

Small time convergence of subordinators with regularly or slowly varying canonical measure

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Abstract

We consider subordinators $X_\alpha = (X_\alpha(t))_{t \geq 0}$ in the domain of attraction at 0 of a stable subordinator $(S_\alpha(t))_{t \geq 0}$ (where $\alpha \in (0, 1)$); thus, with the property that $\overline{\Pi}_\alpha$, the tail function of the canonical measure of X_α , is regularly varying of index $-\alpha \in (-1, 0)$ as $x \downarrow 0$. We also analyse the boundary case, $\alpha = 0$, when $\overline{\Pi}_\alpha$ is slowly varying at 0. When $\alpha \in (0, 1)$, we show that $(t \overline{\Pi}_\alpha(X_\alpha(t)))^{-1}$ converges in distribution, as $t \downarrow 0$, to the random variable $(S_\alpha(1))^\alpha$. This latter random variable, as a function of α , converges in distribution as $\alpha \downarrow 0$ to the inverse of an exponential random variable. We prove these convergences, also generalised to functional versions (convergence in $\mathbb{D}[0, 1]$), and to trimmed versions, whereby a fixed number of its largest jumps up to a specified time are subtracted from the process. The $\alpha = 0$ case produces convergence to an extremal process constructed from ordered jumps of a Cauchy subordinator. Our results generalise random walk and stable process results of Darling, Cressie, Kasahara, Kotani and Watanabe.

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1. Introduction

A classic result of Lévy [17] is that stable laws with index $\alpha \in (0, 2)$ constitute the entire class of possible non-normal limit laws of a normed and centred random walk in \mathbb{R} . Random walks with such behaviour are said to be in the domain of attraction of the corresponding stable distribution.

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A significant connection, going back to Doeblin [8], and expanded on by Feller [9,10], was to use Karamata's regular variation theory to characterise random walks in domains of attraction by regularly varying conditions on the tail of the distribution of the increments of the random walk. With an appropriate interpretation, the boundary case $\alpha = 2$ also corresponds to a stable law, namely the normal distribution, and the corresponding domain of attraction can also be characterised with regular variation-related conditions.

What of the other boundary case, $\alpha = 0$? Cressie [6] showed that if V_α is a $\text{Stable}(\alpha)$ random variable with index $\alpha \in (0, 2)$ and shift constant γ , then $|V_\alpha - \gamma|^\alpha$ converges in distribution as $\alpha \downarrow 0$ to the reciprocal of an exponential random variable. Kasahara [13], in a result he attributes to Kotani, extended this in the following way: let $(T_\alpha(t))_{t \geq 0}$ be a positive stable process of index $\alpha \in (0, 1)$, i.e., a subordinator, having Lévy triplet of the form $(0, 0, \alpha x^{-\alpha-1} dx \mathbf{1}_{\{x>0\}})$. Then

$$((T_\alpha(t))^\alpha)_{t \geq 0} \xrightarrow{D} (e_t)_{t \geq 0}, \text{ as } \alpha \downarrow 0, \quad (1.1)$$

where \xrightarrow{D} denotes convergence in the Skorohod J_1 topology, and (e_t) is an extremal process with marginal distributions

$$\mathbb{P}(e_{t_1} \leq x_1, \dots, e_{t_n} \leq x_n) = \prod_{i=1}^n e^{-t_i/x_i},$$

for $0 \leq t_1 < \dots < t_n$ and $0 < x_1 < \dots < x_n$. We refer to Resnick [21] for background information on extremal processes.

For each $t > 0$, e_t has the distribution of the reciprocal of an exponential random variable, so (1.1) represents an extended version of the Cressie [6] result for subordinators. The identity

$$e_t \stackrel{D}{=} \sup_{0 < s \leq t} \Delta \xi_s, \quad (1.2)$$

for each $t > 0$, also holds, where $(\xi_t)_{t \geq 0}$ is a Cauchy subordinator, i.e., a Lévy process with triplet $(0, 0, x^{-2} dx \mathbf{1}_{\{x>0\}})$, and jump process $\Delta \xi_t := \xi_t - \xi_{t-}$, $t > 0$.

When $0 < \alpha < 2$, the tail of the increment distribution of a random walk in the domain of attraction of a $\text{Stable}(\alpha)$ distribution is regularly varying at ∞ with index $-\alpha$. So for the boundary case, $\alpha = 0$, it is natural to consider a slowly varying tail. In this case affine norming and centering of the random walk cannot lead to a finite nondegenerate limit random variable, but a transformation, whereby the tail of the increment distribution is applied as a function to the random walk, and norming is by the sample size, produces as a limiting random variable the reciprocal of an exponential random variable. This was proved by Darling [7] in a 1-dimensional version, and, subsequent to this, Watanabe [23] proved finite dimensional convergence of the random walk interpolated to a function in $\mathbb{D}[0, 1]$. In a later paper, Kasahara [14] further proved J_1 convergence of the interpolated function in $\mathbb{D}[0, 1]$.

In view of this background, the continuous time environment is a natural one in which to consider results like these, and the aim of the present paper is, firstly to transfer from random walk versions to Lévy processes, in which the convergence is for small time parameter, rather than large time, and, secondly, to generalise the results to trimmed versions of Lévy processes. By “trimming” we mean removing a fixed number of large jumps of the processes. This is natural in the random walk context, because the slowly varying, heavy tails are associated with large jumps (“outliers”) in the random walk, and it is interesting in the process context as the effect of a measure slowly varying at 0 is previously little explored. Apart from these aspects, some quite interesting analytical differences occur between the small and large time situations.

Thus our basic assumption will be of the kind that a generic Lévy process $(Y(t))_{t \geq 0}$ with triplet $(\gamma_Y, \sigma_Y^2, \Pi_Y(dy))$ (see Sato [22], p. 38, for the definition), is in a non-normal domain of attraction at small times, by which we mean there exist non-stochastic functions $a_t \in \mathbb{R}$ and $b_t > 0$ such that

$$\frac{Y(t) - a_t}{b_t} \xrightarrow{D} V, \text{ as } t \downarrow 0, \quad (1.3)$$

where V is an almost surely (a.s.) finite, non-degenerate, non-normal random variable. Conditions on the Lévy measure for (1.3) to hold (in small time) are in Theorem 2.3 of Maller and Mason [19], and give that (1.3) is equivalent to the two-sided tail of the Lévy measure Π_Y of Y being regularly varying at 0 with index $\alpha \in (0, 2)$, together with a balance condition on the right and left tails of Π_Y at 0. The limit random variable V in (1.3) is a $\text{Stable}(\alpha)$ random variable, having Lévy triplet of the form $(\gamma, 0, cx^{-\alpha-1}dx\mathbf{1}_{\{x>0\}})$, where $\alpha \in (0, 2)$, and $\gamma \in \mathbb{R}$ and $c > 0$ are shift and scale constants which may be varied by choice of a_t and b_t .

Maller and Mason [19] remark that (1.3) can be extended to functional convergence in $\mathbb{D}[0, 1]$; that is,

$$\left(\frac{Y(\lambda t) - \lambda a_t}{b_t} \right)_{0 < \lambda \leq 1} \rightarrow (V(\lambda))_{0 < \lambda \leq 1}, \text{ as } t \downarrow 0, \quad (1.4)$$

weakly with respect to the Skorohod J_1 topology, where $(V(\lambda))_{0 < \lambda \leq 1}$ is a Lévy process such that $V(1) \stackrel{D}{=} V$. In Buchmann et al. [5], (1.4) was further extended to a functional theorem for a trimmed version of Y , a result we will need in the proof of Theorem 3.1.

The case of a slowly varying tail for Π seems not to have been considered before, in our context (but see Kevei and Mason [15] and Ipsen et al. [12] for limits of ratios of large jumps of subordinators in this case). Although stated in (1.3) and (1.4) for general Lévy processes, from now on we restrict ourselves to subordinators. Some discussion relevant to this is given at the end of the next section.

2. Notation and statement of results

All processes will be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Since the index α will be a variable in our results, we have to indicate its presence in the notation. We have tried to come up with a notation that is minimal but clear and conveys the necessary information.

For each $\alpha \geq 0$ let $(X_\alpha(t))_{t \geq 0}$ be a driftless subordinator with canonical measure $\Pi_\alpha(dx)$, where Π_α has tail $\overline{\Pi}_\alpha(x) := \Pi_\alpha\{(x, \infty)\}$, $x > 0$, satisfying $\overline{\Pi}_\alpha(x) = x^{-\alpha} L_\alpha(x) \mathbf{1}_{\{x>0\}}$, with $0 \leq \alpha < 1$, and $L_\alpha(x)$ a function slowly varying as $x \downarrow 0$. For the $\alpha = 0$ case, simply write $X(t) := X_0(t)$ and $\Pi := \Pi_0$. In this case, $L_0(x)$ is assumed to be nonincreasing with $L_0(0+) = \infty$. Since the processes $X_\alpha(t)$ are subordinators, α is necessarily restricted to $[0, 1)$.

