t\exp(-p\phi^n_t)\,dt,$$

By using the inequalities $|e^{-a}-e^{-b}|\le|a-b|$ valid for $a,b\ge 0$ and $|ab-cd|\le|b|\cdot|a-c|+|c|\cdot|b-d|$ and also using the boundedness of $r^{\alpha.\beta.}$, $c^{\alpha\beta}$, and $f^{\alpha\beta}$ we easily conclude that

$$|I_{1n}(T)|\le N(1+T)E\int_0^{\tau^n\wedge T}\big[|f^n_t-pf^n_t|+|c^n_t-pc^n_t|\big]\,dt.$$

Observe that, if $\alpha^n_t\in A_1$, then

$$|f^n_t-pf^n_t|\le W_f(|x^n-y^n_t|),$$

where W_f is the modulus of continuity of $f^{\alpha\beta}(x)$ with respect to x uniform with respect to α,β. A similar estimate holds for $|c^n_t-pc^n_t|$ in which W_c is the modulus of continuity of $c^{\alpha\beta}(x)$. Furthermore,

$$E\int_0^{\tau^n\wedge T}I_{\alpha^n_t\in A_2}\,dt\le e^{T/\delta}E\int_0^{\tau^n}I_{\alpha^n_t\in A_2}e^{-\phi^n_t}\,dt\le Ne^{T/\delta}/K,$$

where the last inequality is due to (5.6). Hence,

$$|I_{1n}(T)|\le N(1+T)^2E[W_c+W_f]\left(\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|\right)+Ne^{T/N}/K.$$

We may and will assume that $W_c(r)$ and $W_f(r)$ are concave functions on $[0,\infty)$, so that

$$|I_{1n}(T)|\le N(1+T)^2[W_c+W_f]\left(E\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|\right)+Ne^{T/N}/K.$$

Next use the fact that as follows from Lemma 5.2

$$E\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|\le Ne^{NT}/\sqrt{K}.$$

Then we conclude that

$$|I_{1n}(T)|\le N(1+T)^2[W_c+W_f]\left(Ne^{NT}/\sqrt{K}\right)+Ne^{T/\delta}/K. \tag{5.8}$$

While estimating $I_{2n}(T)$ we again use the boundedness of the data and use Remark 2.1 and by Itô's formula obtain that

$$\begin{aligned}|I_{2n}(T)|&\le NEI_{\tau^n\ge\gamma^n\wedge T}\int_{\gamma^n\wedge T}^{\tau^n}[r^n_t]^2\,dt\le NEI_{\tau^n\ge\gamma^n\wedge T}G(x^n_{\gamma^n\wedge T})\\&\le NEI_{\tau^n\ge\gamma^n\wedge T}|G(x^n_{\gamma^n\wedge T})-G(y^n_{\gamma^n\wedge T})|+NEI_{\tau^n\ge\gamma^n\wedge T}|G(y^n_{\gamma^n\wedge T})|\\&\le NE\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|+NEI_{\gamma^n>T}G(y^n_T).\end{aligned}$$

By Lemma 5.1 of [10]

$$EI_{\gamma^n>T}G(y^n_T)\le Ne^{-T/N}.$$

Next,

$$\begin{aligned}|I_{3n}(T)| &\le NEI_{\gamma^n\ge\tau^n\wedge T}\int_{\tau^n\wedge T}^{\gamma^n}[r^n_t]^2\,dt\le NEI_{\gamma^n\ge\tau^n\wedge T}G(y^n_{\tau^n\wedge T})\\ &\le NE\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|+NEI_{\gamma^n\ge\tau^n\wedge T}G(x^n_{\tau^n\wedge T})\\ &\le NE\sup_{t\le\tau^n\wedge T}|x^n_t-y^n_t|+NEI_{\tau^n>T}G(x^n_T).\end{aligned}$$

We use again Lemma 5.1 of [10] and conclude that, for $K\ge1$, (5.7) is less than

$$w(T,K):=N(1+T)^2[W_c+W_f]\left(Ne^{N_1T}/\sqrt{K}\right)+Ne^{N_1T}/\sqrt{K}+Ne^{-T/N_2}.$$

