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On the connected components of the support of super Brownian motion and of its exit measure

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Abstract

Tribe proved in a previous paper that a typical point of the support of super Brownian motion considered at a fixed time is a.s. disconnected from the others when the space dimension is greater than or equal to 3. We give here a simpler proof of this result based on Le Gall's Brownian snake. This proof can then be adapted in order to obtain an analogous result for the support of the exit measure of the super Brownian motion from a smooth domain of $\mathbb{R}^d$ when d is greater than or equal to 4.

Keywords: Super Brownian motion; Exit measure; Brownian snake

0 Introduction

Let x be a fixed point in $\mathbb{R}^d$ and $(Z_t, t \geqslant 0)$ a super Brownian motion in $\mathbb{R}^d$ starting at δ_x (see for instance Dawson (1991) for the definition and the main properties of this process). We can heuristically describe this process by an infinite system of branching particles (when the lifetime of each particle tends to 0), Z_t being a measure (uniformly) distributed over the set of the positions of the particles alive at time t (see Dynkin (1991a) for such a description). Then, the process (Z_t) is a continuous measure-valued Markov process. Now let Ω be a domain in $\mathbb{R}^d$ which contains x. We can define the exit measure of the super Brownian motion from Ω, denoted by X^Ω (this notion has been introduced in Dynkin, 1991a, b). X^Ω is a random measure on the boundary $\partial\Omega$. Using the precedent description, we can say that X^Ω is supported by the set of the positions of the previous particles when they leave Ω for the first time.

The purpose of this work is to study the support of super Brownian motion and of its exit measure. Some results on the Hausdorff measure of the support of super Brownian motion have been given in Perkins (1989) and Le Gall and Perkins (1995), and on the Hausdorff dimension of the support of the exit measure in Abraham and Le Gall (1994). Moreover, Tribe (1991) proved a result on the connected components of the support of super Brownian motion. Here, we will first give a new (simpler) proof

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of Tribe's result (the key ideas are nonetheless the same). The main tool will be the "Brownian snake" defined by Le Gall (1993, 1994) which allows us to construct the super Brownian motion and the exit measure X^{Ω}. Using this new approach, we can obtain an analogous result for the connected components of the support of the exit measure.

The first part of this paper deals with the Brownian snake. Let us fix $x \in \mathbb{R}^d$. The Brownian snake is a Markov process W with values in the space $\mathscr{W}_x$ of the stopped paths in $\mathbb{R}^d$ starting at x. ζ_s is a reflected Brownian motion in $\mathbb{R}_+$. The law of the process $(W_s, s\geqslant 0)$ knowing the lifetime process $(\zeta_s, s\geqslant 0)$ can be described in an informal way: when ζ_s is decreasing, the path W_s is erased on a small length; when ζ_s is increasing, the path W_s is extended with a small independent Brownian path. The Brownian snake is in fact a particular Markovian parametrization of the set of the paths of the historical process of the super Brownian motion (see Dawson and Perkins (1991), Dynkin (1991a, b), or Le Gall (1993, 1994) for the definition of the historical process).

The second part studies the support of super Brownian motion at time 1. If we denote by $(Z_t, t\geqslant 0)$ a super Brownian motion in $\mathbb{R}^d$ starting at δ_x, Tribe proved the following result: for $d\geqslant 3$, $Z_1(\mathrm{d}y)$-a.e., the connected component of the support of Z_1 which contains the point y is the singleton $\{y\}$. The key tools here are a "Palm formula" for the Brownian snake and some estimates of the hitting probabilities of small balls by the super Brownian motion (see Dawson et al. (1989) for these estimates).

This proof can then be adapted for the exit measure from a domain Ω (denoted by X^{Ω}). The result for this random measure is: if $d\geqslant 4$, X^{Ω}-a.e., the connected component of the support of X^{Ω} which contains the point y is $\{y\}$. Here, some assumptions have to be made on Ω: we suppose it to be bounded and connected and its boundary to be smooth enough ($\mathscr{C}^2$). Then, the Euclidean boundary coincides with the Martin boundary, and, at every boundary point, there exists an outer tangent sphere with constant radius $r_0 > 0$. The proof is again based on a Palm formula for X^{Ω} (Le Gall, 1994) and the estimates of the hitting probabilities of small balls of $\partial\Omega$ (Abraham and Le Gall, 1994).

Notation

The indicator function of a set $A\subset \mathbb{R}^d$ will be $\mathbf{1}_A$.

The set of finite measures on $\mathbb{R}^d$ will be denoted by $\mathscr{M}(\mathbb{R}^d)$.

For every finite measure m of $\mathscr{M}(\mathbb{R}^d)$, $Supp(m)$ will represent the closed support of m.

If Ω is a domain of $\mathbb{R}^d$, we will denote by $\partial\Omega$ its boundary.

The ball of $\mathbb{R}^d$ centered at x with radius r will be denoted by $B(x,r)$ while we will denote by $B_{\partial\Omega}(x,r)$ the ball in $\partial\Omega$ centered at x with radius r :

$$B_{\partial\Omega} = \{y \in \partial\Omega,\ |x-y| < r\}.$$

Finally, the set of continuous functions from E to F will be denoted by $\mathscr{C}(E,F)$.

1. The Brownian snake

We consider the following state spaces:

$$\mathscr{W} = \{(f,\zeta) \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}^d) \times \mathbb{R}_+;\ \forall t \geqslant \zeta,\ f(t) = f(\zeta)\}$$

and

$$\mathscr{W}_x = \{(f,\zeta) \in \mathscr{W};\ f(0) = x\}.$$

For $w = (f,\zeta) \in \mathscr{W}$, we write $\widehat{w} = f(\zeta)$. To simplify some notations, we will often write, for $w = (f,\zeta) \in \mathscr{W}$, $w(t)$ instead of $f(t)$.

For every $x \in \mathbb{R}^d$, we denote by Π_x the law of the Brownian motion in $\mathbb{R}^d$ starting at x and, for $a \geqslant 0$, we denote by Π_x^a the law of the Brownian motion in $\mathbb{R}^d$ starting at x and stopped at time a.

For a path $w_0 = (f_0,\zeta_0)$ of $\mathscr{W}_x$, and $a \in [0,\zeta_0]$, $b \geqslant a$, there exists a unique probability measure $Q_{a,b}^{w_0}$ on $\mathscr{W}_x$ which verifies :

(i) $Q_{a,b}^{w_0}(\zeta = b) = 1$.

(ii) $\forall s \leqslant a$, $f(s) = f_0(s)$, $\quad Q_{a,b}^{w_0}$-a.s.

(iii) The law under $Q_{a,b}^{w_0}$ of $\big(f(s+a), s \geqslant 0\big)$ is $\Pi_{f_0(a)}^{b-a}$.

