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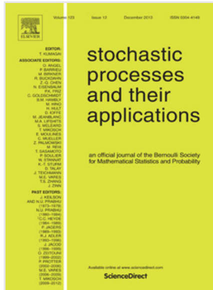


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Emmanuelle Clément, Arnaud Gloter

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# Local Asymptotic Mixed Normality property for discretely observed stochastic differential equations driven by stable Lévy processes

Emmanuelle Clément* Arnaud Gloter†

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revised version

**Abstract**

We prove the Local Asymptotic Mixed Normality property from high frequency observations, of a continuous time process solution of a stochastic differential equation driven by a pure jump Lévy process. The process is observed on the fixed time interval $[0,1]$ and the parameter appears in the drift coefficient only. We compute the asymptotic Fisher information and find that the rate in the LAMN property depends on the behavior of the Lévy measure near zero. The proof of this result contains a sharp study of the asymptotic behavior, in small time, of the transition probability density of the process and of its logarithm derivative.

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**Key words**: Local Asymptotic Mixed Normality Property, Lévy process, stable process, Malliavin calculus for jump processes.

## 1 Introduction

An important concept in parametric estimation is the Local Asymptotic Mixed Normality property introduced by Jeganathan in a series of papers ([11], [12]), which permits to extend the Le Cam and Hajek's results (see [8], [16]) to situations where the local Asymptotic Normality does not hold. Let

*Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050, 5 Bld Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, France.

†Université d'Évry Val d'Essonne, Laboratoire de Mathématiques et Modélisation d'Evry, UMR 8071, 23 Boulevard de France, 91037 Évry Cedex, France. This research benefited by the support of the 'Marchés en mutation', Fédération Bancaire Française.

$\{E_n, \mathcal{E}_n, (P_n^\theta)_{\theta\in\Theta\subset\mathbb{R}^d}\}$ be a statistical experiment, we say that the LAMN property holds at $\theta$ with information matrix $I(\theta)$ and rate $u_n$ ($u_n$ tends to zero as $n$ goes to infinity) if

$$\log \frac{dP_n^{\theta+u_n h}}{dP_n^\theta} = h^T I_n(\theta)^{1/2} N_n - \frac{1}{2} h^T I_n(\theta) h + o_{P_n^\theta}(1)$$

where $(N_n, I_n(\theta))$ converges in law (under $P_n^\theta$) to $(N, I(\theta))$ with $N$ a standard gaussian vector independent of $I(\theta)$, and $I(\theta) > 0$ a.e. The LAN property is obtained when the information matrix $I(\theta)$ is non random.

If the LAMN property is satisfied at $\theta$, then from the Hajek's convolution theorem, we know that for any regular estimator $\hat{\theta}_n$ such that

$$u_n^{-1}[\hat{\theta}_n - \theta] \Rightarrow Z_\theta \quad \text{(in law under } P_n^\theta\text{)},$$

$Z_\theta$ admits the decomposition $Z_\theta = I(\theta)^{-1/2} N + R$ with $N$ a standard gaussian vector and $R$ independent of $N$ conditionally on $I(\theta)$. As a consequence, the minimal asymptotic estimation error is a mixed normal variable with variance $I(\theta)^{-1}$.

In this paper, we consider the statistical experiment $\{\mathbb{R}^n, \mathcal{B}_n, (P_n^\theta)_{\theta\in\Theta\subset\mathbb{R}}\}$, corresponding to the observation of a Lévy driven stochastic equation at discrete times $t_i = \frac{i}{n}$, for $1 \leq i \leq n$. More precisely, we observe $(X^\theta_{\frac{i}{n}})_{1\leq i\leq n}$, where $(X_t^\theta)_{t\in[0,1]}$ is a continuous time process depending on an unknown real parameter $\theta$.

Lévy models are widely used in mathematical finance (see [3]) and there is a large literature concerning the estimation of the parameters and the LAN property, of a translated Lévy process

$$X_t^\theta = \theta_1 t + \theta_2 L_t, \quad \theta = (\theta_1, \theta_2),$$

see for example Aït-Sahalia and Jacod [1] [2], Masuda [17], Kawai and Masuda [13], [14]. In this case, the statistical study is based on the fact that the density of $X_t^\theta$ can be expressed as a function of the density of $L_t$. In particular, it is proved that when $L_t$ is a $\alpha$-stable process, with $\alpha \in (0, 2)$, the parameter $\theta_1$ is estimated with rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ and Fisher information $\int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$, where $\varphi_\alpha(u)$ is the density of $L_1$. The parameter $\theta_2$ is estimated with the usual rate $1/\sqrt{n}$.

Here, we intend to consider the more general stochastic equation

$$X_t^\theta = x_0 + \int_0^t b(X_s^\theta, \theta) ds + L_t \tag{1}$$

where $(L_t)_{t\in[0,1]}$ is a pure jump Lévy process, and focus on the estimation of the drift parameter. When $(X_t^\theta)_t$ is solution of (1), the transition density of $X_t^\theta$ is unknown, and the link between the density of $L_t$ and the density of $X_t^\theta$ is not clear. This complicates the statistical study considerably and to our knowledge, there are no results about the asymptotic behavior of the log-likelihood of the discretized process $(X_{i/n}^\theta)_{1\leq i\leq n}$.

In this paper, we prove the LAMN property based on the observations $(X_{\frac{i}{n}}^\theta)_i$ where $(X_t^\theta)_{t\in[0,1]}$ is solution of (1), with rate $u_n = n^{1/2-1/\alpha}$, when the Lévy measure of $(L_t)$ is an $\alpha$-stable Lévy measure near zero, with $\alpha \in (1,2)$. The case $\alpha \in (0,1]$ requires a more technical study and is not considered in this paper. The main result is obtained through a representation of the transition density of $X_t^\theta$, using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [4]. The recourse to the Malliavin calculus to prove the LAMN property, in a high frequency data setting, has been initiated by Gobet [7] for diffusion processes. However, the situation given by (1) is completely different. Indeed, for diffusion processes, it is well known that one can not estimate the drift parameter from the observation of the process on a fixed time interval.

Besides the statistical application, a main contribution of this paper is to precise the asymptotic behavior of the transition density of $X_t^\theta$, in small time, and of its logarithm derivative with respect to the parameter.

The paper is organized as follows. The main results are stated in Section 2. Section 3 gives some representations of the transition density and its logarithm derivative, using the Malliavin calculus proposed in [4] and Section 4 studies their asymptotic behavior. The proof of the LAMN property is given in Sections 5 and 6. We stress on the fact that contrarily to [7], this proof does not require some lower bounds for the density of $X_t^\theta$. Section 7 contains some more technical proof.

## 2 Main results

We consider the real process $(X_t^\theta)$ defined on the time interval $[0,1]$ by equation (1), where $(L_t)$ is a centered Lévy process defined on a filtered space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_t, P)$. We assume that the Lévy measure of $(L_t)$ is absolutely continuous with respect to the Lebesgue measure and admits a density $F(z)$ given on $\mathbb{R}^*$ by

$$F(z) = \frac{1}{|z|^{\alpha+1}}\tau(z), \tag{2}$$

where $\alpha \in (1,2)$ and $\tau$ is a non negative smooth function equal to 1 on $[-1,1]$, vanishing on $[-2,2]^c$ and such that $0 \leq \tau \leq 1$. The introduction of the truncation function $\tau$ in the density of the Lévy measure is a technical tool to ensure the integrability of $|L_t|^p$, $\forall p \geq 1$, and that our variables are in the domain of the Malliavin operators.

We assume that the function $b$ is bounded, with bounded derivatives up to order three with respect to both variables $x$ and $\theta$. Under these assumptions, we know that for $t > 0$, $X_t^\theta$ admits a density (see Bichteler, Gravereaux and Jacod [4], Ishikawa and Kunita [9], Picard [18], Fournier and Printemps [5] for weaker assumptions on $b$), moreover this density admits a derivative with respect to the parameter $\theta$.

We are interested in the statistical properties of the process $(X_t^\theta)$, based on the discrete time observations $(X_{\frac{i}{n}}^\theta)_{i=0,\dots,n}$. Before stating our main results, we introduce some more notations. We denote by $p_{\frac{1}{n}}^\theta(x,y)$ the transition density of the homogenous Markov chain $(X_{\frac{i}{n}}^\theta)_{i=0,\dots,n}$ and by $P_n^\theta$ the law of the vector $(X_{\frac{1}{n}}^\theta, \dots, X_1^\theta)$.

In all the paper, for a function $f$ depending on both variables $(x,\theta)$, we denote by $f'$ the derivative of $f$ with respect to the variable $x$ and by $\dot{f}$ the derivative of $f$ with respect to the parameter $\theta$.

We first give an asymptotic expansion of the log-likelihood ratio.

**Theorem 1** *Let $u_n = n^{\frac{1}{2}-\frac{1}{\alpha}}$. We have :*

$$\log \frac{dP_n^{\theta+u_n h}}{dP_n^\theta}(X_{\frac{1}{n}}^\theta, \dots, X_1^\theta) = hJ_n(\theta)^{\frac{1}{2}} N_n(\theta) - \frac{h^2}{2} J_n(\theta) + o_P(1), \tag{3}$$

*with :*

$$J_n(\theta) = u_n^2 \sum_{i=0}^{n-1} E\left((\xi_{i,n}^\theta)^2 | \mathcal{G}_{i/n}\right)$$
$$N_n(\theta) = J_n(\theta)^{-\frac{1}{2}} u_n \sum_{i=0}^{n-1} \xi_{i,n}^\theta$$
$$\xi_{i,n}^\theta = \frac{\dot{p}_{\frac{1}{n}}^\theta}{p_{\frac{1}{n}}^\theta}(X_{\frac{i}{n}}^\theta, X_{\frac{i+1}{n}}^\theta).$$

We can precise the asymptotic behavior of $J_n(\theta)$ and $N_n(\theta)$. Let $\varphi_\alpha$ be the density of $L_1^\alpha$, where $(L_t^\alpha)$ is a centered $\alpha$-stable Lévy process whose Lévy measure is $\frac{\mathrm{d}z}{|z|^{1+\alpha}}$. We define the following quantity which will be the asymptotic information of the statistical model :

$$\mathcal{I}_\theta = \int_0^1 \dot{b}(X_s^\theta, \theta)^2 ds \times \int_{\mathbb{R}} \frac{\varphi_\alpha'(u)^2}{\varphi_\alpha(u)} du. \tag{4}$$

It is worth to note that this information does not depend on the truncation function $\tau$, but depends on $\alpha$ through the Fisher information of the translated $\alpha$-stable process. A numerical study of the variations of $\int_{\mathbb{R}} \frac{\varphi_\alpha'(u)^2}{\varphi_\alpha(u)} du$ with respect to $\alpha$ is available in the book [3] p.385.

**Theorem 2** *With the notations of Theorem 1, the following convergences hold :*

$$J_n(\theta) \xrightarrow{n\to\infty} \mathcal{I}_\theta, \text{ in probability,} \tag{5}$$

$$\forall \varepsilon > 0, \quad \sum_{i=0}^{n-1} u_n^2 E\left[\left(\xi_{i,n}^\theta\right)^2 1_{\{u_n|\xi_{i,n}^\theta|\geq\varepsilon\}}\right] \xrightarrow{n\to\infty} 0. \tag{6}$$

**Theorem 3** *We have the convergence in law*

$$J_n(\theta)^{\frac{1}{2}} N_n(\theta) = u_n \sum_{i=0}^{n-1} \xi_{i,n}^\theta \xrightarrow{n\to\infty} \mathcal{N}(0, \mathcal{I}_\theta), \tag{7}$$

*where the limit variable is conditionally Gaussian (recall the definition of $\mathcal{I}_\theta$ in* (4)*), and the convergence is stable with respect to $\mathcal{G}_1$.*

The stable convergence in law (7) and the convergence in probability (5) yield the convergence of the couple $(J_n(\theta), N_n(\theta))$:

$$(J_n(\theta), N_n(\theta)) \xrightarrow{n\to\infty} (\mathcal{I}_\theta, N), \quad \text{in law,}$$

where $N$ is a standard gaussian variable independent of $\mathcal{I}_\theta$.

As a consequence of the asymptotic expansion given in Theorem 1 and the preceding limit theorems, we deduce the LAMN property.

**Corollary 1** *The family $(P_n^\theta)$ satisfies the LAMN property with rate $u_n = n^{\frac{1}{2}-\frac{1}{\alpha}}$, and information $\mathcal{I}_\theta$ given by* (4)*.*

Let us stress that the rate of convergence depends on $\alpha$. When $\alpha$ tends to 2, the rate $u_n$ degenerates. This reflects the situation of a stochastic differential equation driven by a Brownian motion, where the drift coefficient cannot be estimated from the observation of the process on a finite time interval.

The proofs of Theorem 1, Theorem 2 and Theorem 3 will be given in the next sections. They rely on the pointwise convergence of the transition density $p_{\frac{1}{n}}^\theta(x_0, y)$ and its derivative with respect to $\theta$ that will be studied in Section 4. These asymptotic behaviors are precised below, after a time rescaling. Let $q^{n,\theta,x_0}$ be the density of the rescaled variable $n^{\frac{1}{\alpha}}(X_{1/n}^\theta - x_0)$. One can verify that $n^{\frac{1}{\alpha}}(X_{1/n}^\theta - x_0)$ equal in law to $Y_1^{n,\theta,x_0}$ the solution of the equation

$$Y_t^{n,\theta,x_0} = n^{\frac{1}{\alpha}-1} \int_0^t b(x_0 + n^{-1/\alpha} Y_s^{n,\theta,x_0}, \theta) \mathrm{d}s + L_t^{n,\alpha}, \tag{8}$$

with $(L_t^{n,\alpha})$ equal in law to $(n^{1/\alpha} L_{t/n})$.

The connection between the two densities is given by :

$$p^{\theta}_{1/n}(x_0, y) = n^{\frac{1}{\alpha}} q^{n,\theta,x_0}(n^{\frac{1}{\alpha}}(y - x_0)). \tag{9}$$

The next result precises the asymptotic behavior of $q^{n,\theta,x_0}$ and $\dot{q}^{n,\theta,x_0}$ as well as the limit of the Fisher information carried by the observation of $Y_1^{n,\theta,x_0}$,

$$I^{n,\theta,x_0} = E\left[\left(\frac{\dot{q}^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}{q^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}\right)^2\right]. \tag{10}$$

**Proposition 1** *For all $(x_0, u) \in \mathbb{R}^2$, we have*

$$\begin{aligned}
&i) \quad q^{n,\theta,x_0}(u) \xrightarrow{n\to\infty} \varphi_\alpha(u),\\
&ii) \quad n^{1-1/\alpha}\dot{q}^{n,\theta,x_0}(u) \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)\varphi'_\alpha(u),\\
&iii) \quad n^{2-2/\alpha} I^{n,\theta,x_0} \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right] = \dot{b}(x_0,\theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du.
\end{aligned}$$

The proof of this convergence result is based on the representation of the density $q^{n,\theta,x_0}$ and its derivative using the Malliavin calculus for jump processes. This is developed in the next sections.

**Remark 1** *We can observe that the rescaling $n^{\frac{1}{\alpha}}(X^\theta_{1/n} - x_0)$ is convenient for $\alpha \in (1,2)$ only. Indeed, for $\alpha \leq 1$ it is clear by (8) that the rescaled increment is not close to a Lévy process. The case $\alpha \leq 1$ is more complicated and requires some further expansions for the variable $X^\theta_{1/n}$.*

**Remark 2** *As we have already stressed, the introduction of the truncation function $\tau$ in the compensator of the Lévy measure ensures that our processes are in the domain of the Malliavin operators. If we do not assume that $F$ has a compact support, we can decompose $L_t$ as $L_t^1 + L_t^2$, where $L_t^1$ is a truncated $\alpha$-stable process and $L_t^2$ has a finite Lévy measure, equal to zero in the neighbourhood of zero. Then by adding the process $(L_t^2)_{t\in[0,1]}$ to the observations $(X^\theta_{i/n})_i$, we can prove the LAMN property for this new statistical experiment, with the same information $\mathcal{I}_\theta$, since conditionally on $(L_t^2)_t$ we are exactly in the studied preceding framework between the jump times of $(L_t^2)$. This permits to deduce that the minimal asymptotic variance in estimating $\theta$ from the observations $(X^\theta_{i/n})_i$ (the realistic case) is greater that $\mathcal{I}_\theta^{-1}$.*

# 3 Representation of the transition density via Malliavin calculus

The aim of this section is to represent $q^{n,\theta,x_0}$ and $\frac{\dot{q}^{n,\theta,x_0}}{q^{n,\theta,x_0}}$ as an expectation, using the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [4]. Due to the singularity of the

Lévy measure of $(L_t)$ at zero, we are not exactly in the same context, and we define in the next section an integration by parts setting adapted to the study of equation (8).