Our development goes as follows. For each $\alpha \in (0, 1)$, $X_\alpha(t)$ will be assumed to be in the domain of attraction of a positive $\text{Stable}(\alpha)$ distribution as $t \downarrow 0$. So the process $(X_\alpha(\lambda t))_{0 < \lambda \leq 1}$ will converge in $\mathbb{D}[0, 1]$, as $t \downarrow 0$, after norming, to a $\text{Stable}(\alpha)$ subordinator $(S_\alpha(\lambda))_{\lambda \geq 0}$ having Lévy triplet of the form $(\gamma_\alpha, 0, c_\alpha x^{-1-\alpha})$ with $0 < \alpha < 1$ and constants $\gamma_\alpha \in \mathbb{R}$ and $c_\alpha > 0$ to be specified later. We will show that this convergence implies that $(t \overline{\Pi}_\alpha(X_\alpha(t\lambda)))^{-1}$ converges to $(S_\alpha(\lambda))^\alpha$ in $\mathbb{D}[0, 1]$, and that, in turn, this latter process itself converges in distribution, as $\alpha \downarrow 0$, to the largest jump up till time λ of a $\text{Stable}(1)$ process with measure $x^{-2}dx\mathbf{1}_{\{x>0\}}$; i.e., a Cauchy process. We denote this process as $(\xi_t)_{t \geq 0}$, consistent with the notation in (1.2).

These results are included in our main theorem, Theorem 2.1, set out in diagrammatic form below. It deals, not just with the processes mentioned, but also with “trimmed” versions of them.

To introduce trimmed processes, write $(\Delta X_\alpha(t) := X_\alpha(t) - X_\alpha(t-))_{t>0}$, with $\Delta X_\alpha(0) = 0$, for the jump process of X_α , and $\Delta X_\alpha^{(1)}(t) \geq \Delta X_\alpha^{(2)}(t) \geq \dots$ for the ordered jumps at time $t > 0$. Since $\Pi\{(0, \infty)\} = \infty$, there are infinitely many positive jumps, a.s., in any finite time interval $[0, t]$, $t > 0$, so the $\Delta X_\alpha^{(r)}(t)$ are positive a.s. for all $t > 0$, and $\lim_{t \downarrow 0} \Delta X_\alpha^{(r)}(t) = 0$ a.s. for all $r \in \mathbb{N}$. (Throughout, let $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.) The r -trimmed process is defined to be $X_\alpha(t)$ minus its r largest jumps, at a given time t :

$$^{(r)}X_\alpha(t) := X_\alpha(t) - \sum_{i=1}^r \Delta X_\alpha^{(i)}(t), \quad r \in \mathbb{N}, \quad t > 0 \quad (2.1)$$

(and we set $^{(0)}X_\alpha(t) \equiv X_\alpha(t)$). Detailed definitions and properties of this kind of ordering and trimming are given in Buchmann et al. [4], where the $\Delta X_\alpha(t)$, positive in our case, are identified with the points of a Poisson point process on $[0, \infty)$.

We similarly denote the ordered jumps up till time λ of the Cauchy process $(\xi_\lambda)_{\lambda \geq 0}$ with jump process $(\Delta \xi_\lambda)_{\lambda \geq 0}$ as $\Delta \xi_\lambda^{(1)} \geq \Delta \xi_\lambda^{(2)} \geq \dots$. And define the r -trimmed process $^{(r)}S_\alpha(\lambda)$ in the obvious way.

Theorem 2.1. *For each $\alpha \in [0, 1)$ let $(X_\alpha(t))$ be a driftless subordinator whose tail measure $\overline{\Pi}_\alpha$ is regularly varying at zero with exponent $-\alpha$ and satisfies $\overline{\Pi}_\alpha(0+) = \infty$; and for each $r \in \mathbb{N}_0$ let $^{(r)}X_\alpha(t)$ be the trimmed version of $(X_\alpha(t))$ defined in (2.1). When $\alpha = 0$ assume in addition*

$$\lim_{t \downarrow 0} \frac{^{(r)}X_0(t)}{\Delta X_0^{(r+1)}(t)} = 1, \quad \text{a.s.}, \quad (2.2)$$

where “a.s.” denotes almost sure convergence. Assume that the limit random variable S_α of $(X_\alpha(t))$, after norming but without centering, is a Lévy process with canonical triplet $(\alpha/(1 - \alpha), 0, \alpha x^{-1-\alpha} dx)$. Then for all $r \in \mathbb{N}_0$ we have the following convergences in distribution, as $t \downarrow 0$, and as $\alpha \downarrow 0$, with respect to the Skorohod J_1 -topology and the parameter $\lambda \in (0, 1]$:

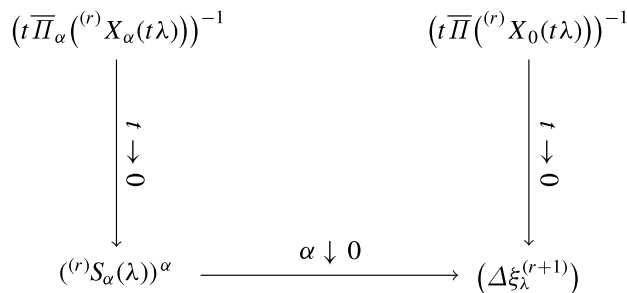


Fig. 1. Main Convergence Diagram. The upper nodes represent processes in $0 < \lambda \leq 1$, indexed by $t > 0$. The lower nodes represent processes in $0 < \lambda \leq 1$. The index $r \in \mathbb{N}_0$ indicates the order of trimming. The vertical arrows indicate process convergence of the upper node processes as $t \downarrow 0$ to the lower node processes for each $\alpha \in (0, 1)$ on the left, and with α set equal to 0 on the right. The horizontal arrow indicates process convergence of the left lower node process as $\alpha \downarrow 0$ to the right lower node process.

Remarks. (i) Some comment on Fig. 1 is in order. Since $\overline{\Pi}_\alpha(0+) = \infty$ (i.e., Π is of “infinite activity”) for each $\alpha \geq 0$, and $\lim_{t \downarrow 0} ^{(r)}X_\alpha(t) = 0$ a.s., we have $\lim_{t \downarrow 0} \overline{\Pi}_\alpha(^{(r)}X_\alpha(t)) = \infty$ a.s., and under the regularly varying (at 0) assumption we impose on $\overline{\Pi}_\alpha$, it turns out that multiplying by t is the correct scaling to get a nondegenerate limit law for $\overline{\Pi}_\alpha(^{(r)}X_\alpha(t))$ as $t \downarrow 0$. It is then convenient to consider the limit of the reciprocal of $t \overline{\Pi}_\alpha(^{(r)}X_\alpha(t))$ as we do in the topmost

entries of Fig. 1 because it produces the trimmed stable in the upright orientation as we see in the bottom left entry of the figure, thereby providing a direct generalisation of the Kotani result in (1.1). Taking the function $\overline{\Pi}_\alpha({}^{(r)}X_\alpha(t))$ of ${}^{(r)}X_\alpha(t)$ is a natural way of generalising the Darling [7] result for random walks, but it is clear that some quite different considerations enter in; note for example that $\overline{\Pi}$ slowly varying at zero reflects a mild singularity, while $\alpha \in (0, 1)$ is steeper — whereas, at infinity, a slowly varying $\overline{\Pi}$ betokens a very heavy tailed random walk.

(ii) The appearance of the almost sure condition (2.2) among the other weak convergence results is at first surprising. We discuss this in more detail after the proof of Theorem 4.1. A necessary and sufficient condition for it in the case $r = 0$ is given in (4.32).

(iii) Since (1.3) and (1.4) hold for general Lévy processes, it is natural to ask if there are versions of the convergences in Theorem 2.1 for (necessarily centred) general Lévy processes, other than subordinators. We have not investigated in detail whether this can be done, but the results for subordinators are certainly of interest in themselves, (i) as being generalisations of non-negative random walk versions which have appeared in the literature discussed in Section 1, and, (ii) because subordinators and their jumps play a prominent role for example in the theory of Poisson–Dirichlet distributions initiated by Kingman [16], which is not geared to the application of general Lévy processes. A further interesting point is that the Kingman Poisson–Dirichlet development relates at its heart to the *small time* behaviour of the stable subordinators, such as we consider here. For other related results see Ipsen et al. [11].

(iv) The reason for the particular choice of the constants in the limit Lévy measure in Theorem 2.1 will be clarified in the proof of Theorem 3.1. In fact the shift constant $\alpha/(1 - \alpha)$ becomes irrelevant as $\alpha \downarrow 0$, as is otherwise apparent in our modification of Kasahara’s argument in Section 5.

3. Convergence of $X_\alpha(t)$ as $t \downarrow 0$, for fixed $\alpha \in (0, 1)$

In this section we prove the lefthand vertical convergence in Fig. 1. Here the parameter α does not vary; the convergence is as $t \downarrow 0$, for fixed α .

Theorem 3.1. Fix $\alpha \in (0, 1)$ and let $(X_\alpha(t))$ be a driftless subordinator whose tail measure $\overline{\Pi}_\alpha$ is regularly varying at zero with exponent $-\alpha$, and is normed so that its limit process S_α is a Lévy subordinator with canonical triplet as in Theorem 2.1, namely, $(\alpha/(1 - \alpha), 0, \alpha x^{-1-\alpha} dx)$. For each $r \in \mathbb{N}$ let $({}^{(r)}X_\alpha(t))$ be the trimmed process defined in (2.1). Then

$$\left(\frac{1}{t \overline{\Pi}_\alpha({}^{(r)}X_\alpha(t\lambda))} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (({}^{(r)}S_\alpha(\lambda))^\alpha)_{0 < \lambda \leq 1}, \text{ as } t \downarrow 0, \quad (3.1)$$

with respect to the J_1 -topology.

