Available online at www.sciencedirect.com
SCIENCE DIRECT®
stochastic processes and their applications
ELSEVIER Stochastic Processes and their Applications 115 (2005) 1979–2005
www.elsevier.com/locate/spa

The noisy voter-exclusion process ☆

Paul Jung*

Department of Mathematics, Cornell University, 440 N Aurora St, Ithaca, NY 14850, USA

Received 27 August 2002; received in revised form 4 March 2003; accepted 14 July 2005
Available online 9 August 2005

Abstract

The symmetric exclusion process and the voter model are two interacting particle systems for which a dual finite particle system allows one to characterize its invariant measures. Adding spontaneous births and deaths to the two processes still allows one to use the dual process to obtain information concerning the original process. This paper introduces the noisy voter-exclusion process which generalizes these processes by allowing for all of these interactions to take place. The dual process is used to characterize its invariant measures under various circumstances. Finally, an ergodic theorem for a related process is proved using the coupling method.

Keywords: Voter model; Exclusion process; Interacting particle system; Invariant measure

1. Introduction

The voter model is an interacting particle system introduced independently by Clifford and Sudbury [2] and Holley and Liggett [7]. In particular it is a spin system (see [12]) with rates given by

$$c(x,\eta) = \begin{cases} \sum_y q_v(x,y)\eta(y) & \text{if } \eta(x)=0, \\ \sum_y q_v(x,y)[1-\eta(y)] & \text{if } \eta(x)=1, \end{cases}$$

where $q_v(x,y) \geqslant 0$ and $\sup_x \sum_y q_v(x,y) < \infty$ for $x, y \in \mathscr{S}$.

☆ Research supported in part by NSF grant DMS-00-70465.
*Tel.: +1 6072727042.
E-mail address: pauljung@gmail.com.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2005.07.001

To describe the voter model in a more intuitive manner let $\mathscr{S}$ be a countable set for which a voter resides at each site in the set. The voter at site x waits an exponential time with mean $[\sum_y q_v(x,y)]^{-1}$ at which point it chooses one of its neighbors with probability $q_v(x,y)/\sum_z q_v(x,z)$ and subsequently takes the opinion (either 1 or 0) of y.

Schwartz [15] introduced the β–δ process, a particle system which modifies the well-known symmetric exclusion process with transition rates $q_{\mathrm{e}}(x,y)$, by allowing a birth with exponential rate $\beta(x)$ when there is a 0 at site x, and a death with rate $\delta(x)$ when there is a 1 at site x.

Define the transition rates $q(x,y) = q_{\mathrm{e}}(x,y) + q_v(x,y)$, and let $q_x = \sum_y q(x,y)$. We combine the voter model and the β–δ process to obtain a new process which much satisfy the following: (a) $\mathscr{S}$ is irreducible with respect to $q(x,y)$, (b) $q_{\mathrm{e}}(x,y) = q_{\mathrm{e}}(y,x)$, (c) $\sup_x q_x < \infty$, and (d) $\inf_x q_x > 0$. Add to this the transition rates $\beta(x)$ and $\delta(x)$ where $\sup_x(\beta(x)+\delta(x)) < \infty$. Condition (d) is not necessary, but it is convenient for the purposes of our discussion. We will call such a process a noisy voter-exclusion process (NVE process). The NVE process is a particular example of a spin system with stirring, also known as a reaction–diffusion process. In the physics literature these processes are known as having Glauber–Kawasaki dynamics.

In the setting of the NVE process, the voter at x waits an exponential time with mean q_x at which point it again chooses a neighbor with probability $q(x,y)/q_x$, but now the voter decides to either switch places with y with probability $q_{\mathrm{e}}(x,y)/[q_{\mathrm{e}}(x,y)+q_v(x,y)]$ or, as before, take the opinion of y with probability $q_v(x,y)/[q_{\mathrm{e}}(x,y)+q_v(x,y)]$. In addition to this, a voter at x with opinion 0 decides to spontaneously change its opinion to 1 with exponential rate $\beta(x)$, and a voter at x with opinion 1 spontaneously changes its opinion to 0 with rate $\delta(x)$.

Let

$$\eta_x(u) = \begin{cases} \eta(u) & \text{if } u \neq x, \\ 1-\eta(u) & \text{if } u = x, \end{cases}$$

and

$$\eta_{xy}(u) = \begin{cases} \eta(y) & \text{if } u = x, \\ \eta(x) & \text{if } u = y, \\ \eta(u) & \text{if } u \neq x, y. \end{cases}$$

Using the results of [12, Chapter I], the generator for an NVE process is given by the closure of the following operator on $\mathscr{D}$, the set of all functions on $\{0,1\}^{\mathscr{S}}$ that depend on finitely many coordinates:

$$\Omega f(\eta) = \sum_{\eta(x)=1,\eta(y)=0} q_{\mathrm{e}}(x,y)[f(\eta_{xy}) - f(\eta)] + \sum_x c(x,\eta)[f(\eta_x) - f(\eta)],$$

where

$$c(x,\eta) = \begin{cases} \beta(x) + \sum_y q_v(x,y)\eta(y) & \text{if } \eta(x) = 0, \\ \delta(x) + \sum_y q_v(x,y)[1-\eta(y)] & \text{if } \eta(x) = 1. \end{cases}$$

We will call the corresponding semigroup $S(t)$.

If $\beta(x) = \delta(x) \equiv 0$ then we will say that we have a voter-exclusion process. One may also refer to this as the voter model with stirring. A previous study [1] has been done concerning the ergodic theory of the voter-exclusion process in the case where $\mathscr{S} = \mathbb{Z}$ and $q_e(x,y)$ is not necessarily symmetric, but there is no overlap with the results of this paper.

If $q_e(x,y) \equiv 0$ then we just get the noisy voter model. Granovsky and Madras [6] study some important equilibrium functionals and critical values of the noisy voter model, but only for the case where β and δ are constant. We, on the other hand, will study the invariant measures of the NVE process where $\beta(x)$ and $\delta(x)$ are in general not constant.

In Chapters V and VIII of [12], one can find a complete characterization of the extremal invariant measures and their domains of attraction for the voter model [7] and the symmetric exclusion process [9,17], respectively. Schwartz [15] does the same for the β–δ process. These results are all based upon the existence of a certain dual finite particle process [16] and a certain monotonicity concerning this dual process. In particular, $S(t)\hat{v}_\alpha(A)$ defined below is nondecreasing in t for the voter model and nonincreasing in t for the symmetric exclusion process. For the NVE process, a dual still exists, however, there is no monotonicity concerning the dual so we will have to use other techniques in order to classify the invariant measures under various conditions. Assume throughout that $q_v(x,y) > 0$ for some $x, y \in \mathscr{S}$ since all other cases have been studied by [15].

We start with some definitions. Let $\mathscr{P}$ denote the set of probability measures on $X = \{0,1\}^{\mathscr{S}}$. The set $\mathscr{I}$ will denote the invariant measures for a given NVE process, and $\mathscr{I}_e$ will be its extreme points.

If we denote the set of nonnegative harmonic functions bounded by 1 on $\mathscr{S}$ as

$$\mathscr{H} = \left\{ \alpha : \mathscr{S} \to [0,1] \text{ such that } \sum_y q(x,y)\alpha(y) = q_x\alpha(x) \text{ for all } x \right\},$$

then we can define v_α to be the product measure on X with marginals $v_\alpha\{\eta : \eta(x) = 1\} = \alpha(x)$. Let $\mu_\alpha = \lim_{t\to\infty} v_\alpha S(t)$. Theorem 1.4 below will show that these limits exist.

Let $\mathscr{S}_n = \mathscr{S}^n \backslash \{\vec{x} : x_i = x_j \text{ for some } i<j\}$. If $E_t = (x_t, y_t) \in \mathscr{S}_2$ is the finite, two particle exclusion process with transition rates $q(x,y)$ then define the functions q_v and q_e on $\mathscr{S}_2$ by $q_v(E_t) = q_v(x_t,y_t) + q_v(y_t,x_t)$ and $q_e(E_t) = q_e(x_t,y_t) + q_e(y_t,x_t) = 2q_e(x_t,y_t)$.

Suppose $X(t)$ and $Y(t)$ are independent continuous time Markov chains on $\mathscr{S}$ with transition rates $q(x,y)$ and denote $p_t(x,y) = P^x(X(t) = y)$. Let $\Lambda = \{\omega | \int_0^\infty \beta(X(t)) + \delta(X(t))\,\mathrm{d}t < \infty\}$. For $\alpha \in \mathscr{H}$, $\alpha(X(t))$ is a bounded martingale so $\lim_{t\to\infty} \alpha(X(t))$ exists with probability one. We can define an equivalence relation R on $\mathscr{H}$ by

$$\alpha_1 R \alpha_2 \quad \text{if} \lim_{t\to\infty} [\alpha_1(X(t)) - \alpha_2(X(t))] = 0 \text{ almost surely on } \Lambda.$$

$\mathscr{H}_R$ is any set of representatives of the equivalence classes determined by R.

Let $\mathscr{E}$ be the following event:

$$\{\text{there exists } t_n \to \infty \text{ such that } X(t_n) = Y(t_n)\}.$$

Then we will say that $\mathscr{H}^*$ is the set of all $\alpha \in \mathscr{H}$ such that

$$P^{\{x,y\}}\left(\lim_{t\to\infty} \alpha(X(t)) = 0 \text{ or } 1 \text{ on } \mathscr{E}\right) = 1 \quad \text{for all } x, y \in \mathscr{S},$$

and $\mathscr{H}^*_R$ is again the set of equivalence classes on $\mathscr{H}^*$.

Define the following function on $\mathscr{S}^2$,

$$g(x,y) = P^{(x,y)}[X(t) = Y(t) \text{ for some } t > 0].$$

Note that if $g(x,y) = 1$ for some $(x,y) \in \mathscr{S}_2$ then by irreducibility $g(x,y) \equiv 1$ (For more detail concerning this see [12, Lemma VIII.1.18]).

We are now in a position to state the theorems:

Theorem 1.1. *An NVE process is ergodic if and only if*

$$P^x\left[\int_0^\infty \beta(X(t)) + \delta(X(t))\,\mathrm{d}t = \infty\right] = 1 \quad \textit{for all } x \in \mathscr{S}. \tag{1}$$

Theorem 1.2. *Suppose* $\mu \in \mathscr{P}$ and δ_0, δ_1 *are the point masses on all* 0'*s and all* 1'*s. Assume that* (1) *does not hold and that*

$$P^E\left[\int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right] = 1 \quad \textit{for all } E \in \mathscr{S}_2. \tag{2}$$

Then

(a) $\lim_{t\to\infty} \delta_0 S(t) = \mu^0$ *and* $\lim_{t\to\infty} \delta_1 S(t) = \mu^1$ *exist,*
(b) $\mathscr{I}_\mathrm{e} = \{\mu^0, \mu^1\}$, *and*
(c) $\lim_{t\to\infty} \mu S(t) = \lambda\mu^1 + (1-\lambda)\mu^0$ *if and only if*

$$\lim_{t\to\infty} \sum_y p_t(x,y)\mu\{\eta : \eta(y) = 1\} = \lambda \quad \textit{for all } x \in \mathscr{S}. \tag{3}$$

We will say that the transition rates $q(x,y)$ on $\mathbb{Z}^d$ have finite range N if $q(x,y) = 0$ when $|x-y| > N$. In order to show that (2) is not an unreasonable condition the following corollary gives circumstances under which (2) holds.