### 3.1 Integration by parts setting

In this section, we consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, P)$ endowed with a Poisson random measure $\mu$ on $[0,1] \times E$, where $E$ is an open subset of $\mathbb{R}$, with compensator $\nu$ given by $d\nu = dt \times g(z)dz$ on $[0,1] \times E$. We denote by $\tilde{\mu}$ the compensated measure and we are interested to study the regularity of the density of $Y_1^\theta$, where the process $(Y_t^\theta)$ is solution of :

$$Y_t^\theta = \int_0^t a(Y_s^\theta, \theta)ds + \int_0^t \int_E z\tilde{\mu}(ds, dz). \tag{11}$$

This is the framework of Bichteler, Gravereaux and Jacod [4], except that $g$ is not assumed to be equal to one and consequently the Malliavin operators have to be defined accordingly.

We make the following assumptions.

**H**: a) We assume that $a$ is bounded with bounded derivatives up to order three with respect to both variables.

b) We assume that $g \geq 0$ on $E$, $\mathcal{C}^1$ on $E$ and that

$$\forall p \geq 2, \quad \int_E |z|^p g(z)dz < \infty.$$

We first precise the Malliavin operators $L$ and $\Gamma$ and their basic properties (see Bichteler, Gravereaux, Jacod, [4] Chapter IV, sections 8-9-10). For a test function $f : [0,1] \times E \mapsto \mathbb{R}$ ($f$ is measurable, $\mathcal{C}^2$ with respect to the second variable, with bounded derivatives, and $f \in \cap_{p\geq 1}\mathbf{L}^p(\nu)$), we set $\mu(f) = \int_0^1 \int_E f(t,z)\mu(dt,dz)$. We introduce an auxiliary function $\rho : E \mapsto (0,\infty)$ such that $\rho$ is derivable and $\rho$, $\rho'$ and $\rho\frac{g'}{g}$ belong to $\cap_{p\geq 1}\mathbf{L}^p(g(z)dz)$. With these notations, we define the Malliavin operator $L$, on simple functional $\mu(f)$, in the following way :

$$L(\mu(f)) = \frac{1}{2}\mu\left(\rho' f' + \rho\frac{g'}{g}f' + \rho f''\right), \tag{12}$$

where $f'$ and $f''$ are the derivatives with respect to the second variable. For $\Phi = F(\mu(f_1), \ldots, \mu(f_k))$, with $F$ of class $\mathcal{C}^2$, we set

$$L\Phi = \sum_{i=1}^k \frac{\partial F}{\partial x_i}(\mu(f_1), \ldots, \mu(f_k))\, L(\mu(f_i)) + \frac{1}{2}\sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(\mu(f_1), \ldots, \mu(f_k))\mu(\rho f_i' f_j'). \tag{13}$$

These definitions permit to construct a linear operator on a space $D \subset \cap_{p\geq 1}\mathbf{L}^p$ whose basic properties are the following :

i) $L$ is self-adjoint : $\forall \Phi, \Psi \in D$, we have $E\Phi L\Psi = EL\Phi\Psi$.

ii) $L\Phi^2 \geq 2\Phi L\Phi$.

iii) $EL\Phi = 0$.

We associate to $L$, the symmetric bilinear operator $\Gamma$ :

$$\Gamma(\Phi, \Psi) = L(\Phi\Psi) - \Phi L\Psi - \Psi L\Phi. \tag{14}$$

If $f$ and $h$ are two test functions, we have :

$$\Gamma(\mu(f), \mu(h)) = \mu\left(\rho f' h'\right), \tag{15}$$

The operators $L$ and $\Gamma$ satisfy the chain rule property :

$$LF(\Phi) = F'(\Phi)L\Phi + \frac{1}{2}F''(\Phi)\Gamma(\Phi, \Phi), \tag{16}$$

$$\Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi). \tag{17}$$

Moreover we have the inequality

$$|\Gamma(\Phi, \Psi)| \leq \Gamma(\Phi, \Phi)^{1/2}\Gamma(\Psi, \Psi)^{1/2}. \tag{18}$$

These operators permit to establish the following integration by parts formula (see [4] Proposition 8-10 p.103).

**Proposition 2** *For $\Phi$ and $\Psi$ in $D$, and $f$ bounded with bounded derivatives up to order two, we have*

$$Ef'(\Phi)\Psi\Gamma(\Phi, \Phi) = Ef(\Phi)(-2\Psi L\Phi - \Gamma(\Phi, \Psi)).$$

*Moreover, if $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p\geq 1}\mathbf{L}^p$, we have*

$$Ef'(\Phi)\Psi = Ef(\Phi)\mathcal{H}_\Phi(\Psi), \tag{19}$$

*with*

$$\mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)). \tag{20}$$

**Remark 3** *The expression of the operator $L$ given in (12) is derived from [4] in the following way. We come back to $g = 1$ by a change of variable. Assuming that there exists $G$, one to one and such that $|G'(z)| = g(z)$, we define the Poisson random measure $\hat{\mu}$ on $[0, 1] \times G(E)$, for a test function $\hat{f}$, as $\hat{\mu}(\hat{f}) = \mu(\hat{f}(., G(.)))$. It is easy to verify that the compensator $\hat{\nu}$ of $\hat{\mu}$ is given by $d\hat{\nu} = dtdu$ on*

$[0,1] \times G(E)$. *Now, the Malliavin operator* $L$*, associated to* $\hat{\mu}$*, is defined on the simple functionals by ([4] p.112) :*

$$L(\hat{\mu}(\hat{f})) = \frac{1}{2}\hat{\mu}(\hat{\rho}'\hat{f}' + \hat{\rho}\hat{f}''), \tag{21}$$

*where* $\hat{\rho}$ *is an auxiliary function. Since* $\mu(f) = \hat{\mu}(\hat{f})$ *with* $\hat{f} = f(., G^{-1}(.))$*, we obtain* $L(\mu(f)) = \frac{1}{2}\hat{\mu}(\tilde{f})$*, where* $\tilde{f} = \hat{\rho}'\hat{f}' + \hat{\rho}\hat{f}''$*. With the choice* $\hat{\rho}(G(z)) = g^2(z)\rho(z)$ *and recalling that* $G'(z)^2 = g(z)^2$*, we obtain after some calculation* $\tilde{f}(t,z) = \rho'(G^{-1}(z))f'(t, G^{-1}(z)) + \rho(G^{-1}(z))\frac{g'}{g}(G^{-1}(z))f'(t, G^{-1}(z)) + \rho(G^{-1}(z))f''(t, G^{-1}(z))$*. This permits to deduce the expression* (12) *using again the relation* $\mu(f) = \hat{\mu}(f(., G^{-1}(.))$*.*

### 3.2 Representation of the density of $Y_1^\theta$

The integration by parts setting of the preceding section permits to derive the existence of the density of $Y_1^\theta$ given by (11), and gives a representation of this density as an expectation. Following Bichteler, Gravereaux, Jacod [4] (section 10, p130), we can prove that $\forall t > 0$, the variable $Y_t^\theta$, solution of (11), belongs to the domain of the operator $L$, and we can compute $LY_t^\theta$ and $\Gamma(Y_t^\theta, Y_t^\theta)$.

**Lemma 1** *There are versions of the processes* $(LY_t^\theta)_{t\in[0,1]}$ *and* $(U_t^\theta)_t = (\Gamma(Y_t^\theta, Y_t^\theta))_t$ *that are solutions of the linear equations:*

$$LY_t^\theta = \int_0^t a'(Y_s^\theta, \theta)LY_s^\theta ds + \frac{1}{2}\int_0^t a''(Y_s^\theta, \theta)U_s^\theta ds + \frac{1}{2}\int_0^t \int_E \left(\rho'(z) + \rho(z)\frac{g'(z)}{g(z)}\right)\mu(ds, dz), \tag{22}$$

$$U_t^\theta = 2\int_0^t a'(Y_s^\theta, \theta)U_s^\theta ds + \int_0^t \int_E \rho(z)\mu(ds, dz). \tag{23}$$

The proof of this result is based essentially on the linearity and chain rule property of the operators $L$ and $\Gamma$ (see equations (16) and (17)). In particular, one has $L(\int_0^t a(Y_s^\theta, \theta)ds) = \int_0^t a'(Y_s^\theta, \theta)LY_s^\theta ds + \frac{1}{2}\int_0^t a''(Y_s^\theta, \theta)\Gamma(Y_s^\theta, Y_s^\theta)ds$.

**Theorem 4** *Let us denote by* $q^\theta$ *the density of* $Y_1^\theta$*. We assume that the auxiliary function* $\rho$ *satisfies:*

$$\liminf_{u\to\infty} \frac{1}{\ln u}\int_E 1_{\{\rho(z)\geq 1/u\}}g(z)dz = +\infty. \tag{24}$$

*Then we have :*

$$q^\theta(u) = E(1_{\{Y_1^\theta \geq u\}}\mathcal{H}_\theta(1)), \tag{25}$$

*with*

$$\mathcal{H}_\theta(1) := \mathcal{H}_{Y_1^\theta}(1) = \frac{\Gamma(Y_1^\theta, \Gamma(Y_1^\theta, Y_1^\theta))}{\Gamma(Y_1^\theta, Y_1^\theta)^2} - 2\frac{LY_1^\theta}{\Gamma(Y_1^\theta, Y_1^\theta)} \tag{26}$$

**Remark 4** *The assumption* (24) *is a non degeneracy assumption which ensures the existence and integrability of* $\Gamma(Y_1^\theta, Y_1^\theta)^{-1}$.

**Proof** We apply the integration by parts formula (19) with $f$ a regularization of the Dirac mass, $\Phi = Y_1^\theta$ and $\Psi = 1$. So we just have to verify that, assuming (24), $U_1^\theta = \Gamma(Y_1^\theta, Y_1^\theta)$ is invertible and that $\frac{1}{U_1^\theta} \in \cap_{p\geq 1}\mathbf{L}^p$.

From Lemma 1, solving equation (23), we obtain

$$U_1^\theta = e^{\int_0^1 2a'(Y_s^\theta,\theta)ds}\int_0^1\int_E e^{-\int_0^s 2a'(Y_u^\theta,\theta)du}\rho(z)\mu(ds,dz).$$

Since $a'$ is bounded and $\rho > 0$, we deduce:

$$U_1^\theta \geq C\int_0^1\int_E \rho(z)\mu(ds,dz),$$

where $C$ is a non negative constant. We set $I_t(\rho) = \int_0^t\int_E \rho(z)d\mu(s,z)$, and we just have to prove that $\forall p \geq 1$, $E\frac{1}{I_1(\rho)^p} < \infty$.

We remark that $\forall \lambda > 0, \quad p \geq 1$,

$$\frac{1}{\lambda^p} = C_p\int_0^\infty u^{p-1}e^{-\lambda u}du,$$

where $C_p$ is a non negative constant depending on $p$. So we deduce from Fubini Theorem :

$$E\frac{1}{I_1(\rho)^p} = C_p\int_0^\infty u^{p-1}E(e^{-uI_1(\rho)})du.$$

But from the classical exponential formula for Poisson measures, we have

$$Ee^{-uI_1(\rho)} = e^{-\int_E(1-e^{-u\rho(z)})g(z)dz}.$$

We finally obtain:

$$E\frac{1}{I_1(\rho)^p} = C_p\int_0^\infty u^{p-1}e^{-\int_E(1-e^{-u\rho(z)})g(z)dz}du.$$

From assumption (24), we conclude easily that $E\frac{1}{I_1(\rho)^p} < \infty$.

$\square$

To complete the result of Theorem 4, we give an expression for $\Gamma(Y_t^\theta, \Gamma(Y_t^\theta, Y_t^\theta))$.

**Lemma 2** *There is a version of* $(W_t^\theta)_t = (\Gamma(Y_t^\theta, U_t^\theta))_t$ *which is solution of the linear equation :*

$$W_t^\theta = 3\int_0^t a'(Y_s^\theta,\theta)W_s^\theta ds + 2\int_0^t a''(Y_s^\theta,\theta)(U_s^\theta)^2 ds + \int_0^t\int_E \rho(z)\rho'(z)\mu(ds,dz). \quad (27)$$

We turn now to the study of the derivative of $q^\theta$ with respect to the parameter $\theta$.

We first remark that $(Y_t^\theta)_t$ admits a derivative with respect to $\theta$, denoted by $(\dot{Y}_t^\theta)_t$, solution of $\dot{Y}_t^\theta = \int_0^t \{a'(Y_s^\theta,\theta)\dot{Y}_s^\theta + \dot{a}(Y_s^\theta,\theta)\}\mathrm{d}s$. This can be easily establish, assuming **H**, observing that for all $p \geq 1$, $E\sup_{t\in[0,1]} |Y_t^{\theta+h} - Y_t^\theta - h\dot{Y}_t^\theta|^p = o(|h|^p)$ as $h$ tends to zero. This result is straightforward here since $\theta$ only appears in the drift part of $(Y_t^\theta)$ (see Theorem 5.24 p.51 in [4] for a more general result).

By iterating the integration by parts formula, since $Y_1^\theta$ admits a derivative with respect to $\theta$, one can prove the existence and the continuity in $\theta$ of $\dot{q}^\theta$. Moreover, we can represent $\frac{\dot{q}^\theta}{q^\theta}$ as a conditional expectation.

**Theorem 5** *Under the assumptions of Theorem 4, we have :*

$$\frac{\dot{q}^\theta}{q^\theta}(u) = E(\mathcal{H}_\theta(\dot{Y}_1^\theta)|Y_1^\theta = u), \tag{28}$$

*where*

$$\mathcal{H}_\theta(\dot{Y}_1^\theta) := \mathcal{H}_{Y_1^\theta}(\dot{Y}_1^\theta) = -2\dot{Y}_1^\theta \frac{LY_1^\theta}{U_1^\theta} + \dot{Y}_1^\theta \frac{W_1^\theta}{(U_1^\theta)^2} - \frac{\Gamma(Y_1^\theta, \dot{Y}_1^\theta)}{U_1^\theta}. \tag{29}$$

*$LY_1^\theta$ and $U_1^\theta$ are given in Lemma 1, $W_1^\theta$ is computed in Lemma 2 and the process $(V_t^\theta) = (\Gamma(Y_t^\theta, \dot{Y}_t^\theta))$ is solution of*

$$V_t^\theta = 2\int_0^t a'(Y_s^\theta,\theta)V_s^\theta ds + \int_0^t U_s^\theta[\dot{a}'(Y_s^\theta,\theta) + a''(Y_s^\theta,\theta)\dot{Y}_s^\theta]ds. \tag{30}$$

**Proof** Let $f$ be a smooth function, by differentiating $\theta \mapsto Ef(Y_1^\theta)$, and using the integration by parts formula (19), we obtain

$$\begin{aligned}\int f(u)\dot{q}^\theta(u)du &= Ef'(Y_1^\theta)\dot{Y}_1^\theta,\\ &= Ef(Y_1^\theta)\mathcal{H}_\theta(\dot{Y}_1^\theta)\\ &= Ef(Y_1^\theta)E(\mathcal{H}_\theta(\dot{Y}_1^\theta)|Y_1^\theta)\\ &= \int f(u)E(\mathcal{H}_\theta(\dot{Y}_1^\theta)|Y_1^\theta = u)q^\theta(u)du.\end{aligned}$$

This gives the expression of $\dfrac{\dot{q}^\theta}{q^\theta}(u)$. The expression of the weight $\mathcal{H}_\theta(\dot{Y}_1^\theta)$ follows from (20) and the basic properties of the operator $\Gamma$. The expression (30) follows from (11) and the fact that $\dot{Y}^\theta$ is solution to $\dot{Y}_t^\theta = \int_0^t \{a'(Y_s^\theta,\theta)\dot{Y}_s^\theta + \dot{a}(Y_s^\theta,\theta)\}\mathrm{d}s$.

□

### 3.3 Application to the representation of the density of the rescaled process

We apply the preceding results to study the asymptotic behavior of $q^{n,\theta,x_0}$, and $\frac{\dot{q}^{n,\theta,x_0}}{q^{n,\theta,x_0}}$, as $n$ goes to infinity, where $q^{n,\theta,x_0}$ is the density of $Y_1^{n,\theta,x_0}$ defined by (8).

We can observe that the process $(L_t^{n,\alpha})$, governing (8), and equal in law to $(n^{1/\alpha}L_{t/n})$, is a centered Lévy process with Lévy measure $F_n(z) = \frac{1}{|z|^{1+\alpha}}\tau(\frac{z}{n^{1/\alpha}})$ where $\tau$ is a non negative function equal to 1 on $[-1,1]$, vanishing on $[-2,2]^c$ and satisfying $0 \leq \tau \leq 1$. This clearly suggests that when $n$ growths, the process $(L_t^{n,\alpha})_t$ becomes close to an $\alpha$-stable process. For the sequel, it will be convenient to construct the family of Lévy processes $(L_t^{n,\alpha})_t$, for $n \geq 1$, on a common probability space where the limiting $\alpha$-stable process exists as well, and where the convergence holds true in a pathwise sense.

Let us consider $\mu^e(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u)$ a Poisson measure on $[0,\infty) \times \mathbb{R}^* \times [0,1]$ with compensating measure $\nu^e(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u) = \mathrm{d}t\frac{\mathrm{d}z}{|z|^{1+\alpha}}\mathrm{d}u$. This measure corresponds to the jump measure of an $\alpha$-stable process, where each jumps is marked with an uniform variable on $[0,1]$.