In what follows, define the generalised inverse function of a monotonically decreasing function g by $g^\leftarrow(x) := \inf\{y > 0 : g(y) \leq x\}$, for $x > 0$.

Proof of Theorem 3.1. Fix $\alpha \in (0, 1)$ and assume $\overline{\Pi}_\alpha$ is regularly varying at zero with exponent $-\alpha$ and $(X_\alpha(t))$ has drift zero. We will first prove that

$$\left(\frac{X_\alpha(t\lambda)}{\overline{\Pi}_\alpha^\leftarrow(1/t)} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (S_\alpha(\lambda))_{0 < \lambda \leq 1}, \text{ as } t \downarrow 0, \quad (3.2)$$

in $\mathbb{D}[0, 1]$, where the limit process S_α is a subordinator with canonical triplet $(\alpha/(1 - \alpha), 0, \alpha x^{-1-\alpha} dx)$. The regular variation implies that (1.3) holds with $Y(t)$ replaced by $X_\alpha(t)$.

This follows from Theorem 2.3 in Maller and Mason [19], in which we wish to specify the choices of a_t and b_t so as to get the limit process with the Lévy triplet specified in the statement of the theorem.

We modify the definition of the norming function in Theorem 2.3 of Maller and Mason [19]; pp. 334–335) to

$$b_t := \inf \left\{ x \in (0, 1) : (2 - \alpha)x^{-2} \int_0^x y \overline{\Pi}_\alpha(y) dy \leq t^{-1} \right\}, \quad t > 0$$

(the constant 2 in Maller and Mason [19] is replaced by $2 - \alpha$). From Karamata's theorem (e.g., Thm. 1.5.11 in Bingham et al. [3]) we have $(2 - \alpha)x^{-2} \int_0^x y \overline{\Pi}_\alpha(y) dy \sim \overline{\Pi}_\alpha(x)$ as $x \downarrow 0$, so our definition of b_t as the inverse function to $(2 - \alpha)x^{-2} \int_0^x y \overline{\Pi}_\alpha(y) dy$ means that it can be taken equivalently as $\overline{\Pi}_\alpha^\leftarrow(t^{-1})$, as in (3.2).

The centering function a_t for the required convergence is given by Maller and Mason [19] as $a_t = t\nu(b_t)$, where $\nu(x)$ is the truncated mean function defined (in our subordinator case) by $\nu(x) = \gamma - \int_{x < y \leq 1} y \Pi_\alpha(dy)$. Since $\alpha < 1$, $\nu(0+) = \gamma - \int_{0 < y \leq 1} y \Pi_\alpha(dy)$ is finite, and equals the drift of $X_\alpha(t)$, which we assumed is zero. Hence, $a_t = t \int_{0 < y \leq x} y \Pi_\alpha(dy)$, and an integration by parts and another application of Karamata's theorem give

$$\lim_{t \downarrow 0} \frac{a_t}{b_t} = \frac{\alpha}{1 - \alpha}.$$

So the canonical triplet of the limit process is as we have it in Theorem 3.1.

We can transfer this working from $X_\alpha(t)$ to the trimmed process $(^r)X_\alpha(t)$ using Theorem 3 of Buchmann et al. [5]. From (3.2) we can then conclude that, for the same norming function $\overline{\Pi}_\alpha^\leftarrow(1/t)$,

$$\left(\frac{(^r)X_\alpha(t\lambda)}{\overline{\Pi}_\alpha^\leftarrow(1/t)} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (^r)S_\alpha(\lambda)_{0 < \lambda \leq 1}, \quad \text{as } t \downarrow 0, \text{ for } r \in \mathbb{N}, \quad (3.3)$$

with respect to the J_1 -topology, where the limit process is defined in terms of a Lévy process with the triplet specified in the statement of the theorem. The convergence in (3.3) together with the regular variation of $\overline{\Pi}_\alpha$ and the fact that $\lim_{t \downarrow 0} (^r)X_\alpha(t\lambda) = 0$ a.s. additionally implies

$$\left(\frac{\overline{\Pi}_\alpha(\overline{\Pi}_\alpha^\leftarrow(1/t))}{\overline{\Pi}_\alpha(^r)X_\alpha(t\lambda)} \right)_{0 < \lambda \leq 1} \xrightarrow{D} \left((^r)S_\alpha(\lambda)^\alpha \right)_{0 < \lambda \leq 1}, \quad (3.4)$$

by application of the following Lemma 3.2, and (3.4) implies (3.1), thereby completing the proof of Theorem 3.1. \square

Lemma 3.2. Suppose $\overline{\Pi}_\alpha$ is regularly varying at zero with exponent $-\alpha$, $\alpha > 0$. Then for two functions $f_t > 0$ and $g_t > 0$ on $[0, \infty)$ with $\lim_{t \downarrow 0} f_t = \lim_{t \downarrow 0} g_t = 0$, we have $\lim_{t \downarrow 0} \overline{\Pi}_\alpha(g_t) / \overline{\Pi}_\alpha(f_t) = c^\alpha$ if and only if $\lim_{t \downarrow 0} f_t / g_t = c \in (0, \infty)$.

Proof of Lemma 3.2. This is a straightforward application of Potter's bounds, see for example Theorem 1.5.6 of Bingham et al. [3]. We omit the details. \square

4. Convergence of $X_t = X_0(t)$ as $t \downarrow 0$, Case $\alpha = 0$

Next we prove the righthand vertical convergence in Fig. 1. The process $X(t) = X_0(t)$ is now assumed to have tail $\overline{\Pi}(x)$ slowly varying as $x \downarrow 0$, and the results in this section formally

correspond to the case $\alpha = 0$. So we drop the subscript α and write X_t rather than $X(t)$ throughout this section. Keep $r \in \mathbb{N}_0$ fixed. Recall that $\Delta \xi_\lambda^{(1)} \geq \Delta \xi_\lambda^{(2)} \geq \dots$ are the ordered jumps, up till time λ , of ξ_λ . The main result for this section is:

Theorem 4.1. *Suppose X_t is a driftless subordinator whose Lévy measure Π has tail $\overline{\Pi}$ slowly varying at zero. Assume (2.2) in addition. Then*

$$\left(\frac{1}{t \overline{\Pi}^{(r)}(X_{t\lambda})} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (\Delta \xi_\lambda^{(r+1)})_{0 < \lambda \leq 1}, \text{ as } t \downarrow 0,$$

with respect to the J_1 -topology.

Proof of Theorem 4.1 proceeds by way of some lemmas and propositions. The first lemma proves convergence in the supremum norm of the difference of two quantities to 0, which is stronger than J_1 convergence.

Lemma 4.2. *Assume the conditions of Theorem 4.1, including (2.2). Then for each $r \in \mathbb{N}_0$*

$$\sup_{0 < \lambda \leq 1} \left| \frac{1}{t \overline{\Pi}^{(r)}(X_{t\lambda})} - \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \right| \xrightarrow{P} 0, \text{ as } t \downarrow 0. \quad (4.1)$$

Proof of Lemma 4.2. Hold $r \in \mathbb{N}_0$ fixed throughout. Since X_t is a subordinator, its jumps are positive, and so ${}^{(r)}X_{t\lambda} \geq \Delta X_{t\lambda}^{(r+1)}$ for $t > 0$ and $\lambda \in (0, 1]$. Thus $1/\overline{\Pi}^{(r)}(X_{t\lambda}) \geq 1/\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})$, and for (4.1) it suffices to prove that for all $y > 0$ and $\eta > 0$ there exists $t_0 = t_0(y, \eta) > 0$ such that $t \in (0, t_0)$ implies

$$\mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{1}{t \overline{\Pi}^{(r)}(X_{t\lambda})} - \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \right) > y \right) < \eta. \quad (4.2)$$

Take $K > 0$. The left hand side of (4.2) equals

$$\mathbb{P} \left(\sup_{0 < \lambda \leq 1} \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \left(\frac{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{t \overline{\Pi}^{(r)}(X_{t\lambda})} - 1 \right) > y \right),$$

and this is bounded above by

$$\begin{aligned} \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \leq K, \sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}^{(r)}(X_{t\lambda})} - 1 \right) > \frac{y}{K} \right) \\ + \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} > K \right). \end{aligned} \quad (4.3)$$

We bound the first probability in (4.3) by ignoring the first supremum in it. To deal with the remaining part of that term, we need to invoke (2.2). This condition implies that there is an event $\Omega_1 \subseteq \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that, for $\omega \in \Omega_1$ and $\delta > 0$, there exists $t_1 \in (0, t_0)$ such that for $t \in (0, t_1)$ we have $W_t^{(r)} := {}^{(r)}X_t / \Delta X_t^{(r+1)} < 1 + \delta$, and thus $\sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} < 1 + \delta$. Hence, we can find $t_2 \in (0, t_1)$ such that, for $t \in (0, t_2)$,

$$\mathbb{P} \left(\sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} > 2 \right) < \frac{\eta}{3}.$$