Corollary 1.3. *Let* $\mathscr{S} = \mathbb{Z}^d$, $q_\mathrm{e}(x,y) = q_\mathrm{e}(0, y-x)$, *and* $q_v(x,y) = q_v(0, y-x)$. *Suppose* $X(t) - Y(t)$ *is recurrent and* $q_\mathrm{e}(x,y)$ *has finite range N. Then* $\mathscr{I}_\mathrm{e} = \{\mu^0, \mu^1\}$ *and for* $\mu \in \mathscr{P}$, $\lim_{t\to\infty} \mu S(t) = \lambda\mu^1 + (1-\lambda)\mu^0$ *if and only if* (3) *holds.*

Theorem 1.4. (a) μ_α *exists for all* $\alpha \in \mathscr{H}$, *and* $\mu_{\alpha_1} = \mu_{\alpha_2}$ *if and only if* $\alpha_1 R \alpha_2$.

(b) *If* $g(x,y) < 1$ *for some* $x, y \in \mathscr{S}$ *and*

$$P^E\left[\int_0^\infty q_{\mathrm{e}}(E_t)\,\mathrm{d}t = \infty\right] = 0 \quad \textit{for some } E \in \mathscr{S}_2 \tag{4}$$

then $\mathscr{I}_{\mathrm{e}} = \{\mu_\alpha : \alpha \in \mathscr{H}^*_R\}$.

(c) *If* $q(x,y) = q(y,x)$ *for all* $x, y \in \mathscr{S}$ *and*

$$P^E\left[\int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right] = 0 \quad \textit{for some } E \in \mathscr{S}_2 \tag{5}$$

then $\mathscr{I}_{\mathrm{e}} = \{\mu_\alpha : \alpha \in \mathscr{H}_R\}$.

The condition that $g(x,y) < 1$ for some $x, y \in \mathscr{S}$ is not needed in part (a), but we put it there because if $g \equiv 1$ then we are left with the situation in Theorem 1.2. It should also be remarked that if $q(x,y) = q(y,x)$ and $g(x,y) < 1$ for some $(x,y) \in \mathscr{S}_2$ then Lemma VIII.1.23 in [12] implies that (4) and (5) are satisfied. On the other hand when $q(x,y) = q(y,x)$, we claim that $g \equiv 1$ implies that $X(t)$ is recurrent so that $\beta(x) + \delta(x) > 0$ for some x gives us (1). To prove the claim use the Chapman–Kolmogorov equation to get

$$\begin{aligned} p_{2t}(x,x) &= \sum_y p_t(x,y)p_t(y,x) \\ &= \sum_y [p_t(x,y)]^2 = P^{(x,x)}[X_1(t) = X_2(t)]. \end{aligned}$$

So if $X(t)$ is transient then $g(x,y) < 1$ for some $x, y \in \mathscr{S}$ since

$$\int_0^\infty P^{(x,x)}[X_1(t) = X_2(t)]\,\mathrm{d}t < \infty$$

(This argument will be made more explicit by Lemma 3.1).

Theorem 1.5. *Suppose* $\mu \in \mathscr{P}$ *and that* $E^{(x,y)}g(X(t), Y(t)) \to 0$ *for some* $x, y \in \mathscr{S}$. *If*

$$\lim_{t\to\infty} \sum_y p_t(x,y)\mu\{\eta : \eta(y) = 1\} = \alpha(x) \quad \textit{and} \tag{6}$$

$$\lim_{t\to\infty} \sum_{u,v} p_t(x,u)p_t(x,v)\mu\{\eta : \eta(u) = \eta(v) = 1\} = \alpha^2(x) \quad \textit{for all } x \in \mathscr{S} \tag{7}$$

then $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$. *A necessary and sufficient condition for* $\lim_{t\to\infty} \mu S(t) = \mu_\alpha$ *is that*

$$\lim_{s\to\infty} \limsup_{t\to\infty} \int_X \left\{ \sum_x p_s(w,x) P^x[A] \sum_y p_t(x,y)[\eta(y) - \alpha(y)] \right\}^2 \mathrm{d}\mu(\eta) = 0 \tag{8}$$

We should mention two instances for which $g(x,y) < 1$ for some $x, y \in \mathscr{S}$ implies

$$E^{(x,y)}g(X(t), Y(t)) \to 0 \quad \text{for some } x, y \in \mathscr{S}. \tag{9}$$

Firstly, if $q(x,y)$ is symmetric then as stated in the comments following Theorem 1.4, Lemma VIII.1.18 in [12] gives (9). Secondly, if the only bounded harmonic functions are constants then Corollary II.7.3 in [12] together with Proposition 5.19 in [8] give (9). We also note here that condition (8) is equivalent to (6) and (7) when $P^x[\Lambda] = 1$ for all $x \in \mathscr{S}$.

Corollary 1.6. *If* $g(x,y) < 1$ *for some* $x, y \in \mathscr{S}$ *and* $\mathscr{H} = \{\alpha : \alpha \in [0,1]\}$, *then* $\mathscr{I}_{\mathrm{e}} = \{\mu_\alpha : \alpha \in \mathscr{H}\}$.

The proofs of the above theorems appear in Section 4. The above theorems give partial results concerning the invariant measures and their respective domains of attraction for certain NVE processes. Clearly there are NVE processes which are not covered by these theorems. Examples of these situations include the process on $\mathbb{Z}^2$ where $q_{\mathrm{e}}(x,y)$ is translation invariant, $\beta(x) = \delta(x) \equiv 0$, $q_v(x,y) = 0$ outside of a finite set, and $q_v(x,y)$ is not symmetric. A more interesting example is provided in V.1.6 of [12]; in fact using Liggett's example we can create similar examples to show that there exist NVE processes which do not satisfy (4) yet have $g(x,y) < 1$ for some $x, y \in \mathscr{S}$. Section 5 discusses how one might go about proving a general result that would include the exceptions we have just mentioned.

We now turn to a discussion of a slightly more general process. In particular, modify the NVE process by allowing for exclusion rates where $q_{\mathrm{e}}(x,y) \neq q_{\mathrm{e}}(y,x)$. Call such a process a generalized NVE process. It should be noted that not requiring the symmetry of $q_{\mathrm{e}}(x,y)$ really does change the nature of the process. We will state two main reasons for this. Firstly, the properties of the dual finite particle system that allow us to prove the above theorems no longer exist. Secondly, the results for the asymmetric case are completely different; in fact it is known that Theorems 1.4 and 1.5 and Corollary 1.6 do not hold in general when $q_{\mathrm{e}}(x,y)$ is not symmetric. We can, however, prove certain things about the generalized NVE process in specific cases using methods other than duality.

In Section 6 we prove an ergodic theorem for the case where $q_v(x,y) \equiv 0$ using the coupling method. When $q_v(x,y) \equiv 0$ we will call the process a noisy exclusion process. We will also show in this final section that Theorem 1.1 does not hold in general when $q_{\mathrm{e}}(x,y)$ is not symmetric.

The main result of Section 6 is an extension, in the case where $\mathscr{S} = \mathbb{Z}^d$ and the transition rates have finite range, of [15] ergodic theorem which is exactly Theorem 1.1 when $q_v(x,y) \equiv 0$. Before we state the theorem we need the following definitions:

$$T_n = \{x \in \mathbb{Z}^d : |x_i| \leqslant n \text{ for all } i\}.$$

$$T_n^N = T_{n+N} \backslash T_n.$$

Theorem 1.7. *Suppose* η_t *is a noisy exclusion process with transition rates* $q_{\mathrm{e}}(x,y)$ *irreducible with respect to* $\mathbb{Z}^d$ *and having finite range N. Let* $\{b_l\}$ *be a nonnegative sequence satisfying* (a) $\sum b_l = \infty$ *if* $d = 1$ *and* (b) $\lim_{l\to\infty} l b_l = \infty$ *if* $d \geqslant 2$. *If* $p(l)$ *is a nonnegative function on* $\mathbb{N}$ *satisfying* $p(l+1) \geqslant p(l) + N$ *and is bounded by* kl^k *for some* $k > 0$, *and if* β, δ *satisfy* $\beta(x) + \delta(x) \geqslant b_l$ *for all* $x \in T^N_{p(l)}$ *and* $\beta(x) = \delta(x) = 0$ *otherwise, then* η_t *is ergodic.*

For some simple examples to see the applicability of Theorem 1.7 set $N = 1$ and let $p(l)$ be an arithmetic sequence e.g. $k, 2k, 3k, \ldots$ Suppose $\beta(x) = \delta(x) = 1$ for all $\|x\| = nk, n \in \mathbb{N}$ with $\|\cdot\|$ being the l^∞ norm and $\beta(x) = \delta(x) = 0$ otherwise. Then the theorem tells us that the noisy exclusion process is ergodic. Note that if $k = 1$ and $\beta(x) = \delta(x) = 1 + \delta$ for $\delta > 0$ then the $M < \varepsilon$ Theorem in Section I.4 of [12] also gives us ergodicity for doubly stochastic transition kernels. If $k > 1$ then the $M < \varepsilon$ Theorem in general gives us no information. Also, Theorem 1.7 allows us to let $\beta(x) + \delta(x) \to 0$ whereas the $M < \varepsilon$ Theorem again gives no information in such a circumstance. We should however mention here that if $q_e(x, y)$ is symmetric and $k = 1$, a version of the $M < \varepsilon$ Theorem proven in [5] allows for $\beta(x) + \delta(x) \to 0$, but once again, Ferarri's theorem gives no information in the case where $k > 1$.

2. The dual process: a finite particle system

In order to prove the theorems we will need many lemmas. The lemma in this section which concerns the dual process is the most important and is in fact the reason that we are able to prove anything about these processes. Its proof follows that of Theorem VIII.1.1 in [12]. Before stating and proving Lemma 2.1 we will need some more definitions.

Let Y be the class of all finite subsets of $\mathscr{S}$ excluding the empty set. The semi-dual process A_t is a continuous time Markov chain on Y such that the particles in A_t move independently on $\mathscr{S}$ according to the motions of the independent $X_i(t)$ processes except that transitions to sites that are already occupied are handled in the following way: If a particle at x attempts to move to y which is already occupied then the transition is either suppressed with probability $q_e(x, y)/[q_e(x, y) + q_v(x, y)]$ or the two particles coalesce and move together thereafter with probability $q_v(x, y)/[q_e(x, y) + q_v(x, y)]$. In particular $|A_t| \leqslant |A_{t+s}|$ for all $s \geqslant 0$.

Now let Y^* be defined by adding to Y a cemetery state, Δ, and the empty set, $\emptyset$. We define the process A_t^* starting in a state $A \in Y$ to move just as A_t does except that in addition A_t^* goes to $A_t^* \backslash \{x\}$ at rate $\beta(x)$ if $x \in A_t^*$ and A_t^* goes to the cemetery state Δ at rate $\sum_{x \in A_t^*} \delta(x)$. We will call A_t^* the dual process. Define D to be the event that A_t^* is never in the state Δ.

If $\mu \in \mathscr{P}$ and $A \in Y$, then define

$$\hat{\mu}(A) = \mu\{\eta : \eta(x) = 1 \text{ for all } x \in A\}.$$

Extend this function to Y^* by letting $\hat{\mu}(\Delta) = 0$ and $\hat{\mu}(\emptyset) = 1$.

Lemma 2.1. *Extend the domain of $\eta \in X$ by letting $\eta(\Delta) = 0$. If $A \in Y$ then for all $t \geqslant 0$*

$$P^\eta[\{\eta_t = 1 \ on\ A\}] = P^A[\{\eta = 1 \ on\ A_t^*\} \cup \{A_t^* = \emptyset\}].$$

Proof. Let

$$u_\eta(t, A) = P^\eta[\{\eta_t = 1 \text{ on } A\} \cup \{A = \emptyset\}] = S(t)H(\cdot, A)(\eta),$$

where for $A \neq \emptyset$

$$H(\eta, A) = \prod_{x \in A} \eta(x) = \begin{cases} 1 & \text{if } \eta(x) = 1 \text{ for all } x \in A^*, \\ 0 & \text{otherwise,} \end{cases}$$

and $H(\eta, \emptyset) = 1$.