We define the Poisson measures $\mu^{(n)}$, for all $n \geq 1$, and $\mu$ by setting:

$$\forall A \subset [0,\infty) \times \mathbb{R},\ \mu^{(n)}(A) = \int_{[0,\infty)} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t,z) 1_{\{u \leq \tau(\frac{z}{n^{1/\alpha}})\}} \mu^e(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u), \tag{31}$$

$$\forall A \subset [0,\infty) \times \mathbb{R},\ \mu(A) = \int_{[0,\infty)} \int_{\mathbb{R}} \int_{[0,1]} 1_A(t,z) \mu^e(\mathrm{d}t, \mathrm{d}z, \mathrm{d}u). \tag{32}$$

By simple computations, one can check that the compensator of the measure $\mu^{(n)}(\mathrm{d}t, \mathrm{d}z)$ is $\nu^{(n)}(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t \times \tau(\frac{z}{n^{1/\alpha}})\frac{\mathrm{d}z}{|z|^{1+\alpha}} = \mathrm{d}t \times F_n(z)\mathrm{d}z$ and the compensator of $\mu(\mathrm{d}t, \mathrm{d}z)$ is $\nu(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t \times \frac{\mathrm{d}z}{|z|^{1+\alpha}}$. Remark that, since $\tau(z) = 1$ for $|z| \leq 1$, the measures $\mu^{(n)}(\mathrm{d}s, \mathrm{d}z)$ and $\mu(\mathrm{d}s, \mathrm{d}z)$ coincide on the set $\{(s,z) \mid |z| \leq n^{1/\alpha}\}$.

We now define the stochastic processes associated to these random measures,

$$L_t^\alpha = \int_0^t \int_{[-1,1]} z\{\mu(\mathrm{d}s, \mathrm{d}z) - \nu(\mathrm{d}s, \mathrm{d}z)\} + \int_0^t \int_{[-1,1]^c} z\mu(\mathrm{d}s, \mathrm{d}z) \tag{33}$$

$$L_t^{n,\alpha} = \int_0^t \int_{\mathbb{R}} z\{\mu^{(n)}(\mathrm{d}s, \mathrm{d}z) - \nu^{(n)}(\mathrm{d}s, \mathrm{d}z)\} = \int_0^t \int_{|z| \leq 2n^{1/\alpha}} z\{\mu^{(n)}(\mathrm{d}s, \mathrm{d}z) - \nu^{(n)}(\mathrm{d}s, \mathrm{d}z)\} \tag{34}$$

By construction, the process $L^\alpha$ is a centered $\alpha$-stable process, and the process $L^{n,\alpha}$ is equal in law to the process $(n^{1/\alpha}L_{t/n})_t$, since they are based on random measures with the same compensators. Remark that the jumps of $L_t^{n,\alpha}$ with size smaller than $n^{1/\alpha}$ exactly coincide with the jumps of $L^\alpha$ with size smaller than $n^{1/\alpha}$. On the other hand, the process $L^{n,\alpha}$ has no jump with a size greater than $2n^{1/\alpha}$.

Using that the measures $\mu$ and $\mu^{(n)}$ coincide on the subsets of $\{(t,z); |z| \leq n^{1/\alpha}\}$, and that, on $|z| \leq n^{1/\alpha}$, the function $\tau(\frac{z}{n^{1/\alpha}})\frac{1}{|z|^{1+\alpha}} = \frac{1}{|z|^{1+\alpha}}$ is symmetric, we can write:

$$L_t^{n,\alpha} = \int_0^t \int_{[-1,1]} z\{\mu(\mathrm{d}s,\mathrm{d}z) - \nu(\mathrm{d}s,\mathrm{d}z)\} + \int_0^t \int_{1<|z|<n^{1/\alpha}} z\mu(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{n^{1/\alpha}\leq|z|\leq 2n^{1/\alpha}} z\{\mu^n(\mathrm{d}s,\mathrm{d}z) - \nu^n(\mathrm{d}s,\mathrm{d}z)\}. \quad (35)$$

The following simple lemma gives a precise connection between $L^{n,\alpha}$ and the stable process $L^\alpha$.

**Lemma 3** *There exists a sequence $\kappa_n$ with $\kappa_n \xrightarrow{n\to\infty} 0$ such that for all $t \leq 1$,*

$$L_t^{n,\alpha} = L_t^\alpha - t\kappa_n \quad (36)$$

*on the event $\mu\left(\{(t,z) \mid 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}\right) = 0$. Moreover*

$$P\left(\mu\left(\{(t,z) \mid 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}\right) = 0\right) = 1 + O(1/n). \quad (37)$$

**Proof** Let us set $\kappa_n = \int_{n^{1/\alpha}\leq|z|\leq 2n^{1/\alpha}} z\tau(z/n^{1/\alpha})\frac{\mathrm{d}z}{|z|^{1+\alpha}}$ which converges to zero since $\tau$ is bounded and $\alpha > 1$. Now, by comparison of the representations (33) and (35), it is clear that the equation (36) holds true on the event that the supports of the random measures $\mu$ and $\mu^{(n)}$ do not intersect $\{(t,z) \mid 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}$. Since, by construction, the support of $\mu^{(n)}$ is included in the support of $\mu$, we see that (36) holds true on the event $\mu\left(\{(t,z) \mid 0 \leq t \leq 1, |z| \geq n^{1/\alpha}\}\right) = 0$. Finally, the probability of the latter event is $\exp\left(-\int_0^1 \int_{|z|\geq n^{1/\alpha}} \frac{\mathrm{d}z}{|z|^{1+\alpha}}\mathrm{d}t\right)$ which converges to 1 at rate $1/n$ as stated. □

In the following, we will assume that the process $Y^{n,\theta,x_0}$ is solution of

$$Y_t^{n,\theta,x_0} = n^{\frac{1}{\alpha}-1}\int_0^t b(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)\mathrm{d}s + L_t^{n,\alpha}, \quad (38)$$

where $L_t^{n,\alpha}$ is given by (34). We are in the framework of section 3.2, with $g(z) = \frac{1}{|z|^{1+\alpha}}\tau(\frac{z}{n^{1/\alpha}})$ and the auxiliary function $\rho$ can be chosen as $\rho(z) = z^4\tau(2z)$.

**Proposition 3** *Let $q^{n,\theta,x_0}$ be the density of $Y_1^{n,\theta,x_0}$, we have :*

$$q^{n,\theta,x_0}(u) = E(1_{\{Y_1^{n,\theta,x_0}\geq u\}}\mathcal{H}_\theta^n(1)), \quad (39)$$

*with*

$$\mathcal{H}_\theta^n(1) := \mathcal{H}_{Y_1^{n,\theta,x_0}}(1) = \widehat{\mathcal{H}}_\theta^n(1) + \mathcal{R}_\theta^n(1). \quad (40)$$

*The main term* $\widehat{\mathcal{H}}_\theta^n(1)$ *is given by*

$$\widehat{\mathcal{H}}_\theta^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-3} \rho'(z)\rho(z)\mu(ds,dz)}{\mathcal{E}_1^n \left(\int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-2}\rho(z)\mu(ds,dz)\right)^2} - \frac{\int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-1} \left[\rho'(z) - \frac{(1+\alpha)\rho(z)}{z}\right]\mu(ds,dz)}{\mathcal{E}_1^n \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-2}\rho(z)\mu(ds,dz)} \tag{41}$$

*where*

$$\mathcal{E}_t^n = \exp\left(n^{-1}\int_0^t b'(x_0 + n^{-1/\alpha} Y_s^{n,\theta,x_0}, \theta)\,ds\right), \tag{42}$$

*and the remainder term satisfies the upper bound,*

$$|\mathcal{R}_\theta^n(1)| \leq C\frac{1}{n^{1+\frac{1}{\alpha}}}, \tag{43}$$

*where* $C$ *is some deterministic constant.*

**Proof**

We apply the integration by parts formula given in Theorem 4 to $Y_1^{n,\theta,x_0}$. The non degeneracy assumption is verified by choosing $\rho(z) = z^4\tau(2z)$. We obtain :

$$q^{n,\theta,x_0}(u) = E(1_{\{Y_1^{n,\theta,x_0} \geq u\}}\mathcal{H}_\theta^n(1)),$$

with

$$\mathcal{H}_\theta^n(1) = \frac{\Gamma(Y_1^{n,\theta,x_0}, \Gamma(Y_1^{n,\theta,x_0}, Y_1^{n,\theta,x_0}))}{\Gamma^2(Y_1^{n,\theta,x_0}, Y_1^{n,\theta,x_0})} - 2\frac{LY_1^{n,\theta,x_0}}{\Gamma(Y_1^{n,\theta,x_0}, Y_1^{n,\theta,x_0})}.$$

The random variables appearing in the weight $\mathcal{H}_\theta^n(1)$ can be computed explicitely. Let us denote by $U_t^{n,\theta} = \Gamma[Y_t^{n,\theta,x_0}, Y_t^{n,\theta,x_0}]$, and $W_t^{n,\theta} = \Gamma[Y_t^{n,\theta,x_0}, U_t^{n,\theta}]$. Then applying the results of Lemma 1 and Lemma 2 we have,

$$U_t^{n,\theta} = \frac{2}{n}\int_0^t U_s^{n,\theta} b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \rho(z)\mu^{(n)}(\mathrm{d}s, \mathrm{d}z),$$

$$\begin{aligned} L(Y_t^{n,\theta,x_0}) = \frac{1}{n}\int_0^t b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta) L(Y_s^{n,\theta,x_0})\mathrm{d}s + \frac{1}{2n^{1+1/\alpha}}\int_0^t b''(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)U_s^{n,\theta}\mathrm{d}s \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}} (\rho'(z) + \rho(z)\frac{F_n'(z)}{F_n(z)})\mu^{(n)}(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

$$\begin{aligned} W_t^{n,\theta} = \frac{3}{n}\int_0^t b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)W_s^{n,\theta}\mathrm{d}s + \frac{2}{n^{1+1/\alpha}}\int_0^t b''(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)(U_s^{n,\theta})^2\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}} \rho(z)\rho'(z)\mu^{(n)}(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

These linear equations can be resolved explicitly using $\mathcal{E}_t^n$ given by (42). By simple computations, we find

$$U_1^{n,\theta} = (\mathcal{E}_1^n)^2 \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-2} \rho(z) \mu(\mathrm{d}s, \mathrm{d}z), \tag{44}$$

where we used that the measures $\mu^{(n)}(\mathrm{d}s, \mathrm{d}z)$ and $\mu(\mathrm{d}s, \mathrm{d}z)$ coincide on $(s,z) \in [0,1] \times [-1,1]$ and that the support of $\rho$ is included in $[-1,1]$. By analogous computations we get,

$$\begin{aligned} L(Y_1^{n,\theta,x_0}) = \frac{\mathcal{E}_1^n}{2} \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-1} \left[ \rho'(z) - \frac{(1+\alpha)\rho(z)}{z} \right] \mu(\mathrm{d}s, \mathrm{d}z) \\ + \frac{\mathcal{E}_1^n}{2n^{1+1/\alpha}} \int_0^1 b''(x_0 + n^{-1/\alpha} Y_s^{n,\theta,x_0}, \theta) U_s^{n,\theta} (\mathcal{E}_s^n)^{-1} \mathrm{d}s \end{aligned} \tag{45}$$

where we have used that $\frac{F_n'(z)}{F_n(z)} = -\frac{1+\alpha}{z}$ on the support of $\rho$. Solving the equation for $W_1^{n,\theta}$ yields

$$\begin{aligned} W_1^{n,\theta} = (\mathcal{E}_1^n)^3 \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-3} \rho'(z) \rho(z) \mu(\mathrm{d}s, \mathrm{d}z) \\ + \frac{(\mathcal{E}_1^n)^3}{2n^{1+1/\alpha}} \int_0^1 b''(x_0 + n^{-1/\alpha} Y_s^{n,\theta,x_0}, \theta) (U_s^{n,\theta})^2 (\mathcal{E}_s^n)^{-3} \mathrm{d}s. \end{aligned} \tag{46}$$

Based on these expressions and recalling that

$$\mathcal{H}_\theta^n(1) = \frac{W_1^{n,\theta}}{(U_1^{n,\theta})^2} - 2 \frac{L(Y_1^{n,\theta,x_0})}{U_1^{n,\theta}}, \tag{47}$$

we deduce, after some calculus, the decomposition (40), where the leading term is

$$\widehat{\mathcal{H}}_\theta^n(1) = \frac{\int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-3} \rho'(z) \rho(z) \mu(\mathrm{d}s, \mathrm{d}z)}{\mathcal{E}_1^n \left( \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-2} \rho(z) \mu(\mathrm{d}s, \mathrm{d}z) \right)^2} - \frac{\int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-1} \left[ \rho'(z) - \frac{(1+\alpha)\rho(z)}{z} \right] \mu(\mathrm{d}s, \mathrm{d}z)}{\mathcal{E}_1^n \int_0^1 \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-2} \rho(z) \mu(\mathrm{d}s, \mathrm{d}z)}$$

and, using that $b$ is bounded with bounded derivatives and $\frac{U_s^{n,\theta}}{U_1^{n,\theta}}$ is bounded for $0 \le s \le 1$, the remainder term satisfies the upper bound

$$|\mathcal{R}_\theta^n(1)| \le \frac{C}{n^{1+1/\alpha}}$$

where $C$ is some deterministic constant.

$\square$

In a similar way, we give an expansion of $\frac{\dot{q}^{n,\theta,x_0}}{q^{n,\theta,x_0}}(u)$.

**Proposition 4** *We have :*

$$\frac{\dot{q}^{n,\theta,x_0}}{q^{n,\theta,x_0}}(u) = E(\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta,x_0}) | Y_1^{n,\theta,x_0} = u), \tag{48}$$

*with*

$$\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0}) := \mathcal{H}_{Y_1^{n,\theta,x_0}}(\dot{Y}_1^{n,\theta,x_0}) = \dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}^n_\theta(1) + \mathcal{R}^n_\theta(\dot{Y}_1^{n,\theta,x_0}), \tag{49}$$

*where* $\left|\mathcal{R}^n_\theta(\dot{Y}_1^{n,\theta,x_0})\right| \leq Cn^{-1}$ *and* $\widehat{\mathcal{H}}^n_\theta(1)$ *is given in* (41).

**Proof** Using successively the Theorem 5 and the equation (47), we have

$$\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0}) = \dot{Y}_1^{n,\theta,x_0}\left(\frac{W_1^{n,\theta}}{(U_1^{n,\theta})^2} - 2\frac{LY_1^{n,\theta,x_0}}{U_1^{n,\theta}}\right) - \frac{\Gamma(Y_1^{n,\theta,x_0},\dot{Y}_1^{n,\theta,x_0})}{U_1^{n,\theta}} = \dot{Y}_1^{n,\theta,x_0}\mathcal{H}^n_\theta(1) - \frac{\Gamma(Y_1^{n,\theta,x_0},\dot{Y}_1^{n,\theta,x_0})}{U_1^{n,\theta}},$$

where $U_1^{n,\theta}$ is given by (44). For the computation of $V_1^{n,\theta} = \Gamma(Y_1^{n,\theta,x_0},\dot{Y}_1^{n,\theta,x_0})$, we use (30), this gives

$$V_1^{n,\theta} = (\mathcal{E}_1^n)^2\int_0^1 (\mathcal{E}_s^n)^{-2}U_s^{n,\theta}\left(\frac{1}{n}\dot{b}'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta) + \dot{Y}_s^{n,\theta,x_0}\frac{1}{n^{1+1/\alpha}}b''(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\right)ds. \tag{50}$$

The expression of $\dot{Y}_1^{n,\theta,x_0}$ is explicitly given by,

$$\dot{Y}_1^{n,\theta,x_0} = n^{\frac{1}{\alpha}-1}\mathcal{E}_1^n\int_0^1(\mathcal{E}_s^n)^{-1}\dot{b}(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)ds. \tag{51}$$

Using these expressions, we deduce the bounds

$$\left|\dot{Y}_1^{n,\theta,x_0}\right| \leq Cn^{\frac{1}{\alpha}-1} \tag{52}$$

$$\left|V_1^{n,\theta}\right| \leq Cn^{-1}\left|U_1^{n,\theta}\right|$$

Combining this with the Proposition 3, the result follows.

□

# 4 Asymptotic behaviour of the transition density

In this section we study the asymptotic behaviour of $q^{n,\theta,x_0}$, the density of $Y_1^{n,\theta,x_0}$, solution of (38). We will establish some stronger versions of Proposition 1.

## 4.1 Pointwise convergence

The following two propositions will imply the results of Proposition 1 i) and ii).