Then for $t \in (0, t_2)$ we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}^{(r)} X_{t\lambda}} - 1 \right) > \frac{y}{K} \right) \\ & \leq \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K}, \sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} \leq 2 \right) \\ & \quad + \mathbb{P} \left(\sup_{0 < \lambda \leq 1} W_{t\lambda}^{(r)} > 2 \right) \\ & \leq \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K} \right) + \frac{\eta}{3}. \end{aligned} \quad (4.4)$$

The slow variation of $\overline{\Pi}$ implies there exists $x_0 > 0$ such that, for all $x \in (0, x_0]$, $\overline{\Pi}(x)/\overline{\Pi}(2x) - 1 \leq y/K$. Further, notice that $\{\Delta X_t^{(r+1)} \leq x_0\}$ implies $\{\sup_{0 < \lambda \leq 1} \Delta X_{t\lambda}^{(r+1)} \leq x_0\}$, and thus, when $\Delta X_t^{(r+1)} \leq x_0$,

$$\left\{ \sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) \leq \frac{y}{K} \right\}.$$

Hence, the probability on the righthand side of (4.4) can be estimated as

$$\begin{aligned} & \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}(2\Delta X_{t\lambda}^{(r+1)})} - 1 \right) > \frac{y}{K} \right) \leq \mathbb{P} \left(\sup_{0 < \lambda \leq 1} \Delta X_{t\lambda}^{(r+1)} > x_0 \right) \\ & = \mathbb{P}(\Delta X_t^{(r+1)} > x_0). \end{aligned} \quad (4.5)$$

Since $\lim_{t \downarrow 0} \Delta X_t^{(r)} = 0$ a.s., there exists $t_3 \in (0, t_2)$ such that the righthand side of (4.5) does not exceed $\eta/3$, for $t \in (0, t_3)$.

To estimate the second probability on the righthand side of (4.3), we will use that there exists $K > 0$ and $t_4 \in (0, t_3)$ such that, for $t \in (0, t_4)$,

$$\mathbb{P} \left(\sup_{0 < \lambda \leq 1} \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} > K \right) \leq \mathbb{P} \left(\frac{1}{t \overline{\Pi}(\Delta X_t^{(r+1)})} > K \right) \leq \frac{\eta}{3}. \quad (4.6)$$

This holds because, as a special case of the convergence in Proposition 4.3, $1/t \overline{\Pi}(\Delta X_t^{(r+1)})$ converges to a finite positive random variable; we defer proof of (4.6) till then.

Accepting (4.6), then, we can combine (4.3) with (4.4), (4.5) and (4.6) to get, for $t \in (0, t_4)$,

$$\mathbb{P} \left(\sup_{0 < \lambda \leq 1} \left(\frac{1}{t \overline{\Pi}^{(r)} X_{t\lambda}} - \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \right) > y \right) \leq 3 \left(\frac{\eta}{3} \right) = \eta.$$

Since η is arbitrary this completes the proof of (4.2), and of Lemma 4.2. \square

Now write

$$\frac{1}{t \overline{\Pi}^{(r)} X_{t\lambda}} = \left(\frac{1}{t \overline{\Pi}^{(r)} X_{t\lambda}} - \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \right) + \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}.$$

By Lemma 4.2 the first summand converges to zero in probability uniformly in $0 < \lambda \leq 1$. Thus, the processes

$$\left(\frac{1}{t \overline{\Pi}^{(r)} X_{t\lambda}} \right)_{0 < \lambda \leq 1} \quad \text{and} \quad \left(\frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})} \right)_{0 < \lambda \leq 1}$$

have the same limit in distribution as $t \downarrow 0$. So to complete the proof of [Theorem 4.1](#) it remains only to prove the following proposition.

Proposition 4.3. *Assume the conditions of [Theorem 4.1](#), including [\(2.2\)](#). Then, for all $r \in \mathbb{N}$, as $t \downarrow 0$,*

$$\left(\frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r)})} \right)_{0 < \lambda \leq 1} \xrightarrow{D} (\Delta \xi_{\lambda}^{(r)})_{0 < \lambda \leq 1}, \text{ in } \mathbb{D}[0, 1]. \quad (4.7)$$

We prove this in a classical way, first establishing finite dimensional (“fidi”) convergence, then tightness of the process on the left of [\(4.7\)](#). This is done in the next two subsections.

4.1. Proof of fidi convergence in [Proposition 4.3](#)

Define the following random variables

$$Z_{r,t,\lambda} := \frac{1}{t \overline{\Pi}(\Delta X_{t\lambda}^{(r)})}, \quad r \in \mathbb{N}, \quad t > 0, \quad \lambda > 0, \quad (4.8)$$

and note that $Z_{r,t,\lambda}$ is nondecreasing in λ . Recall that $\Delta \xi_{\lambda}^{(1)} \geq \Delta \xi_{\lambda}^{(2)} \geq \dots$ are the ordered jumps, at time λ , of the Cauchy process $(\xi_{\lambda})_{\lambda \geq 0}$ having Lévy measure $x^{-2} dx \mathbf{1}_{\{x > 0\}}$. Let $\lambda_1 < \dots < \lambda_n$. We aim to show

$$\begin{aligned} \lim_{t \downarrow 0} \mathbb{P}(Z_{r,t,\lambda_1} \leq y_1, \dots, Z_{r,t,\lambda_n} \leq y_n) \\ = \mathbb{P}(\Delta \xi_{\lambda_1}^{(r)} \leq y_1, \dots, \Delta \xi_{\lambda_n}^{(r)} \leq y_n), \quad n, r \in \mathbb{N}, \end{aligned} \quad (4.9)$$

wherein it is sufficient to restrict ourselves to values $0 < y_1 < \dots < y_n$, since $\{Z_{r,t,\lambda_i} \leq y_i\} \supseteq \{Z_{r,t,\lambda_j} \leq y_j\}$ whenever $i < j$ and $y_i \geq y_j$.

For formal reasons let $\lambda_0 := 0$ and $y_{n+1} := \infty$, and introduce triangular arrays of random variables $(V_{\ell,j})_{1 \leq \ell \leq j \leq n}$ and $(\tilde{V}_{\ell,j,t})_{1 \leq \ell \leq j \leq n, t \geq 0}$ by setting

$$V_{\ell,j} := \#\{s \in (\lambda_{\ell-1}, \lambda_{\ell}] : \Delta \xi_s \in (y_j, y_{j+1}]\}$$

and

$$\tilde{V}_{\ell,j,t} := \#\left\{s \in (t\lambda_{\ell-1}, t\lambda_{\ell}] : \Delta X_s \in \left(\overline{\Pi}^{\leftarrow}\left((ty_j)^{-1}\right), \overline{\Pi}^{\leftarrow}\left((ty_{j+1})^{-1}\right)\right]\right\},$$

for $t > 0$ and pairs ℓ, j fulfilling $1 \leq \ell \leq j \leq n$. The events $\{\Delta \xi_{\lambda_i}^{(r)} \leq y_i\}$ and $\{\sum_{\ell=1}^i \sum_{j=i}^n V_{\ell,j} \leq r-1\}$ are equal. This can be seen as follows. By the definition of $V_{\ell,j}$ we have that $\sum_{j=i}^n V_{\ell,j} = \#\{s \in (\lambda_{\ell-1}, \lambda_{\ell}] : \Delta \xi_s > y_i\}$. Thus,

$$\begin{aligned} \sum_{\ell=1}^i \sum_{j=i}^n V_{\ell,j} &= \sum_{\ell=1}^i \#\{s \in (\lambda_{\ell-1}, \lambda_{\ell}] : \Delta \xi_s > y_i\} \\ &= \#\{s \in (0, \lambda_i] : \Delta \xi_s > y_i\}. \end{aligned}$$

Hence, $\sum_{\ell=1}^i \sum_{j=i}^n V_{\ell,j} \leq r-1$ holds if and only if $\#\{s \in (0, \lambda_i] : \Delta \xi_s > y_i\} \leq r-1$, which is equivalent to $\{\Delta \xi_{\lambda_i}^{(r)} \leq y_i\}$.

We assert that the event on the right hand side of [\(4.9\)](#) can be written as a finite union of disjoint events, each of which is the intersection of a finite number of events of the form

$\{V_{\ell,j} = \kappa_{\ell,j}\}$. Here the $(\kappa_{\ell,j})_{1 \leq \ell \leq j \leq n}$ are triangular arrays of non-negative integers in which the $V_{\ell,j}$ and $\tilde{V}_{\ell,j,t}$ take values. To verify that assertion, define

$$B_{r,n,i} := \left\{ \kappa = (\kappa_{\ell,j})_{1 \leq \ell \leq j \leq n} : \sum_{\ell=1}^i \sum_{j=i}^n \kappa_{\ell,j} \leq r-1 \right\}.$$

Assume that for a given tuple $\kappa = (\kappa_{\ell,j})$ we have that $\{V_{\ell,j} = \kappa_{\ell,j}\}$ for all pairs ℓ, j with $1 \leq \ell \leq j \leq n$. Then $\sum_{\ell=1}^i \sum_{j=i}^n V_{\ell,j} \leq r-1$ holds if and only if $\kappa \in B_{r,n,i}$. On the other hand, that the event $\{V_{\ell,j} = \kappa_{\ell,j}\}$ holds simultaneously for all pairs ℓ, j with $1 \leq \ell \leq j \leq n$, can also be written as $\bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \kappa_{\ell,j}\}$. This implies

$$\begin{aligned} \{\Delta \xi_{\lambda_i}^{(r)} \leq y_i\} &= \left\{ \sum_{\ell=1}^i \sum_{j=i}^n V_{\ell,j} \leq r-1 \right\} \\ &= \bigcup_{\kappa=(\kappa_{\ell,j}) \in B_{r,n,i}} \bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \kappa_{\ell,j}\}. \end{aligned}$$