For every $t > 0$ and every $\zeta \geqslant 0$, we define a probability measure on $\mathbb{R}^2$ by

$$\begin{aligned} \theta_t^\zeta(\mathrm{d}a\,\mathrm{d}b) = {} & \frac{2(\zeta+b-2a)}{(2\pi t^3)^{1/2}} \exp\left\{-\frac{(\zeta+b-2a)^2}{2t}\right\} \mathbf{1}_{\{0<a<b\wedge\zeta\}} \mathrm{d}a\,\mathrm{d}b \\ & + 2(2\pi t)^{-1/2} \exp\left\{-\frac{(\zeta+b)^2}{2t}\right\} \mathbf{1}_{\{0<b\}} \delta_{(0)}(\mathrm{d}a)\mathrm{d}b. \end{aligned}$$

$\theta_t^\zeta(\mathrm{d}a\,\mathrm{d}b)$ is the law of $(\inf_{[0,t]} \beta_u, \beta_t)$ where β is a reflected Brownian motion in $\mathbb{R}$, starting at ζ. We then define, for every $w = (f,\zeta) \in \mathscr{W}$, the kernels $Q_t(w,\mathrm{d}w')$ by

$$Q_t(w,\mathrm{d}w') = \int \theta_t^\zeta(\mathrm{d}a\,\mathrm{d}b) Q_{a,b}^f(\mathrm{d}w').$$

Theorem 1.1. (Le Gall, 1993, 1994). *There exists a homogeneous continuous $\mathscr{W}_x$-valued strong Markov process $W_s = (f_s,\zeta_s)$ whose transition kernels are $Q_t(w,\mathrm{d}w')$.*

The process (ζ_s) is a reflected Brownian motion in $\mathbb{R}_+$. Moreover, the conditional law of the process (W_s) knowing (ζ_s) is the one of an inhomogeneous Markov process with transition kernel between times u and v:

$$R_{u,v}^{(\zeta_s)}(w,\mathrm{d}w') = Q_{\inf\{\zeta_r,\, u \leqslant r \leqslant v\},\zeta_v}(\mathrm{d}w').$$

We may suppose that W is defined on the canonical space $\mathscr{C}(\mathbb{R}_+,\mathscr{W}_x)$. For $w \in \mathscr{W}_w$, we denote by $\mathbb{Q}_w$ the law of W starting at the path w. We also denote by $\mathbb{Q}_w^*$ the law of this process stopped when the process (ζ_s) reaches 0. Under $\mathbb{Q}_w^*$, the process (ζ_s) is distributed as a Brownian motion starting at ζ_w stopped when it reaches 0.

For every $x \in \mathbb{R}^d$, we denote by $\mathbf{x}$ the trivial path of lifetime 0 reduced to the point x. It is easy to see that $\mathbf{x}$ is regular for the Markov process $(W,\mathbb{Q}_w)$. We may consequently define the excursion measure away from $\mathbf{x}$ of the process W. We denote this measure by $\mathbb{N}_x$. This measure plays a very important role in the following. It is characterized, up to a multiplicative constant, by:

Proposition 1.2. (Le Gall, 1994). (i) *The law under* $\mathbb{N}_x$ *of the process* (ζ_s) *is the Itô measure of positive excursions of the standard Brownian motion.*

(ii) $W_0 = \mathbf{x}$, $\mathbb{N}_x$*-a.e.*

(iii) *The conditional distribution under* $\mathbb{N}_x$ *of* (W_s) *knowing* (ζ_s) *is as in Theorem* 1.1.

We will suppose that $\mathbb{N}_x$ is normalized such that

$$\mathbb{N}_x\left(\sup_{s\geqslant 0}\zeta_s > \varepsilon\right) = \frac{1}{2\varepsilon}.$$

In order to understand the meaning of the kernels (Q_t), we can describe the conditional joint distribution of the couple (W_{s_1}, W_{s_2}) under $\mathbb{N}_x$ knowing (ζ_s) for $s_1 \leqslant s_2$: the paths are two Brownian paths which coincide until time $\inf_{s_1\leqslant t\leqslant s_2}\zeta_t$ and then behave independently; each one is respectively stopped at ζ_{s_1} and ζ_{s_2}.

We can construct the super Brownian motion thanks to the Brownian snake W. Let us first denote by $(l_s^a,\ a\geqslant 0,\ s\geqslant 0)$ the local time at level a of the process (ζ_s) and, for a fixed $\rho > 0$, we set

$$\sigma = \inf\{s\geqslant 0,\ l_s^0 > 4\rho\}.$$

We can then define an $\mathcal{M}(\mathbb{R}^d)$-valued process (Z_t) by setting

$$\forall t\geqslant 0, \qquad \int Z_t(\mathrm{d}\omega)\phi(\omega) = \frac{1}{4}\int_0^\sigma \mathrm{d}l_s^t\phi(\widehat{W}_s)$$

for any nonnegative measurable function ϕ on $\mathbb{R}^d$.

Theorem 1.3. (Le Gall, 1993). *Under* $\mathbb{Q}_\mathbf{x}$ *the process* (Z_t) *is a super Brownian motion starting at* $\rho\,\delta_x$.

We also define the measures X_t under $\mathbb{N}_x$ by the formula

$$\forall t\geqslant 0, \qquad \langle X_t, \varphi\rangle = \int_0^\infty \mathrm{d}l_s^t\varphi(\widehat{W}_s),$$

where l_s^t represents the local time at level t at time s of ζ. If Ξ represents the law under $4\mathbb{N}_x$ of the process $(\frac{1}{4}X_t,\ t\geqslant 0)$ defined on $\mathscr{C}(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^d))$, the measures Z and X are linked by the following result (this result derives directly from the excursion theory for the process (W_s)): we denote by $v = (v_t,\ t\geqslant 0)$ the generic element of $\mathscr{C}(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^d))$. Then,

Proposition 1.4. *Let* N *be a Poisson point measure on* $\mathscr{C}(\mathbb{R}_+, \mathcal{M}(\mathbb{R}^d))$ *with intensity* $\rho\,\Xi(\mathrm{d}v)$. *The law under* $\mathbb{Q}_\mathbf{x}$ *of* Z *coincides with the law of the process*

$$\left(\int v_t N(\mathrm{d}v),\ t\geqslant 0\right).$$

In other words, we can say that Ξ is the canonical measure of the super Brownian motion starting at δ_x (see Dawson and Perkins (1991), Section. 2 for the definition of the canonical measure).

We will now present a result that links the behavior of the process (W_s) under $\mathbb{Q}_x$ to the excursion measures $\mathbb{N}_x$. Let us fix $w_0 = (f_0, \zeta_0) \in \mathscr{W}_x$. We set

$$T = \inf\{s > 0,\ \zeta_s = 0\},$$
$$\check{\zeta}_s = \inf_{0 \leqslant u \leqslant s} \zeta_u$$

and we denote by $]\alpha_i, \beta_i[$, $i \in I$, the interval excursions of $\zeta - \check{\zeta}$ away from 0 before time T. For every $i \in I$ and $s \geqslant 0$, we set

$$W_s^i(t) = W_{(\alpha_i + s) \wedge \beta_i}(\zeta_{\alpha_i} + t), \quad t \geqslant 0.$$

We then have the following characterization:

Proposition 1.5. (Le Gall, 1994). *The measure*

$$\sum_{i \in I} \delta_{(\zeta_{\alpha_i}, W^i)}(\mathrm{d}t\, \mathrm{d}\kappa)$$

is under $\mathbb{Q}^*_{w_0}$ a Poisson point measure on $[0, \zeta_0] \times \mathscr{C}(\mathbb{R}_+, \mathscr{W}_x)$ with intensity

$$2\,\mathrm{d}t\, \mathbb{N}_{w_0(t)}(\mathrm{d}\kappa).$$

One of the main tools that we will use in the following is a scale property for the process (W_s). Let $\lambda \in \mathbb{R}_+^*$ and $x_0 \in \mathbb{R}^d$. We define the mapping $\phi_\lambda^{x_0}$ from $\mathscr{C}(\mathbb{R}_+, \mathscr{W}_{x_0})$ to $\mathscr{C}(\mathbb{R}_+, \mathscr{W}_{x_0})$ by $\phi_\lambda^{x_0}((w_u)_{u \geqslant 0}) = w'$ where

$$\forall u \geqslant 0, \quad w'_u = \left(f'_u(\cdot), \zeta'_u\right)$$

with

$$f'_u(\cdot) = \lambda \left[f_{u/\lambda^4}\left(\frac{\cdot}{\lambda^2}\right) - x_0 \right] + x_0,$$
$$\zeta'_u = \lambda^2 \zeta_{u/\lambda^4}.$$