For each $A \in Y$, $H(\cdot, A) \in \mathscr{D}$ so we have

$$\begin{aligned}
\Omega H(\cdot, A)(\eta) = & \sum_{\eta(x)=1, \eta(y)=0} q_{\mathrm{e}}(x, y)[H(\eta_{xy}, A) - H(\eta, A)] \\
& + \sum_{x,y:\eta(x) \neq \eta(y)} q_v(x, y)[H(\eta_x, A) - H(\eta, A)] \\
& + \sum_x [\beta(x)(1 - \eta(x)) + \delta(x)\eta(x)][H(\eta_x, A) - H(\eta, A)] \\
= & \frac{1}{2} \sum_{x,y} q_{\mathrm{e}}(x, y)[H(\eta_{xy}, A) - H(\eta, A)] \\
& + \sum_{x \in A, y \in \mathscr{S}} q_v(x, y) H(\eta, A \backslash \{x\})[1 - 2\eta(x)] \\
& \times \{\eta(x)[1 - \eta(y)] + \eta(y)[1 - \eta(x)]\} \\
& + \sum_{x \in A} \beta(x)[H(\eta, A \backslash \{x\}) - H(\eta, A)] + \sum_{x \in A} \delta(x)[H(\eta, \Delta) - H(\eta, A)] \\
= & \frac{1}{2} \sum_{x,y} q_{\mathrm{e}}(x, y)[H(\eta, A_{xy}) - H(\eta, A)] \\
& + \sum_{x \in A, y \in \mathscr{S}} q_v(x, y) H(\eta, A \backslash \{x\})[\eta(y) - \eta(x)] \\
& + \sum_{x \in A} \beta(x)[H(\eta, A \backslash \{x\}) - H(\eta, A)] + \sum_{x \in A} \delta(x)[H(\eta, \Delta) - H(\eta, A)] \\
= & \sum_{x \in A, y \notin A} q_{\mathrm{e}}(x, y)[H(\eta, A_{xy}) - H(\eta, A)] \\
& + \sum_{x \in A, y \in \mathscr{S}} q_v(x, y)[H(\eta, (A \backslash \{x\}) \cup \{y\}) - H(\eta, A)] \\
& + \sum_{x \in A} \beta(x)[H(\eta, A \backslash \{x\}) - H(\eta, A)] + \sum_{x \in A} \delta(x)[H(\eta, \Delta) - H(\eta, A)].
\end{aligned}$$

Here A_{xy} is obtained from A in the same way that η_{xy} is obtained from η. The symmetry of $q_{\mathrm{e}}(x, y)$ is used in second and fourth steps above.

By Theorem I.2.9 in [12]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} u_\eta(t, A) &= \Omega S(t) H(\cdot, A)(\eta) \\
&= \sum_{x \in A, y \notin A} q_{\mathrm{e}}(x, y)[S(t)H(\cdot, A_{xy})(\eta) - S(t)H(\cdot, A)(\eta)]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{x\in A, y\in\mathscr{S}} q_v(x,y)[S(t)H(\cdot,(A\backslash\{x\})\cup\{y\})(\eta)-S(t)H(\cdot,A)(\eta)] \\
&+\sum_{x\in A}\beta(x)[S(t)H(\cdot,A\backslash\{x\})(\eta)-S(t)H(\cdot,A)(\eta)] \\
&+\sum_{x\in A}\delta(x)[S(t)H(\cdot,\Delta)(\eta)-S(t)H(\cdot,A)(\eta)] \\
=&\sum_{x\in A, y\notin A} q_{\mathrm{e}}(x,y)[u_\eta(t,A_{xy})-u_\eta(t,A)] \\
&+\sum_{x\in A, y\in\mathscr{S}} q_v(x,y)[u_\eta(t,(A\backslash\{x\})\cup\{y\})-u_\eta(t,A)] \\
&+\sum_{x\in A}\beta(x)[u_\eta(t,A\backslash\{x\})-u_\eta(t,A)]+\sum_{x\in A}\delta(x)[u_\eta(t,\Delta)-u_\eta(t,A)].
\end{aligned}
$$

For each $A \in Y$, the unique solution to this system of differential equations with initial condition $H(\eta, A)$ is

$$E^A H(\eta, A_t^*) = P^A[\{\eta = 1 \text{ on } A_t^*\} \cup \{A_t^* = \emptyset\}]$$

(See [4, Theorem 1.3]). $\square$

3. Preliminary lemmas

The first five lemmas are adaptations of lemmas proved by Schwartz [15]. We omit the proofs of Lemmas 3.1, 3.2, and 3.5 since they are the same as found in Schwartz [15] except for perhaps a change in notation.

Suppose $\mathscr{E}_t$ is a continuous time nonexplosive jump process on a countable set $\mathscr{N}$ and let $\mathscr{E}_k$ be the imbedded discrete-time Markov chain. The transition rates of $\mathscr{E}_t$ are given by Q_{xy}. For $\mathscr{L} \subset \mathscr{N}$ define

$$Q_{\mathscr{L}}(x) = \sum_{y\in\mathscr{L}, y\neq x} Q_{xy}.$$

Lemma 3.1. *Assume there exist constants* $0 < \alpha_1 < \alpha_2 < \infty$ *such that for each* $x \in \mathscr{N}$, $\alpha_1 \leqslant Q_{\mathscr{N}}(x) \leqslant \alpha_2$. *Then almost surely*

$$
\begin{aligned}
\left\{\omega \,\middle|\, \int_0^\infty Q_{\mathscr{L}}(\mathscr{E}_t)\,\mathrm{d}t = \infty\right\} &= \{\omega | \mathscr{E}_k \in \mathscr{L} \textit{ infinitely often}\} \\
&\subset \{\omega | \mathscr{E}_t \in \mathscr{L} \textit{ for some } t\}.
\end{aligned}
$$

Lemma 3.2. *Assume* $0 < \sup_x(\beta(x) + \delta(x)) < \infty$. *Then* (1) *holds if and only if*

$$P^A[A_t^* = \emptyset \textit{ or } A_t^* = \Delta \textit{ eventually}] = 1$$

for all $A \in Y$.

For the next lemma define the function

$$h(A) = P^A(|A_t| < |A| \text{ for some } t > 0) \quad \text{for } A \in Y$$

which is in some sense a voter model analog of the function $g(x, y)$.

Lemma 3.3. *If* (2) *holds then* $P^A(|A_t| = 1$ *eventually*$) = 1$ *for all* $A \in Y$.

Proof. We first prove the case for which A_t starts in a two particle state $|A| = 2$.

Take $\mathscr{E}_t$ in Lemma 3.1 to be A_t, and let $\mathscr{L}$ be the set of states such that $|A_t| = 1$. We then interpret $Q_{\mathscr{L}}(A_t)$ as the rate at which A_t jumps to a one particle state. If $A_t = \{x\}$ then $Q_{\mathscr{L}}(A_t)$ is just q_x. Now suppose that $|A_t| = 2$ for all t. Then A_t is exactly E_t defined above to be the two particle exclusion process with respect to $q(x, y)$. Therefore

$$\int_0^\infty q_v(E_t)\,\mathrm{d}t = \int_0^\infty Q_{\mathscr{L}}(A_t)\,\mathrm{d}t = \infty$$

and by Lemma 3.1, $|A_t| = 1$ eventually, a contradiction. We have thus proved the case where $|A| = 2$.

For the general case suppose $|A| \geqslant 2$. Couple B_t, a semi-dual process starting from a two particle state $|B| = 2$, with A_t so that $B_t \subset A_t$. In order to do this let A_t and B_t move as usual except when a particle tries to move with rate $q_e(x, y)$ to an occupied site, instead of the motion being "excluded", let the two particles switch places. Of course this is the same motion as before, just a different way of thinking of it.

Using the coupling we have now that $h(A) = 1$ for all $|A| \geqslant 2$. Thus with probability one, $|A|$ decreases for all $|A| \geqslant 2$ which proves the lemma. □

Recall that D is the event where A_t^* is never in the state Δ.

Lemma 3.4. *If* $\beta(x) \equiv 0$ *then*

$$\lim_{t\to\infty} E^{\{x\}} P^{A_t}[D^c, \Lambda] = 0 \quad \textit{for all } x \in \mathscr{S}.$$

Proof. Let $\mathscr{E}_t = (X(t), \zeta(t))$ be a Markov jump process on $\mathscr{N} = \mathscr{S} \times \{0, 1, 2, \ldots\}$ with jump rates $Q_{(x,n),(y,0)} = q(x, y)$ and $Q_{(x,n),(x,n+1)} = \delta(x)$. Let $\mathscr{L} = \mathscr{S} \times \{1, 2, \ldots\}$ so that $Q_{\mathscr{L}}((x, n)) = \delta(x)$. We then have that

$$\begin{aligned}\lim_{t\to\infty} E^{\{x\}} P^{A_t}[D^c, \Lambda] &= \lim_{t\to\infty} P^x[\mathscr{E}_s \text{ jumps to } \mathscr{L} \text{ after time } t, \Lambda] \\ &= P^x[\mathscr{E}_k \in \mathscr{L} \text{ infinitely often}, \Lambda].\end{aligned}$$

But the right-hand side is equal to 0 by Lemma 3.1 completing the proof. □

We will need three definitions in stating the next lemma and in proving Theorem 3.9. Before stating the definitions we ask the reader to think of $\mu\{\eta : \eta(X(t)) = 0\}$ as a family of random variables (indexed by t) on the space of paths. We then have

$$\mathscr{P}' = \{\mu \in \mathscr{P} : \lim_{t\to\infty} \mu\{\eta : \eta(X(t)) = 0\} = 1 \text{ almost surely on } \Lambda^c\}.$$

$$\mathscr{H}' = \{\alpha \in \mathscr{H} : \lim_{t\to\infty} \alpha(X(t)) = 0 \text{ almost surely on } \Lambda^c\}.$$

If $S(t)$ is the semigroup for an NVE process then let $S'(t)$ be the semigroup for the same process except that $\beta(x) = \delta(x) \equiv 0$.

For part (b) of the following lemma we couple A_t and A_t^* so that they move together until the first time that $A_t^* = \Delta$ or $|A_t^*| < |A_t|$.

Lemma 3.5. (a) *$\mathscr{H}'$ is a set of class representatives for the equivalence relation R on $\mathscr{H}$.*

(b) *If we extend the state space of A_t to include Δ and $\emptyset$ as absorbing states then* $\lim_{t\to\infty} P^{A_t^*}[A_s^* \neq A_s$ *for some* $s \geqslant 0] = 0$ *almost surely.*

(c) *Suppose that $\beta(x) \equiv 0$. If $\mu \in \mathscr{I}$ or if $\mu = \lim_{t\to\infty} \nu S'(t)$ exists for $\nu \in \mathscr{I}$, then $\mu \in \mathscr{P}'$.*

Define E_t^n to be the finite exclusion process on n particles starting in the state A where $|A| = n$. To be consistent with our previous definition of E_t we will leave the superscript off if $n = 2$ so that $E_t = E_t^2$ and $|E| = 2$.