Now let $A_{r,n} := \bigcap_{i=1}^n B_{r,n,i}$, so that $A_{r,n}$ denotes the set of tuples $\kappa = (\kappa_{\ell,j})$ whose components satisfy $\sum_{\ell=1}^i \sum_{j=i}^n \kappa_{\ell,j} \leq r-1$ for $1 \leq i \leq n$. Then

$$\begin{aligned} \{\Delta \xi_{\lambda_1}^{(r)} \leq y_1, \dots, \Delta \xi_{\lambda_n}^{(r)} \leq y_n\} &= \bigcap_{i=1}^n \{\Delta \xi_{\lambda_i}^{(r)} \leq y_i\} = \\ &= \bigcap_{i=1}^n \bigcup_{\kappa=(\kappa_{\ell,j}) \in B_{r,n,i}} \bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \kappa_{\ell,j}\} = \bigcup_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \kappa_{\ell,j}\}. \end{aligned} \quad (4.10)$$

The same construction holds with $\tilde{V}_{\ell,j,t}$ in place of $V_{\ell,j}$, which means we can relate $\{Z_{r,t,\lambda_i} \leq y_i\}$ to $\{\sum_{\ell=1}^i \sum_{j=i}^n \tilde{V}_{\ell,j,t} \leq r-1\}$ using the same sets $B_{r,n,i}$. Thus

$$\{Z_{r,t,\lambda_1} \leq y_1, \dots, Z_{r,t,\lambda_n} \leq y_n\} = \bigcup_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \bigcap_{1 \leq \ell \leq j \leq n} \{\tilde{V}_{\ell,j,t} = \kappa_{\ell,j}\} \quad (4.11)$$

for the same sets $A_{r,n}$.

Due to the Poisson nature of the jumps of the processes Z and ξ in (4.10) and (4.11), counts of the numbers of points falling in disjoint subrectangles are independent; in particular, the events $\{V_{\ell,j} = \kappa_{\ell,j}\}$ are independent for all pairs ℓ, j , $1 \leq \ell \leq j \leq n$, and the same is true for the events $\{\tilde{V}_{\ell,j,t} = \kappa_{\ell,j}\}$. Furthermore, the events

$$\bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \kappa_{\ell,j}\} \quad \text{and} \quad \bigcap_{1 \leq \ell \leq j \leq n} \{V_{\ell,j} = \bar{\kappa}_{\ell,j}\}$$

are disjoint if $\kappa_{\ell,j} \neq \bar{\kappa}_{\ell,j}$ for at least one tuple (ℓ, j) , and the same is true for the tilde version also.

Thus, (4.10) and (4.11) imply

$$\mathbb{P}(\Delta \xi_{\lambda_1}^{(r)} \leq y_1, \dots, \Delta \xi_{\lambda_n}^{(r)} \leq y_n) = \sum_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \prod_{1 \leq \ell \leq j \leq n} \mathbb{P}\{V_{\ell,j} = \kappa_{\ell,j}\} \quad (4.12)$$

and

$$\mathbb{P}(Z_{r,t,\lambda_1} \leq y_1, \dots, Z_{r,t,\lambda_n} \leq y_n) = \sum_{\kappa=(\kappa_{\ell,j}) \in A_{r,n}} \prod_{1 \leq \ell \leq j \leq n} \mathbb{P}\{\tilde{V}_{\ell,j,t} = \kappa_{\ell,j}\}.$$

Hence, to prove (4.9), it remains only to show that for all $m \in \mathbb{N}_0$ the probabilities of the elementary events $\{\tilde{V}_{\ell,j,t} = m\}$ converge to the probabilities of the events $\{V_{\ell,j} = m\}$ as $t \downarrow 0$. If we define $N_I : \mathbb{R}^+ \rightarrow \mathbb{N}$ by

$$N_I(x) := \#\{s \in I : \Delta X_s > x\}, \quad (4.13)$$

where I is any subinterval of $(0, \infty)$, and set

$$\gamma_{j,t} := \overline{\Pi}(\overline{\Pi}^\leftarrow((ty_j)^{-1})) - \overline{\Pi}(\overline{\Pi}^\leftarrow((ty_{j+1})^{-1})),$$

then we can write

$$\begin{aligned} \tilde{V}_{\ell,j,t} &= N_{t(\lambda_{\ell-1}, \lambda_\ell]}(\overline{\Pi}(\overline{\Pi}^\leftarrow(ty_{j+1})^{-1})) - N_{t(\lambda_{\ell-1}, \lambda_\ell]}(\overline{\Pi}(\overline{\Pi}^\leftarrow(ty_j)^{-1})) \\ &\sim \text{Pois}(t(\lambda_\ell - \lambda_{\ell-1})\gamma_{j,t}). \end{aligned}$$

Noting further that

$$\lim_{t \downarrow 0} t\gamma_{j,t} = \frac{1}{y_j} - \frac{1}{y_{j+1}},$$

which follows easily from the slow variation of $\overline{\Pi}(x)$ at 0 and the relation $\overline{\Pi}(\overline{\Pi}^\leftarrow(x)) \leq x < \overline{\Pi}(\overline{\Pi}^\leftarrow(x-))$, $x > 0$, the convergence of the probabilities of the elementary events finally follows from

$$\begin{aligned} \lim_{t \downarrow 0} \mathbb{P}(\tilde{V}_{\ell,j,t} = m) &= \lim_{t \downarrow 0} e^{-t(\lambda_\ell - \lambda_{\ell-1})\gamma_{j,t}} \cdot \frac{(\lambda_\ell - \lambda_{\ell-1})^m}{m!} \cdot (t\gamma_{j,t})^m \\ &= e^{-(\lambda_\ell - \lambda_{\ell-1})(1/y_j - 1/y_{j+1})} \cdot \frac{(\lambda_\ell - \lambda_{\ell-1})^m}{m!} \cdot \left(\frac{1}{y_j} - \frac{1}{y_{j+1}}\right)^m \\ &= \mathbb{P}(V_{\ell,j} = m), \end{aligned}$$

for all pairs ℓ, j fulfilling $1 \leq \ell \leq j \leq n$. With this, we have completed the proof of finite dimensional convergence in Proposition 4.3. \square

4.2. Proof of tightness in Proposition 4.3

Recall the $Z_{r,t,\lambda}$ defined in (4.8), which are positive and nondecreasing in λ for each $r \in \mathbb{N}$ and $t > 0$, and have the convergence behaviour described in Proposition 4.3. In this subsection we show:

Proposition 4.4. *Assume Π has tail $\overline{\Pi}$ slowly varying at zero. Then for all $r \in \mathbb{N}$ the process $((t\overline{\Pi}(\Delta X_{t\lambda}^{(r)}))^{-1})_{0 < \lambda \leq 1}$ is tight in $\mathbb{D}[0, 1]$ as $t \downarrow 0$.*

Proof of Proposition 4.4. We use Theorem 15.3 of Billingsley [2], where the result is only stated for discrete time but can immediately be generalised to continuous time as in the next lemma.

Lemma 4.5. *For each $r \in \mathbb{N}$ the process $(Z_{r,t,\lambda})_{0 < \lambda \leq 1}$ indexed by $t > 0$ is tight in $\mathbb{D}[0, 1]$ as $t \downarrow 0$ if and only if the following conditions hold:*

(i)

$$\lim_{y \rightarrow \infty} \limsup_{t \downarrow 0} \mathbb{P} \left(\sup_{0 < \lambda \leq 1} Z_{r,t,\lambda} > y \right) = 0; \quad (4.14)$$

(ii) for all $y > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{t \downarrow 0} \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_\delta} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right) = 0, \quad (4.15)$$

where

$$A_\delta := \{ \lambda_1, \lambda_2 \in (0, 1) : \lambda_1 \leq \lambda_2, \lambda_2 - \lambda_1 \leq \delta \}; \quad (4.16)$$

(iii) for all $y > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{t \downarrow 0} \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in [0, \delta]} |Z_{r,t,\lambda_2} - Z_{r,t,\lambda_1}| > y \right) = 0; \quad (4.17)$$

(iv) for all $y > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{t \downarrow 0} \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in [1-\delta, 1]} |Z_{r,t,\lambda_2} - Z_{r,t,\lambda_1}| > y \right) = 0. \quad (4.18)$$

In what follows we prove (4.14), (4.15), (4.17) and (4.18) in sequence, keeping $r \in \mathbb{N}$ fixed.

Proof of Condition (i): The probability in the lefthand side of (4.14) is

$$\begin{aligned} \mathbb{P} \left(\sup_{0 < \lambda \leq 1} Z_{r,t,\lambda} > y \right) &= \mathbb{P}(\Delta X_t^{(r)} > \overline{H}^{\leftarrow}((ty)^{-1})) \\ &\leq \mathbb{P}(\Delta X_t^{(1)} > \overline{H}^{\leftarrow}((ty)^{-1})) \\ &= 1 - \mathbb{P} \left(N_{[0,t)} \left(\overline{H}^{\leftarrow}((ty)^{-1}) \right) = 0 \right) \\ &= 1 - \exp \left(-t \overline{H} \left(\overline{H}^{\leftarrow}((ty)^{-1}) \right) \right) \\ &\leq 1 - \exp(-1/y). \end{aligned} \quad (4.19)$$

(Recall the definition of N_t in (4.13)). The last inequality in (4.19) follows from the fact that $\overline{H} \left(\overline{H}^{\leftarrow}(x) \right) \leq x$, $x > 0$. Letting $y \rightarrow \infty$ in (4.19) gives (4.14).