**Proposition 5** *Let $(\theta_n)_{n\geq 1}$ be a sequence of parameters such that $\theta_n \xrightarrow{n\to\infty} \theta$. For all $(x_0,u)\in\mathbb{R}^2$, we have $q^{n,\theta_n,x_0}(u) \xrightarrow{n\to\infty} \varphi_\alpha(u)$. Moreover*

$$\sup_{u\in\mathbb{R}}\sup_n q^{n,\theta_n,x_0}(u) < \infty. \tag{53}$$

**Proposition 6** *Let $(\theta_n)_{n\geq 1}$ be a sequence of parameters such that $\theta_n \xrightarrow{n\to\infty} \theta$. For all $(x_0,u)\in\mathbb{R}^2$, we have $\sqrt{n}u_n\dot{q}^{n,\theta_n,x_0}(u) \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)\varphi'_\alpha(u)$. Moreover*

$$\sup_u\sup_n \left| n^{1-1/\alpha}\dot{q}^{n,\theta_n,x_0}(u)\right| < \infty. \tag{54}$$

### 4.1.1 Proof of Proposition 5

From the Proposition 3, the expression for the density of $Y_1^{n,\theta,x_0}$ at some point $u$ and with $\theta_n\in\Theta$ is given by equation (39)

$$q^{n,\theta_n,x_0}(u) = E[1_{[u,\infty)}(Y_1^{n,\theta_n,x_0})\mathcal{H}^n_{\theta_n}(1)],$$

where $\mathcal{H}^n_{\theta_n}(1) = \widehat{\mathcal{H}}^n_{\theta_n}(1) + \mathcal{R}^n_{\theta_n}(1)$, with $\widehat{\mathcal{H}}^n_{\theta_n}(1)$ given by (41) and $\mathcal{R}^n_{\theta_n}(1)$ bounded by (43).

Let us note

$$\mathcal{H}_{L^\alpha}(1) = \frac{\int_0^1\int_{\mathbb{R}}\rho'(z)\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)}{\left(\int_0^1\int_{\mathbb{R}}\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)\right)^2} - \frac{\int_0^1\int_{\mathbb{R}}\left[\rho'(z) - \frac{(1+\alpha)\rho(z)}{z}\right]\mu(\mathrm{d}s,\mathrm{d}z)}{\int_0^1\int_{\mathbb{R}}\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)}, \tag{55}$$

then from (42), and the boundedness of $b'$, it is clear that $\widehat{\mathcal{H}}^n_{\theta_n}(1)$ converges almost surely to $\mathcal{H}_{L^\alpha}(1)$.

Using again the boundedness of $b'$ and the fact that $\rho$ is a non negative function, we deduce the upper bound

$$\left|\widehat{\mathcal{H}}^n_{\theta_n}(1)\right| \leq C\left(\frac{\int_0^1\int_{\mathbb{R}}|\rho'(z)|\,\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)}{\left(\int_0^1\int_{\mathbb{R}}\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)\right)^2} + \frac{\int_0^1\int_{\mathbb{R}}\left[|\rho'(z)| + \frac{(1+\alpha)\rho(z)}{|z|}\right]\mu(\mathrm{d}s,\mathrm{d}z)}{\int_0^1\int_{\mathbb{R}}\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)}\right) \tag{56}$$

for some constant $C>0$. Using that $\rho$, $|\rho'|$ and $z\mapsto\frac{\rho(z)}{z}$ belongs to $\bigcap_{p\geq 1}\mathbf{L}^p(|z|^{-1-\alpha}\,\mathrm{d}z)$ we get $E\left[\left(\int_0^1\int_{\mathbb{R}}|\rho'(z)|\,\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)\right)^p\right]<\infty$, $E\left[\left(\int_0^1\int_{\mathbb{R}}\left[|\rho'(z)| + \frac{(1+\alpha)\rho(z)}{|z|}\right]\mu(\mathrm{d}s,\mathrm{d}z)\right)^p\right]<\infty$ for all $p\geq 1$. Since $\rho$ satisfies the non degeneracy assumption (24), $[\int_0^1\int_{\mathbb{R}}\rho(z)\mu(\mathrm{d}s,\mathrm{d}z)]^{-1}$ belongs to $\bigcap_{p\geq 1}\mathbf{L}^p$, as a consequence we deduce from (56) that $\sup_n\left|\widehat{\mathcal{H}}^n_{\theta_n}(1)\right|^p$ is integrable for all $p\geq 1$. Applying the dominated convergence Theorem, we deduce that

$$\widehat{\mathcal{H}}^n_{\theta_n}(1) \xrightarrow[\mathbf{L}^p]{n\to\infty} \mathcal{H}_{L^\alpha}(1), \quad ,\forall p\geq 1. \tag{57}$$

The Lemma 3 implies that $L_1^{n,\alpha}$ converges to $L_1^\alpha$ in probability. From the boundedness of $b$ and equation (38) we deduce that $Y_1^{n,\theta_n,x_0}$ converges in probability to $L_1^\alpha$. Then, an easy computation, using that $P(L_1^\alpha = u) = 0$, shows the convergence in probability

$$1_{[u,\infty)}(Y_1^{n,\theta_n,x_0}) \xrightarrow[\mathbf{P}]{n\to\infty} 1_{[u,\infty)}(L_1^\alpha). \tag{58}$$

Moreover from the boundedness property of the variables, the latter convergence holds in $\mathbf{L}^p$ sense, $\forall p \geq 1$.

Using (39), (40), (43), (57), (58) we get,

$$q^{n,\theta_n,x_0}(u) \xrightarrow{n\to\infty} E\left[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)\right]. \tag{59}$$

To finish the proof of the convergence, it remains to show that the right-hand side of (59) is a representation for $\varphi_\alpha(u)$, the density of an $\alpha$-stable process. This is done in Lemma 4 below.

Remark that, we easily get from (39), (40), (43), and (57) that $\sup_{u\in\mathbb{R}} \sup_n q^{n,\theta_n,x_0}(u) < \infty$.

$\square$

**Lemma 4** *We have*

$$\varphi_\alpha(u) = E\left[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)\right]. \tag{60}$$

**Proof** The relation (60) could be formally obtained by Malliavin computations for the stable measure. However Malliavin computation in the setting of a stable process does not immediately enter the framework developed by [4], since the amplitude of the (big) jumps are not $L^p$ random variables for all $p$. Hence, we prefer to give another proof.

Let us denote by $\varphi^n(u)$ the density of the variable $L_1^{n,\alpha}$. We apply the results (53) and (59), in the situation where the drift function $b \equiv 0$, for which $Y_1^{n,\theta,x_0} = L_1^{n,\alpha}$. This yields,

$$\varphi^n(u) \xrightarrow{n\to\infty} E\left[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)\right] := \psi(u), \tag{61}$$

$$\sup_u \sup_n \varphi^n(u) < \infty. \tag{62}$$

Assume by contradiction that, for some $u$, we have $\psi(u) \neq \varphi_\alpha(u)$. From the fact that $P(L_1^\alpha = u) = 0$, it can be seen that $\psi$ is continuous at the point $u$. Hence, one can find a continuous, compactly supported, function $f$ such that $\int f(x)\psi(x)\mathrm{d}x \neq \int f(x)\varphi_\alpha(x)\mathrm{d}x$.

On the one hand we have,

$$E[f(L_1^{n,\alpha})] = \int f(x)\varphi^n(x)\mathrm{d}u \xrightarrow{n\to\infty} \int f(x)\psi(x)\mathrm{d}x, \tag{63}$$

where we have used the dominated convergence Theorem with (61)–(62). On the other hand, we write

$$E[f(L_1^{n,\alpha})] = E[f(L_1^{\alpha} - \kappa_n)1_{\{L_1^{n,\alpha}=L_1^{\alpha}-\kappa_n\}}] + E[f(L_1^{n,\alpha})1_{\{L_1^{n,\alpha}\neq L_1^{\alpha}-\kappa_n\}}],$$

where we have used the notations of Lemma 3. Moreover, by Lemma 3, we have $P(L_1^{n,\alpha} = L_1^{\alpha} - \kappa_n) \xrightarrow{n\to\infty} 1$. We deduce that,

$$E[f(L_1^{n,\alpha})] \xrightarrow{n\to\infty} E[f(L_1^{\alpha})] = \int f(x)\varphi_\alpha(x)\mathrm{d}x \tag{64}$$

This last convergence result clearly contradicts (63). □

### 4.1.2 Proof of Proposition 6

First we write a representation as an expectation for $\dot{q}^{n,\theta,x_0}$. Let $f$ be a smooth, non negative and compactly supported function. Differentiating the relation $E[f(Y_1^{n,\theta,x_0})] = \int f(u)q^{n,\theta,x_0}\mathrm{d}u$, we get,

$$E[f'(Y_1^{n,\theta,x_0})\dot{Y}_1^{n,\theta,x_0}] = \int f(u)\dot{q}^{n,\theta,x_0}(u)\mathrm{d}u.$$

Using the integration by parts formula (19) , we obtain

$$\int f(u)\dot{q}^{n,\theta,x_0}(u)\mathrm{d}u = E[f(Y_1^{n,\theta,x_0})\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta,x_0})], \tag{65}$$

where $\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta,x_0})$ is given by (49) :

$$\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta,x_0}) = \dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1) + \mathcal{R}_\theta^n(\dot{Y}_1^{n,\theta,x_0}),$$

with $\left|\mathcal{R}_\theta^n(\dot{Y}_1^{n,\theta,x_0})\right| \leq Cn^{-1}$ and $\widehat{\mathcal{H}}_\theta^n(1)$ given in (41).

Using (65), we get

$$\left|\int f(u)\dot{q}^{n,\theta,x_0}(u)\mathrm{d}u - E[f(Y_1^{n,\theta,x_0})\dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1)]\right| \leq E\left[f(Y_1^{n,\theta,x_0})\left|\mathcal{R}_\theta(\dot{Y}_1^{n,\theta,x_0})\right|\right] \leq Cn^{-1}E[f(Y_1^{n,\theta,x_0})]$$

Applying the integration by parts formula, we deduce

$$\left|\int f(u)\dot{q}^{n,\theta,x_0}(u)\mathrm{d}u - E\left[F(Y_1^{n,\theta,x_0})\mathcal{H}_{Y_1^{n,\theta,x_0}}\left(\dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1)\right)\right]\right| \leq Cn^{-1}E[f(Y_1^{n,\theta,x_0})]$$

where $F$ denotes a primitive function of $f$ and $\mathcal{H}_{Y_1^{n,\theta,x_0}}\left(\dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1)\right)$ is defined by (20). If $f$ converges to a Dirac mass at some point $u$, we deduce,

$$\left|\dot{q}^{n,\theta,x_0}(u) - E\left[1_{[u,\infty)}(Y_1^{n,\theta,x_0})\mathcal{H}_{Y_1^{n,\theta,x_0}}\left(\dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1)\right)\right]\right| \leq Cn^{-1}q^{n,\theta,x_0}(u). \tag{66}$$

Thus we need to study $\lim_n n^{1-1/\alpha}E\left[1_{[u,\infty)}(Y_1^{n,\theta_n,x_0})\mathcal{H}_{Y_1^{n,\theta_n,x_0}}\left(\dot{Y}_1^{n,\theta_n,x_0}\widehat{\mathcal{H}}_{\theta_n}^n(1)\right)\right]$. Actually, the main step is to show that

$$n^{1-1/\alpha}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}\left(\dot{Y}_1^{n,\theta_n,x_0}\widehat{\mathcal{H}}_{\theta_n}^n(1)\right) \xrightarrow[\mathbf{L}^p]{n\to\infty} \dot{b}(x_0,\theta)\mathcal{H}^{(2)}, \quad \forall p \geq 1, \tag{67}$$

where $\mathcal{H}^{(2)}$ is some random variable whose expression does not depend on $\theta$ and $b$. This is done in Lemma 10 (see the Section 7). Then, as in the proof of (59), we can deduce from (66)–(67), that

$$n^{1-1/\alpha}\dot{q}^{n,\theta_n,x_0}(u) \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)E\left[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}\right]. \tag{68}$$

Remark that from (53) and (66)–(67), we get

$$\sup_u \sup_n \left|n^{1-1/\alpha}\dot{q}^{n,\theta_n,x_0}(u)\right| < \infty.$$

The proof of the Proposition will be finished if we identify $E\left[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}\right]$ as being equal to $\varphi_\alpha'(u)$. This is done in Lemma 5 below. $\square$

**Lemma 5** *We have for all $u \in \mathbb{R}$,*

$$\varphi_\alpha'(u) = E[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}]. \tag{69}$$

**Proof** Let us consider the situation where $b(x,\theta) = \theta$. In that case, we have $Y_1^{n,\theta,x_0} = n^{\frac{1}{\alpha}-1}\theta + L_1^{\alpha,n}$ and thus the density of $Y_1^{n,\theta,x_0}$ is related to the density of $L_1^{\alpha,n}$ by the relation

$$q^{n,\theta,x_0}(u) = \varphi^n(u - n^{1/\alpha-1}\theta).$$

We can apply the results (54) and (68) in this specific setting. This yields

$$\forall u,\ \varphi^{n\prime}(u - n^{1/\alpha-1}\theta) \xrightarrow{n\to\infty} E[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}], \tag{70}$$

$$\sup_{u,n}\left|\varphi^{n\prime}(u - n^{1/\alpha-1}\theta)\right| < \infty. \tag{71}$$

Let us denote $\chi(u) = E[1_{[u,\infty)}(L_1^\alpha)\mathcal{H}^{(2)}]$ and assume by contradiction that $\chi \neq \varphi_\alpha'$. Using the continuity of $u \mapsto \chi(u)$, there exists a smooth, compactly supported function, $f$ such that $\int \chi(u)f(u)\mathrm{d}u \neq \int \varphi_\alpha'(u)f(u)\mathrm{d}u$.

Now, on the one hand we have

$$\int \varphi^{n\prime}(u - n^{1/\alpha-1}\theta) f(u)\mathrm{d}u \xrightarrow{n\to\infty} \int \chi(u) f(u)\mathrm{d}u, \tag{72}$$

where we have used the dominated convergence theorem, together with (70)–(71).

On the other hand, we can write

$$\begin{aligned}
\int \varphi^{n\prime}(u - n^{1/\alpha-1}\theta) f(u)\mathrm{d}u \quad &= \quad -\int \varphi^{n}(u - n^{1/\alpha-1}\theta) f'(u)\mathrm{d}u \\
&= \quad -\int \varphi^{n}(u) f'(u + n^{1/\alpha-1}\theta)\mathrm{d}u \\
&= \quad -E[f'(L_1^{\alpha,n} + n^{1-1/\alpha})] \\
&\xrightarrow{n\to\infty} \quad -E[f'(L_1^{\alpha})] = -\int \varphi_\alpha(u) f'(u)\mathrm{d}u \qquad (73)\\
&\qquad = \int \varphi'_\alpha(u) f(u)\mathrm{d}u \qquad (74)
\end{aligned}$$

where the convergence (73) is obtained in the same way as (64). Clearly (74) contradicts (72), and the lemma is proved.

□

### 4.2 Fisher information

We study now the asymptotic properties of the Fisher information defined by (10) corresponding to the observation of the random variable $Y_1^{n,\theta,x_0}$. We recall that it is given by

$$I^{n,\theta,x_0} = E\left[\left(\frac{\dot{q}^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}{q^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}\right)^2\right]$$

We will show a stronger version of the Proposition 1 iii).

**Proposition 7** *Let $(\theta_n)$ be a sequence such that $\theta_n \xrightarrow{n\to\infty} \theta$, we have*

*i)*

$$n^{2-2/\alpha} I^{n,\theta_n,x_0} \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right] = \dot{b}(x_0,\theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du,$$

*and this convergence is uniform with respect to $x_0$,*

*ii)* $\sup_{n,x_0,\theta} n^{2-2/\alpha} I^{n,\theta,x_0} = \sup_{n,x_0,\theta} n^{2-2/\alpha} E\left[\frac{\dot{q}^{n,\theta,x_0}(Y_1^{n,\theta,x_0})^2}{q^{n,\theta,x_0}(Y_1^{n,\theta,x_0})^2}\right] < \infty.$

The proof of this proposition is based on the following lemma, which is related to a continuity property with respect to the conditioning variable, in a conditional expectation.