The measure $\mathbb{N}_x$ verifies the following scale property:

Proposition 1.6. (Le Gall, 1994).

$$\phi_\lambda^{x_0}(\mathbb{N}_x) = \lambda^2 \mathbb{N}_{x_0 + \lambda(x - x_0)}.$$

We now give a "Palm formula" for the measure X_1 which is a particular case of the results of Dawson and Perkins (1991). Here, we use a formulation using the Brownian snake (see Le Gall (1991) for such a formulation):

Theorem 1.7. *Let F be a nonnegative measurable function on $\mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d)$. Then,*

$$\mathbb{N}_x\left[\int X_1(\mathrm{d}y) F(y, X_1)\right] = \int \Pi_x^1(\mathrm{d}\gamma) \mathbb{E}^{(\gamma)}\left(F\left(\gamma(1), \int \mathscr{N}(\mathrm{d}t\, \mathrm{d}\kappa) X_{1-t}(\kappa)\right)\right),$$

where, for every $\gamma \in \mathscr{W}_x$, $\mathscr{N}$ *is under* $\mathbb{P}^{(\gamma)}$ *a Poisson point measure on* $[0,1] \times \mathscr{C}(\mathbb{R}_+, \mathscr{W})$ *with intensity*

$$4\mathrm{d}t\, \mathbb{N}_{\gamma(t)}(\mathrm{d}\kappa).$$

Finally, we will use in Section 2 the estimates of the hitting probabilities of small balls. These results have been obtained in Dawson et al. (1989, Theorem 3.1(a)). Again, we give here a formulation in terms of the Brownian snake.

Proposition 1.8. *There exists a constant* C_1 *such that, for* $d \geqslant 3$, $x, y \in \mathbb{R}^d$ *and* $\varepsilon > 0$,

$$\mathbb{N}_x\big(Supp(X_1) \cap B(y,\varepsilon) \neq \emptyset\big) \leq C_1(\varepsilon^{d-2} \wedge 1).$$

Let us note that the constant C_1 does not depend on x, y.

We also need the "probability" under the excursion measure that the super Brownian motion starting at δ_x leaves a ball centered at x. This result easily derives from the scale property (Proposition 1.6).

Proposition 1.9. *There exists a constant* $C_2 > 0$ *such that, for every* $R > 0$,

$$\begin{aligned}
&\mathbb{N}_x\Bigg(\Big(\bigcup_{t \geqslant 0} Supp(X_t)\Big) \cap B(x,R)^{\mathrm{c}} \neq \emptyset\Bigg)\\
&\quad = \mathbb{N}_x\Big(\{W_s(t),\ s \geqslant 0,\ t \in [0,\zeta_s]\} \cap B(x,R)^{\mathrm{c}} \neq \emptyset\Big)\\
&\quad = \mathbb{N}_x\big(\{\widehat{W}_s,\ s \geqslant 0\} \cap B(x,R)^{\mathrm{c}} \neq \emptyset\big)\\
&\quad = C_2 R^2.
\end{aligned}$$

Let Ω be a domain of $\mathbb{R}^d$ which contains x. We will now consider the exit measure of the super Brownian motion from Ω. The definition of this measure has been given by Dynkin (1991a, b). We give here a presentation via the Brownian snake: let us fix $w \in \mathscr{W}_x$, we set

$$\tau(w) = \inf\{t \in [0,\zeta],\ w(t) \notin \Omega\}$$

with the usual convention $\inf \emptyset = +\infty$.

For $s \geqslant 0$, we define

$$\gamma_s = \big(\zeta_s - \tau(W_s)\big)_+.$$

Proposition 1.10. (Le Gall, 1994). *For every* $s \geqslant 0$, *we set*

$$A_s = \inf\left\{u,\ \int_0^u \mathbf{1}_{\{\tau(W_r) < \zeta_r\}} \mathrm{d}v > s\right\} = \inf\left\{u,\ \int_0^u \mathbf{1}_{\{\gamma_r > 0\}} \mathrm{d}v > s\right\}.$$

Then $A_s < \infty$, $\mathbb{Q}_{w_0}$-a.s. *and the process* $\Gamma_s = \gamma_{A_s}$ *is a reflected Brownian motion in* $\mathbb{R}_+$ *starting at* $(\zeta_0 - \tau(W_0))_+$.

Let us denote by $L^a(s)$ the local time at level a at time s of the reflected Brownian motion Γ_s. We then set

$$L_s^{\Omega} = L^0\left(\int_0^s \mathbf{1}_{\{\tau(W_u)<\zeta_u\}}\mathrm{d}u\right).$$

L_s^{Ω} only increases when $\tau(W_s) = \zeta_s$. It is easy to see that L_s^{Ω} can also be defined under $\mathbb{Q}^*_{w_0}$ and under $\mathbb{N}_x$. We then define under $\mathbb{N}_x$ the exit measure X^{Ω} by

$$\langle X^{\Omega}, \varphi\rangle = \int_0^{\sigma} \mathrm{d}L_s^{\Omega}\varphi(\widehat{W}_s)$$

for every nonnegative measurable function φ on Ω. Thanks to the support property of $\mathrm{d}L_s^{\Omega}$, it is clear that X^{Ω} is supported by $\partial\Omega$.

A "Palm formula" for the random measure X^{Ω} has been proved by Le Gall in (1994). We denote by Π_x^{Ω} the law of the Brownian motion in $\mathbb{R}^d$ starting at x and stopped when it leaves Ω, seen as a probability on $\mathscr{W}_x$.

Theorem 1.11. *Let F be a nonnegative measurable function on $\mathbb{R}^d \times \mathscr{M}(\mathbb{R}^d)$. Then,*

$$\mathbb{N}_x\left[\int X^{\Omega}(\mathrm{d}y)F(y,X^{\Omega})\right] = \int \Pi_x^{\Omega}(\mathrm{d}\gamma)\mathbb{E}^{(\gamma)}\left(F\left(\widehat{\gamma}, \int \mathscr{N}(\mathrm{d}t\,\mathrm{d}\kappa)X^{\Omega}(\kappa)\right)\right),$$

where, for every $\gamma \in \mathscr{W}_x$, $\mathscr{N}$ is under $\mathbb{P}^{(\gamma)}$ a Poisson point measure on $\mathbb{R}_+ \times \mathscr{C}(\mathbb{R}_+, \mathscr{W})$ with intensity

$$4\mathbf{1}_{\{t\leqslant\zeta_\gamma\}}\mathrm{d}t\,\mathbb{N}_{\gamma(t)}(\mathrm{d}\kappa).$$

The estimates of the hitting probabilities of small balls on the boundary $\partial\Omega$ have been obtained in Abraham and Le Gall (1994):

Theorem 1.12. *For $d\geqslant 4$, $x \in \Omega$, $y \in \partial\Omega$ and $\varepsilon \in]0, 1/2[$,*

$$\mathbb{N}_x\left(\mathrm{Supp}(X^{\Omega}) \cap B_{\partial\Omega}(y,\varepsilon) \neq \emptyset\right) \leqslant C(x)\varepsilon^{d-3}.$$

We will give a partial proof of this result in Section 3 because we need the exact form of the function $C(x)$.