Proof of Condition (ii): In the following, keep $y > 0$ and $\eta > 0$ fixed, and take $\lambda_0 \in (0, 1)$ such that $1 - e^{-2\lambda_0/y} < \eta/2$. Recall A_δ in (4.16) and define

$$A_\delta^{\leq}(\lambda_0) := \{ \lambda_1, \lambda_2 \in A_\delta : \lambda_1 \leq \lambda_0 \} \text{ and } A_\delta^{>}(\lambda_0) := \{ \lambda_1, \lambda_2 \in A_\delta : \lambda_1 > \lambda_0 \}.$$

Decompose the probability in the lefthand side of (4.15) as

$$\begin{aligned} &\mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_\delta} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right) \\ &\leq \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_\delta^{\leq}(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right) \\ &\quad + \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_\delta^{>}(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right). \end{aligned} \quad (4.20)$$

In the first summand on the righthand side of (4.20), $\lambda_1 \leq \lambda \leq \lambda_2 \leq \lambda_1 + \delta \leq \lambda_0 + \delta$, so the probability is bounded above by

$$\mathbb{P}\left(\sup_{\lambda \in [\lambda_1, \lambda_0 + \delta]} Z_{r,t,\lambda} > y\right) \leq 1 - e^{-(\lambda_0 + \delta)/y}, \quad (4.21)$$

just as in (4.19). When δ is chosen less than λ_0 , the righthand side is less than $1 - e^{-2\lambda_0/y} < \eta/2$.

Next we estimate the second summand on the righthand side of (4.20). In it, $\lambda_1 > \lambda_0$. Take $\lambda \in [\lambda_1, \lambda_2]$ and $\gamma_1, \gamma_2 > 0$, and set

$$\Gamma_t := \{\gamma_1 \leq Z_{r,t,\lambda_0} \leq Z_{r,t,\lambda} \leq \gamma_2\}.$$

Now (4.7) implies $Z_{r,t,\lambda} \xrightarrow{D} \Delta \xi_\lambda^{(r)}$ for each $\lambda \in (0, 1]$ as $t \downarrow 0$. The Cauchy process $(\xi_\lambda)_{\lambda \geq 0}$ has Lévy measure $x^{-2} dx \mathbf{1}_{\{x > 0\}}$, so the number of jumps exceeding $x > 0$ up till time λ is Poisson with expectation λ/x . Thus

$$\begin{aligned} \mathbb{P}(\Delta \xi_\lambda^{(r)} \leq x) &= \mathbb{P}(\#\{s \in (0, \lambda) : \Delta \xi_s > x\} \leq r - 1) \\ &= e^{-\lambda/x} \sum_{j=0}^{r-1} \frac{(\lambda/x)^j}{j!}. \end{aligned}$$

This defines a proper distribution with no mass at 0: $\mathbb{P}(\Delta \xi_\lambda^{(r)} = 0) = 0$, which is continuous as $x \downarrow 0$. Thus we can choose $\gamma_1 > 0$ small enough, $\gamma_2 > 0$ large enough and t_0 small enough so that, for all $t \in (0, t_0)$,

$$\mathbb{P}(\Gamma_t^c) = \mathbb{P}(Z_{r,t,\lambda_0} < \gamma_1) + \mathbb{P}(Z_{r,t,\lambda} > \gamma_2) < \eta/2. \quad (4.22)$$

The next task is to show that, for any $t > 0$ and κ, μ with $\lambda_0 < \kappa < \mu \leq 1$,

$$\begin{aligned} \{Z_{r,t,\mu} - Z_{r,t,\kappa} > y\} \cap \Gamma_t &= \left\{ \frac{1}{\overline{\Pi}(\Delta X_{t\mu}^{(r)})} - \frac{1}{\overline{\Pi}(\Delta X_{t\kappa}^{(r)})} \geq ty \right\} \cap \Gamma_t \\ &\subseteq \left\{ \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\kappa}^{(r)}}{\Delta X_{t\lambda_0}^{(r)} \overline{\Pi}(\Delta X_t^{(r)})} \geq ty \right\} \cap \Gamma_t. \end{aligned} \quad (4.23)$$

We again apply Potter's bounds, see Theorem 1.5.6 of Bingham et al. [3], where the theorem is stated for functions slowly varying at infinity but can be immediately transferred to functions slowly varying at zero. In one of its forms it states that for a function L slowly varying at zero there exists $T > 0$ such that

$$\min \left\{ \frac{u}{v}, \frac{v}{u} \right\} < \frac{L(u)}{L(v)} < \max \left\{ \frac{u}{v}, \frac{v}{u} \right\}, \quad (4.24)$$

for all $u, v \in (0, T]$. On Γ_t we have $Z_{r,t,\lambda_0} = 1/(t \overline{\Pi}(\Delta X_{t\lambda_0}^{(r)})) \leq \gamma_1$, so

$$\overline{\Pi}^{\leftarrow} \left(\frac{1}{\gamma_1 t} \right) \leq \Delta X_{t\lambda_0}^{(r)} \leq \Delta X_{t\kappa}^{(r)}. \quad (4.25)$$

Choosing $0 < t \leq 1/(\gamma_1 \overline{\Pi}(T))$, we have $\overline{\Pi}^{\leftarrow}(1/(\gamma_1 t)) \leq T$, so by (4.24),

$$\frac{\overline{\Pi}(\Delta X_{t\mu}^{(r)})}{\overline{\Pi}(\Delta X_{t\kappa}^{(r)})} \geq \frac{\Delta X_{t\kappa}^{(r)}}{\Delta X_{t\mu}^{(r)}}.$$

This yields

$$\begin{aligned} \frac{1}{\overline{\Pi}(\Delta X_{t\mu}^{(r)})} - \frac{1}{\overline{\Pi}(\Delta X_{t\kappa}^{(r)})} &\leq \frac{1}{\overline{\Pi}(\Delta X_{t\mu}^{(r)})} - \left(\frac{\Delta X_{t\kappa}^{(r)}}{\Delta X_{t\mu}^{(r)}} \right) \frac{1}{\overline{\Pi}(\Delta X_{t\mu}^{(r)})} \\ &= \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\kappa}^{(r)}}{\Delta X_{t\mu}^{(r)} \overline{\Pi}(\Delta X_{t\mu}^{(r)})} \\ &\leq \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\kappa}^{(r)}}{\Delta X_{t\lambda_0}^{(r)} \overline{\Pi}(\Delta X_t^{(r)})}, \end{aligned}$$

on the event Γ_t , when $0 < t \leq 1/(\gamma_1 \overline{\Pi}(T))$. With this inequality we have proved the inclusion in (4.23).

Continuing from (4.23), argue from (4.25) that, on Γ_t ,

$$\left\{ \frac{\Delta X_{t\mu}^{(r)} - \Delta X_{t\kappa}^{(r)}}{\Delta X_{t\lambda_0}^{(r)} \overline{\Pi}(\Delta X_t^{(r)})} > ty \right\} \subseteq \left\{ \Delta X_{t\mu}^{(r)} - \Delta X_{t\kappa}^{(r)} > \frac{y}{\gamma_2} \overline{\Pi}^{\leftarrow} \left(\frac{1}{t\gamma_1} \right) \right\} \quad (4.26)$$

(here note too that $1/\overline{\Pi}(\Delta X_t^{(r)}) = tZ_{r,t,1} \leq t\gamma_2$ on Γ_t). For the following, set $a_t := (y/\gamma_2) \overline{\Pi}^{\leftarrow}(1/(t\gamma_1))$. Applying (4.26) once for $\mu := \lambda$ and $\kappa := \lambda_1$, and once for $\mu := \lambda_2$ and $\kappa := \lambda$, we obtain

$$\begin{aligned} &\mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_{\delta}^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y; \Gamma_t \right) \\ &\leq \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_{\delta}^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min(\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)}, \Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)}) > a_t; \Gamma_t \right). \end{aligned}$$

For given λ_1, λ_2 , the event

$$\left\{ \sup_{\lambda \in [\lambda_1, \lambda_2]} \min(\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)}, \Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)}) > a_t \right\} \quad (4.27)$$

requires that there exist at least two points $s_1, s_2 \in (\lambda_1, \lambda_2]$ such that $\Delta X_{ts_1} > a_t$ and $\Delta X_{ts_2} > a_t$. To see this, assume there is no point $s \in (\lambda_1, \lambda_2]$ with $\Delta X_{ts} > a_t$. Then $\Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ and thus $\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ and $\Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)} \leq a_t$ hold for any $\lambda \in (\lambda_1, \lambda_2]$. This is not possible under (4.27). If there is only one point $s \in (\lambda_1, \lambda_2]$ with $\Delta X_{ts} > a_t$, then for any $\lambda \in (\lambda_1, \lambda_2]$ we have that either $\Delta X_{t\lambda}^{(r)} - \Delta X_{t\lambda_1}^{(r)} \leq a_t$ or $\Delta X_{t\lambda_2}^{(r)} - \Delta X_{t\lambda}^{(r)} \leq a_t$, also not possible under (4.27). Hence we deduce

$$\begin{aligned} &\mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_{\delta}^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y; \Gamma_t \right) \\ &\leq \mathbb{P} \left(\exists \lambda \in [\lambda_0, 1 - \delta] : N_{[t\lambda, t(\lambda + \delta))}(a_t) \geq 2; \Gamma_t \right). \end{aligned} \quad (4.28)$$

Now define intervals $I_{k,t,\delta} := [t(\lambda_0 + k\delta), t(\lambda_0 + (k+2)\delta))$, for $t > 0$, $\delta > 0$ and $k \in \mathbb{N}$. Note that the length of each of these intervals is $2t\delta$. Further, define the integers $k_{\delta} := \lceil (1 - \lambda_0)/\delta \rceil$.