**Lemma 6** *Let $(\theta_n)_{n\geq 1}$ be a sequence such that $\theta_n \xrightarrow{n\to\infty} \theta$. Then, the following convergence holds uniformly with respect to $x_0$,*

$$n^{2-2/\alpha} E\left[E[\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta_n,x_0}) \mid Y_1^{n,\theta_n,x_0}]^2\right] \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[E[\mathcal{H}_{L^\alpha}(1) \mid L_1^\alpha]^2\right],$$

*where $\mathcal{H}_{L^\alpha}(1)$ is given by (55) and $L_1^\alpha$ by (33).*

**Proof** Let us recall the crucial decomposition given in (49), $\mathcal{H}_\theta^n(\dot{Y}_1^{n,\theta,x_0}) = \dot{Y}_1^{n,\theta,x_0}\widehat{\mathcal{H}}_\theta^n(1) + \mathcal{R}_\theta^n(\dot{Y}_1^{n,\theta,x_0})$ where $\left|\mathcal{R}_\theta^n(\dot{Y}_1^{n,\theta,x_0})\right| \leq Cn^{-1}$. From the fact that $\sup_{x_0}\sup_{s\in[0,1]}|\mathcal{E}_s^n - 1| + \left|(\mathcal{E}_s^n)^{-1} - 1\right| \xrightarrow{n\to\infty} 0$ and the explicit expression of $\dot{Y}_1^{n,\theta,x_0}$ given in (51), we easily get

$$\sup_{x_0}\left|n^{1-1/\alpha}\dot{Y}_1^{n,\theta_n,x_0} - \dot{b}(x,\theta)\right| \xrightarrow[\text{a.s.}]{n\to\infty} 0.$$

From the expressions (41) and (55) it can be seen that $\sup_{x_0}\left|\widehat{\mathcal{H}}_{\theta_n}^n(1) - \mathcal{H}_{L^\alpha}(1)\right| \xrightarrow[\text{a.s.}]{n\to\infty} 0$. We deduce that almost surely, one has the convergence

$$\sup_{x_0}\left|n^{1-1/\alpha}\mathcal{H}_{\theta_n}^n(\dot{Y}_1^{n,\theta_n,x_0}) - \dot{b}(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\right| \xrightarrow{n\to\infty} 0. \tag{75}$$

Moreover using the upper bound (56) with (49) again, we can apply the dominated convergence Theorem and see that the convergence (75) holds in $\mathbf{L}^p$-norm for all $p \geq 1$. Now, we can write

$$\begin{aligned}
&E\left[\left|E[n^{1-1/\alpha}\mathcal{H}_{\theta_n}^n(\dot{Y}_1^{n,\theta_n,x_0}) \mid Y_1^{n,\theta,n,x_0}] - E[\dot{b}(x_0,\theta)\mathcal{H}_{L^\alpha}(1) \mid Y_1^{n,\theta,n,x_0}]\right|^2\right] \\
&\leq E\left[E[\left|n^{1-1/\alpha}\mathcal{H}_{\theta_n}^n(\dot{Y}_1^{n,\theta_n,x_0}) - \dot{b}(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\right|^2 \mid Y_1^{n,\theta,n,x_0}]\right] \\
&= E\left[\left|n^{1-1/\alpha}\mathcal{H}_{\theta_n}^n(\dot{Y}_1^{n,\theta_n,x_0}) - \dot{b}(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\right|^2\right]
\end{aligned}$$

converges to zero uniformly with respect to $x_0$. In turns, it gives the uniform convergence

$$n^{2-2/\alpha}E\left[E[\mathcal{H}_{\theta_n}^n(\dot{Y}_1^{n,\theta_n,x_0}) \mid Y_1^{n,\theta_n,x_0}]^2\right] - \dot{b}(x_0,\theta)^2 E\left[E[\mathcal{H}_{L^\alpha}(1) \mid Y_1^{n,\theta_n,x_0}]^2\right] \xrightarrow{n\to\infty} 0.$$

Hence, the proposition will be proved as soon as we show the uniform convergence with respect to $x_0$,

$$E\left[E[\mathcal{H}_{L^\alpha}(1) \mid Y_1^{n,\theta_n,x_0}]^2\right] - E\left[E[\mathcal{H}_{L^\alpha}(1) \mid L_1^\alpha]^2\right] \xrightarrow{n\to\infty} 0. \tag{76}$$

This is a delicate part of the proof, since it amounts to compare the conditional expectation of a variable with respect to the two different variables $Y_1^{n,\theta_n,x_0}$ and $L_1^\alpha$. First, we reduce the situation to the case where the random variable in the expectation is bounded. Let $K > 0$ and denote by

$x \mapsto \chi_K(x)$ a smooth truncation function with $\chi_K(x) = 0$ for $|x| \geq K$, $\chi_K(x) = 1$ for $|x| \leq K/2$ and $0 \leq \chi_K \leq 1$. Using that $E[\mathcal{H}_{L^\alpha}(1)^2] < \infty$, one can see that (76) is implied by the following convergence for all $K > 0$,

$$\sup_{x_0} \left| E\left[ E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid Y_1^{n,\theta_n,x_0}]^2 \right] - E\left[ E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid L_1^\alpha]^2 \right] \right| \xrightarrow{n\to\infty} 0. \tag{77}$$

Let us denote by $\eta_n$ and $\eta$ the measurable functions such that,

$$E\left[ \mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid Y_1^{n,\theta_n,x_0} \right] = \eta_n(Y_1^{n,\theta_n,x_0}),$$
$$E\left[ \mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid L_1^\alpha \right] = \eta(L_1^\alpha).$$

With these notations, the condition (77) writes

$$\sup_{x_0} \left| E[\eta_n(Y_1^{n,\theta_n,x_0})^2] - E[\eta(L_1^\alpha)^2] \right| \xrightarrow{n\to\infty} 0 \tag{78}$$

Using Proposition 10 in Section 7.2, we know that

$$\sup_{x_0} E\left[ \left| \eta(Y_1^{n,\theta_n,x_0}) - \eta_n(Y_1^{n,\theta_n,x_0}) \right| \right] \xrightarrow{n\to\infty} 0.$$

Since $|\eta_n|$ and $|\eta|$ are bounded by the constant $K$, we deduce $\left\| \eta(Y_1^{n,\theta_n,x_0}) - \eta_n(Y_1^{n,\theta_n,x_0}) \right\|_2 \xrightarrow{n\to\infty} 0$ uniformly with respect to $x_0$. This yields

$$\sup_{x_0} \left| E[\eta_n(Y_1^{n,\theta_n,x_0})^2] - E[\eta(Y_1^{n,\theta_n,x_0})^2] \right| \xrightarrow{n\to\infty} 0$$

Now, applying (105) in Corollary 2 with the bounded function $\eta^2$ yields (78), and the lemma is proved.

□

We can now prove the main result of the Section.

**Proof of Proposition 7.** i) First we remark that we have the representation

$$\frac{\varphi_\alpha'(u)}{\varphi_\alpha(u)} = E\left[ \mathcal{H}_{L_1^\alpha}(1) \mid L_1^\alpha = u \right]. \tag{79}$$

Indeed, by considering the specific model $b(x,\theta) = \theta$, we obtain, $Y_1^{n,\theta,x_0} = L_1^{n,\alpha} + n^{1/\alpha-1}\theta$, $\dot{Y}_1^{n,\theta,x_0} = n^{1/\alpha-1}$, $\mathcal{H}_\theta^n(\dot{Y}^{n,\theta,x_0}) = n^{1/\alpha-1}\mathcal{H}_{L^\alpha}(1)$, $q^{n,\theta,x_0}(u) = \varphi^n(u - n^{1/\alpha-1}\theta)$. Using (65), we get for any smooth function $f$,

$$\int f(u)\varphi^{n\prime}(u - n^{1/\alpha-1}\theta)n^{1/\alpha-1}\mathrm{d}u = E[f(L_1^{n,\alpha} + n^{1/\alpha-1}\theta)n^{1/\alpha-1}\mathcal{H}_{L^\alpha}(1)]$$

From the convergence results (74) and the smoothness of $f$, we get $\int f(u)\varphi\prime_\alpha(u)\mathrm{d}u = E[f(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)]$, and we deduce (79).

Next, we have from Proposition 4

$$\begin{aligned}
n^{2-2/\alpha} I^{n,\theta_n,x_0} &= n^{2-2/\alpha} E\left[ E\left[ \mathcal{H}^n_{\theta_n}(Y_1^{n,\theta_n,x_0}) \Big| Y^{n,\theta_n,x_0} \right]^2 \right], \\
&\xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[ E[\mathcal{H}_{L^\alpha}(1) \mid L_1^\alpha]^2 \right], \text{ from Lemma 6}, \\
&= \dot{b}(x_0,\theta)^2 E\left[ \frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2} \right], \text{ from (79)},
\end{aligned}$$

which proves the first part of the proposition.

ii) Using successively the Proposition 4 and Jensen inequality, we get

$$I^{n,\theta,x_0} = E[E[\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0}) \mid Y^{n,\theta,x_0}]^2] \leq E[\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0})^2]$$

But it is clear from (49), (52) and (56) that $n^{1-1/\alpha}\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0})$ is bounded in $\mathbf{L}^p$ norm independently of $n,\ \theta,\ x_0$, for any $p \geq 1$. $\square$

# 5 Proof of the asymptotic expansion of the likelihood (Theorems 1–2)

This section is devoted to the proof of the asymptotic expansion for the log-likelihood function, established in the Theorem 1. The proof is based essentially on the $L^2$-regularity property of the transition density $p^\theta_{1/n}(x,y)$ and on the result of Theorem 2. Indeed, from Jeganathan [11], the following four conditions A1–A4 are sufficient to get the expansion (3) of Theorem 1.

We recall the notation $\xi^\theta_{i,n} = \frac{\dot{p}^\theta_{\frac{1}{n}}}{p^\theta_{\frac{1}{n}}}(X^\theta_{\frac{i}{n}}, X^\theta_{\frac{i+1}{n}})$.

**A1. $L^2$-regularity**

$$\sum_{j=1}^n E\left[ \int_{\mathbb{R}} \left\{ p^{\theta+u_n h}_{\frac{1}{n}}(X^\theta_{\frac{j-1}{n}}, y)^{1/2} - p^\theta_{\frac{1}{n}}(X^\theta_{\frac{j-1}{n}}, y)^{1/2} - \frac{1}{2} h u_n \frac{\dot{p}^\theta_{\frac{1}{n}}(X^\theta_{\frac{j-1}{y}}, y)}{p^\theta_{\frac{1}{n}}(X^\theta_{\frac{j-1}{y}}, y)^{1/2}} \right\}^2 \mathrm{d}y \right] \xrightarrow{n\to\infty} 0.$$

**A2.**

$$J_n(\theta) = u_n^2 \sum_{i=0}^{n-1} E\left[ (\xi^\theta_{i,n})^2 | \mathcal{G}_{i/n} \right] \xrightarrow{n\to\infty} \mathcal{I}_\theta \quad (>0\ a.e.), \text{ in probability},$$

**A3.**

$$\forall \varepsilon > 0, \quad \sum_{i=0}^{n-1} u_n^2 E\left[ \left( \xi^\theta_{i,n} \right)^2 1_{\{u_n|\xi^\theta_{i,n}| \geq \varepsilon\}} \right] \xrightarrow{n\to\infty} 0.$$

**A4.**

$$\sup_n u_n^2 \sum_{i=0}^{n} E(\xi_{i,n}^\theta)^2 \leq C, \text{ for a strictly positive constant } C$$

The condition A1 is proved in Section 5.1 below. The conditions A2 and A3 coincide with the Theorem 2, which is proved in Section 5.2 below. The condition A4 is immediate from the Proposition 7 ii), since $E(\xi_{i,n}^\theta)^2 = EI^{n,\theta,X_{i/n}^\theta}$ and $nu_n^2 = n^{2-2/\alpha}$.

Note that these conditions does not imply the stable convergence in law (7) since in our framework the filtration $(\mathcal{G}_{\frac{i}{n}})_i$ does not satisfy the nested condition. The proof of the stable convergence in law will be given in Section 6.

## 5.1 Proof of the $L^2$ regularity condition

**Proposition 8** *Set $u_n = n^{1/2-1/\alpha}$, we have*

$$\sum_{j=1}^{n} E\left[\int_{\mathbb{R}} \left\{ p_{\frac{1}{n}}^{\theta+u_n h}(X_{\frac{j-1}{n}}^\theta, y)^{1/2} - p_{\frac{1}{n}}^\theta(X_{\frac{j-1}{n}}^\theta, y)^{1/2} - \frac{1}{2} h u_n \frac{\dot{p}_{\frac{1}{n}}^\theta(X_{\frac{j-1}{y}}^\theta, y)}{p_{\frac{1}{n}}^\theta(X_{\frac{j-1}{y}}^\theta, y)^{1/2}} \right\}^2 dy \right] \xrightarrow{n\to\infty} 0. \qquad (80)$$

**Proof** Recall that $q^{n,\theta,x_0}$ is the density of the rescaled process $(X_{1/n}^\theta - x_0)n^{1/\alpha}$. One has the simple relation $p_{\frac{1}{n}}^\theta(x,y) = n^{1/\alpha} q^{n,\theta,x}[n^{1/\alpha}(y-x)]$, and proving (80) amounts to show the convergence to zero of the following quantity,

$$\sum_{j=1}^{n} E\Big[n^{1/\alpha} \int_{\mathbb{R}} \{q^{n,\theta+u_n h, X_{\frac{j-1}{n}}^\theta}(n^{1/\alpha}(y - X_{\frac{j-1}{n}}^\theta))^{1/2} - q^{n,\theta,X_{\frac{j-1}{n}}^\theta}(n^{1/\alpha}(y - X_{\frac{j-1}{n}}^\theta))^{1/2}$$
$$- \frac{1}{2} h u_n \frac{\dot{q}^{n,\theta,X_{\frac{j-1}{n}}^\theta}(n^{1/\alpha}(y - X_{\frac{j-1}{n}}^\theta))}{q^{n,\theta,X_{\frac{j-1}{n}}^\theta}(n^{1/\alpha}(y - X_{\frac{j-1}{n}}^\theta))^{1/2}} \}^2 \mathrm{d}y\Big].$$

By a simple change of variable, it is equivalent to show

$$n^{-1} \sum_{j=1}^{n} E\Big[\int_{\mathbb{R}} \{n^{1/2}[q^{n,\theta+u_n h, X_{\frac{j-1}{n}}^\theta}(u)^{1/2} - q^{n,\theta,X_{\frac{j-1}{n}}^\theta}(u)^{1/2}] - \frac{1}{2} h n^{1/2} u_n \frac{\dot{q}^{n,\theta,X_{\frac{j-1}{n}}^\theta}(u)}{q^{n,\theta,X_{\frac{j-1}{n}}^\theta}(u)^{1/2}} \}^2 \mathrm{d}u\Big] \xrightarrow{n\to\infty} 0. \qquad (81)$$

Let us denote

$$f_n(x,u) = n^{1/2}[q^{n,\theta+u_n h,x}(u)^{1/2} - q^{n,\theta,x}(u)^{1/2}],$$
$$g_n(x,u) = \frac{1}{2} n^{1/2} u_n h \frac{\dot{q}^{n,\theta,x}(u)}{q^{n,\theta,x}(u)^{1/2}}.$$

Let us admit temporarily that the three following properties hold true :

1) There exists a function $f$ such that,

$$\forall x,u,\quad f_n(x,u) \xrightarrow{n\to\infty} f(x,u),$$
$$g_n(x,u) \xrightarrow{n\to\infty} f(x,u).$$

2) We have for all $x$,

$$\limsup_n \int_{\mathbb{R}} f_n(x,u)^2 \mathrm{d}u \le \int_{\mathbb{R}} f(x,u)^2 \mathrm{d}u,$$
$$\limsup_n \int_{\mathbb{R}} g_n(x,u)^2 \mathrm{d}u \le \int_{\mathbb{R}} f(x,u)^2 \mathrm{d}u.$$

3) We have

$$\sup_{x,n} \int_{\mathbb{R}} f_n(x,u)^2 \mathrm{d}u < \infty, \tag{82}$$
$$\sup_{x,n} \int_{\mathbb{R}} g_n(x,u)^2 \mathrm{d}u < \infty. \tag{83}$$

Admitting these three points, we can prove (81). Let $\varepsilon > 0$, we first show a uniform polynomial decay for $y \mapsto p_t^\theta(x_0,y)$, when $t \ge \varepsilon$. Using Theorem 4, we have $p_t^\theta(x_0,y) = E[1_{\{X_t^\theta \ge y\}} \mathcal{H}_{X_t^\theta}(1)]$, where $\mathcal{H}_{X_t^\theta}(1)$ is given by (20). But, it can be seen that $\sup_{\varepsilon\le t\le 1} E[\Gamma(X_t^\theta, X_t^\theta)^{-p}]$ is bounded for any $p \ge 1$. Then, one can easily deduce that $\sup_{\varepsilon \le t \le 1} E\left[\left|\mathcal{H}_{X_t^\theta}(1)\right|^p\right] < \infty$. From (1) and the fact that the Lévy measure of $L$ has a compact support we deduce $\sup_{t\in[0,1]} E[|X_t^\theta|^p] < \infty$. Using the Markov inequality, we deduce that $\sup_{\varepsilon\le t\le 1} p_t^\theta(x_0,y) \le \frac{C}{1+y^2}$ for some $C > 0$.