Lemma 3.6. *If* (5) *holds and $q(x, y) = q(y, x)$ then*

$$P^A\left[\int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,\mathrm{d}t = \infty\right] = 0 \quad \textit{for all } A \in Y.$$

Proof. Suppose $A = \{x_1, \dots, x_n\}$. Let $E_t^{\{i,j\}}$ be the two particle exclusion process starting from $\{x_i, x_j\}$. We will show there exists a multiple coupling of the processes E_t^n and $E_t^{\{i,j\}}$ for $0 \leqslant i < j \leqslant n$ such that

$$\{E_t^n\} \subset \bigcup_{0\leqslant i<j\leqslant n} \{E_t^{\{i,j\}}\}. \tag{10}$$

Let $X_i(t)$ be a process equal in distribution to $X(t)$. The key to seeing why (10) is true is noticing that there exists a way to couple $X_i(t)$ and $X_j(t)$ so that whenever one tries to coalesce with the other, they simply switch places. This can be done since $q(x, y) = q(y, x)$. With that said, it is clear that we can couple the $X_i(t)$'s with E_t^n so that

$$\{E_t^n\} = \{X_1(t), \dots, X_n(t)\}.$$

Here the processes $X_i(t)$ start at x_i and are clearly not independent of each other.

For each $E_t^{\{i,j\}}$ we can label one particle first class and the other particle second class. We can now think of the evolution of $E_t^{\{i,j\}}$ in the following way. If a second class particle tries to go to a site occupied by a first class particle, it is not allowed to do so. However, if a first class particle attempts to move to a site occupied by a second class particle, the two particles switch places. With this evolution a first class particle is equal in distribution to $X(t)$. By choosing the first class particles to have the paths of the $X_i(t)$ processes above it is clear that (10) holds.

Suppose now that

$$P^A\left[\int_0^\infty \sum_{E\subset E_t^n} q_v(E)\,\mathrm{d}t = \infty\right] > 0 \quad \text{for some } A \in Y.$$

In light of (10), it must be that

$$P^E\left[\int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right] > 0 \quad \text{for some } E \in \mathscr{S}_2.$$

By irreducibility

$$P^E\left[\int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right] > 0 \quad \text{for all } E \in \mathscr{S}_2. \qquad \square$$

Lemma 3.7. *If $q(x,y) = q(y,x)$ and (5) holds then $h(E_t^n) \to 0$ almost surely for all initial states $A \in \mathscr{S}_n$.*

Proof. By Lemma 3.6,

$$P^A\left[\int_0^\infty \sum_{E \subset E_t^n} q_v(E)\,\mathrm{d}t = \infty\right] = 0 \quad \text{for all } A \in Y. \tag{11}$$

Let E_k^n be the imbedded Markov chain for the process E_t^n starting with initial state A. Let Ω be the path space for E_k^n and let $\mathscr{M}$ be the probability measure on Ω for our process. Choose $\varepsilon > 0$. If there exists a set $F \subset \Omega$ such that $\mathscr{M}(F) > 0$ and $h(E_k^n) > \varepsilon$ infinitely often on F then it must be that

$$\sum_{k=0}^\infty \sum_{E \subset E_k^n} q_v(E) = \infty$$

almost surely on F since whenever $\sum_{k=0}^\infty \sum_{E \subset E_k^n} q_v(E) < \infty$ it must be that $h(E_k^n) > \varepsilon$ finitely many times.

We claim that

$$\left\{\omega \,\middle|\, \int_0^\infty \sum_{E \subset E_t^n} q_v(E)\,\mathrm{d}t = \infty\right\} = \left\{\omega \,\middle|\, \sum_{k=0}^\infty \sum_{E \subset E_k^n} q_v(E) = \infty\right\} \tag{12}$$

almost surely. To see this define τ_k to be the kth jump time of E_t^n. Now note that

$$\int_0^\infty \sum_{E \subset E_t^n} q_v(E)\,\mathrm{d}t = \sum_{k=0}^\infty \sum_{E \subset E_k^n} q_v(E)[\tau_{k+1} - \tau_k].$$

By our assumptions $E[\tau_{k+1} - \tau_k]$ and $\mathrm{Var}[\tau_{k+1} - \tau_k]$ are bounded above and below uniformly in k. Since $[\tau_{k+1} - \tau_k | E_1^n, E_2^n, \ldots]$ are independent, Kolmogorov's three series theorem proves the claim.

Since (12) contradicts (11) we have shown that $h(E_k^n) \to 0$ almost surely. This however implies that $h(E_t^n) \to 0$ almost surely. $\square$

Suppose V_t is the dual process for the voter model with rates $q(x,y)$ starting from the set A. If we couple A_t and V_t so that they move together as much as possible then we can define the function

$$f(A) = P^A[A_t \neq V_t \text{ for some } t > 0].$$

Again, $f(A)$ plays much the same role as $h(A)$ and $g(x, y)$.

Lemma 3.8. *If* (4) *holds then* $E^A f(A_t) \to 0$ *for all* $A \in Y$.

Proof. We prove first the case where $|A| \leqslant 2$. Let $\mathscr{E}_t = (A_t, \zeta(t))$ be a Markov jump process on $\mathscr{N} = (\mathscr{S}_2 \cup \mathscr{S}) \times \{0, 1, 2, \ldots\}$ with jump rates (i) $Q_{(A,n),(B,0)}$ equal to the jump rate from A to B of the semi-dual process and (ii) $Q_{(A,n),(A,n+1)} = q_e(A)$ if $|A| = 2$. Let $\mathscr{L} = \mathscr{S}_2 \times \{1, 2, \ldots\}$ so when $|A| = 2$, $Q_{\mathscr{L}}((A, n)) = q_e(A)$ and when $|A| = 1$, $Q_{\mathscr{L}}((A, n)) = 0$. We then have that

$$\begin{aligned} \lim_{t\to\infty} E^A f(A_t) &= \lim_{t\to\infty} P^A[\mathscr{E}_s \text{ jumps to } \mathscr{L} \text{ after time } t] \\ &= P^A[\mathscr{E}_k \in \mathscr{L} \text{ infinitely often}]. \end{aligned}$$

Since (4) holds, Lemma 3.1 implies that the right-hand side is 0.

Now suppose $|A| > 2$. Change the coupling of the $X_i(t)$ processes that we used in Lemma 3.6 by letting $X_i(t)$ and $X_j(t)$ switch places at rate $q_e(X_i(t), X_j(t))$ and coalesce and move together thereafter at rate $q_v(X_i(t), X_j(t))$. Again, we are allowed to do this since $q_e(x, y) = q_e(y, x)$. With this new coupling we can couple the $X_i(t)$'s with A_t so that

$$\{A_t\} = \{X_1(t), \ldots, X_n(t)\}.$$

As in Lemma 3.6, we use the idea of first class particles along with the fact that $X_i(t)$ can be coupled with $E_t^{\{i,j\}}$ so that $\{X_i(t)\} \subset \{E_t^{\{i,j\}}\}$, we have that the proof for $|A| \leqslant 2$ implies the proof for all $A \in Y$. $\square$

The next theorem is actually a special case of Theorem 1.4. We prove this special case right now in order make the proof of the general case easier to read.

Theorem 3.9. *Suppose* $q_e(x, y) \equiv 0$.
(a) μ_α *exists for all* $\alpha \in \mathscr{H}$, *and* $\mu_{\alpha_1} = \mu_{\alpha_2}$ *if and only if* $\alpha_1 R \alpha_2$.
(b) $\mathscr{I}_e = \{\mu_\alpha : \alpha \in \mathscr{H}_R^*\}$.

Proof. The proof is virtually the same as that of Theorem 1.3 in [15], but it is included here for completeness. We will, however, leave out some repetitive details.

Let $\mathscr{J}$ represent the set of invariant measures for the case where $\beta(x) = \delta(x) \equiv 0$, in other words the voter model. In Chapter V of [12], it is shown that $\mathscr{J}_e = \{\mu_\alpha : \alpha \in \mathscr{H}^*\}$. Consider a certain subset of $\mathscr{J}$, namely

$$\mathscr{J}' = \left\{\mu \in \mathscr{J} : \lim_{t\to\infty} \mu\{\eta : \eta(X(t)) = 0\} = 1 \text{ almost surely on } \Lambda^c\right\}.$$

The main part of the proof is showing that there exists a bijective affine map between $\mathscr{J}'$ and $\mathscr{I}$. To avoid confusion, we will put a bar over the extremal invariant measures of the pure voter model so that we have $\mathscr{J}_e = \{\bar{\mu}_\alpha : \alpha \in \mathscr{H}^*\}$.

In order to do this we will first consider the case where $\beta(x) \equiv 0$, but $\delta(x) \geqslant 0$. We start by coupling the semi-dual process A_t with n independent processes

$X_1(t), \ldots, X_n(t)$ which start from $A = \{x_1, \ldots, x_n\}$ and are equal in distribution to $X(t)$. In particular, couple the processes so that $A_t \subset \{X_1(t), \ldots, X_n(t)\}$. Let $X_i^*(t)$ be the dual process starting from $\{x_i\}$ and henceforth define $T(t)$ to be the semigroup for the voter model.

By coupling the processes A_t^* and A_t so that they move together as much as possible, it is clear that for any measure $\mu \in \mathscr{P}$ and any $A \in Y$, $S(t)\hat{\mu}(A) \leqslant T(t)\hat{\mu}(A)$. Thus if $\mu \in \mathscr{I}$ and $v \in \mathscr{J}'$ then $\hat{\mu}(A) \leqslant T(t)\hat{\mu}(A)$ and $S(t)\hat{v}(A) \leqslant \hat{v}(A)$. Applying the respective semigroups once more to both these inequalities gives $T(s)\hat{\mu}(A) \leqslant T(t+s)\hat{\mu}(A)$ and $S(t+s)\hat{v}(A) \leqslant S(s)\hat{v}(A)$ so that $\lim_{t\to\infty} \mu T(t)$ and $\lim_{t\to\infty} vS(t)$ exist by monotonicity and duality.

Now take $\mu_1 \in \mathscr{J}'$. Let $\lim_{t\to\infty} \mu_1 S(t) = \mu_2$ and define the map $\sigma(\mu_1) = \mu_2$. We will show that σ is an affine bijection from $\mathscr{J}'$ to $\mathscr{I}$.

Since $\mu_1 \in \mathscr{P}'$, it follows that

$$\lim_{t\to\infty} |T(t)\hat{\mu}_1(A) - S(t)\hat{\mu}_1(A)| \leqslant P^A\left[\bigcup_{1\leqslant i\leqslant n} \{X_i^*(t) = \Delta \text{ eventually}, \Lambda\}\right]$$

$$\leqslant \sum_{i=1}^{n} P^{\{x_i\}}[D^c, \Lambda].$$

By the definition of μ_2 and by the fact that $\mu_1 \in \mathscr{J}$

$$|\hat{\mu}_1(A) - \hat{\mu}_2(A)| \leqslant \sum_{i=1}^{n} P^{\{x_i\}}[D^c, \Lambda].$$

Applying $T(t)$ to both sides of this last inequality and passing to the limit gives

$$\lim_{t\to\infty} |\hat{\mu}_1(A) - T(t)\hat{\mu}_2(A)| \leqslant \lim_{t\to\infty} \sum_{i=1}^{n} E^{\{x_i\}} P^{A_t}[D^c, \Lambda].$$

Lemma 3.4 says that the right-hand side above is equal to 0 so that $\lim_{t\to\infty} \mu_2 T(t) = \mu_1$.This proves that σ is injective. If we think of $X^*(t) = \Delta$ as an absorbing state where $X^*(t)$ continually jumps to Δ at exponential rate one then a similar argument using Lemma 3.5 (c) shows σ to be surjective. To see that σ is affine note simply that if $\mu_1, v_1 \in \mathscr{J}'$ then

$$\lim_{t\to\infty} (\lambda\mu_2 + (1-\lambda)v_2)S(t) = \lambda\mu_1 + (1-\lambda)v_1.$$

We have thus far shown that there exists an affine bijection between $\mathscr{J}'$ and $\mathscr{I}$ for the case $\beta \equiv 0$. For the general case we compare the process η_t with birth rates $\beta(x)$ and death rates $\delta(x)$ to a similar process $\tilde{\eta}_t$ having the same transition rates except that the death rates are now $\tilde{\delta}(x) = \beta(x) + \delta(x)$ and the birth rates are identically 0. Let the associated dual process, semigroup, and set of invariant measures for $\tilde{\eta}_t$ be $\tilde{A}_t^*$, $\tilde{S}(t)$, and $\tilde{\mathscr{I}}$.