For given $\delta > 0$ and $\lambda \in [\lambda_0, 1 - \delta]$ there exists $k \in [0, k_{\delta}] \cap \mathbb{N}$ such that $\lambda \in [\lambda_0 + k\delta, \lambda_0 + (k+1)\delta)$, hence $t\lambda \in [t(\lambda_0 + k\delta), t(\lambda_0 + (k+1)\delta))$. This implies $[t\lambda, t(\lambda + \delta)) \subseteq [t(\lambda_0 + k\delta), t(\lambda_0 + (k+1)\delta)) = I_{k,t,\delta}$ for the same k , so for each interval $[t\lambda, t(\lambda + \delta))$ there exists $k \in [0, k_{\delta}] \cap \mathbb{N}$ such that $[t\lambda, t(\lambda + \delta)) \subset I_{k,t,\delta}$.

Thus,

$$\{\exists \lambda \in [\lambda_0, 1 - \delta] : N_{[t\lambda, t(\lambda+\delta))}(a_t) \geq 2\} \subseteq \bigcup_{k=0}^{k_\delta} \{N_{I_{k,t,\delta}}(a_t) \geq 2\}. \quad (4.29)$$

The intervals $I_{k,t,\delta}$ are constructed in such a way that every second interval is disjoint from the preceding one. Thus the events $\{N_{I_{2k-1,t,\delta}}(a_t) > 2\}$ for $k \in [1, \lceil k_\delta/2 \rceil] \cap \mathbb{N}$ are mutually independent, as are the events $\{N_{I_{2k,t,\delta}}(a_t) > 2\}$ for $k \in [0, \lfloor k_\delta/2 \rfloor] \cap \mathbb{N}_0$. Accordingly, write the righthand side of (4.29) as

$$\bigcup_{k=0}^{k_\delta} \{N_{I_{k,t,\delta}}(a_t) \geq 2\} = \bigcup_{k=0}^{\lfloor k_\delta/2 \rfloor} \{N_{I_{2k,t,\delta}}(a_t) \geq 2\} \cup \bigcup_{k=1}^{\lceil k_\delta/2 \rceil} \{N_{I_{2k-1,t,\delta}}(a_t) \geq 2\},$$

and combine this with (4.28) and (4.29) to get

$$\begin{aligned} & \mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A_\delta^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} \geq y; \Gamma_t\right) \\ & \leq \mathbb{P}\left(\bigcup_{k=0}^{\lfloor k_\delta/2 \rfloor} \{N_{I_{2k,t,\delta}}(a_t) \geq 2\}\right) + \mathbb{P}\left(\bigcup_{k=1}^{\lceil k_\delta/2 \rceil} \{N_{I_{2k-1,t,\delta}}(a_t) \geq 2\}\right) \\ & = 2 - \mathbb{P}\left(\bigcup_{k=0}^{\lfloor k_\delta/2 \rfloor} \{N_{I_{2k,t,\delta}}(a_t) \leq 1\}\right) - \mathbb{P}\left(\bigcup_{k=1}^{\lceil k_\delta/2 \rceil} \{N_{I_{2k-1,t,\delta}}(a_t) \leq 1\}\right). \end{aligned}$$

The events $\{N_{I_{2k,t,\delta}}(a_t) \leq 1\}$ with $k \in [0, \lfloor k_\delta/2 \rfloor] \cap \mathbb{N}_0$ are mutually independent as are the events $\{N_{I_{2k-1,t,\delta}}(a_t) \leq 1\}$ with $k \in [0, \lceil k_\delta/2 \rceil] \cap \mathbb{N}$. These imply

$$\begin{aligned} & \mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A_\delta^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} \geq y; \Gamma_t\right) \\ & \leq 2 - \left(\prod_{k=0}^{\lfloor k_\delta/2 \rfloor} \mathbb{P}(N_{I_{2k,t,\delta}}(a_t) \leq 1)\right) - \left(\prod_{k=1}^{\lceil k_\delta/2 \rceil} \mathbb{P}(N_{I_{2k-1,t,\delta}}(a_t) \leq 1)\right) \\ & = 2(1 - \mathbb{P}^{[k_\delta/2]+1}(N_{I_{0,t,\delta}}(a_t) \leq 1)). \end{aligned} \quad (4.30)$$

Here the last equality follows from the fact that each of the intervals has the same length and thus the probabilities $\mathbb{P}(N_{I_{k,t,\delta}}(a_t) \leq 1)$ are equal for all $k \in [0, k_\delta] \cap \mathbb{N}_0$. Furthermore

$$\mathbb{P}(N_{I_{0,t,\delta}}(a_t) \leq 1) = (1 + 2t\delta\overline{\Pi}(a_t))e^{-2t\delta\overline{\Pi}(a_t)},$$

thus,

$$\mathbb{P}^{[k_\delta/2]+1}(N_{I_{0,t,\delta}}(a_t) \leq 1) = (1 + \delta c_t)^{[k_\delta/2]+1} e^{-\delta([k_\delta/2]+1)c_t}, \quad (4.31)$$

where $c_t := 2t\overline{\Pi}(a_t)$. Letting $t \downarrow 0$, so that

$$c_t = 2t\overline{\Pi}(a_t) = 2t\overline{\Pi}(t\overline{\Pi}^\leftarrow(1/(t\gamma_1))y/\gamma_2) \rightarrow 2/\gamma_1,$$

followed by $\delta \downarrow 0$, so that $([k_\delta/2] + 1)\delta \rightarrow (1 - \lambda_0)/2$, shows that the righthand side of (4.31) tends to $e^{(1-\lambda_0)/\gamma_1} e^{-(1-\lambda_0)/\gamma_1} = 1$. Then we deduce from (4.30) that

$$\limsup_{t \downarrow 0} \mathbb{P}\left(\sup_{\lambda_1, \lambda_2 \in A_\delta^>(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min\{Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda}\} > y; \Gamma_t\right)$$

tends to 0 as $\delta \downarrow 0$. Combining this with (4.21) and (4.22) yields that

$$\lim_{\delta \downarrow 0} \limsup_{t \downarrow 0} \mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in A_{\delta}^c(\lambda_0)} \sup_{\lambda \in [\lambda_1, \lambda_2]} \min \{ Z_{r,t,\lambda} - Z_{r,t,\lambda_1}, Z_{r,t,\lambda_2} - Z_{r,t,\lambda} \} > y \right)$$

is less than η , and since η is arbitrary this proves (4.15).

Proof of Condition (iii): The probability in the lefthand side of (4.17) can be written as

$$\mathbb{P} \left(\sup_{\lambda_1, \lambda_2 \in [0, \delta]} |Z_{r,t,\lambda_2} - Z_{r,t,\lambda_1}| > y \right),$$

and this is no larger than $\mathbb{P}(Z_{r,t,\delta/2} > y)$. Using a similar calculation as in (4.19), there exists $t_5 > 0$ such that for all $t \in (0, t_5)$

$$\mathbb{P}(Z_{r,t,\delta} > y) \leq 1 - e^{-\delta/y}.$$

For fixed $y > 0$ and $\delta > 0$ small enough this is no larger than η .

Proof of Condition (iv): The probability in the lefthand side of (4.18) is no larger than $\mathbb{P}(Z_{r,t,1} - Z_{r,t,1-\delta} > y)$. Just as in the proof of Condition (ii) there exists $t_6 > 0$ such that for $t \in (0, t_6)$

$$\begin{aligned} \mathbb{P}(Z_{r,t,1} - Z_{r,t,1-\delta} > y) &\leq \mathbb{P}(N_{[t(1-\delta), t]}(a_t) \geq 1) + \mathbb{P}(\Gamma_t^c) \\ &\leq 1 - e^{-\delta t \overline{\Pi}(a_t)} + \eta \\ &\leq 1 - e^{-2\delta/\gamma_1} + \eta. \end{aligned}$$

For $\delta > 0$ small enough this is no larger than 2η , so the proof of Condition (iv) is complete, and this finally completes the proof of Proposition 4.4. \square

Remarks. (i) We remark incidentally that the slow variation of $\overline{\Pi}(x)$ at 0 is equivalent to a weak version of (2.2), namely that ${}^{(r)}X_t / \Delta X_t^{(r+1)} \xrightarrow{P} 1$ as $t \downarrow 0$ for $r \in \mathbb{N}_0$ (see Buchmann et al. [4]).