Thus, (5.5) yields

$$v_K(x^n)\le1/n+v^{p\alpha^n_\cdot\boldsymbol{\beta}(p\alpha^n_\cdot)}(x^n)+w(T,K).$$

Hence

$$v_K(x^n)\le\sup_{\alpha_\cdot\in\mathfrak{A}}v^{\alpha_\cdot\boldsymbol{\beta}(\alpha_\cdot)}(x^n)+w(T,K)+1/n.$$

Owing to the arbitrariness of $\boldsymbol{\beta}\in\mathbb{B}$ we have

$$v_K(x^n)\le v(x^n)+w(T,K)+1/n,$$

and the arbitrariness of x^n yields that for $K\ge1$

$$\sup_{\bar D}(v_K-v)\le w(T,K),\tag{5.9}$$

which leads to the desired result after first letting $K\to\infty$ and then $T\to\infty$. The theorem is proved.

Remark 5.1. Assume that $c^{\alpha\beta}(x)$ and $f^{\alpha\beta}(x)$ are Hölder continuous with respect to x with exponent $\kappa\in(0,1]$ and constant independent of α and β. Then by taking T such that $e^{N_1T}=K^{1/4}$ we see that, for $K\ge1$, the left-hand side of (5.9) is dominated by

$$N(1+\ln K)^2K^{-\kappa/4}+NK^{-1/(4N_1N_2)}.$$

Hence, there is a $\chi\in(0,1]$ such that the left-hand side of (5.9) is dominated by $NK^{-\chi}$ for $K\ge1$. Thus, we have justified a claim made in Section 5 of [12].

6. Proof of Theorem 2.1

The properties of P listed before Theorem 2.2 or just the construction of P in [9] yield that there is a set A_2, having no common points with A, and bounded continuous functions $\sigma^\alpha=\sigma^{\alpha\beta}$, $b^\alpha=b^{\alpha\beta}$, $c^\alpha=c^{\alpha\beta}$ (independent of x and β), and $f^{\alpha\beta}\equiv0$ defined on A_2 such that the assumptions in Section 2 are satisfied perhaps with different constants δ and K_0 and for $a^\alpha:=a^{\alpha\beta}=(1/2)\sigma^\alpha(\sigma^\alpha)^*$ we have

$$P[u](x)=\sup_{\alpha\in A_2}\left[a^\alpha_{ij}D_{ij}u(x)+b^\alpha_iD_iu(x)-c^\alpha u(x)\right].\tag{6.1}$$

Use the notation from Section 5 and observe that

$$\begin{aligned}&\max(H[u](x), P[u](x) - K)\\&\quad = \max\Big\{\sup_{\alpha\in A_1}\inf_{\beta\in B}[L^{\alpha\beta}u(x) + f^{\alpha\beta}(x)], \sup_{\alpha\in A_2}\inf_{\beta\in B}[L^{\alpha\beta}u(x) + f^{\alpha\beta}(x) - K]\Big\}\\&\quad = \sup_{\alpha\in\hat{A}}\inf_{\beta\in B}\big[L^{\alpha\beta}u(x) + f_K^{\alpha\beta}(x)\big] \quad (f_K^{\alpha\beta}(x) = f^{\alpha\beta}(x)I_{\alpha\in A_1} - KI_{\alpha\in A_2}),\end{aligned}$$

where the first equality follows from the definition of $H[u]$, (6.1), and the fact that $L^{\alpha\beta}$ is independent of β for $\alpha \in A_2$. It follows by Theorems 2.2 and 4.1 that $u_K = v_K$ and by Theorem 5.1 that in $\bar{G}$

$$v = \lim_{K\to\infty} u_K,$$

where the right-hand side is indeed independent of the probability space, filtration, and the choice of w, r, π. Since the above convergence is uniform, v is continuous in $\bar{G}$. The theorem is proved.

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