Couple the two dual processes so that they make the same transitions except when a particle in A_t^* dies off due to a $\beta(x)$ jump, then $\tilde{A}_t^*$ goes to the state Δ. Since $\tilde{S}(t)\hat{\mu}(A) \leqslant S(t)\hat{\mu}(A)$, we can repeat the monotonicity arguments used above to show that for $v_1 \in \mathscr{I}$ and $v_2 \in \tilde{\mathscr{I}}$, the limits $\lim_{t\to\infty} v_1\tilde{S}(t) = v_2$ and $\lim_{t\to\infty} v_2 S(t) = v_1$

exist. If we can show that

$$\lim_{t\to\infty} E^A P^{A_t^*}[A_s^* \neq \tilde{A}_s^* \text{for some } s \geqslant 0] = 0 \quad \text{for all } A \in Y \tag{13}$$

and similarly that

$$\lim_{t\to\infty} E^A P^{\tilde{A}_t^*}[A_s^* \neq \tilde{A}_s^* \text{ for some } s \geqslant 0] = 0 \quad \text{for all } A \in Y \tag{14}$$

then we can also show that the map $\lim_{t\to\infty} v_2 S(t) = v_1 = \tilde{\sigma}(v_2)$ is an affine bijection between $\tilde{\mathscr{I}}$ and $\mathscr{I}$. If we extend the state space of A_t as in Lemma 3.5 (b) then the following inequalities combined with Lemma 3.5 (b) prove (13) and (14):

$$\begin{aligned} &P^{\tilde{A}_t^*}[A_s^* \neq \tilde{A}_s^* \text{ for some } s \geqslant 0] \leqslant P^{A_t^*}[A_s^* \neq \tilde{A}_s^* \text{ for some } s \geqslant 0] \\ &\quad \leqslant P^{A_t^*}[A_s^* \neq A_s \text{ for some } s \geqslant 0]. \end{aligned}$$

Our desired affine bijection from $\mathscr{J}'$ to $\mathscr{I}$ is just $\tilde{\sigma} \circ \sigma$. We are now ready to prove the two parts of the theorem. We start with part (a).

To prove μ_α exists we need only show

$$\lim_{t\to\infty} v_\alpha S(t) = \lim_{t\to\infty} \lim_{s\to\infty} \lim_{r\to\infty} v_\alpha T(r)\tilde{S}(s)S(t). \tag{15}$$

Let $\bar{\mu}_\alpha = \lim_{r\to\infty} v_\alpha T(r)$ and let $\tilde{\bar{\mu}} = \lim_{s\to\infty} \bar{\mu}\tilde{S}(s)$. We have already argued that these limits exist. Applying $S(t)$ and passing to the limit in the following inequalities proves (15).

$$\begin{aligned} &\lim_{t\to\infty} |S(t)\hat{v}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\ &\quad \leqslant \lim_{t\to\infty} |S(t)\hat{v}_\alpha(A) - \tilde{S}(t)\hat{\bar{\mu}}_\alpha(A)| + \lim_{t\to\infty} |\hat{\tilde{\bar{\mu}}}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\ &\quad \leqslant \lim_{t\to\infty} |S(t)\hat{v}_\alpha(A) - T(t)\hat{v}_\alpha(A)| + \lim_{t\to\infty} |\hat{\bar{\mu}}_\alpha(A) - \tilde{S}(t)\hat{\bar{\mu}}_\alpha(A)| \\ &\qquad + \lim_{t\to\infty} |\hat{\tilde{\bar{\mu}}}_\alpha(A) - S(t)\hat{\tilde{\bar{\mu}}}_\alpha(A)| \\ &\quad \leqslant 3P^A[A_s^* \neq A_s \text{ for some } s \geqslant 0]. \end{aligned}$$

Suppose now that $\lim_{t\to\infty} v_{\alpha_1} S(t) = \lim_{t\to\infty} v_{\alpha_2} S(t)$. We have

$$\begin{aligned} \hat{v}_{\alpha_i}(\{X(s)\}) &= \lim_{t\to\infty} E^{X(s)} \hat{v}_{\alpha_i}(\{X^*(t)\}) \\ &= P^{X(s)}[X^*(t) = \emptyset \text{ eventually}] \\ &\quad + E^{X(s)}\left(\lim_{t\to\infty} \hat{v}_{\alpha_i}(\{X^*(t)\}) 1_{\{X^*(t)\neq\emptyset\,\forall\, t, D\}}\right). \end{aligned}$$

But since $P^{X(s)}(\{X^*(t) \neq \emptyset \,\forall\, t, D\}) \to 1$ on Λ_x by the arguments given for Lemma 3.4 and since $E^{X(s)}(\lim_{t\to\infty} \hat{v}_{\alpha_i}(\{X(t)\})) = \alpha_i(X(s))$, then it follows that $\alpha_1 R \alpha_2$.

For the opposite direction if we assume that $\alpha_1 R \alpha_2$, then

$$\begin{aligned}\lim_{t\to\infty}(v_{\alpha_1}S(t)-v_{\alpha_2}S(t))(A) &= \lim_{t\to\infty} E^A(\hat{v}_{\alpha_1}(A_t^*)) - \lim_{t\to\infty} E^A(\hat{v}_{\alpha_2}(A_t^*)) \\ &= \lim_{t\to\infty} E^A\left(\prod_{x\in A_t^*}\alpha_1(x)\right) \\ &\quad - \lim_{t\to\infty} E^A\left(\prod_{x\in A_t^*}\alpha_2(x)\right) = 0.\end{aligned}$$

For part (b) it is enough to show that the extreme points of $\mathscr{I}'$ are $\{\bar{\mu}_\alpha \in \mathscr{I} : \alpha \in \mathscr{H}^* \cap \mathscr{H}'\}$. Then applying (15) along with Lemma 3.5 (a) completes the proof. To prove $\mathscr{I}'_{\mathrm{e}} = \{\bar{\mu}_\alpha \in \mathscr{I} : \alpha \in \mathscr{H}^* \cap \mathscr{H}'\}$ note that if $\lambda\pi_1 + (1-\lambda)\pi_2 = \mu \in \mathscr{I}'_{\mathrm{e}}$ for $\pi_1, \pi_2 \in \mathscr{I}$ then $\pi_1, \pi_2 \in \mathscr{I}'$ and hence $\pi_1 = \pi_2 = \mu$. Therefore $\mu \in \mathscr{I}_{\mathrm{e}} \cap \mathscr{I}' = \{\bar{\mu}_\alpha : \alpha \in \mathscr{H}^* \cap \mathscr{H}'\}$. On the other hand if $\alpha \in \mathscr{H}^*$ and $\bar{\mu}_\alpha \in \mathscr{I}'$, then $\bar{\mu}_\alpha$ is an extreme point of $\mathscr{I}'$. $\square$

4. Proofs of the theorems

Proof of Theorem 1.1. Suppose condition (1) holds. By Lemma 2.1 we need only show that for any two measures $\mu_1, \mu_2 \in \mathscr{P}$, the limits $\lim_{t\to\infty} S(t)\hat{\mu}_i(A)$ exist and are equal for all $A \in Y$. But Lemma 3.2 implies that

$$\lim_{t\to\infty} S(t)\hat{\mu}_i(A) = P^A[A_t^* = \emptyset \text{ eventually}]$$

which is independent of μ_i proving one direction of the theorem.

For the opposite direction suppose that (1) does not hold. Lemma 3.2 implies that $P^A[A_t^* = \emptyset \text{ or } A_t^* = \Delta \text{ eventually}] < 1$ for some $A \in Y$. Therefore

$$\lim_{t\to\infty} S(t)\hat{\delta}_1(A) = P^A[A_t^* = \emptyset \text{ eventually}] + P^A[A_t^* \neq \emptyset\, \forall\, t, D]$$

is not equal to

$$\lim_{t\to\infty} S(t)\hat{\delta}_0(A) = P^A[A_t^* = \emptyset \text{ eventually}]$$

for some $A \in Y$ showing that the process is not ergodic. $\square$

Proof of Theorem 1.2. By Lemma 2.1, $\lim_{t\to\infty} \delta_1 S(t)$ exists since

$$\lim_{t\to\infty} S(t)\hat{\delta}_1(A) = 1 - P^A[D^c].$$

Similarly, $\lim_{t\to\infty} \delta_0 S(t)$ exists since

$$\lim_{t\to\infty} S(t)\hat{\delta}_0(A) = P^A[A_t = \emptyset \text{ eventually}]$$

completing the proof of part (a).

Consider now part (b). By Lemma 3.3 and a coupling argument it can be seen that if $\lim_{t\to\infty} E^x \hat{\mu}(\{X(t)\})$ exists, it is independent of our choice of x. So now using Lemmas 2.1 and 3.3 together with the strong Markov property, we have that if the following limits then

$$\begin{aligned}\lim_{t\to\infty} S(t)\hat{\mu}(A) &= \lim_{t\to\infty} E^A \hat{\mu}(A_t^*)\\ &= P^A[A_t^* = \emptyset \text{ eventually}] + P^A\left[\lim_{t\to\infty} |A_t^*| = 1, D\right]\\ &\quad \times \lim_{t\to\infty} E^x \hat{\mu}(\{X(t)\}). \end{aligned} \tag{16}$$

If $\mu \in \mathscr{I}$ then the limits on the left exist. Since we have assumed that (1) does not hold then $P^A[\lim_{t\to\infty} |A_t^*| = 1, D] > 0$ so that the last limit on the right-hand side exists.

Let $\lambda = \lim_{t\to\infty} E^x \hat{\mu}(\{X(t)\})$ and consider the invariant measure $\mu^\lambda = \lambda\mu^1 + (1-\lambda)\mu^0$. We have that for all $A \in Y$,

$$\hat{\mu}^\lambda(A) = E^A \hat{\mu}^\lambda(A_t^*) = P^A[A_t^* = \emptyset \text{ eventually}] + \lambda P^A\left[\lim_{t\to\infty} |A_t^*| = 1, D\right].$$

Since $P^A[A_t^* = \emptyset$ eventually$]$ and $P^A[\lim_{t\to\infty} |A_t^*| = 1, D]$ do not depend on μ or μ^λ, comparing the above equation with (16) gives us $\mu = \mu^\lambda$ showing that every invariant measure is a mixture of μ^1 and μ^0. This proves part (b).

Part (c) follows from the above arguments together with the fact that $\lim_{t\to\infty} E^x \hat{\mu}(\{X(t)\})$ is independent of our choice of x. $\square$

Proof of Corollary 1.3. We need only show that the recurrence of $Z(t) = X(t) - Y(t)$ implies (2).

Let R be the set of all $y \in \mathbb{Z}^d$ such that $|y| \leqslant N$. By our assumptions we can choose $z \in \mathscr{S}$ so that $q_v(0,z) > 0$. If $E_t = \{x_t, y_t\}$ is the two particle exclusion process then we will say that $E_t = z$ if $x_t - y_t = z$ and $E_t \in R$ if $|x_t - y_t| \leqslant N$.