2. The connected components of the support of super Brownian motion

As mentioned in the introduction, we will give in this section a new proof of Tribe's result (Tribe, 1991, Theorem 1):

Theorem 2.0. *For $d\geqslant 3$, if we denote by Comp(y) the connected component of Supp(Z_1) which contains y, we obtain a.s.*

$$\mathit{Comp}(y) = \{y\}, \quad Z_1(\mathrm{d}y)\text{-a.e.}$$

In fact, to prove this result, it suffices to prove an analogous result for the measure X_1:

Theorem 2.1. *For $d \geqslant 3$, if we denote by $C(y)$ the connected component of $Supp(X_1)$ which contains y, we obtain, $\mathbb{N}_x$-a.e.*

$$C(y) = \{y\}, \quad X_1(\mathrm{d}y)\text{-a.e.}$$

Indeed, using the relationship between the measures Z and X of Proposition 1.4, we obtain

$$Z_1 = \sum_{i=1}^{N} X_1^i,$$

where N is distributed according to a Poisson law of parameter

$$\mathbb{N}_x(X_1 \neq 0) = \mathbb{N}_x(\sup \zeta > 1) = 1/2$$

and where the $(X^i)_{i \leqslant N}$ are independent processes distributed according to the law of the measure-valued process X under $\mathbb{N}_x(\cdot \mid X_1 \neq 0)$. Let $y \in Supp(Z_1)$. Then, $y \in Supp(X_1^i)$ for some $i \leqslant N$. Let us suppose, for instance, that it is for $i = 1$. Using the estimates of Proposition 1.8, and making ε tend to 0, we obtain

$$\forall z \in \mathbb{R}^d, \quad \mathbb{N}_x\big(z \in Supp(X_1)\big) = 0.$$

Thanks to the independence of the X^i and to the Fubini theorem, it follows that, a.s., $X_1^1(\mathrm{d}y)$-a.e.,

$$\forall i \in \{2, \dots, N\}, \quad y \notin Supp(X_1^i).$$

So, $X_1^1(\mathrm{d}y)$-a.e., there exists $\varepsilon > 0$ such that

$$\mathrm{Comp}(y) \cap B(y, \varepsilon) = C_1(y) \cap B(y, \varepsilon),$$

where $C_1(y)$ represents the connected component of $Supp(X_1^1)$ which contains y. Therefore, Theorem 2.0 is a consequence of Theorem 2.1.

Proof of Theorem 2.1. Let $\varepsilon \in]0, 1[$. We will first estimate a quantity linked to the event "the super Brownian motion considered at time 1 does not charge an annulus of radii ε and 2ε centered at a typical point of the support of X_1". This quantity is

$$\mathbb{N}_x \left[\int X_1(\mathrm{d}y) \mathbf{1}_{\{Supp(X_1) \cap Ann(y, \varepsilon, 2\varepsilon) = \emptyset\}} \right],$$

where

$$Ann(y, \varepsilon, 2\varepsilon) = \{z \in \mathbb{R}^d \mid \varepsilon < |y - z| < 2\varepsilon\}.$$

The relationship between this quantity and the theorem is that we want to show that

$$\mathbb{N}_x \left(\int X_1(\mathrm{d}y) \mathbf{1}_{\{C(y) \neq \{y\}\}} \right) = 0.$$

Let $\varepsilon_k = 2^{-k}$. Then,

$$\begin{aligned}
&\mathbb{N}_x\left(\int X_1(\mathrm{d}y)\mathbf{1}_{\{C(y)\neq\{y\}\}}\right)\\
&\leqslant \mathbb{N}_x\left(\int X_1(\mathrm{d}y)\liminf_{k\to\infty}\mathbf{1}_{\{Ann(y,\varepsilon_k,2\varepsilon_k)\cap Supp(X_1)\neq\emptyset\}}\right)\\
&= 1-\mathbb{N}_x\left(\int X_1(\mathrm{d}y)\limsup_{k\to\infty}\mathbf{1}_{\{Ann(y,\varepsilon_k,2\varepsilon_k)\cap Supp(X_1)=\emptyset\}}\right)\\
&= 1-\int \Pi_x^1(\mathrm{d}\gamma)\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty}\left\{Ann(y,\varepsilon_k,2\varepsilon_k)\right.\right.\\
&\qquad \left.\left.\cap Supp\left(\int \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X_{1-t}(\kappa)\right)=\emptyset\right\}\right)
\end{aligned}$$

using the equality $\mathbb{N}_x(\langle X_1,1\rangle)=1$ and Theorem 1.7. To prove that this latter integral is equal to 1, we will first prove, thanks to the following Lemmas 2.2 and 2.3, that it is bounded below by a positive constant and then we will apply a zero–one law.

Let us set $E=\mathscr{C}(\mathbb{R}_+,\mathscr{W})$.

Lemma 2.2. *There exists a constant $C_3>0$ independent of ε and γ such that*

$$\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[0,1-\varepsilon^2]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X_{1-t}(\kappa)\right)\cap B(\gamma(1),2\varepsilon)=\emptyset\right)\geqslant C_3.$$

Proof. Using the exponential formula for the Poisson point measure,

$$\begin{aligned}
&\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[0,1-\varepsilon^2]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)\right)\cap B(\gamma(1),2\varepsilon)=\emptyset\right)\\
&=\exp\left\{-\int_0^{1-\varepsilon^2}4\,\mathrm{d}s\,\mathbb{N}_{\gamma(s)}\left(Supp(X_{1-s})\cap B(\gamma(1),2\varepsilon)\neq\emptyset\right)\right\}\\
&=\exp\left\{-4\int_{\varepsilon^2}^1\mathrm{d}s\,\mathbb{N}_{\gamma(1-s)}\left(Supp(X_s)\cap B(\gamma(1),2\varepsilon)\neq\emptyset\right)\right\}.
\end{aligned}$$

Using Proposition 1.6, we obtain for $s\geqslant 0$,

$$\begin{aligned}
&\mathbb{N}_{\gamma(1-s)}\left(Supp(X_s)\cap B(\gamma(1),2\varepsilon)\neq\emptyset\right)\\
&=\frac{1}{s}\mathbb{N}_{\gamma(1-s)/\sqrt{s}}\left(Supp(X_1)\cap B\left(\frac{\gamma(1)}{\sqrt{s}},\frac{2\varepsilon}{\sqrt{s}}\right)\neq\emptyset\right)\\
&\leqslant\frac{C_1}{s}\left(\left(\frac{2\varepsilon}{\sqrt{s}}\right)^{d-2}\wedge 1\right)
\end{aligned}$$

using Proposition 1.8. The evaluated quantity is consequently bounded below by

$$\exp\left\{-4C_1(2\varepsilon)^{d-2}\int_{\varepsilon^2}^1 \frac{\mathrm{d}s}{s^{d/2}}\right\} \geqslant C_3$$

for a constant $C_3 > 0$. □

Lemma 2.3. *There exists a constant $C_4 > 0$ independent of ε such that, if γ verifies*

$$\gamma([1-\varepsilon^2, 1]) \subset B(\gamma(1), \varepsilon/2), \tag{2.1}$$

then,

$$\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[1-\varepsilon^2,1]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X_{1-t}(\kappa)\right) \cap B(\gamma(1),\varepsilon)^{\mathrm{c}} = \emptyset\right) \geqslant C_4.$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[1-\varepsilon^2,1]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X_{1-t}(\kappa)\right) \cap B(\gamma(1),\varepsilon)^{\mathrm{c}} = \emptyset\right) \\
&\quad = \exp\left\{-\int_{1-\varepsilon^2}^1 4\,\mathrm{d}t\, \mathbb{N}_{\gamma(t)}\left(Supp(X_{1-t}) \cap B(\gamma(1),\varepsilon)^{\mathrm{c}} \neq \emptyset\right)\right\} \\
&\quad \geqslant \exp\left\{-\int_{1-\varepsilon^2}^1 4\,\mathrm{d}t\, \mathbb{N}_{\gamma(t)}\left(Supp(X_{1-t}) \cap B(\gamma(t),\varepsilon/2)^{\mathrm{c}} \neq \emptyset\right)\right\}
\end{aligned}$$