Then, we split the right-hand side of (81) on the following way

$$\begin{aligned}
&n^{-1}\sum_{j=1}^{n} \int_{\mathbb{R}} E[\{f_n(X^\theta_{\frac{j-1}{n}},u) - g_n(X^\theta_{\frac{j-1}{n}},u)\}^2]\mathrm{d}u \\
&\quad= n^{-1}\sum_{j=1}^{\lfloor n\varepsilon\rfloor} E\left[\int_{\mathbb{R}} \{f_n(X^\theta_{\frac{j-1}{n}},u) - g_n(X^\theta_{\frac{j-1}{n}},u)\}^2\mathrm{d}u\right] + n^{-1}\sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} E\left[\int_{\mathbb{R}} \{f_n(X^\theta_{\frac{j-1}{n}},u) - g_n(X^\theta_{\frac{j-1}{n}},u)\}^2\mathrm{d}u\right] \\
&\quad\le n^{-1}\sum_{j=1}^{\lfloor n\varepsilon\rfloor} \sup_{x,n}\int_{\mathbb{R}} 2[f_n(x,u)^2 + g_n(x,u)^2]\mathrm{d}u + n^{-1}\sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} E\left[\int_{\mathbb{R}} \{f_n(X^\theta_{\frac{j-1}{n}},u) - g_n(X^\theta_{\frac{j-1}{n}},u)\}^2\mathrm{d}u\right] \\
&\quad\le \varepsilon C' + n^{-1}\sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} E\left[\int_{\mathbb{R}} \{f_n(X^\theta_{\frac{j-1}{n}},u) - g_n(X^\theta_{\frac{j-1}{n}},u)\}^2\mathrm{d}u\right], \quad \text{by (82)–(83)}, \\
&\quad\le \varepsilon C' + n^{-1}C\sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} \int_{\mathbb{R}}\int_{\mathbb{R}} \{f_n(y,u) - g_n(y,u)\}^2\mathrm{d}u\frac{\mathrm{d}y}{1+y^2},\ \text{using } p_t^\theta(x_0,y) \le \frac{C}{1+y^2}, \\
&\quad= \varepsilon C' + C\frac{n-\lfloor n\varepsilon\rfloor}{n}\int_{\mathbb{R}}\int_{\mathbb{R}} \{f_n(y,u) - g_n(y,u)\}^2\mathrm{d}u\frac{\mathrm{d}y}{1+y^2}.
\end{aligned}$$

From Lemma 7, the conditions 1) and 2) imply that $\int_{\mathbb{R}}\{f_n(y,u)-g_n(y,u)\}^2\mathrm{d}u \xrightarrow{n\to\infty} 0$. The condition 3) is sufficient to apply the dominated convergence Theorem and find that $\int_{\mathbb{R}}\int_{\mathbb{R}}\{f_n(y,u)-g_n(y,u)\}^2\mathrm{d}u\frac{\mathrm{d}y}{1+y^2}$ converges to zero as $n\to\infty$. Hence, we have proved the proposition, up to the fact that we need to check the validity of the conditions 1), 2) and 3).

We start with the proof of the property 1). From Propositions 5–6, we see that $g_n(x,u) \xrightarrow{n\to\infty} f(x,u) := \frac{1}{2}h\dot{b}(x,\theta)\frac{\varphi_\alpha'(u)}{\varphi_\alpha(u)^{1/2}}$. Using the mean value theorem, we can write $f_n(x,u)=\frac{1}{2}n^{1/2}u_nh\frac{\dot{q}^{n,\theta_n,x}(u)}{q^{n,\theta_n,x}(u)^{1/2}}$, for some $\theta_n\in[\theta,\theta+u_nh]$. Using again the Propositions 5–6, we get $f_n(x,u)\xrightarrow{n\to\infty} f(x,u)$.

We now prove the property 2). Recalling that $u_n=n^{1/2-1/\alpha}$ and (10), we have $\int_{\mathbb{R}}g_n(x,u)^2\mathrm{d}u = \frac{h^2}{4}n^{2-2/\alpha}I^{n,\theta,x}$. From the Proposition 7, we get $\int_{\mathbb{R}}g_n(x,u)^2\mathrm{d}u \xrightarrow{n\to\infty} \int_{\mathbb{R}}f(x,u)^2\mathrm{d}u$, for all $x$.

Using $f_n(x,u)=\frac{1}{2}n^{1/2}\int_\theta^{\theta+u_nh}\frac{\dot{q}^{n,s,x}(u)}{q^{n,s,x}(u)^{1/2}}\mathrm{d}s$. We write

$$
\begin{aligned}
\int_{\mathbb{R}} f_n(x,u)^2\mathrm{d}u = \frac{n}{4}\left\|\int_\theta^{\theta+u_nh}\frac{\dot{q}^{n,s,x}(\cdot)}{q^{n,s,x}(\cdot)^{1/2}}\mathrm{d}s\right\|_2^2 &\leq \frac{n}{4}\left(\int_\theta^{\theta+u_nh}\left\|\frac{\dot{q}^{n,s,x}(\cdot)}{q^{n,s,x}(\cdot)^{1/2}}\right\|_2\mathrm{d}s\right)^2 \\
&= \frac{h^2}{4}nu_n^2\left(\int_0^1\left(\int_{\mathbb{R}}\frac{\dot{q}^{n,\theta+su_nh,x}(u)^2}{q^{n,\theta+su_nh,x}(u)}\mathrm{d}u\right)^{1/2}\mathrm{d}s\right)^2 \\
&= \frac{h^2}{4}\left(\int_0^1(n^{2-2/\alpha}I^{n,\theta+su_nh,x})^{1/2}\mathrm{d}s\right)^2 \qquad (84)\\
&\xrightarrow{n\to\infty} \frac{h^2}{4}\dot{b}(x,\theta)^2\int_{\mathbf{R}}\frac{\varphi_\alpha'(u)^2}{\varphi_\alpha(u)}\mathrm{d}u = \int_{\mathbb{R}}f^2(x,u)\mathrm{d}u,
\end{aligned}
$$

where, in the last line, we have used the Proposition 7 for the convergence of $n^{2-2/\alpha}I^{n,\theta+su_nh,x}$ and the application of the dominated convergence Theorem.

We end the proof of the Proposition by showing the property 3). From (84) and Proposition 7 we get (82). The bound (83) is deduced by Proposition 7 as well. ☐

**Lemma 7** *Assume that $(f_n)_n$, $(g_n)_n$ are two sequences of real functions such that:*

*1) There exists $f\in\mathbf{L}^2(\mathbb{R})$ such that $f_n(u)\xrightarrow{n\to\infty} f(u)$ and $g_n(u)\xrightarrow{n\to\infty} f(u)$ for almost every $u$.*

*2) We have $\limsup_n\int_{\mathbb{R}}f_n(u)^2du\leq\int_{\mathbb{R}}f(u)^2du$ and $\limsup_n\int_{\mathbb{R}}g_n(u)^2du\leq\int_{\mathbb{R}}f(u)^2du$.*

*Then,*

$$\int_{\mathbb{R}}(f_n(u)-g_n(u))^2du \xrightarrow{n\to\infty} 0.$$

**Proof** We write $(f_n(u)-g_n(u))^2\leq 2f_n(u)^2+2g_n(u)^2$ and thus $2f_n(u)^2+2g_n(u)^2-(f_n(u)-g_n(u))^2\geq 0$.

Applying Fatou's lemma to this non negative function, we get

$$\begin{aligned}\int_{\mathbb{R}} 4f(u)^2 \mathrm{d}u &\leq \liminf_n \int_{\mathbb{R}} [2f_n(u)^2 + 2g_n(u)^2 - (f_n(u) - g_n(u))^2]\mathrm{d}u \\ &\leq \limsup_n \int_{\mathbb{R}} [2f_n(u)^2 + 2g_n(u)^2]\mathrm{d}u - \limsup_n \int_{\mathbb{R}} (f_n(u) - g_n(u))^2 \mathrm{d}u.\end{aligned}$$

This yields the inequality $\limsup_n \int_{\mathbb{R}} (f_n(u) - g_n(u))^2 \mathrm{d}u \leq \limsup_n \int_{\mathbb{R}} 2f_n(u)^2 \mathrm{d}u + \limsup_n \int_{\mathbb{R}} 2g_n(u)^2 \mathrm{d}u - \int_{\mathbb{R}} 4f(u)^2 \mathrm{d}u \leq 0$, and thus the lemma follows. □

## 5.2 Proof of Theorem 2

**Proof** First, we use that $\frac{\dot{p}^\theta_{1/n}(x,y)}{p^\theta_{1/n}(x,y)} = \frac{\dot{q}^{n,\theta,x}(n^{1/\alpha}(y-x))}{q^{n,\theta,x}(n^{1/\alpha}(y-x))}$, and as a result of the Markov property for the process $X^\theta$ and (10), we have

$$E[(\xi^\theta_{i,n})^2 \mid \mathcal{G}_{i/n}] = I^{n,\theta,X^\theta_{i/n}}.$$

From the Proposition 7, we know that the quantity

$$\sup_{0 \leq i \leq n-1} \left| n u_n^2 I^{n,\theta,X^\theta_{i/n}} - \dot{b}(X^\theta_{i/n},\theta)^2 \int_{\mathbb{R}} \frac{\varphi'(u)^2}{\varphi(u)} \mathrm{d}u \right| = \sup_{0 \leq i \leq n-1} \left| n^{2-2/\alpha} I^{n,\theta,X^\theta_{i/n}} - \dot{b}(X^\theta_{i/n},\theta)^2 \int_{\mathbb{R}} \frac{\varphi'(u)^2}{\varphi(u)} \mathrm{d}u \right|$$

converges to zero as $n \to \infty$. Then the convergence (5) is a consequence of the convergence of a Riemann sum.

To prove (6), we use again the relation $\frac{\dot{p}^\theta_{1/n}(x,y)}{p^\theta_{1/n}(x,y)} = \frac{\dot{q}^{n,\theta,x}(n^{1/\alpha}(y-x))}{q^{n,\theta,x}(n^{1/\alpha}(y-x))}$ and the Markov property to get, $E[|\xi_{i,n}|^k \mid X^\theta_{i/n} = x] = E\left[\left|\frac{\dot{q}^{n,\theta,x}(Y^{n,\theta,x}_1)}{q^{n,\theta,x}(Y^{n,\theta,x}_1)}\right|^k\right]$, for any $k \geq 1$. It then follows from Proposition 4 that,

$$E[|\xi_{i,n}|^k \mid X^\theta_{i/n} = x] = E\left[\left| E\left[\mathcal{H}^n_\theta(\dot{Y}^{n,\theta,x}_1) \mid Y^{n,\theta,x}_1\right]\right|^k\right] \leq E\left[\left|\mathcal{H}^n_\theta(\dot{Y}^{n,\theta,x}_1)\right|^k\right],$$

where we used the Jensen inequality in the last step. As seen in the proof of Proposition 7, the random variables $n^{1-1/\alpha}\mathcal{H}^n_\theta(\dot{Y}^{n,\theta,x}_1)$ are bounded in $\mathbf{L}^k$-norm independently of $n$ and $x$. From this, we deduce

$$\sup_{0 \leq i \leq n-1} n^{k-k/\alpha} E[|\xi_{i,n}|^k] \leq C(k), \ \forall k \geq 1,$$

where the $C(k)$ are some finite constants. It can be classically checked that the previous control, for instance with $k = 4$, is sufficient to imply the Lindeberg's condition (6). □

# 6 Stable central limit theorem

This section is devoted to the proof of the stable convergence in law stated in Theorem 3.

**Proof** Since $u_n = n^{1/2-1/\alpha}$, we have

$$u_n \sum_{i=0}^{n-1} \xi_{i,n}^{\theta} = n^{-1/2} \sum_{i=0}^{n-1} n^{1-1/\alpha} \frac{\dot{p}^{\theta}_{\frac{1}{n}}(X^{\theta}_{\frac{i}{n}}, X^{\theta}_{\frac{i+1}{n}})}{p^{\theta}_{\frac{1}{n}}(X^{\theta}_{\frac{i}{n}}, X^{\theta}_{\frac{i+1}{n}})}.$$

The Theorem 3 is an immediate consequence of the Lemmas 8–9 below. □

**Lemma 8** *Set*

$$\eta_{i,n} = n^{1-1/\alpha} \frac{\dot{p}^{\theta}_{\frac{1}{n}}(X^{\theta}_{\frac{i}{n}}, X^{\theta}_{\frac{i+1}{n}})}{p^{\theta}_{\frac{1}{n}}(X^{\theta}_{\frac{i}{n}}, X^{\theta}_{\frac{i+1}{n}})} - \dot{b}(X^{\theta}_{\frac{i}{n}}, \theta) \frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)},$$

*then we have* $n^{-1/2} \sum_{i=0}^{n-1} \eta_{i,n} \xrightarrow{n\to\infty} 0$.

**Proof** Using Lemma 9 in [6], it is sufficient to show :

$$n^{-1/2} \sum_{i=0}^{n-1} \left| E[\eta_{i,n} \mid \mathcal{G}_{i/n}] \right| \xrightarrow[\mathbf{P}]{n\to\infty} 0, \tag{85}$$

$$n^{-1} \sum_{i=0}^{n-1} E[\eta_{i,n}^2 \mid \mathcal{G}_{i/n}] \xrightarrow[\mathbf{P}]{n\to\infty} 0, \tag{86}$$

We start by the proof of (85). Since a score function has an expectation equal to zero, and $L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$ is independent of $\mathcal{G}_{i/n}$, we deduce that

$$E[\eta_{i,n} \mid \mathcal{G}_{i/n}] = -\dot{b}(X^{\theta}_{\frac{i}{n}}, \theta) E\left[ \frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)} \right]$$

But, since $(L_t)_t$ has stationary increments, the law of $n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})$ is the same as the law of $L_1^{n,\alpha}$. We know from Lemma 3, that $P(L_1^{n,\alpha} + \kappa_n \neq L_1^\alpha) = O(1/n)$, thus

$$E[\eta_{i,n} \mid \mathcal{G}_{i/n}] = -\dot{b}(X^{\theta}_{\frac{i}{n}}, \theta) E\left[ \frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)} \right] + \left\| \frac{\varphi'_\alpha}{\varphi_\alpha} \right\|_\infty O(n^{-1}),$$

where we used that $\frac{\varphi'_\alpha}{\varphi_\alpha}$ is bounded (see e.g. Theorem 7.3.2 in [15]). Using $E\left[\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right] = \int_{\mathbb{R}} \varphi'_\alpha(u) \mathrm{d}u = 0$, we deduce $\left|[\eta_{i,n} \mid \mathcal{G}_{i/n}]\right| \leq C n^{-1}$ for some constant $C$ and (85) follows.

We now prove (86). Recalling the definition (10), we have

$$E[(\eta_{i,n})^2 \mid \mathcal{G}_{\frac{i}{n}}] = n^{2-2/\alpha} I^{n,\theta,X^\theta_{\frac{i}{n}}} + \dot{b}(X^\theta_{\frac{i}{n}},\theta)^2 E\left[\frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)^2}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)^2}\right]$$
$$- 2E\left[n^{1-1/\alpha}\frac{\dot{p}^\theta_{\frac{1}{n}}(X^\theta_{\frac{i}{n}}, X^\theta_{\frac{i+1}{n}})}{p^\theta_{\frac{1}{n}}(X^\theta_{\frac{i}{n}}, X^\theta_{\frac{i+1}{n}})}\dot{b}(X_{\frac{i}{n}},\theta)\frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)} \;\middle|\; \mathcal{G}_{\frac{i}{n}}\right] \quad (87)$$

With a method analogous to the proof of (85), we can show that $E\left[\frac{\varphi'_\alpha(n^{1/\alpha}(L_{(i+1)/n}-L_{i/n})+\kappa_n)^2}{\varphi_\alpha(n^{1/\alpha}(L_{(i+1)/n}-L_{i/n})+\kappa_n)^2}\right] = E\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] + o(1)$. From Proposition 7, it appears that the first two terms in the right-hand side of (87) are asymptotically close to the same quantities, and that (85) is proved as soon as we show the following control, uniformly with respect to $i$,

$$E\left[n^{1-1/\alpha}\frac{\dot{p}^\theta_{\frac{1}{n}}(X^\theta_{\frac{i}{n}}, X^\theta_{\frac{i+1}{n}})}{p^\theta_{\frac{1}{n}}(X^\theta_{\frac{i}{n}}, X^\theta_{\frac{i+1}{n}})}\dot{b}(X_{\frac{i}{n}},\theta)\frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) + \kappa_n)} \;\middle|\; \mathcal{G}_{\frac{i}{n}}\right]$$
$$= \dot{b}(X^\theta_{\frac{i}{n}},\theta)^2 E\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] + o(1) \quad (88)$$

Using the notations of Section 4, we define $d^{n,\theta,x_0} = E\left[n^{1-1/\alpha}\frac{\dot{q}^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}{q^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}\dot{b}(x_0,\theta)\frac{\varphi'_\alpha(L_1^{n,\alpha}+\kappa_n)}{\varphi_\alpha(L_1^{n,\alpha}+\kappa_n)}\right]$, so that the left-hand side of (88) reduces, from the Markov property, to $d^{n,\theta,X^\theta_{i/n}}$. From (38) and $\kappa_n \to 0$, we have $\left\|Y_1^{n,\theta,x_0} - L_1^{n,\alpha} + \kappa_n\right\|_\infty \xrightarrow{n\to\infty} 0$. Using the fact that $\frac{\varphi'_\alpha}{\varphi_\alpha}$ has a bounded derivative (see e.g. Theorem 7.3.2 in [15]) together with Cauchy- Schwarz inequality and Proposition 7 ii), we can deduce that,

$$d^{n,\theta,x_0} = E\left[n^{1-1/\alpha}\frac{\dot{q}^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}{q^{n,\theta,x_0}(Y_1^{n,\theta,x_0})}\dot{b}(x_0,\theta)\frac{\varphi'_\alpha(Y_1^{n,\theta,x_0})}{\varphi_\alpha(Y_1^{n,\theta,x_0})}\right] + o(1),$$

where the $o(1)$ term is uniform with respect to $x_0$. Using the Proposition 4, we have

$$d^{n,\theta,x_0} = E\left[n^{1-1/\alpha}\mathcal{H}^n_\theta(\dot{Y}_1^{n,\theta,x_0})\dot{b}(x_0,\theta)\frac{\varphi'_\alpha(Y_1^{n,\theta,x_0})}{\varphi_\alpha(Y_1^{n,\theta,x_0})}\right] + o(1).$$