Since $Z(t)$ is recurrent, $Z(t)$ jumps to 0 infinitely often and therefore $X(t)$ and $Y(t)$ meet infinitely often. If there are infinitely many jumps of $Z(t)$ to 0 caused by the $q_v(x,y)$ rates then (2) automatically holds by arguments similar to those of Lemma 3.1. Thus we will henceforth assume that there are infinitely many jumps of $Z(t)$ to 0 caused by the $q_e(x,y)$ rates giving us

$$P^{\{x,y\}}(Z(t) \in R \text{ for some } t > 0) = 1 \quad \text{for all } x, y \in \mathscr{S}.$$

By coupling $Z(t)$ and E_t together until the first time that $X(t)$ and $Y(t)$ meet, we have in fact that

$$P^{\{x,y\}}(E_t \in R \text{ for some } t > 0) = 1 \quad \text{for all } \{x,y\} \in \mathscr{S}_2.$$

If E_k is the embedded discrete-time Markov Chain for E_t then the above equation implies that

$$P^{\{x,y\}}(E_k \in R \text{ infinitely often}) = 1 \quad \text{for all } \{x,y\} \in \mathscr{S}_2.$$

For a fixed $\bar{t}>0$ let $m = \min_{x\in R}\{P^x(E_{\bar{t}} = z)\}$. Since our process is irreducible $m>0$ therefore

$$P^{\{x,y\}}(E_k = z \text{ infinitely often}) = 1 \quad \text{for all } \{x,y\} \in \mathscr{S}_2. \tag{17}$$

Now by the same argument given to show (12) in the proof of Lemma 3.7,

$$\left\{\omega \middle| \int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right\} = \left\{\omega \middle| \sum_{k=0}^\infty q_v(E_k) = \infty\right\}$$

almost surely. By (17) we get that (2) holds as desired. □

Proof of Theorem 1.4. For part (a) we prove only the case where $\beta \equiv 0$; the general case follows the proof of Theorem 3.9. Again let $T(t)$ be the semigroup for the voter model with rates $q(x,y)$. Chapter V in [12] tells us $\lim_{t\to\infty} v_\alpha T(t) = \bar{\mu}_\alpha$ exists for all $\alpha \in \mathscr{H}$. By coupling the dual of our process together with the dual of the voter model so that they move together as much as possible, it is clear that $S(t)\hat{\bar{\mu}}_\alpha(A) \leqslant T(t)\hat{\bar{\mu}}_\alpha(A) = \hat{\bar{\mu}}_\alpha(A)$. Applying $S(s)$ to both sides gives $S(t+s)\hat{\bar{\mu}}_\alpha(A) \leqslant S(s)\hat{\bar{\mu}}_\alpha(A)$. This monotonicity shows that $\lim_{t\to\infty} \bar{\mu}_\alpha S(t)$exists. By previous arguments we also have that

$$\lim_{t\to\infty} v_\alpha S(t) = \lim_{t\to\infty} \lim_{s\to\infty} v_\alpha T(s) S(t).$$

Concerning the rest of the proof we will only prove part (b) since the proof of part (c) is basically the same as that of (b) except for replacing the use of Lemma 3.8 with Lemma 3.7. Just as in the proof of Theorem 3.9 the general idea is to show that there exists a bijective, affine map σ between $\mathscr{I}$ and $\mathscr{J}$ where $\mathscr{J}_{\mathrm{e}} = \{\lim_{t\to\infty} v_\alpha T(t) : \alpha \in \mathscr{H}^*\}$.

For part (b) we will prove only the case where $\beta(x) = \delta(x) \equiv 0$ so that $A_t^* = A_t$. The general result follows from the arguments laid out in the proof of Theorem 3.9 except for a slight change in the independent processes $X_1(t), \ldots, X_n(t)$ starting from $A = \{x_1, \ldots, x_n\}$. For the proof here we must use the coupling of the $X_i(t)$ processes that we used in the proof of Lemma 3.8 instead of letting them be independent. We now prove the case $\beta(x) = \delta(x) \equiv 0$.

Take $\mu \in \mathscr{I}$ and suppose that both A_t and V_t start with initial set A. By coupling the two processes so that A_t contains V_t, we see that

$$|S(t)\hat{\mu}(A) - T(t)\hat{\mu}(A)| \leqslant f(A) = P^A[A_t \neq V_t \text{ for some } t>0].$$

By the invariance of μ

$$|\hat{\mu}(A) - T(t)\hat{\mu}(A)| \leqslant f(A) \tag{18}$$

so that

$$|T(s)\hat{\mu}(A) - T(t+s)\hat{\mu}(A)| \leqslant T(s)f(A).$$

By Lemma 3.8 and the fact that $S(s)f(A) \to 0$ implies that $T(s)f(A) \to 0$, the right-hand side goes to 0. This in turn shows that $\lim_{t\to\infty} T(t)\hat{\mu}(A)$ exists. The duality of the voter model which is a special case of Lemma 2.1, implies that $\lim_{t\to\infty} \mu T(t) = v$ exists and is invariant for the voter model with rates $q(x,y)$.

By passing to the limit in (18)

$$|\hat{\mu}(A) - \hat{v}(A)| \leqslant f(A).$$

Hence Lemma 3.8 tells us $\lim_{t\to\infty} vS(t) = \mu$.

For $\mu \in \mathscr{I}$, if we define $\sigma(\mu) = \lim_{t\to\infty} \mu T(t) = v$, then the above arguments have shown that σ is injective. A similar arguments proves σ maps onto $\mathscr{J}$. To see that it is affine note that

$$\lim_{t\to\infty} (\lambda\mu_1 + (1-\lambda)\mu_2)T(t) = \lambda v_1 + (1-\lambda)v_2.$$

We now conclude the proof of the case $\beta(x) = \delta(x) \equiv 0$ by showing that for $\bar{\mu}_\alpha = \lim_{t\to\infty} v_\alpha T(t)$,

$$\lim_{t\to\infty} v_\alpha S(t) = \lim_{t\to\infty} \bar{\mu}_\alpha S(t).$$

Applying $S(s)$ to the following inequality and passing to the limit proves the above equation.

$$\begin{aligned}&\lim_{t\to\infty} |S(t)\hat{v}_\alpha(A) - S(t)\hat{\bar{\mu}}_\alpha(A)| \\ &\quad \leqslant \lim_{t\to\infty} |S(t)\hat{v}_\alpha(A) - T(t)\hat{v}_\alpha(A)| + \lim_{t\to\infty} |\hat{\bar{\mu}}_\alpha(A) - S(t)\hat{\bar{\mu}}_\alpha(A)| \leqslant 2f(A). \qquad \square\end{aligned}$$

Proof of Theorem 1.5. Putting $A = \{x_1, \ldots, x_n\}$ let $W_n(t)\hat{\mu}(A) = E^A\hat{\mu}(\{X_1(t), \ldots, X_n(t)\})$ be the semigroup for n independent processes. Then the assumptions of the theorem tell us $W_2(t)g(x,y) \to 0$ so that

$$P^{\{x,y\}}[X(t) = Y(t) \text{ infinitely often}] = 0. \tag{19}$$

The proof that (8) is necessary and sufficient for $\lim_{t\to} \mu S(t) = \mu_\alpha$ is proven in Theorem 8.7 in [15]. The only thing to note is that the assumption that $X(t)$ is transient is needed only to show that when $q(x,y) = q(y,x)$, (19) holds.

The rest of the proof is similar to the proof of Theorems V.1.9 in [12]. Assume that μ satisfies (6) and (7). By Lemma 2.1 and the definition of μ_α, it suffices to show that for each $A \in Y$,

$$\lim_{t\to\infty} E^A\hat{\mu}(A_t^*) = \lim_{t\to\infty} E^A \prod_{x\in A_t^*} \alpha(x), \tag{20}$$

where we make the convention that $\alpha(\Delta) = 0$ and $\prod_{x\in\emptyset} \alpha(x) = 1$.

Conditions (6) and (7) are equivalent to the assertion that for each $x \in \mathscr{S}$

$$\sum_y p_t(x,y)\eta(y)$$

converges in probability to $\alpha(x)$ with respect to μ. This in turn is equivalent to

$$\lim_{t\to\infty} E^{\{x_1,\ldots,x_n\}}\hat{\mu}(\{X_1(t), \ldots, X_n(t)\}) = \prod_{i=1}^n \alpha(x_i), \tag{21}$$

where the $X_i(t)$ are all independent.

Let τ_1 be the first time that either $X_i(t) = X_j(t)$ for some $1 \leqslant i < j \leqslant n$, $A_t^* = \Delta$, or $|A_t^*|$ decreases. Still putting $A = \{x_1, \ldots, x_n\}$, let τ_2 be the first time starting from $A_{\tau_1}^*$

that any of the three events described above occur unless $A^*_\tau = \Delta$ in which case we will let $\tau_2 = \infty$. Continuing in this way we can define τ_k for all $k \geqslant 1$.

By (21) and the strong Markov property, if the limits below exist then

$$\begin{aligned}\lim_{t\to\infty} E^A[\hat{\mu}(A^*_t), \tau_1 = \infty] &= \lim_{t\to\infty} E^A[\hat{\mu}(\{X_1(t), \dots, X_n(t)\}), \tau_1 = \infty] \\ &= \prod_{i=1}^n \alpha(x_i) - \lim_{t\to\infty} E^A[\hat{\mu}(\{X_1(t), \dots, X_n(t)\}), \tau_1 < \infty] \\ &= \prod_{i=1}^n \alpha(x_i) - E^A \lim_{t\to\infty} E^{(X_1(\tau_1),\dots,X_n(\tau_1))} \\ &\quad \times [\hat{\mu}(\{X_1(t), \dots, X_n(t)\}), \tau_1 < \infty] \\ &= \lim_{t\to\infty} E^A\left[\prod_{x\in A^*_t} \alpha(x), \tau_1 = \infty\right] \\ &= \lim_{t\to\infty} E^A\left[\prod_{x\in A_t} \alpha(x), \tau_1 = \infty\right]. \end{aligned} \tag{22}$$

By the convergence theorem for bounded submartingales $\lim_{t\to\infty} \prod_{x\in A_t} \alpha(x)$ exists almost surely so by the dominated convergence theorem the above limits exist.

Using the strong Markov property once more we can get

$$\begin{aligned}\lim_{t\to\infty} E^A[\hat{\mu}(A^*_t), \tau_1 < \infty] &= E^A \lim_{t\to\infty} (E^{A^*_{\tau_1}}[\hat{\mu}(A^*_t), \tau_1 < \infty, \tau_2 = \infty]) \\ &\quad + E^A \lim_{t\to\infty} (E^{A^*_{\tau_1}}[\hat{\mu}(A^*_t), \tau_2 < \infty]). \end{aligned} \tag{23}$$

But by the argument given for (22) the first term on the right-hand side above equals

$$\begin{aligned}&E^A \lim_{t\to\infty} E^{A^*_{\tau_1}}\left[\prod_{x\in A^*_t} \alpha(x), \tau_1 < \infty, \tau_2 = \infty\right] \\ &\quad = \lim_{t\to\infty} E^A\left[\prod_{x\in A_t} \alpha(x), \tau_1 < \infty, \tau_2 = \infty\right] P[A^*_{\tau_1} \neq \emptyset \neq \Delta] + P[A^*_{\tau_1} = \emptyset].\end{aligned}$$

The second term on the right-hand side of (23) equals

$$E^A \lim_{t\to\infty} (E^{A^*_{\tau_2}}[\hat{\mu}(A^*_t), \tau_2 < \infty, \tau_3 = \infty]) + E^A \lim_{t\to\infty} (E^{A^*_{\tau_2}}[\hat{\mu}(A^*_t), \tau_3 < \infty]).$$

Since (19) holds we have that $P[\tau_k = \infty \text{ for some } k] = 1$. By repeated use of the arguments above it follows that (20) holds. $\square$

Proof of Corollary 1.6. Again, we prove only the case $\beta(x) = \delta(x) \equiv 0$ so that $A^*_t = A_t$. The general case follows from above arguments.