thanks to condition (2.1). Then, using the scale property of Proposition 1.6, we get

$$\begin{aligned}
&\mathbb{N}_{\gamma(t)}\left(Supp(X_{1-t}) \cap B(\gamma(t),\varepsilon/2)^{\mathrm{c}} \neq \emptyset\right) \\
&\quad = \mathbb{N}_0\left(Supp(X_{1-t}) \cap B(0,\varepsilon/2)^{\mathrm{c}} \neq \emptyset\right) \\
&\quad = \frac{1}{1-t}\mathbb{N}_0\left(Supp(X_1) \cap B\left(0, \frac{\varepsilon}{2\sqrt{1-t}}\right)^{\mathrm{c}} \neq \emptyset\right) \\
&\quad \leqslant \frac{4C_2}{\varepsilon^2}
\end{aligned}$$

according to the estimate of Proposition 1.9. We therefore find that the considered probability is bounded below by

$$\exp\left\{-\int_{1-\varepsilon^2}^1 4\,\mathrm{d}t \frac{4C_2}{\varepsilon^2}\right\} = C_4 > 0. \quad \square$$

We now use the zero–one law for the Brownian motion reversed at time 1 to easily obtain that

$$\Pi_x^1\left(\limsup_{k\to\infty}\left\{\gamma([1-\varepsilon_k^2, 1]) \subset B(\gamma(1), \varepsilon_k/2)\right\}\right) = 1.$$

Let us fix γ in that limit event. Then there exists a subsequence (ε'_k) of (ε_k) such that

$$\gamma([1-\varepsilon'^2_k,1]) \subset B(\gamma(1),\varepsilon'_k/2),$$

for every $k \in \mathbb{N}$.

For every $\delta > 0$, let $\mathscr{G}_\delta$ be the σ-algebra generated by the Poisson point measure $\mathcal{N}$ considered on the subset $[1-\delta,1] \times E$. Thanks to the independence property of Poisson point measures, it is clear that the σ-algebra

$$\mathscr{G}_{0+} = \bigcap_{\delta>0} \mathscr{G}_\delta$$

is $\mathbb{P}^{(\gamma)}$-trivial. Moreover, the event

$$A_\delta = \limsup_{k\to\infty}\left\{Supp\left(\int_{[1-\delta,1]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X_{1-t}(\kappa)\right)\cap Ann(\gamma(1),\varepsilon'_k,2\varepsilon'_k)=\emptyset\right\}$$

which is $\mathscr{G}_\delta$-measurable coincides $\mathbb{P}^{(\gamma)}$-a.s. with

$$A = \limsup_{k\to\infty}\left\{Supp\left(\int_{[0,1]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X_{1-t}(\kappa)\right)\cap Ann(\gamma(1),\varepsilon'_k,2\varepsilon'_k)=\emptyset\right\}.$$

Indeed, we only have a finite number of atoms (t_i,κ_i) of $\mathcal{N}$ such that $t_i < 1-\delta$ and $X_{1-t_i}(\kappa_i) \neq 0$, and we again use the property $\mathbb{N}_x(z \in Supp(X_a)) = 0$ for every $x, z \in \mathbb{R}^d$, $a < 0$. According to Lemmas 2.2, 2.3 and the choice of γ and (ε'_k),

$$\begin{aligned}
\mathbb{P}^{(\gamma)}(A_\delta) &= \mathbb{P}^{(\gamma)}(A)\\
&\geqslant \limsup_{k\to\infty}\left(\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[0,1-\varepsilon'^2_k]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)\right)\cap B(\gamma(1),2\varepsilon'_k)=\emptyset\right)\right.\\
&\quad\left.\times\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[1-\varepsilon'^2_k,1]\times E}\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)\right)\cap B(\gamma(1),\varepsilon'_k)^{\mathrm{c}}=\emptyset\right)\right)\\
&\geqslant C_3C_4 > 0.
\end{aligned}$$

If δ_p is a sequence which decreases toward 0, the event $\lim\uparrow A_{\delta_p} (\overset{\text{a.s.}}{=} A)$ is in the σ-algebra $\mathscr{G}_{0+}$ and we conclude that

$$\mathbb{P}^{(\gamma)}(A) = \mathbb{P}^{(\gamma)}(\lim\uparrow A_{\delta_p}) = 1.$$

We have consequently proved that

$$\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty}\left\{Supp\left(\int\mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X_{1-t}(\kappa)\right)\cap Ann(\gamma(1),\varepsilon_k,2\varepsilon_k)=\emptyset\right\}\right)=1$$

$\Pi^1_x(\mathrm{d}\gamma)$-a.s., which leads to

$$\mathbb{N}_x\text{-a.e., } X_1(\mathrm{d}y)\text{-a.e., } C(y)=\{y\}. \qquad \square$$

3. On the connected components of the support of the exit measure of super Brownian motion

Let Ω be a domain in $\mathbb{R}^d$. We suppose that Ω is connected, bounded and regular ($\mathscr{C}^2$). As before, we denote by E the set of continuous functions from $\mathbb{R}_+$ to $\mathscr{W}$ and by X^Ω the exit measure of the super Brownian motion out of Ω. We present here a result on the connected components of the support of X^Ω.

Theorem 3.1. *We suppose $d \geqslant 4$. If we denote by $C(y)$ the connected component of the support of X^Ω which contains y, then, for every $x \in \Omega$, $\mathbb{N}_x$-a.e.,*

$$C(y) = \{y\}, \quad X^\Omega(\mathrm{d}y)\text{-a.e.}$$

Remark: In Abraham and Le Gall (1994), it is proved that, if $d = 2$, X^Ω is absolutely continuous with respect to the surface measure on $\partial\Omega$ and consequently Theorem 3.1 is false for $d = 2$. On the contrary, for $d = 3$, the measure X^Ω is singular (although the Hausdorff measure of its support is 2). Yet, the problem studied here is still open in that case.

As mentioned in the introduction, we will use some estimates of the hitting probabilities of small balls on the boundary $\partial\Omega$ for the process (W_s) given in Abraham and Le Gall (1994). Nevertheless, we need a little more precise result here:

Lemma 3.2. *There exists a constant $C_5 > 0$ such that*

$$\mathbb{N}_x\Big(X^\Omega\big(B_{\partial\Omega}(y_0,\varepsilon)\big) > 0\Big) \leqslant C_5\rho(x)^{1-d}\varepsilon^{d-3},$$

where $\rho(x)$ represents the distance between x and $\partial\Omega$.

Proof. Let us first suppose that $\rho(x) > 2\varepsilon$. The proof of Theorem 3.1 of Abraham and Le Gall (1994) gives

$$\mathbb{N}_x\Big(X^\Omega\big(B_{\partial\Omega}(y_0,\varepsilon)\big) > 0\Big) \leqslant \frac{C_5}{\varepsilon^2}\Pi_x\big(B_\tau \in B_{\partial\Omega}(y_0,5\varepsilon)\big),$$

where

$$\tau = \inf\{t > 0,\ B_t \notin \Omega\}.$$

The upper bound

$$\Pi_x\big(B_\tau \in B_{\partial\Omega}(y_0,5\varepsilon)\big) \leqslant \left(\frac{\varepsilon}{\rho(x)}\right)^{d-1}$$

is then easy to prove.