(ii) A necessary and sufficient condition for (2.2) in the case $r = 0$ is given in Maller [18]; namely, (2.2) is equivalent to

$$\int_{0^+} \left(\frac{\overline{\Pi}(\delta x)}{\overline{\Pi}(x)} - 1 \right) \frac{\Pi(dx)}{\overline{\Pi}(x)} < \infty, \quad (4.32)$$

for all $\delta \in (0, 1)$; and this is also equivalent to

$$\lim_{t \downarrow 0} \frac{\Delta X_t^{(2)}}{\Delta X_t^{(1)}} = 0, \text{ a.s.}$$

In (4.32) the integrand $\overline{\Pi}(\delta x)/\overline{\Pi}(x) - 1$ tends to 0 when $\overline{\Pi}$ is slowly varying, but (4.32) requires more than this — that the integrand tends to 0 quickly enough, by comparison with the infinite measure $\Pi(dx)/\overline{\Pi}(x)$, for the integral in (4.32) to converge. Thus, (2.2) is indeed a stronger condition than the slow variation of $\overline{\Pi}(x)$ at 0.

For general r , a conjecture for an equivalent condition to (2.2) is

$$\int_{0^+} \left(\frac{\overline{\Pi}(\delta x)}{\overline{\Pi}(x)} - 1 \right)^{r+1} \frac{\Pi(dx)}{\overline{\Pi}(x)} < \infty,$$

for all $\delta \in (0, 1)$. As the proof that (2.2) is equivalent to (4.32) in the case $r = 0$ is already very technical, the proof for larger r would certainly be more involved, and would require substantial extra analysis.

(iii) The almost sure condition (2.2) may seem anomalous in the midst of the other weak convergence conditions, but it is not excessive in context. It follows from the proof of Lemma 4.2, via (4.4), that, with

$$R_{t\lambda} := \frac{\overline{\Pi}(\Delta X_{t\lambda}^{(r+1)})}{\overline{\Pi}^{(r)}X_{t\lambda}} \geq 1,$$

we have

$$\lim_{t \downarrow 0} \mathbb{P}\left(\sup_{0 < \lambda \leq 1} R_{t\lambda} > 1 + \varepsilon\right) = 0,$$

for all $\varepsilon > 0$. Hence, for given $\varepsilon > 0$, $\delta > 0$, and t small enough, $t \leq t_0(\varepsilon, \delta)$,

$$\mathbb{P}(R_{t\lambda} > 1 + \varepsilon \text{ for some } \lambda \in (0, 1]) \leq \delta.$$

But this implies

$$\mathbb{P}(R_s > 1 + \varepsilon \text{ for some } s \leq t) \leq \delta$$

whenever $t \leq t_0$. Hence (2.2) implies

$$\lim_{t \downarrow 0} \frac{\overline{\Pi}(\Delta X_t^{(r+1)})}{\overline{\Pi}^{(r)}X_t} = 1, \text{ a.s.} \quad (4.33)$$

This condition is a crucial intermediate step in the proof of Theorem 4.1 and we can ask if, conversely, it implies (2.2). (4.33) is close to (2.2) but we believe does not imply it in general. (Note that the converse part of Lemma 3.2 is not true for $\alpha = 0$; take for example $\overline{\Pi}(x) = |\log(1/x)|$, $f_t = t|\log t|$, $g_t = t$, for $0 < x, t < 1$.)

To attempt to construct an example where (4.33) holds but (2.2) does not, consider a situation where $\overline{\Pi}$ is slowly varying at 0 and

$$\liminf_{t \downarrow 0} \frac{\Delta X_t^{(r+1)}}{{}^{(r)}X_t} = c, \text{ a.s., for some } c \in (0, 1) \text{ and } r \in \mathbb{N}. \quad (4.34)$$

Then by the slow variation of $\overline{\Pi}$ we would have

$$\lim_{t \downarrow 0} \frac{\overline{\Pi}(\Delta X_t^{(r+1)})}{\overline{\Pi}^{(r)}X_t} \geq \lim_{t \downarrow 0} \frac{\overline{\Pi}(c {}^{(r)}X_t)}{\overline{\Pi}^{(r)}X_t} = 1, \text{ a.s.,}$$

in which case (4.33) holds, but (2.2) does not. If (4.34) were possible it would confirm that we have to impose some side condition such as (2.2).

To complete this discussion we would need to exhibit a subordinator with canonical measure slowly varying at 0 having the behaviour in (4.34). To do this would require substantial further analysis. Here we merely say that we think it is very likely the case that there exist such subordinators. This is because an analogous behaviour has been shown for sums of i.i.d. non-negative random variables with slowly varying tail distribution in Pruitt [20, Thm. 2], under certain conditions. We assume that a similar behaviour can be expected at small times for subordinators.

5. Convergence of the trimmed stable as $\alpha \downarrow 0$

In this section, to complete Fig. 1 we prove the analogue of the Kasahara [13] result, that the process $({}^{(r)}S_\alpha(\lambda))^\alpha$ converges to $(\Delta \xi_\lambda^{(r+1)})$ in $\mathbb{D}[0, 1]$ as $\alpha \downarrow 0$, for each $r \in \mathbb{N}_0$, where $(\Delta \xi_\lambda^{(r+1)})$

is the $(r + 1)$ -st largest up till time λ of the jumps $(\Delta\xi_\lambda)$ of a Cauchy process $(\xi_\lambda)_{0 < \lambda \leq 1}$. First suppose $r = 0$, and define

$$(T_\alpha(\lambda))_{0 < \lambda \leq 1} = \left(\sum_{0 < s \leq \lambda} (\Delta\xi_s)^{1/\alpha} \right)_{0 < \lambda \leq 1}.$$

Using the exponential formula on p. 8 of Bertoin [1] and that the Lévy measure of the Cauchy process is $y^{-2}dy\mathbf{1}_{\{y>0\}}$, we can calculate

$$\begin{aligned} \mathbb{E}e^{-\theta T_\alpha(\lambda)} &= \exp\left(-\lambda \int_0^\infty (1 - e^{-\theta y^{1/\alpha}})y^{-2}dy\right) \\ &= \exp\left(-\lambda \int_0^\infty (1 - e^{-\theta x})\alpha x^{-\alpha-1}dx\right), \text{ for } \theta > 0. \end{aligned} \quad (5.1)$$

This shows that $T_\alpha(\lambda)$ has Lévy triplet precisely $(0, 0, \alpha x^{-\alpha-1}dx\mathbf{1}_{\{x>0\}})$.

Since raising to the power $1/\alpha$ does not change the order of the jumps, we can write

$$({}^{(r)}T_\alpha(\lambda))^\alpha = \left(\sum_{i \geq r+1} (\Delta\xi_\lambda^{(i)})^{1/\alpha} \right)^\alpha, \quad (5.2)$$

where ${}^{(r)}T_\alpha$ is the r -trimmed version of T_α defined in the obvious way. Using a classical argument¹ we can show that when $\alpha \downarrow 0$ each term in the process on the righthand side of (5.2) converges *surely* (i.e., for each $\omega \in \Omega$) to

$$\sup_{i \geq r+1} \Delta\xi_\lambda^{(i)} = \Delta\xi_\lambda^{(r+1)}.$$

Consequently, also the process on the righthand side of (5.2) converges surely to the process $(\Delta\xi_\lambda^{(r+1)})$. This of course also implies convergence in distribution.

As a last step, by comparing the Lévy triplet of $S_\alpha(\lambda)$ with the one specified by (5.1), we see that

$$({}^{(r)}S_\alpha(\lambda)) = ({}^{(r)}T_\alpha(\lambda)) + \frac{\alpha\lambda}{1-\alpha}.$$

Raising both sides to power α and taking limits as $\alpha \downarrow 0$ yields

$$\begin{aligned} \lim_{\alpha \downarrow 0} ({}^{(r)}S_\alpha(\lambda))^\alpha &= \lim_{\alpha \downarrow 0} \left(({}^{(r)}T_\alpha(\lambda)) + \frac{\lambda\alpha}{1-\alpha} \right)^\alpha \\ &= \lim_{\alpha \downarrow 0} ({}^{(r)}T_\alpha(\lambda))^\alpha \cdot \lim_{\alpha \downarrow 0} \left(1 + \frac{\alpha}{1-\alpha} \cdot \frac{\lambda}{({}^{(r)}T_\alpha(\lambda))} \right)^\alpha, \end{aligned} \quad (5.3)$$

for any $\omega \in \Omega$ for which the denominator is positive; and such ω form an event with probability 1. Then since the last factor on the righthand side of (5.3) tends to 1 as $\alpha \downarrow 0$, we obtain the required result. \square

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¹ When $a_r \geq a_{r+1} \geq \dots > 0$ and $\sum_{i \geq r} a_i < \infty$, then

$$\alpha \log\left(\sum_{i \geq r} a_i^{1/\alpha}\right) = \log a_r + \alpha\left(1 + \sum_{i > r} (a_i/a_r)^{1/\alpha}\right).$$

Take $\alpha < 1$ and choose $i_0(r) > r$ so that $(a_{i_0}/a_r)^{1/\alpha-1} < 1$. Then the second term on the righthand is less than $\alpha(i_0 - r + \sum_{i > i_0} a_i/a_r) \rightarrow 0$ as $\alpha \downarrow 0$.

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