Take $\mu \in \mathscr{I}$ and again let $W_n(t)$ be the semigroup for n independent random walks $\vec{X}(t) = (X_1(t), \dots, X_n(t))$. Couple A_t and $\vec{X}(t)$ so that they move together until the first time that two coordinates of $\vec{X}(t)$ meet. We then have that

$$|S(t)\hat{\mu}(A) - W_n(t)\hat{\mu}(A)| \leqslant g(A). \tag{24}$$

Since $S(t)\hat{\mu}(A) = \hat{\mu}(A)$ then

$$|W_n(s)\hat{\mu}(A) - W_n(t+s)\hat{\mu}(A)| \leqslant W_n(s)g(A).$$

Corollary II.7.3 in [12] tells us that $\vec{X}(t)$ has no nonconstant bounded harmonic functions. By Proposition 5.19 in [8] $W_n(t)g(A) \to 0$ so that $\lim_{t\to\infty} W_n(t)\hat{\mu}(A)$ exists and is harmonic for the random walk $\vec{X}(t)$ on $\mathscr{S}^n$. Such harmonic functions are constant so we can write

$$\lim_{t\to\infty} W_n(t)\hat{\mu}(A) = \alpha_n \quad \text{for } |A| = n.$$

The proof of Theorem 2.6 in [14] shows that there exists a random variable G taking values in $[0, 1]$ with moment sequence α_n. Since $\alpha_n \leqslant 1$ we can use Carleman's condition to show that the random variables $\sum_y p_t(x, y)\eta(y)$ with respect to the measure μ converge in distribution to G.

If γ is the probability measure on $[0, 1]$ for G, let $\mu_\gamma = \int_0^1 \mu_\alpha \gamma(\mathrm{d}\alpha)$. Using the arguments presented in Theorem 1.5 we can show that for each $A \in Y$,

$$\lim_{t\to\infty} E^A \hat{\mu}(A_t) = \lim_{t\to\infty} EG^{|A_t|} = \lim_{t\to\infty} E^A \hat{\mu}_\gamma(A_t).$$

Thus $\mu = \mu_\gamma$ and is hence a mixture of the measures $\{\mu_\alpha : \alpha \in [0, 1]\}$. By Theorem 1.5, each measure μ_α has a different domain of attraction proving that $\mathscr{I}_\mathrm{e} = \{\mu_\alpha : \alpha \in [0, 1]\}$. $\square$

5. Further results

The brief discussion below shows how one might adapt [15] and Chapter V in [12] in order to obtain a general result. Let

$$\hat{\mathscr{E}} = \left\{\omega : \int_0^\infty q_v(E_t)\,\mathrm{d}t = \infty\right\}.$$

In the introduction we argued that $\lim_{t\to\infty} \alpha(X(t))$ exists almost surely so we can define $\hat{\mathscr{H}}$ to be the set of all $\alpha \in \mathscr{H}$ such that

$$\lim_{t\to\infty} \alpha(X(t)) = 0 \text{ or } 1 \quad \text{a.s. on } \hat{\mathscr{E}},$$

where $X(t)$ starts from x if $E_0 = \{x, y\}$. For those that are keeping track, $\hat{\mathscr{E}}$ and $\hat{\mathscr{H}}$ are analogous to $\mathscr{E}$ and $\mathscr{H}^*$.

Following [15] and Chapter V in [12], we conjecture that $\mathscr{I}_\mathrm{e} = \{\mu_\alpha : \alpha \in \hat{\mathscr{H}}_R\}$. In order to prove this one would have to generalize Theorem 1.5 and show that for $\mu \in \mathscr{I}_\mathrm{e}$

$$\hat{\mu}(\{X(t), Y(t)\}) \to \hat{\mu}(\{X(t)\})\hat{\mu}(\{Y(t)\}). \tag{25}$$

As mentioned in the introduction, it is the monotonicity of $S(t)\hat{v}_\alpha(A)$ that allows us to do this for the pure voter model or the pure symmetric exclusion process.

If one were to prove (25) and Theorem 1.5 in general, new techniques would be needed.

6. An ergodic theorem for a related process

The proof of Theorem 1.7 requires the following lemma:

Lemma 6.1. *Suppose $\{a_n\}$ is bounded above by $k_1 n^{k_2-1}$ for some $k_1, k_2 > 0$ and that $a_n > 0$ for all n. Then there exists a sequence $\{w_n\}$ such that*

$$\text{(i)} \liminf_{n\to\infty} \frac{a_n}{w_n} = 1 \text{ and (ii) } \limsup_{n\to\infty} \frac{n w_n}{\sum_{l=0}^{n-1} w_l} < \infty.$$

Proof. If for some sequence $\{w_n\}$ we have that $w_l/l^{k_2} \geqslant w_n/n^{k_2}$ for $l \leqslant n$ then

$$\sum_{l=0}^{n-1} w_l \geqslant \sum_{l=0}^{n-1} \frac{w_n}{n^{k_2}} l^{k_2} \geqslant \frac{w_n}{n^{k_2}} \frac{(n-1)^{k_2+1}}{k_2+1}$$

so that condition (ii) holds. So it remains to find a sequence $\{w_n\}$ satisfying condition (i) and the inequality $w_l/l^{k_2} \geqslant w_n/n^{k_2}$ for $l \leqslant n$. Let $w_0 = a_0$ and let $w_n = w_{n-1}$ unless $a_n/n^{k_2} = \min_{l \leqslant n} a_l/l^{k_2}$ in which case we let $w_n = a_n$. Then $w_l/l^{k_2} \geqslant w_n/n^{k_2}$ for $l \leqslant n$. Now since $\{a_n\}$ is bounded above by $k_1 n^{k_2-1}$ it follows that $a_n/n^{k_2} \to 0$ and hence $a_n/n^{k_2} = \min_{l \leqslant n} a_l/l^{k_2}$ infinitely often so that $w_n = a_n$ infinitely often. Therefore (i) is also satisfied by this choice of $\{w_n\}$. $\square$

Using the basic coupling for the exclusion process combined with the basic coupling for spin systems, we have that the basic coupling for a noisy exclusion process has generator

$$\begin{aligned}
\bar{\Omega} f(\eta,\xi) = & \sum_{\substack{\eta(x)=\xi(x)=1 \\ \eta(y)=\xi(y)=0}} q_{\mathrm{e}}(x,y)[f(\eta_{xy},\xi_{xy}) - f(\eta,\xi)] \\
& + \sum_{\substack{\eta(x)=1,\eta(y)=0 \text{ and} \\ \xi(y)=1 \text{ or } \xi(x)=0}} q_{\mathrm{e}}(x,y)[f(\eta_{xy},\xi) - f(\eta,\xi)] \\
& + \sum_{\substack{\xi(x)=1,\xi(y)=0 \text{ and} \\ \eta(y)=1 \text{ or } \eta(x)=0}} q_{\mathrm{e}}(x,y)[f(\eta,\xi_{xy}) - f(\eta,\xi)] \\
& + \sum_{x:\eta(x)\neq\xi(x)} c_1(x,\eta)[f(\eta_x,\xi) - f(\eta,\xi)] \\
& + \sum_{x:\eta(x)\neq\xi(x)} c_2(x,\xi)[f(\eta,\xi_x) - f(\eta,\xi)] \\
& + \sum_{x:\eta(x)=\xi(x)} c(x,\eta,\xi)[f(\eta_x,\xi_x) - f(\eta,\xi)],
\end{aligned}$$

where

$$c_1(x,\eta)=\begin{cases}\beta(x) & \text{when } \eta(x)=0,\\ \delta(x) & \text{when } \eta(x)=1.\end{cases}\quad c_2(x,\xi)=\begin{cases}\beta(x) & \text{when } \xi(x)=0,\\ \delta(x) & \text{when } \xi(x)=1.\end{cases}$$

$$\text{and}\quad c(x,\eta,\xi)=\begin{cases}\beta(x) & \text{when } \eta(x)=\xi(x)=0,\\ \delta(x) & \text{when } \eta(x)=\xi(x)=1.\end{cases}$$

Let $\bar{\mathscr{I}}$ be the set of invariant measures for this coupling.

In order to simplify the notation we define the functions

$$f_x(\eta,\xi)=[1-\eta(x)]\xi(x),\quad h_{yx}(\eta,\xi)=[1-\eta(y)][1-\xi(y)]f_x(\eta,\xi),$$
$$g_{yx}(\eta,\xi)=\eta(y)\xi(y)f_x(\eta,\xi),\quad \text{and}\quad f_{yx}(\eta,\xi)=\eta(y)[1-\xi(y)]f_x(\eta,\xi).$$

In particular, for T a finite subset of $\mathscr{S}$ we have

$$\begin{aligned}\tilde{\Omega}\left(\sum_{x\in T}f_x(\eta,\xi)\right) = &-\sum_{x\in T,y\in\mathscr{S}}(q_{\mathrm{e}}(x,y)+q_{\mathrm{e}}(y,x))f_{yx}(\eta,\xi)\\ &-\sum_{x\in T}(\beta(x)+\delta(x))f_x(\eta,\xi)\\ &+\sum_{x\in T,y\notin T}[q_{\mathrm{e}}(x,y)g_{xy}-q_{\mathrm{e}}(y,x)g_{yx}]\\ &+\sum_{x\in T,y\notin T}[q_{\mathrm{e}}(y,x)h_{xy}-q_{\mathrm{e}}(x,y)h_{yx}].\end{aligned}\tag{26}$$

Proof. Proof of Theorem 1.7 recall that $T_n=\{x\in\mathbb{Z}^d:|x_i|\leqslant n\}$. Couple two noisy exclusion processes, η_t and ξ_t, with $v\in\bar{\mathscr{I}}$ so that

$$\int\bar{\Omega}\left(\sum_{x\in T_n}f_x(\eta_t,\xi_t)\right)\mathrm{d}v=0.$$

If we let $\int f_x(\eta_t,\xi_t)dv=a(x)$ then since $f_{yx}(\eta_t,\xi_t)\geqslant 0$, Eq. (26) gives us

$$\begin{aligned}&\sum_{x\in T_n}(\beta(x)+\delta(x))a(x)\\ &\quad\leqslant\sum_{x\in T_n,y\notin T_n}q_{\mathrm{e}}(x,y)\int(g_{xy}-h_{yx})\,\mathrm{d}v+\sum_{x\in T_n,y\notin T_n}q_{\mathrm{e}}(y,x)\int(h_{xy}-g_{yx})\,\mathrm{d}v\\ &\quad\leqslant\sum_{x\in T_n,y\notin T_n}q_{\mathrm{e}}(x,y)a(y)+\sum_{x\in T_n,y\notin T_n}q_{\mathrm{e}}(y,x)a(y)\\ &\quad\leqslant\frac{(2N+1)^d}{2}\sum_{y\in T_n^N}a(y)+\sum_{y\in T_n^N}a(y)\leqslant C_1\sum_{y\in T_n^N}a(y)\end{aligned}\tag{27}$$

for some constant C_1. If we define

$$a_l = \sum_{y \in T^N_{p(l)}} a(y)$$

then by the inequality $\beta(x) + \delta(x) \geqslant b_l$ for $x \in T^N_{p(l)}$ we can rewrite (27) as

$$\sum_{l=0}^{n-1} b_l a_l \leqslant C_1 a_n. \tag{28}$$

Now suppose $d = 1$ and condition (a) in the theorem holds. Then since $a(x) \leqslant 1$, we have $a_n \leqslant 2N$ for all n. In light of Eq. (28) we then have that $\sum_{l \geqslant 0} b_l a_l < \infty$. On the other hand, if we multiply both sides of (28) by b_n and then sum over n we get

$$\sum_{n \geqslant 0} b_n \sum_{l=0}^{n} b_l a_l \leqslant C_1 \sum_{n \geqslant 0} b_n a_n < \infty.$$

Rewriting the left hand side we get

$$\sum_{n \geqslant 0} b_n \sum_{l=0}^{n} b_l a_l = \sum_{l \geqslant 0} b_l a_l \sum_{n \geqslant l} b_n < \infty.$$

This implies that $b_l a_l = 0$ for all l since condition (a) gives us $\sum b_l = \infty$. So we have $a(x) = \int f_x \,\mathrm{d}v = 0$ for all x so that the marginals of v are exactly the same.