In the case $\rho(x) < 2\varepsilon$, we use Proposition 1.9:

$$\begin{aligned}\mathbb{N}_x\Big(X^\Omega\big(B_{\partial\Omega}(y_0,\varepsilon)\big) > 0\Big) &\leqslant \mathbb{N}_x\Big(\{\widehat{W}s,\ s \geqslant 0\} \cap B\big(x,\rho(x)\big)^{\mathrm{c}} \neq \emptyset\Big)\\ &\leqslant \frac{C_6}{\rho(x)^2} \leqslant C_5\rho(x)^{1-d}\varepsilon^{d-3}. \qquad \square\end{aligned}$$

Proof of Theorem 3.1. The ideas are the same as in the proof of Theorem 2.1. We will consequently first estimate a quantity linked to the event: "the Brownian snake does not visit an annulus of $\partial\Omega$ of radii ε and 2ε centered at a typical point of the support of X^{Ω}". Let us set

$$Ann_{\partial\Omega}(y,\varepsilon,2\varepsilon) = \{z \in \partial\Omega,\ \varepsilon \leqslant |y-z| \leqslant 2\varepsilon\}.$$

Let $(\varepsilon_k) = 2^{-k}$. We want to estimate

$$\mathbb{N}_x \left[\int X^{\Omega}(\mathrm{d}y) \limsup_{k\to\infty} \mathbf{1}_{\{Supp(X^{\Omega}) \cap Ann_{\partial\Omega}(y,\varepsilon_k,2\varepsilon_k)=\emptyset\}} \right].$$

Using the Palm formula of Theorem 1.11, we get that this quantity is equal to

$$\int \Pi_x^{\Omega}(\mathrm{d}\gamma)\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty}\left(Supp\left(\int \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X^{\Omega}(\kappa)\right) \cap Ann_{\partial\Omega}\left(\gamma(\tau),\varepsilon_k,2\varepsilon_k\right) = \emptyset\right)\right).$$

We can then use the measure $\Pi_{x,y}^{\Omega}$ which is the law of Brownian motion starting at x and conditioned to leave Ω at $y \in \partial\Omega$. More precisely, $\Pi_{x,y}^{\Omega}$ is the law of the h-process of Brownian motion started at x associated with the harmonic function $K(\cdot,y)$ where K is a Poisson kernel of Ω (see Doob, (1984, Ch. X)). We have

$$\frac{\mathrm{d}\Pi_{x,y}^{\Omega}}{\mathrm{d}\Pi_x^{\Omega}}_{|\mathscr{F}_t} = \frac{K(\gamma(t),y)}{K(x,y)}$$

and

$$\hat{\gamma} = y, \quad \Pi_{x,y}^{\Omega}(\gamma)\text{-a.s.}$$

If we denote by $\mu_x^{\Omega}(\mathrm{d}y)$ the harmonic measure on $\partial\Omega$, it is easy to see that

$$\Pi_x^{\Omega} = \int \mu_x^{\Omega}(\mathrm{d}y)\Pi_{x,y}^{\Omega}$$

and then the studied quantity is equal to

$$\int \mu_x^{\Omega}(\mathrm{d}y) \int \Pi_{x,y}^{\Omega}(\mathrm{d}\gamma)\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty}\left(Supp\left(\int \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X^{\Omega}(\kappa)\right)\right.\right.$$
$$\left.\left.\cap Ann_{\partial\Omega}\left(y,\varepsilon_k,2\varepsilon_k\right) = \emptyset\right)\right).$$

We then define, under $\Pi_{x,y}^{\Omega}$, $\gamma' \in \mathscr{W}_y$ by

$$\forall t \in \mathbb{R}_+, \quad \begin{cases} \gamma'(t) = \gamma\left((\tau - t)_+\right), \\ \zeta_{\gamma'} = \zeta_\gamma = \tau(\gamma). \end{cases}$$

Let us denote by $\overleftarrow{\Pi}_{y,x}$ the law under $\Pi_{x,y}^{\Omega}$ of the process γ' (this is the law of a $G^{\Omega}(x,\cdot)$-Brownian motion starting at y where G^{Ω} denotes the Green function of the domain Ω). Then, our quantity is equal to

$$\int \mu_x^{\Omega}(\mathrm{d}y) \int \overleftarrow{\Pi}_{y,x}(\mathrm{d}\gamma) \quad \mathbb{P}^{(\gamma)}\Bigg(\limsup_{k\to\infty} \left(Supp\left(\int \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X^{\Omega}(\kappa)\right) \cap Ann_{\partial\Omega}(y,\varepsilon_k,2\varepsilon_k) = \emptyset \right)\Bigg).$$

Lemma 3.3. *There exists a constant $C_7 > 0$ independent of ε such that, if $\gamma \in \mathcal{W}_y$ verifies the condition*

$$\forall t \in [0,\varepsilon^2], \quad \gamma(t) \in B\left(y,\frac{\varepsilon}{2}\right), \tag{3.1}$$

then

$$\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[0,\varepsilon^2]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)\right) \cap B_{\partial\Omega}(y,\varepsilon)^{\mathrm{c}} = \emptyset \right) \geqslant C_7.$$

Proof. We get

$$\begin{aligned} &\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[0,\varepsilon^2]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X^{\Omega}(\kappa)\right) \cap B_{\partial\Omega}(y,\varepsilon)^{\mathrm{c}} = \emptyset \right)\\ &\quad = \exp\left\{ -\int_0^{\varepsilon^2} 4\,\mathrm{d}t \int \mathbb{N}_{\gamma(t)}(\mathrm{d}\kappa) \mathbf{1}_{\{Supp(X^{\Omega})\cap B_{\partial\Omega}(y,\varepsilon)^{\mathrm{c}}\neq\emptyset\}}\right\}\\ &\quad \geqslant \exp\left\{ -\int_0^{\varepsilon^2} 4\,\mathrm{d}t \mathbb{N}_{\gamma(t)}\left(Supp(X^{\Omega}) \cap B\left(\gamma(t),\frac{\varepsilon}{2}\right)^{\mathrm{c}} \neq \emptyset\right)\right\} \end{aligned}$$

using condition (3.1). Proposition 1.9 allows us to bound below this probability by

$$\exp\left\{ -\int_0^{\varepsilon^2} 4\,\mathrm{d}t \frac{4\,C_2}{\varepsilon^2}\right\} \geqslant C_7 > 0. \qquad \square$$

Lemma 3.4. *Let $K > 0$ and $\beta \in]0,1/2[$, with $\beta(1-d) < -1$. There exists a constant $C_8 > 0$ independent of ε and δ such that, if $\delta > 0$, $\varepsilon^2 < \delta$ and if $\gamma \in \mathcal{W}_y$ verifies the condition*

$$\forall t \in \left[1, \frac{\zeta_\gamma \wedge \delta}{\varepsilon^2}\right], \quad \frac{1}{\varepsilon}\rho(\gamma(\varepsilon^2 t)) \geqslant K t^{\beta}, \tag{3.2}$$

then

$$\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[\varepsilon^2,\delta[\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X^{\Omega}(\kappa)\right) \cap B_{\partial\Omega}(y,2\varepsilon) = \emptyset \right) \geqslant C_8 > 0.$$