From the convergence result (75), we deduce that

$$\sup_{x_0}\left|d^{n,\theta,x_0} - \dot{b}(x_0,\theta)^2 E\left[\mathcal{H}_{L^\alpha}(1)\frac{\varphi'_\alpha(Y_1^{n,\theta,x_0})}{\varphi_\alpha(Y_1^{n,\theta,x_0})}\right]\right| \xrightarrow{n\to\infty} 0.$$

From Lemma 3 and (38), we can deduce that,

$$d^{n,\theta,x_0} \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[\mathcal{H}_{L^\alpha}(1)\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right],$$

uniformly with respect to $x_0$. Then, the relation (79) enables to rewrite this convergence as,

$$d^{n,\theta,x_0} \xrightarrow{n\to\infty} \dot{b}(x_0,\theta)^2 E\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right], \text{ uniformly with respect to } x_0.$$

This result implies (88) and hence (86). □

**Lemma 9** *On has the convergence in law,*

$$n^{-1/2}\sum_{i=0}^{n-1}\frac{\varphi'_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})+\kappa_n)}{\varphi_\alpha(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})+\kappa_n)}\dot{b}(X_{\frac{i}{n}},\theta) \xrightarrow{n\to\infty} \mathcal{N}(0,\mathcal{I}_\theta), \tag{89}$$

*where the convergence is stable with respect to $\mathcal{G}_1$.*

**Proof** Let us define the processes,

$$\begin{aligned}
Z_t^n &= \sum_{i=0}^{\lfloor nt\rfloor}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}}),\\
\Gamma_t^n &= n^{-1/2}\sum_{i=0}^{\lfloor nt\rfloor}\frac{\varphi'_\alpha}{\varphi_\alpha}(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})+\kappa_n)\dot{b}(X^\theta_{\frac{i}{n}},\theta),\\
\Gamma_t'^n &= n^{-1/2}\sum_{i=0}^{\lfloor nt\rfloor}\frac{\varphi'_\alpha}{\varphi_\alpha}(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})+\kappa_n).
\end{aligned}$$

We will apply Lemma 2.8 in [10] to prove (89). Indeed, we will show that there exists a Gaussian random variable $\gamma$, independent of $L_1$, such that one has the convergence

$$(\Gamma_1'^n, Z_1^n) \xrightarrow[\text{law}]{n\to\infty} (\gamma, L_1). \tag{90}$$

Then, by application of Lemma 2.8 in [10], there exists a Brownian motion $(\Gamma'_t)_t$ independent of $(L_t)_t$ such that one has the convergence in law for processes $(Z^n,\Gamma^n,\Gamma'^n) \xrightarrow[\text{law}]{n\to\infty} (L,\Gamma,\Gamma')$ where $\Gamma_t = \int_0^t \dot{b}(X_s^\theta,\theta)\mathrm{d}\Gamma'_s$. This exactly implies the lemma, if we show furthermore $var(\Gamma'_1) = var(\gamma) = E\left[\frac{\varphi'_\alpha}{\varphi_\alpha}(L_1^\alpha)^2\right]$.

Let us focus on the derivation of the convergence (90). For $(u,v)\in\mathbb{R}^2$, let us set

$$\chi_n(u,v) = E\left[\exp\left(i\frac{u}{n^{1/2}}\frac{\varphi'_\alpha}{\varphi_\alpha}(n^{1/\alpha}L_{\frac{1}{n}}+\kappa_n)+ivL_{\frac{1}{n}}\right)\right].$$

Using the i.i.d. structure of the increments of the Lévy process $L$, we easily get the following expression about the characteristic function of $(\Gamma_1'^n, Z_1^n)$

$$\log E\left[\exp\left(iu\Gamma_1'^n + ivZ_1^n\right)\right] = n \log \chi_n(u,v). \tag{91}$$

Let us study the asymptotic behaviour of $\chi_n(u,v)$. Using that $\varphi_\alpha'/\varphi_\alpha$ is bounded we get

$$\begin{aligned}
\chi_n(u,v) &= E[e^{ivL_{\frac{1}{n}}}] + \frac{iu}{n^{1/2}} E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(n^{1/\alpha}L_{\frac{1}{n}} + \kappa_n)e^{ivL_{\frac{1}{n}}}\right] \\
&\quad - \frac{u^2}{2n} E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(n^{1/\alpha}L_{\frac{1}{n}} + \kappa_n)^2 e^{ivL_{\frac{1}{n}}}\right] + O(n^{-3/2}) \qquad (92) \\
&:= \chi_n^{(1)}(u,v) + \frac{iu}{n^{1/2}}\chi_n^{(2)}(u,v) - \frac{u^2}{2n}\chi_n^{(3)}(u,v) + O(n^{-3/2}). \qquad (93)
\end{aligned}$$

First, we have

$$\chi_n^{(1)}(u,v) = e^{\psi(v)/n} = 1 + \psi(v)/n + O(n^{-2}), \tag{94}$$

where $\psi(v)$ is the Lévy Kintchine exponent of $L_1$.

We now focus on the term $\chi_n^{(2)}(u,v)$. Using (36)–(37) of Lemma 3, and the fact that $n^{1/\alpha}L_{1/n}$ has the same law as $L_1^{n,\alpha}$, we get

$$\begin{aligned}
\chi_n^{(2)}(u,v) &= E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(L_1^\alpha)e^{i\frac{v(L_1^\alpha - \kappa_n)}{n^{1/\alpha}}}\right] + O(n^{-1}) \\
&= \int_{\mathbb{R}} \varphi_\alpha'(s)e^{i\frac{v(s-\kappa_n)}{n^{1/\alpha}}}\mathrm{d}s + O(n^{-1}) \\
&= \frac{iv}{n^{1/\alpha}}\int_{\mathbb{R}} \varphi_\alpha(s)e^{i\frac{v(s-\kappa_n)}{n^{1/\alpha}}}\mathrm{d}s + O(n^{-1}) \text{ using integration by parts formula} \\
&= O(n^{-1/\alpha}) \qquad (95)
\end{aligned}$$

For the term, $\chi_n^{(3)}(u,v)$ using Lemma 3 again, it is easy to see that

$$\begin{aligned}
\chi_n^{(3)}(u,v) &= E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(L_1^\alpha)^2 e^{i\frac{v(L_1^\alpha - \kappa_n)}{n^{1/\alpha}}}\right] + O(n^{-1}) \\
&\xrightarrow{n\to\infty} E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(L_1^\alpha)^2\right] \qquad (96)
\end{aligned}$$

Collecting together (91)–(96), we have

$$\log E\left[\exp\left(iu\Gamma_1'^n + ivZ_1^n\right)\right] \xrightarrow{n\to\infty} \psi(v) - \frac{u^2}{2}E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(L_1^\alpha)^2\right],$$

and thus the convergence (90) with $\gamma \sim \mathcal{N}\left(0, E\left[\frac{\varphi_\alpha'}{\varphi_\alpha}(L_1^\alpha)^2\right]\right)$. $\square$

# 7 Appendix

## 7.1 Proof of Lemma 10

We prove in this section the following result.

**Lemma 10** *We have for all $p \geq 1$,*

$$n^{1-1/\alpha}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}\left(\dot{Y}_1^{n,\theta_n,x_0}\widehat{\mathcal{H}}_{\theta_n}(1)\right) \xrightarrow[\mathbf{L}^p]{n\to\infty} \dot{b}(x_0,\theta)\mathcal{H}^{(2)},$$

*where $\mathcal{H}^{(2)}$ is a random variable that can be expressed as a functional of the random measure $\mu$ and the function $\rho$.*

**Proof** We first show two intermediate results that are useful for the proof of the lemma.

**Lemma 11** *Let $\xi : \mathbb{R} \to \mathbb{R}$ be a bounded function with support included in $[-1, 1]$, with bounded derivative and such that $\xi \in \mathbf{L}^1(\frac{dz}{|z|^{1+\alpha}})$ and let*

$$I_n(t) = (\mathcal{E}_t^n)^p \int_0^t \int_{\mathbb{R}} (\mathcal{E}_s^n)^{-p}\xi(z)\mu(ds, dz),$$
$$I(t) = \int_0^t \int_{\mathbb{R}} \xi(z)\mu(ds, dz),$$

*where $p \geq 1$ is some real constant. Then, the following convergences hold in $\mathbf{L}^q$-norm for all $q \geq 1$,*

$$I_n(1) \xrightarrow{n\to\infty} I(1),$$
$$\sup_\theta \Gamma(I_n(1) - I(1), I_n(1) - I(1)) \xrightarrow{n\to\infty} 0.$$

**Proof** The convergence of $I_n(1)$ to $I(1)$ is clear since $s \mapsto \mathcal{E}_s^n$ converges uniformly to the constant 1, and is bounded by above and below (recall (42)).

We now focus on bracket $\Gamma(I_n(1) - I(1), I_n(1) - I(1))$. Let us remark that $(I_n(t))_t$ is solution to the linear equation,

$$I_n(t) = p\int_0^t I_n(s)n^{-1}b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)\mathrm{d}s + I(t).$$

We set $W_n(t) = I_n(t) - I(t)$ and $R_n(t) = \Gamma(W_n(t), W_n(t))$. The process $W_n$ satisfies the linear equation $W_n(t) = p\int_0^t n^{-1}[I(s) + W_n(s)]b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}, \theta)\mathrm{d}s$. In turns, it can be seen that the process

$R_n(t)$ is solution to

$$R_n(t) = pn^{-1/\alpha-1}\int_0^t 2(I(s)+W_n(s))b''(x_0+n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\Gamma(Y_s^{n,\theta,x_0},W_n(s))\mathrm{d}s+ \\ n^{-1}p\int_0^t 2b'(x_0+n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)[\Gamma(I(s),W_n(s))+R_n(s)]\mathrm{d}s.$$

Using that

$$\left|\Gamma(Y_s^{n,\theta,x_0},W_n(s))\right| \leq \Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0})^{1/2}\Gamma(W_n(s),W_n(s))^{1/2} \leq \Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0})+\Gamma(W_n(s),W_n(s))$$

and a similar control for $|\Gamma(I(s),W_n(s))|$ we get,

$$R_n(t) \leq Cn^{-1/\alpha-1}\int_0^t(|I(s)|+|W_n(s)|)(\Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0})+R_n(s))\mathrm{d}s+Cn^{-1}\int_0^t[\Gamma(I(s),I(s))+R_n(s)]\mathrm{d}s,$$

where $C$ is some constant depending on $\|b'\|_\infty$, $\|b''\|_\infty$ Now, we recall the control $\Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0}) \leq C\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})$, for $s \leq 1$, and use the controls

$$|I(s)|+|W_n(s)| \leq C\int_0^1\int_{\mathbb{R}}|\xi(z)|\,\mu(\mathrm{d}s,\mathrm{d}z) := I^\star$$
$$\Gamma(I(s),I(s)) \leq \int_0^1\int_{\mathbb{R}}\xi'(z)^2\rho(z)\mu(\mathrm{d}s,\mathrm{d}z) := J^\star.$$

We deduce,

$$R_n(t) \leq C\int_0^t[n^{-1/\alpha-1}I^\star+n^{-1}]R_n(s)\mathrm{d}s+Ct[n^{-1/\alpha-1}I^\star\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})+n^{-1}J^\star].$$

We deduce that

$$R_n(1) \leq n^{-1}C\exp\left(Cn^{-1/\alpha}I^\star+C\right)(n^{-1/\alpha}I^\star\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})+J^\star).$$

Now, $J^\star$ and $\sup_\theta\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})$ have finite moments of any order, bounded independently on $n$ and using the exponential formula for Poisson measure, we have $E\left[\exp\left(Cn^{-1}\int_0^1\int_{\mathbb{R}}|\xi(z)|\,\mu(\mathrm{d}s,\mathrm{d}z)\right)\right] = \exp\left[\int_0^1\int_{\mathbb{R}}(e^{\frac{C|\xi(z)|}{n}}-1)\mathrm{d}t\frac{\mathrm{d}z}{|z|^{1+\alpha}}\right]$, which is finite and bounded independently of $n$. This shows that the exponential moments of $I^\star$ are bounded. We deduce that $R_n(1) \to 0$ in $L^p$ norm, uniformly with respect to the parameter $\theta$, and the lemma follows. □

**Lemma 12** *We have*

$$\frac{\Gamma\left(n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0},n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0}\right)}{\Gamma\left(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0}\right)} \leq Cn^{1-2/\alpha},$$

*where $C$ is some constant independent of $n$, $\theta$, $x_0$.*

**Proof** The process $\dot{Y}^{n,\theta,x_0}$ is solution of

$$\dot{Y}_t^{n,\theta,x_0} = n^{-1}\int_0^t b'(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\dot{Y}_s^{n,\theta,x_0}\mathrm{d}s + n^{1/\alpha-1}\int_0^t \dot{b}(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\mathrm{d}s.$$

From this, we can deduce that $Q_t = \Gamma(\dot{Y}_t^{n,\theta,x_0},\dot{Y}_t^{n,\theta,x_0})$ is solution of the equation,

$$Q_t = \int_0^t 2n^{-1}Q_s b'(x_0+n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\mathrm{d}s + 2n^{-1-1/\alpha}\int_0^t \Gamma(Y_s^{n,\theta,x_0},\dot{Y}_s^{n,\theta,x_0})b''(x_0+n^{-1/\alpha}Y_s^{n,\theta,x_0},\theta)\mathrm{d}s + 2n^{-1}\int_0^t \Gamma(Y_s^{n,\theta,x_0},\dot{Y}_s^{n,\theta,x_0})\dot{b}'(Y_s^{n,\theta,x_0},\theta)\mathrm{d}s$$

Using $\Gamma(Y_s^{n,\theta,x_0},\dot{Y}_s^{n,\theta,x_0}) \leq \Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0}) + \Gamma(\dot{Y}_s^{n,\theta,x_0},\dot{Y}_s^{n,\theta,x_0})$ and the boundedness of the derivatives of $b$, we get,

$$Q_t \leq Cn^{-1}\int_0^t Q_s\mathrm{d}s + Cn^{-1}\int_0^t \Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0})\mathrm{d}s.$$

Now using that $\Gamma(Y_s^{n,\theta,x_0},Y_s^{n,\theta,x_0}) \leq C\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})$ for $s \leq 1$, we deduce by application of Gronwall lemma that,

$$\Gamma(\dot{Y}_1^{n,\theta,x_0},\dot{Y}_1^{n,\theta,x_0}) = Q_1 \leq Cn^{-1}\Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0}).$$

The lemma is proved. □

We are now able to prove the Lemma 10.

From the definition (20) and the basic properties of $(\cdot,\cdot)\mapsto\Gamma(\cdot,\cdot)$, one can check that

$$n^{1-1/\alpha}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\dot{Y}_1^{n,\theta_n,x_0}\widehat{\mathcal{H}}_{\theta_n}^n(1)) = n^{1-1/\alpha}\dot{Y}_1^{n,\theta_n,x_0}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\widehat{\mathcal{H}}_{\theta_n}^n(1)) - n^{1-1/\alpha}\widehat{\mathcal{H}}_{\theta_n}^n(1)\frac{\Gamma(Y_1^{n,\theta_n,x_0},\dot{Y}_1^{n,\theta_n,x_0})}{\Gamma(Y_1^{n,\theta_n,x_0},Y_1^{n,\theta_n,x_0})}.$$

Using $\left|\Gamma(Y_1^{n,\theta,x_0},n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0})\right| \leq \Gamma(Y_1^{n,\theta,x_0},Y_1^{n,\theta,x_0})^{\frac{1}{2}}\Gamma(n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0},n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0})^{\frac{1}{2}}$ with Lemma 12 gives,

$$\frac{\Gamma(Y_1^{n,\theta_n,x_0},n^{1-1/\alpha}\dot{Y}_1^{n,\theta_n,x_0})}{\Gamma(Y_1^{n,\theta_n,x_0},Y_1^{n,\theta_n,x_0})} \leq Cn^{1/2-1/\alpha} \xrightarrow{n\to\infty} 0.$$

We deduce that, for any $p \geq 1$,

$$n^{1-1/\alpha}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\dot{Y}_1^{n,\theta_n,x_0}\widehat{\mathcal{H}}_{\theta_n}^n(1)) = n^{1-1/\alpha}\dot{Y}_1^{n,\theta,x_0}\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\widehat{\mathcal{H}}_{\theta_n}^n(1)) + o_{\mathbf{L}^p}(1).$$

Now, the explicit expression for $\dot{Y}_1^{n,\theta_n,x_0}$ (see (51)) shows that $n^{1-1/\alpha}\dot{Y}_1^{n,\theta_n,x_0}$ converges in $\mathbf{L}^p$-norm to $\dot{b}(x_0,\theta)$.