Suppose now that $d \geqslant 2$ and that condition (b) of the theorem holds. Since $p(l) \leqslant k l^k$ we have that a_n is bounded above by $k_1 n^{k_2 - 1}$ for some k_1, k_2. If we assume that for all n, $a_n > 0$ then by Lemma 6.1, there exists a sequence w_n such that $\liminf a_n / w_n = 1$ and $\limsup n w_n / \sum_{l=0}^{n-1} w_l < \infty$. By condition (b), we have then that

$$\liminf n b_n a_n / w_n = \infty \tag{29}$$

However, we also have that there exists a subsequence $\{n_j\}$ for which

$$\sum_{l=0}^{n_j - 1} b_l a_l \leqslant C_1 a_{n_j} \leqslant C_2 w_{n_j} \leqslant C_3 \frac{\sum_{l=1}^{n_j - 1} w_l}{n_j} \leqslant C_3 \sum_{l=1}^{n_j - 1} \frac{w_l}{l}. \tag{30}$$

Notice now that if the limit of the right hand side is infinite, (29) and (30) contradict each other so that we must have $a_n = 0$ for some n and consequently $a(x) = \int f_x \,\mathrm{d}v = 0$ for all x by irreducibility. If the right hand side is bounded then we can use the argument given above for the case $d = 1$ to show that $a(x) = \int f_x \,\mathrm{d}v = 0$ for all x. In either case we have that the marginals of v are the same, and we thus have ergodicity of the process. □

We now restrict ourselves to the case where $d = 1$ and the transition rates are $q_{\mathrm{e}}(x, x+1) = p > 1/2$ and $q_{\mathrm{e}}(x, x-1) = 1 - p = q < 1/2$ for all x. In order to show the importance of the condition that there exist a sequence b_l satisfying $b_l \leqslant \beta(x) + \delta(x)$ for all $x \in T^N_{p(l)}$, we will find examples of processes on $\mathbb{Z}$ that are not ergodic but satisfy $\sum_x \beta(x) = \infty$.

To start off, consider the case where we have $\beta > 0$ and $\delta > 0$ for a single fixed z and no births and deaths at any other site. Choose c so that $c\pi(z)/(1 + c\pi(z)) = \beta/(\beta + \delta)$ for a reversible measure $\pi(x)$ on $\mathbb{Z}$. The product measure v^c with marginals $v^c\{\eta : \eta(x) = 1\} = c\pi(z)/(1 + c\pi(z))$ is reversible with respect to the exclusion process, and its marginal measure at the site z is reversible with respect to the birth and death process so that v^c is reversible with respect to the noisy exclusion process. The product measure v_ρ with marginals $v_\rho\{\eta : \eta(x) = 1\} = \rho$ where $\rho = \frac{\beta}{\beta+\delta}$, is also invariant with respect to the exclusion process, and again, its marginal measure at the site z is reversible with respect to the birth and death process. So v_ρ is also invariant with respect to the noisy exclusion process.

We have two more invariant measures by starting the process off with initial states δ_0 and δ_1. This is because some subsequence of $\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \delta_1 S(t)\,\mathrm{d}t$ for $T_n \to \infty$ must lie above both of the invariant measures we have constructed above. Similarly some subsequence of $\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \delta_0 S(t)\,\mathrm{d}t$ lies below the two invariant measures.

In order to show that the noisy exclusion process with $\beta(z_i) > 0$ if and only if $\delta(z_i) > 0$ for a finite number of sites $\{z_1, \ldots, z_k\}$ is not ergodic (this is a special case of Proposition 6.2 below) we will need the following coupling for two noisy exclusion processes with the same transition and death rates, but different birth rates. If $\beta_1(x)$ for the process η_t is greater than $\beta_2(x)$ for the process ξ_t for all x then we can couple the two processes in such a way that $\eta_t \geqslant \xi_t$. Formally, we have the coupling given by

$$\begin{aligned}
\bar{\Omega} f(\eta, \xi) = & \sum_{\substack{\eta(x)=\xi(x)=1 \\ \eta(y)=\xi(y)=0}} q_{\mathrm{e}}(x, y)[f(\eta_{xy}, \xi_{xy}) - f(\eta, \xi)] \\
& + \sum_{\substack{\eta(x)=1, \eta(y)=0 \text{ and} \\ \xi(y)=1 \text{ or } \xi(\mathrm{x})=0}} q_{\mathrm{e}}(x, y)[f(\eta_{xy}, \xi) - f(\eta, \xi)] \\
& + \sum_{\substack{\xi(x)=1, \xi(y)=0 \text{ and} \\ \eta(y)=1 \text{ or } \eta(\mathrm{x})=0}} q_{\mathrm{e}}(x, y)[f(\eta, \xi_{xy}) - f(\eta, \xi)] \\
& + \sum_{x:\eta(x) \neq \xi(x)} c_1(x, \eta)[f(\eta_x, \xi) - f(\eta, \xi)] \\
& + \sum_{x:\eta(x) \neq \xi(x)} c_2(x, \xi)[f(\eta, \xi_x) - f(\eta, \xi)] \\
& + \sum_{x:\eta(x)=\xi(x)} c(x, \eta, \xi)[f(\eta_x, \xi_x) - f(\eta, \xi)] \\
& + \sum_{x:\eta(x)=\xi(x)=0} (\beta_1(x) - \beta_2(x))[f(\eta_x, \xi) - f(\eta, \xi)],
\end{aligned}$$

where

$$c_1(x,\eta)=\begin{cases}\beta_1(x) & \text{when } \eta(x)=0,\\ \delta(x) & \text{when } \eta(x)=1.\end{cases}\quad c_2(x,\xi)=\begin{cases}\beta_2(x) & \text{when } \xi(x)=0,\\ \delta(x) & \text{when } \xi(x)=1.\end{cases}$$

$$\text{and}\quad c(x,\eta,\xi)=\begin{cases}\beta_2(x) & \text{when } \eta(x)=\xi(x)=0,\\ \delta(x) & \text{when } \eta(x)=\xi(x)=1.\end{cases}$$

Similarly, we can couple two processes together so that $\eta_t \leqslant \xi_t$ when η_t and ξ_t have the same transition and birth rates, but death rates such that $\delta_1(x) \geqslant \delta_2(x)$ for all x.

Proposition 6.2. *Suppose that* $q_e(x,x+1)=p>\frac{1}{2}$ *and* $q_e(x,x-1)=1-p=q$ *for all* x *and that* $\beta(x)>0$ *if and only if* $\delta(x)>0$. *If there exists a* z *such that* $\beta(x)=0$ *for either all* $x \leqslant z$ *or for all* $x \geqslant z$ *and if there exist* a_1 *and* a_2 *such that* $\frac{a_1\pi(x)}{1+a_1\pi(x)} \leqslant \frac{\beta(x)}{\beta(x)+\delta(x)} \leqslant \frac{a_2\pi(x)}{1+a_2\pi(x)}$ *for all* x *where* $\beta(x)>0$, *then the process is not ergodic.*

Proof. Without loss of generality suppose that $\beta(x)=0$ for all positive x and let $\{z_i\}$ denote the set of points where $\beta(x)>0$. If η_t is the process described in the hypothesis of the proposition, let the process ξ_t be the same as η_t except that we change the death rates ofξ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}=\frac{a_1\pi(z_i)}{1+a_1\pi(z_i)}$ for all $\{z_i\}$. Let the process ζ_t be the same as η_t except that we change the birth rates of ζ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}=\frac{a_2\pi(z_i)}{1+a_2\pi(z_i)}$ for all $\{z_i\}$. We can triple couple ξ_t,η_t, and ζ_t so that $\xi_t \leqslant \eta_t \leqslant \zeta_t$. Since the measure v^{a_1} is invariant for ξ_t and v^{a_2} is invariant for ζ_t, then η_t has an invariant measure μ_1 with $v^{a_1} \leqslant \mu_1 \leqslant v^{a_2}$.

Let $M=\max_i \frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}$. Note that this maximum is achieved since we assumed earlier that $\beta(x)=0$ for all positive x and consequently if there exist an infinite number of z_i's then $\lim_{i\to\infty}\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}=0$. Now let the process ζ_t be the same as η_t except that we change the birth rates of ζ_t so that $\frac{\beta(z_i)}{\beta(z_i)+\delta(z_i)}=M$ for all $\{z_i\}$. Again, we can couple η_t and ζ_t so that $\eta_t \leqslant \zeta_t$. The measure v_M is invariant for ζ_t. So η_t has an invariant measure μ_2 such that $\mu_2 \leqslant v_M$. Since μ_2 is different from μ_1, the process is not ergodic. $\square$

Note that using the above proposition, we can construct examples of nonergodic processes that satisfy all of the hypotheses for Schwartz's ergodic theorem except for $q_e(x,y)=q_e(y,x)$.

Acknowledgements

The author thanks his advisor, Thomas M. Liggett, for suggesting the problem and for his encouragement during the undertaking.

References

[1] V. Belitsky, P.A. Ferrari, M.V. Menshikov, S.Y. Popov, A mixture of the exclusion process and the voter model, Bernoulli 7 (2001) 119–144.

[2] P. Clifford, A. Sudbury, A model for spatial conflict, Biometrika 60 (1973) 581–588.
[4] E.B. Dynkin, Markov Processes, Springer, New York, 1965.
[5] P.A. Ferrari, Ergodicity for spin systems with stirrings, Ann. Probab. 18 (1990) 1523–1538.
[6] B.L. Granovsky, N. Madras, The noisy voter model, Stochastic Process. Appl. 55 (1995) 23–43.
[7] R. Holley, T.M. Liggett, Ergodic theorems for weakly interacting systems and the voter model, Ann. Probab. 3 (1975) 643–663.
[8] J.G. Kemeny, J.L. Snell, A.W. Knapp, Denumerable Markov Chains, second ed., Springer, New York, 1976.
[9] T.M. Liggett, A characterization of the invariant measures for an infinite particle system with interactions, Trans. Amer. Math. Soc. 165 (1973) 471–481.
[12] T.M. Liggett, Interacting Particle Systems, Springer, New York, 1985.
[14] T.M. Liggett, Interacting Particle Systems-An Introduction. Lectures from the School and Conference on Probability Theory, Trieste, Italy, 2002.
[15] D. Schwartz, Ergodic theorems for an infinite particle system with births and deaths, Ann. Probab. 4 (1976) 783–801.
[16] F. Spitzer, Interaction of Markov processes, Adv. in Math. 5 (1970) 246–290.
[17] F. Spitzer, Recurrent random walk of an infinite particle system, Trans. Amer. Math. Soc. 198 (1974) 191–199.