Proof. Using Lemma 3.2, we get

$$\begin{aligned} &\mathbb{P}^{(\gamma)}\left(Supp\left(\int_{[\varepsilon^2,\delta[\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)\right) \cap B_{\partial\Omega}(y,2\varepsilon) = \emptyset \right)\\ &\quad = \exp\left\{ -\int_{\varepsilon^2}^{\delta\wedge\zeta_\gamma} 4\,\mathrm{d}t\, \mathbb{N}_{\gamma(t)}\Big(Supp(X^{\Omega}) \cap B_{\partial\Omega}(\gamma(0),2\varepsilon) \neq \emptyset\Big)\right\} \end{aligned}$$

$$\geqslant \exp\left\{-C_9\int_{\varepsilon^2}^{\delta\wedge\zeta_\gamma} \mathrm{d}t\, \varepsilon^{d-3}\rho\big(\gamma(t)\big)^{1-d}\right\}$$

$$\geqslant \exp\left\{-C_9\varepsilon^{d-1}\int_1^{(\zeta_\gamma\wedge\delta)/\varepsilon^2} \mathrm{d}t\, \rho\big(\gamma(\varepsilon^2 t)\big)^{1-d}\right\}$$

$$\geqslant \exp\left\{-C_9\int_1^{(\zeta_\gamma\wedge\delta)/\varepsilon^2} \mathrm{d}t\, \left(\frac{\rho\big(\gamma(\varepsilon^2 t)\big)}{\varepsilon}\right)^{1-d}\right\}$$

$$\geqslant \exp\left\{-C_{10}\int_1^{\infty} \mathrm{d}t\, t^{\beta(1-d)}\right\} = C_8 > 0$$

thanks to condition (3.2). □

Let us now fix $\beta \in]0, 1/2[$ such that $\beta(1-d) < -1$. We put together conditions (3.1) and (3.2) by setting, for every $\varepsilon > 0$ and $\delta > \varepsilon^2$,

$$A_\varepsilon^\delta = \left\{\gamma \in \mathscr{W}_y;\ \gamma([0,\varepsilon^2]) \subset B\left(y, \frac{\varepsilon}{2}\right);\ \forall t \in \left[1, \frac{\delta\wedge\zeta_\gamma}{\varepsilon^2}\right],\ \frac{1}{\varepsilon}\rho\big(\gamma(\varepsilon^2 t)\big) \geqslant K t^\beta\right\}.$$

Let us note that the sets A_ε^δ are increasing when δ is decreasing.

Lemma 3.5. *We can choose $K > 0$ such that*

$$\overleftarrow{\Pi}_{y,x}\left(\bigcup_{\delta>0}\limsup_{k\to\infty} A_{\varepsilon_k}^\delta\right) = 1.$$

Proof. Let us set

$$A = \bigcup_{\delta>0}\limsup_{k\to\infty} A_{\varepsilon_k}^\delta.$$

According to Doob (1984), the tail σ-algebra generated by γ is $\overleftarrow{\Pi}_{y,x}$-trivial because the function $K(\cdot, y)$ is minimal harmonic. As A belongs to this algebra, we have

$$\overleftarrow{\Pi}_{y,x}(A) = 0 \text{ or } 1.$$

Furthermore, we can say, thanks to the smoothness of $\partial\Omega$, that this probability does not depend on y. Let us denote then by $\overleftarrow{\Pi}_x$ the law of the process reversed in time under the probability Π_x^Ω. We still have the relation

$$\overleftarrow{\Pi}_x = \int \mu_x^\Omega(\mathrm{d}y)\overleftarrow{\Pi}_{y,x}$$

and consequently, to prove that

$$\overleftarrow{\Pi}_{y,x}(A) = 1$$

for every $y \in \partial\Omega$, it suffices to prove that

$$\overleftarrow{\Pi}_x(A) \geqslant b$$

for some strictly positive constant b.

We will first show that the probability of A_ε^δ can be bounded below for $\delta < \delta_1$ by a positive constant independent of ε and δ. Let $n_0 = \sup\{n \in \mathbb{N},\, 2^{2(n+1)} \leqslant \delta/\varepsilon^2\}$ and let $n \leqslant n_0$. We also set, for $\gamma \in \mathscr{W}_y$ and D a domain of $\mathbb{R}^d$,

$$T_D(\gamma) = \inf\{t \geqslant 0,\, \gamma(t) \in D\}.$$

Then,

$$\begin{aligned}
&\overleftarrow{\Pi}_x \left(\exists t \in \left[\varepsilon^2 2^{2n}, \varepsilon^2 2^{2(n+1)}\right[,\ \rho\big(\gamma(t)\big) < K\varepsilon^{1-2\beta} t^\beta\right)\\
&\leqslant \overleftarrow{\Pi}_x \left(\exists t \in \left[\varepsilon^2 2^{2n}, \varepsilon^2 2^{2(n+1)}\right[,\ \rho\big(\gamma(t)\big) < K\varepsilon^{1-2\beta} (\varepsilon^2 2^{2(n+1)})^\beta\right)\\
&\leqslant \overleftarrow{\Pi}_x \left(\exists t \in \left[\varepsilon^2 2^{2n}, \delta\right[,\ \rho\big(\gamma(t)\big) < K\varepsilon 2^{2\beta(n+1)}\right)\\
&\leqslant \sup_{\substack{z \in \Omega \\ \rho(z) = K\varepsilon 2^{2\beta(n+1)}}} \Pi_z^\Omega(\zeta > \varepsilon^2 2^{2n}).
\end{aligned}$$

Then, we use the fact that for every boundary point $y \in \partial\Omega$, there exists an outer tangent sphere of radius r_0 independent of y. As, in order to reach this sphere, the process must exit Ω first, we can bound above $\Pi_z^\Omega(\zeta > \varepsilon^2 2^{2n})$ by the probability that a Brownian motion starting at a distance of $K\varepsilon 2^{2\beta(n+1)}$ of the ball $B(0, r_0)$ reaches that ball after time $\varepsilon^2 2^{2n}$, i.e.

$$\begin{aligned}
&\leqslant \Pi_{r_0 + K\varepsilon 2^{2\beta(n+1)}}(T_{B(0,r_0)} > \varepsilon^2 2^{2n})\\
&= \Pi_1 \left(\frac{\varepsilon^2 2^{2n}}{K^2 \varepsilon^2 2^{4\beta(n+1)}} \leqslant T_{B\left(-\frac{r_0}{K\varepsilon 2^{2\beta(n+1)}}, \frac{r_0}{K\varepsilon 2^{2\beta(n+1)}}\right)} \right)\\
&\leqslant \Pi_1 \left(\frac{2^{2n}}{K^2 2^{4\beta(n+1)}} \leqslant T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta}, \frac{r_0}{K\varepsilon 4^\beta}\right)} \right)\\
&= \Pi_1 \left(\frac{2^{2n}}{K^2 2^{4\beta(n+1)}} \leqslant T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta}, \frac{r_0}{K\varepsilon 4^\beta}\right)} < \infty \right) + \Pi_1 \left(T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta}, \frac{r_0}{K\varepsilon 4^\beta}\right)} = \infty \right).
\end{aligned}$$

To estimate both terms, we use some results of potential theory which can be found in Port and Stone (1978) for instance. For the second term, we use the following result: if $d \geqslant 3$, $r > 0$, $x \in \mathbb{R}^d$ such that $\| x \| \geqslant r$, then

$$\Pi_x(T_{B(0,r)} < \infty) = \left(\frac{r}{\| x \|}\right)^{d-2}.$$

So, we get

$$\begin{aligned}
\Pi_1 \left(T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta}, \frac{r_0}{K\varepsilon 4^\beta}\right)} = \infty \right) &= 1 - \left(\frac{r_0}{r_0 + K\varepsilon 4^\beta}\right)^{d-2}\\
&\leqslant C_{11} K\varepsilon
\end{aligned}$$

for δ_1 small enough.