Thus, we need to prove the convergence of $\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\widehat{\mathcal{H}}^n_{\theta_n}(1))$ to some $\mathcal{H}^{(2)}$, depending only on $\rho$ and $\mu$. Recalling (20) and using basic properties of the operator $\Gamma$, we have

$$\mathcal{H}_{Y_1^{n,\theta_n,x_0}}(\widehat{\mathcal{H}}^n_{\theta_n}(1)) = \frac{-2L(Y_1^{n,\theta_n,x_0})\widehat{\mathcal{H}}^n_{\theta_n}(1)}{\Gamma\left(Y_1^{n,\theta_n,x_0}, Y_1^{n,\theta_n,x_0}\right)} - \widehat{\mathcal{H}}^n_{\theta_n}(1)\Gamma\left(Y_1^{n,\theta_n,x_0}, \frac{1}{\Gamma(Y_1^{n,\theta_n,x_0}, Y_1^{n,\theta_n,x_0})}\right) - \frac{\Gamma\left(Y_1^{n,\theta_n,x_0}, \widehat{\mathcal{H}}_{\theta_n}(1)\right)}{\Gamma\left(Y_1^{n,\theta_n,x_0}, Y_1^{n,\theta_n,x_0}\right)}. \quad (97)$$

The convergence of the first two terms of the right-hand side of this equation follows from computations analogous to the proof of Proposition 5. Indeed, in the proof of Proposition 5, it is shown that $\widehat{\mathcal{H}}^n_{\theta_n}(1)$ converges to $\mathcal{H}_{L^\alpha}(1)$ defined by (55). The convergence of $\Gamma(Y_1^{n,\theta_n,x_0}, Y_1^{n,\theta_n,x_0})$, $\Gamma(Y_1^{n,\theta_n,x_0}, \Gamma(Y_1^{n,\theta_n,x_0}, Y_1^{n,\theta_n,x_0}))$ and $L(Y_1^{n,\theta_n,x_0})$ to quantities independent of $b$ can be obtained by studying their respective explicit expressions (44)–(46).

It remains to study the convergence of $\Gamma(Y_1^{n,\theta_n,x_0}, \widehat{\mathcal{H}}^n_{\theta_n}(1))$. After cumbersome computations relying on (41), (55), the fact that $\left(\int_0^1 \int_{\mathbb{R}} \rho(z) \mathrm{d}\mu(\mathrm{d}s, \mathrm{d}z)\right)^{-1}$ belongs to $\cap_p \mathbf{L}^p$ and Lemma 11 one can show,

$$\Gamma\left(\widehat{\mathcal{H}}^n_{\theta_n}(1) - \mathcal{H}_{L^\alpha}(1), \widehat{\mathcal{H}}^n_{\theta_n}(1) - \mathcal{H}_{L^\alpha}(1)\right) \xrightarrow[\mathbf{L}^p]{n\to\infty} 0, \ \forall p \geq 1.$$

As a consequence, we have $\Gamma(Y_1^{n,\theta_n,x_0}, \widehat{\mathcal{H}}^n_{\theta_n}(1)) = \Gamma(Y_1^{n,\theta_n,x_0}, \mathcal{H}_{L^\alpha}(1)) + o_{\mathbf{L^p}(1)}$. Using (38), we deduce

$$\begin{aligned}
\Gamma(Y_1^{n,\theta_n,x_0}, \widehat{\mathcal{H}}^n_{\theta_n}(1)) &= \Gamma(L_1^{n,\alpha}, \mathcal{H}_{L^\alpha}(1)) + \int_0^1 \Gamma(b(x_0 + n^{-1/\alpha} Y_s^{n,\theta_n,x_0}, \theta), \mathcal{H}_{L^\alpha}(1)) \mathrm{d}s + o_{\mathbf{L^p}(1)} \\
&= \Gamma(L_1^{n,\alpha}, \mathcal{H}_{L^\alpha}(1)) + \int_0^1 n^{-1/\alpha} b'(x_0 + n^{-1/\alpha} Y_s^{n,\theta_n,x_0}) \Gamma(Y_s^{n,\theta_n,x_0}, \mathcal{H}_{L^\alpha}(1)) \mathrm{d}s + o_{\mathbf{L^p}(1)} \\
&= \Gamma(L_1^{n,\alpha}, \mathcal{H}_{L^\alpha}(1)) + o_{\mathbf{L^p}(1)},
\end{aligned}$$

where at the last line we have used that the quantity $\Gamma(Y_s^{n,\theta_n,x_0}, Y_s^{n,\theta_n,x_0})$ is bounded in $\mathbf{L}^p$ norm and that $\Gamma(\mathcal{H}_{L^\alpha}(1), \mathcal{H}_{L^\alpha}(1))$ has finite moments of any order.

Recalling the expressions (34), (55) and using the basic properties of the operator $\Gamma$, one can see that the computation $\Gamma(L_1^{n,\alpha}, \mathcal{H}_{L^\alpha}(1))$ can be reduced to the computation of the $\Gamma$-bracket between simple stochastic integrals. Moreover, since $\rho$ is supported on $[-1, 1]$, such computations show that $\Gamma(L_1^{n,\alpha}, \mathcal{H}_{L^\alpha}(1)) = \Gamma(L_1^{\alpha}, \mathcal{H}_{L^\alpha}(1))$ and we have $\Gamma(Y_1^{n,\theta_n,x_0}, \widehat{\mathcal{H}}^n_{\theta_n}(1)) \xrightarrow{n\to\infty} \Gamma(L_1^{\alpha}, \mathcal{H}_{L^\alpha}(1))$. This ends the proof of the lemma.

□

### 7.2 Regularity of the conditional expectation

Let us recall that we have defined the functions $\eta_n$ and $\eta$ by the relations

$$E\left[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid Y_1^{n,\theta_n,x_0}\right] = \eta_n(Y_1^{n,\theta_n,x_0}), \quad E\left[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1)) \mid L_1^\alpha\right] = \eta(L_1^\alpha).$$

The aim of the section is to show that the function $\eta_n$ and $\eta$ are close in some sense.

We recall that $\kappa_n$ is defined in Lemma 3. Our first result is the following.

**Proposition 9** *There exists a sequence $(\varepsilon_n)_n$, independent of $x_0$ and $\theta$, with $\varepsilon_n \to 0$, such that the following holds true. For all $h$ bounded smooth function,*

$$\left|E[h(Y_1^{n,\theta,x_0})] - E[h(L_1^{n,\alpha}+\kappa_n)]\right| \leq \varepsilon_n \|h\|_\infty \tag{98}$$

$$\left|E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))h(Y_1^{n,\theta,x_0})] - E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))h(L_1^{n,\alpha}+\kappa_n)]\right| \leq \varepsilon_n \|h\|_\infty \tag{99}$$

**Proof** Remark that (98) is a result about the total variation distance between the laws of $Y_1^{n,\theta,x_0}$ and $L_1^{n,\alpha}+\kappa_n$, and (99) will be useful to control difference between the conditional expectations of $\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))$. We shall only prove (99) since (98) can be obtained in a similar way. Remark that we will not fully use the explicit expression of $\mathcal{H}_{L^\alpha}(1)$ in the proof. For the sake of shortness, let us denote $\mathcal{H}^K = \mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))$. The crucial facts about $\mathcal{H}^K$ is that $\left\|\mathcal{H}^K\right\|_\infty \leq K$ and that it is a smooth Malliavin functional, with

$$\Gamma(\mathcal{H}^K,\mathcal{H}^K) \leq c_K^2 \Gamma(\mathcal{H}_{L^\alpha}(1),\mathcal{H}_{L^\alpha}(1)) \text{ is element of } \bigcap_{p\geq 1} \mathbf{L}^p,$$

where $c_K$ is any upper bound of the derivative of $x \mapsto x\chi_K(x)$.

We now prove (99). Let us denote by $H$ any primitive function of $h$. We compute the following expectation using the integration by parts formula (19) in Proposition 2,

$$E[h(L_1^{n,\alpha}+\kappa_n)\mathcal{H}^K] = E\left[H(L_1^{n,\alpha}+\kappa_n)\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right] \tag{100}$$

where $\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)$ is given by (20). Using the definition of $\Gamma$ in (14), we get the following expression for the Malliavin weight

$$\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K) = L\Big(\frac{\mathcal{H}^K}{\Gamma(L_1^{n,\alpha},L_1^{n,\alpha})}\Big)L_1^{n,\alpha} - \frac{L(L_1^{n,\alpha})\mathcal{H}^K}{\Gamma(L_1^{n,\alpha},L_1^{n,\alpha})} - L\Big(\frac{\mathcal{H}^K L_1^{n,\alpha}}{\Gamma(L_1^{n,\alpha},L_1^{n,\alpha})}\Big).$$

By (38) we have $\left|L_1^{n,\alpha} - \kappa_n - Y_1^{n,\theta,x_0}\right| \leq \|b\|_\infty n^{1/\alpha-1} + \kappa_n \xrightarrow{n\to\infty} 0$, hence using that the function $H$ is globally Lipschitz with a constant $\|h\|_\infty$, we deduce from (100) that

$$\left|E[\mathcal{H}^K h(L_1^{n,\alpha} + \kappa_n)] - E\left[H(Y_1^{n,\theta,x_0})\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right]\right| \leq \varepsilon_n \|h\|_\infty E\left[\left|\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right|\right] \tag{101}$$

where $(\varepsilon_n)_n$ is some sequence converging to zero. We now compute $E\left[H(Y_1^{n,\theta,x_0})\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right]$ using successively the self-adjoint property of the operator $L$ and the chain rule, to obtain an I.P.P. formula in a reverse direction :

$$\begin{aligned}
E\left[H(Y_1^{n,\theta,x_0})\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right] &= E\left[H(Y_1^{n,\theta,x_0})\left\{L(\frac{\mathcal{H}^K}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})})L_1^{n,\alpha} - \frac{L(L_1^{n,\alpha})\mathcal{H}^K}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - L(\frac{\mathcal{H}^K L_1^{n,\alpha}}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})})\right\}\right] \\
&= E\left[\frac{\{L(H(Y_1^{n,\theta,x_0})L_1^{n,\alpha}) - H(Y_1^{n,\theta,x_0})L(L_1^{n,\alpha}) - L(H(Y_1^{n,\theta,x_0}))L_1^{n,\alpha}\}\mathcal{H}^K}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}\right] \\
&= E\left[\mathcal{H}^K \frac{\Gamma(L_1^{n,\alpha}, H(Y_1^{n,\theta,x_0}))}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}\right] \\
&= E\left[\mathcal{H}^K h(Y_1^{n,\theta,x_0})\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}\right]. 
\end{aligned} \tag{102}$$

Putting together (101) and (102) we deduce

$$\begin{aligned}
&\left|E\left[\mathcal{H}^K h(L_1^{n,\alpha} + \kappa_n)] - E[\mathcal{H}^K h(Y_1^{n,\theta,x_0})]\right]\right| \\
&\quad \leq \epsilon_n \|h\|_\infty \left\|\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right\|_1 + \left|E\left[\mathcal{H}^K h(Y_1^{n,\theta,x_0})\left\{\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - 1\right\}\right]\right| \\
&\quad \leq \epsilon_n \|h\|_\infty \left\|\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right\|_1 + \|\mathcal{H}^K\|_\infty \|h\|_\infty \left\|\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - 1\right\|_1.
\end{aligned}$$

Hence the proposition will be proved if we show

$$\sup_n \left\|\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)\right\|_1 < \infty, \tag{103}$$

$$\left\|\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - 1\right\|_1 \xrightarrow{n\to\infty} 0. \tag{104}$$

To prove (103), we write from (20)

$$\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K) = \frac{-2L(L_1^{n,\alpha})\mathcal{H}^K}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} + \frac{\mathcal{H}^K}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})^2}\Gamma(L_1^{n,\alpha}, \Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})) - \frac{\Gamma(L_1^{n,\alpha}, \mathcal{H}^K)}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}.$$

From the fact that $\mathcal{H}^K$ is bounded and $\Gamma(\mathcal{H}^K, \mathcal{H}^K)$ admits finite moments, together with the fact that $\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha}))$, $\Gamma(L_1^{n,\alpha}, \Gamma(L_1^{n,\alpha}, L_1^{n,\alpha}))$ and $L(L_1^{n,\alpha})$ do not depend on $n$ (this is due to the choice of the support of $\rho$), we easily see that $\mathcal{H}_{L_1^{n,\alpha}}(\mathcal{H}^K)$ admits moments bounded independently of $n$.

To prove (104), we write by (38),

$$\left|\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - 1\right| = \left|n^{1/\alpha-1}\int_0^1 \frac{\Gamma(L_1^{n,\alpha}, b(x_0 + n^{-1/\alpha}Y_s^{n,\theta,x_0}))}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}\mathrm{d}s\right|$$
$$\leq \frac{n^{-1}\|b'\|_\infty}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})}\int_0^1 \left|\Gamma(L_1^{n,\alpha}, Y_s^{n,\theta,x_0})\right|\mathrm{d}s.$$

Now using that $\left|\Gamma(L_1^{n,\alpha}, Y_s^{n,\theta,x_0}))\right| \leq \Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})^{1/2}\Gamma(Y_s^{n,\theta,x_0}, Y_s^{n,\theta,x_0})^{1/2} \leq C\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})^{1/2}\Gamma(Y_1^{n,\theta,x_0}, Y_1^{n,\theta,x_0})^{1/2}$, we easily derive $\left\|\frac{\Gamma(L_1^{n,\alpha}, Y_1^{n,\theta,x_0})}{\Gamma(L_1^{n,\alpha}, L_1^{n,\alpha})} - 1\right\|_1 = O(n^{-1})$ and thus (104). □

**Corollary 2** *There exists a sequence $(\varepsilon_n)_n$, independent of $x_0$ and $\theta$, with $\varepsilon_n \to 0$, such that the following holds true. For all $h$ bounded smooth function,*

$$\left|E[h(Y_1^{n,\theta,x_0})] - E[h(L_1^\alpha)]\right| \leq \varepsilon_n\|h\|_\infty, \tag{105}$$

$$\left|E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))h(Y_1^{n,\theta,x_0})] - E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))h(L_1^\alpha)]\right| \leq \varepsilon_n\|h\|_\infty. \tag{106}$$

**Proof** From Lemma 3, we know that $P(L_1^{n,\alpha} + \kappa_n = L_1^\alpha) \xrightarrow{n\to\infty} 1$. Hence we deduce (105) from (98). Using that $\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))$ is bounded we deduce (106) from (99). □

**Proposition 10** *We have*

$$\left\|\eta_n(Y_1^{n,\theta,x_0}) - \eta(Y_1^{n,\theta,x_0}))\right\|_1 \xrightarrow{n\to\infty} 0, \tag{107}$$

*and this convergence is uniform with respect to $x_0$, $\theta$.*

**Proof** We estimate the $\mathbf{L}^1$ norm appearing in (107) by duality. Let $\beta : \mathbb{R} \to [-1, 1]$ be a measurable function, we evaluate :

$$\left|E\left[(\eta_n(Y_1^{n,\theta,x_0}) - \eta(Y_1^{n,\theta,x_0}))\beta(Y_1^{n,\theta,x_0})\right]\right|$$
$$\leq \left|E[\eta_n(Y_1^{n,\theta,x_0})\beta(Y_1^{n,\theta,x_0})] - E[\eta(L_1^\alpha)\beta(L_1^\alpha)]\right| + \left|E[\eta(L_1^\alpha)\beta(L_1^\alpha)] - E[\eta(Y_1^{n,\theta,x_0})\beta(Y_1^{n,\theta,x_0})]\right|$$
$$\leq \left|E[\eta_n(Y_1^{n,\theta,x_0})\beta(Y_1^{n,\theta,x_0})] - E[\eta(L_1^\alpha)\beta(L_1^\alpha)]\right| + \varepsilon_n K,$$

where we have used (105) with the choice $h = \eta\beta$, and recalling that $\|\eta\|_\infty \leq K$. From the definition of $\eta(L_1^\alpha)$ and $\eta_n(Y_1^{n,\theta,x_0})$ as conditional expectations, we have

$$\left|E[\eta_n(Y_1^{n,\theta,x_0})\beta(Y_1^{n,\theta,x_0})] - E[\eta(L_1^\alpha)\beta(L_1^\alpha)]\right|$$
$$= \left|E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))\beta(Y_1^{n,\theta,x_0})] - E[\mathcal{H}_{L^\alpha}(1)\chi_K(\mathcal{H}_{L^\alpha}(1))\beta(L_1^\alpha)]\right|$$
$$\leq \varepsilon_n,$$

where we used (106) in the last line.

Collecting the previous computations, we get

$$\sup_{\|\beta\|_\infty \leq 1} \left| E\left[ (\eta_n(Y_1^{n,\theta,x_0}) - \eta(Y_1^{n,\theta,x_0}))\beta(Y_1^{n,\theta,x_0}) \right] \right| \leq (1+K)\varepsilon_n,$$

and thus $\left\| \eta_n(Y_1^{n,\theta,x_0}) - \eta(Y_1^{n,\theta,x_0}) \right\|_1 \leq (1+K)\varepsilon_n$. The proposition is proved. □

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