For the first term, we use the following property: if D is bounded, then

$$\lim_{t\to\infty} r(t)\Pi_x(t < T_D < \infty) = Cap(D)\Pi_x(T_D = \infty),$$

where $Cap(D)$ represents the capacity of D and

$$r(t) = (2\pi)^{d/2}(\mathrm{d}/2 - 1)t^{d/2-1}.$$

Then, we get

$$\Pi_1\left(\frac{2^{2n}}{K^2 2^{4\beta(n+1)}} \leqslant T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta},\frac{r_0}{K\varepsilon 4^\beta}\right)}\right) = u_n$$

and

$$u_n \sim \frac{Cap\left(B\left(-\frac{r_0}{K\varepsilon 4^\beta},\frac{r_0}{K\varepsilon 4^\beta}\right)\right)\Pi_1\left(T_{B\left(-\frac{r_0}{K\varepsilon 4^\beta},\frac{r_0}{K\varepsilon 4^\beta}\right)} = \infty\right)}{C_{12}(K^{-2}2^{2n(1-2\beta)})^{d/2-1}}$$
$$\leqslant C_{13}K^{d-2}2^{-n\alpha}$$

with $\alpha > 0$.

Finally,

$$\overleftarrow{\Pi}_x\left(\exists t \in [\varepsilon^2, \delta\wedge\zeta_\gamma[,\ \rho\big(\gamma(t)\big) \leqslant K\varepsilon^{1-2\beta}t^\beta\right)$$
$$\leqslant \sum_{n=0}^{n_0}\overleftarrow{\Pi}_x\left(\exists t \in [\varepsilon^2 2^{2n}, \varepsilon^2 2^{2(n+1)}[,\ \rho\big(\gamma(t)\big) \leqslant K\varepsilon^{1-2\beta}t^\beta\right)$$
$$\leqslant \sum_{n=0}^{n_0}(u_n + C_{11}K\varepsilon)$$
$$\leqslant (n_0+1)C_{11}K\varepsilon + \sum_{n=0}^{\infty}u_n$$
$$\leqslant C_{14}K|\ln\varepsilon|\varepsilon + C_{15}K^{d-2}$$
$$\leqslant C_{16}K$$

for δ_1 small enough, with C_{16} independent of ε and δ.

Using a scale property, it is easy to prove that

$$\overleftarrow{\Pi}_x\left(\gamma([0,\varepsilon^2])\subset B\big(y,\varepsilon/2\big)\right) \geqslant a > 0.$$

We then obtain, for $K > 0$ small enough,

$$\overleftarrow{\Pi}_x(A^\delta_\varepsilon) \geqslant 1 - \overleftarrow{\Pi}_x\left(\gamma([0,\varepsilon^2]) \cap B\big(y,\varepsilon/2\big)^c \neq \emptyset\right)$$
$$-\overleftarrow{\Pi}_x\left(\exists t \in \left[1, \frac{\delta\wedge\zeta_\gamma}{\varepsilon^2}\right],\ \frac{1}{\varepsilon}\rho\big(\gamma(\varepsilon^2 t)\big) < Kt^\beta\right)$$
$$\geqslant a - C_{16}K \geqslant b > 0.$$

So,

$$\overleftarrow{\Pi}_x\left(\limsup_{k\to\infty} A^{\delta}_{\varepsilon_k}\right) \geqslant \limsup_{k\to\infty} \overleftarrow{\Pi}_x(A^{\delta}_{\varepsilon_k}) \geqslant b$$

and of course

$$\overleftarrow{\Pi}_x\left(\bigcup_{0<\delta<\delta_1} \limsup_{k\to\infty} A^{\delta}_{\varepsilon_k}\right) \geqslant b > 0. \qquad \square$$

We have, as in Section 2,

$$\begin{aligned}
&\mathbb{N}_x\left(\int X^{\Omega}(\mathrm{d}y)\mathbf{1}_{\{C(y)\neq\{y\}\}}\right)\\
&\leqslant 1 - \mathbb{N}_x\left(\int X^{\Omega}(\mathrm{d}y)\limsup_{k\to\infty}\left(\mathbf{1}_{\{Ann_{\partial\Omega}(y,\varepsilon_k,2\varepsilon_k)\cap Supp(X^{\Omega})\neq\emptyset\}}\right)\right)\\
&= 1 - \int \mu_x(\mathrm{d}y)\int \overleftarrow{\Pi}_{y,x}(\mathrm{d}\gamma)\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty}\left(\mathbf{1}_{\{Ann_{\partial\Omega}(y,\varepsilon_k,2\varepsilon_k)\cap Supp(X^{\Omega})\neq\emptyset\}}\right)\right).
\end{aligned}$$

Remark: Here, we use $\mathbb{N}_x(\langle X^{\Omega}, 1\rangle) = 1$.

According to Lemma 3.5, $\overleftarrow{\Pi}_{y,x}$-a.s., there exists $\delta_0 > 0$ and a subsequence (ε'_k) of (ε_k) such that $\gamma \in A^{\delta_0}_{\varepsilon'_k}$ for every $k \in \mathbb{N}$. We then use the following zero–one law:

Lemma 3.6. *Let* $\delta_0 > 0$. *On* $\limsup_{k\to\infty}\{\gamma \in A^{\delta_0}_{\varepsilon_k}\}$, *we have*

$$\mathbb{P}^{(\gamma)}\left(\bigcup_{\delta>0}\limsup_{k\to\infty} B^{\delta}_{\varepsilon_k}\right) = 1,$$

where

$$B^{\delta}_{\varepsilon} = \left\{Supp\left(\int_{[0,\delta]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa)X^{\Omega}(\kappa)\right)\cap Ann_{\partial\Omega}(y,\varepsilon,2\varepsilon) = \emptyset\right\}.$$

Proof. We have proved with Lemmas 3.3 and 3.4 that, if $\gamma \in A^{\delta}_{\varepsilon_k}$,

$$\mathbb{P}^{(\gamma)}(B^{\delta}_{\varepsilon_k}) \geqslant C_{17} > 0.$$

As we supposed that

$$\gamma \in \limsup_{k\to\infty} A^{\delta}_{\varepsilon_k}$$

for all $\delta \leqslant \delta_0$, we also have

$$\mathbb{P}^{(\gamma)}\left(\limsup_{k\to\infty} B^{\delta}_{\varepsilon_k}\right) \geqslant C_{17} > 0$$

for every $\delta \leqslant \delta_0$. But, $B^{\delta}_{\varepsilon} \in \mathcal{G}_{\delta}$ where

$$\mathcal{G}_t = \sigma\left(\mathcal{N}_{|[0,t]\times E}\right).$$

Therefore, the increasing union

$$\bigcup_{\delta>0} \limsup_{k\to\infty} B^{\delta}_{\varepsilon_k}$$

is in the σ-field $\mathscr{G}_{0-}$. As in Section 2, this σ-field is $\mathbb{P}^{(\gamma)}$-trivial. $\square$

With Lemmas 3.5 and 3.6, we obtain that $\overleftarrow{\Pi}_{y,x}(\mathrm{d}\gamma)$-a.s., $\mathbb{P}^{(\gamma)}$-a.s., there exists $\delta > 0$ such that the event

$$\limsup_{k\to\infty} \left\{ Supp\left(\int_{[0,\delta]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X^{\Omega}(\kappa)\right) \cap Ann_{\bar{\Omega}}\big(\gamma(0), \varepsilon_k, 2\varepsilon_k\big) = \emptyset \right\}$$

is realized. It is almost the desired result. It remains for us to prove that the measure

$$\int_{[\delta,\zeta_\gamma]\times E} \mathcal{N}(\mathrm{d}t\,\mathrm{d}\kappa) X^{\Omega}(\kappa)$$

does not charge a small ball of nonzero radius centered at $\gamma(0)$. As in Section 2, this is a consequence of the fact that there exists only a finite number of atoms (t_i, κ_i) of the measure $\mathcal{N}|_{[\delta,\zeta_\gamma]\times E}$ for which $X^{\Omega}(\kappa_i) \neq 0$, and of the property $\mathbb{N}_x\big(\gamma(0) \in Supp(X^{\Omega})\big) = 0$ for every $x \in \Omega$. $\square$

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