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On the asymptotic behaviour of Lévy processes, Part I: Subexponential and exponential processes

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Abstract

We study tail probabilities of the suprema of Lévy processes with subexponential or exponential marginal distributions over compact intervals. Several of the processes for which the asymptotics are studied here for the first time have recently become important to model financial time series. Hence our results should be important, for example, in the assessment of financial risk.

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1. Introduction

In the past decade there has been a great interest to use Lévy processes in mathematical finance, see, e.g., Schoutens [34] for a review. Most of the classes of Lévy processes that feature here, such as generalized z processes (GZ), CGMY processes and generalized hyperbolic processes (GH) have univariate marginal distributions with *semi-heavy tails*.

Recall that a probability distribution is said to have a semi-heavy (upper) tail if it has a probability density function (PDF) f such that

$$f(u) \sim Cu^{\rho}\mathrm{e}^{-\alpha u} \quad \text{as } u \to \infty \text{ for some constants } C, \alpha > 0 \text{ and } \rho \in \mathbb{R}. \tag{1.1}$$

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Owing to the exponential in (1.1) Lévy processes with semi-heavy tails could also be called *exponential processes*. This is a custom that we will adopt later.

We study the (upper) tail behaviour of suprema over compact intervals of Lévy processes $\{\xi(t)\}_{t\geq 0}$ with semi-heavy tails and other related behaviours of the tails of their marginal distributions. More specifically, given a constant $h > 0$ we prove that

$$\sup_{t\in[0,h]} \xi(t) \in \mathcal{C} \Leftrightarrow \xi(h) \in \mathcal{C} \tag{1.2}$$

for different classes of distributions (tail behaviours) $\mathcal{C}$ together with the implication

$$\xi(h) \in \mathcal{C} \Rightarrow \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{0\leq t\leq h} \xi(t) > u\right\} = H \quad \text{exists.} \tag{1.3}$$

The classes of distributions $\mathcal{C}$ we are most interested in for (1.2) and (1.3) are the exponential classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$ which are well-known from the literature, see Definition 2.5. In particular, $\mathcal{L}(\alpha)$ includes all distributions with semi-heavy tails.

The implication (1.3) is known from Braverman and Samorodnitsky [14] for $\mathcal{C} = \mathcal{S}(\alpha)$. We prove (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ under an additional technical condition that always seems to be met in practice. In particular, as $\mathcal{L}(\alpha)$ includes $\mathcal{S}(\alpha)$ and distributions in $\mathcal{S}(\alpha)$ satisfy our technical condition, we complete the result of Braverman and Samorodnitsky [14] with the equivalency (1.2) for $\mathcal{C} = \mathcal{S}(\alpha)$.

It turns out that semi-heavy-tailed Lévy processes with $\rho < -1$ belong to $\mathcal{S}(\alpha)$ while processes with $\rho \geq -1$ belong to $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$. Many of the above-mentioned specific Lévy processes have $\rho \geq -1$, so that (1.2) and (1.3) are established here for them for the first time. We also show that $H > 1$ when $\rho < -1$ unless ξ is a subordinator (a result which does not follow from Braverman and Samorodnitsky [14]) and that $H = 1$ for $\rho \geq -1$.

Distributions in the class $\mathcal{L} \equiv \mathcal{L}(0)$ are called *long-tailed*. As these distributions have tails that decay slower than any exponential they can also be called *subexponential*. The relations (1.2) and (1.3) are known from Willekens [39] for $\mathcal{C} = \mathcal{L}$, see Theorem 4.1. We complete his results by establishing a partial converse to (1.3) for $\mathcal{C} = \mathcal{L}$.

As there is no converse to (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ when $\alpha > 0$ we consider the larger $\mathcal{O}$-exponential class $\mathcal{OL}$ of Bengtsson [8] and Shimura and Watanabe [37], see Definition 2.5. For $\mathcal{C} = \mathcal{OL}$ we prove (1.2) together with the following version of (1.3):

$$\xi(h) \in \mathcal{OL} \Rightarrow \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{0\leq t\leq h} \xi(t) > u\right\} < \infty. \tag{1.4}$$

In addition, we establish a partial converse to (1.4).

From a practical point of view the implication (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$ should be the most interesting of our results. For example, an asset price process $\{S(t)\}_{0\leq t\leq h}$ such as a stock price is often modelled by an exponential Lévy model $S(t) = \mathrm{e}^{\xi(t)}$ where ξ is a Lévy process. Then the risk that S falls below a low level ε is given by

$$\mathbf{P}\left\{\inf_{0\leq t\leq h} S(t) < \varepsilon\right\} = \mathbf{P}\left\{\sup_{0\leq t\leq h} -\xi(t) > -\ln(\varepsilon)\right\} \sim H\,\mathbf{P}\{-\xi(h) > \ln(1/\varepsilon)\}$$

as $\varepsilon \downarrow 0$ provided that $-\xi \in \mathcal{C}$ for a class $\mathcal{C}$ such that (1.3) holds.

To establish (1.3) for the class $\mathcal{L}(\alpha)$ we develop Tauberian results for infinitely divisible distributions which should be of substantial interest in their own.

The paper is organized as follows: In Section 2 we review various classes of subexponential and exponential distributions that feature in the paper.

In Section 3 we develop the mentioned Tauberian results. In particular, they show that $\xi(h) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ when ξ has a semi-heavy-tailed Lévy measure such that $\rho \geq -1$ in (1.1). We also express the tail behaviour of $\xi(h)$ in terms of the Lévy characteristic triple of the process. Note that on a less precise level than our Tauberian results, it is well-known that a Lévy measure with an exponential tail more or less corresponds to an infinitely divisible distribution with an exponential tail with the same exponent, see, e.g., Sato [33], Theorem 25.3.

In Section 4 we prove a partial converse to (1.3) for $\mathcal{C} = \mathcal{L}$. This converse is crucial for our proof that $H > 1$ in (1.3) for $\mathcal{C} = \mathcal{S}(\alpha)$.

In Section 5 we prove (1.2) for $\mathcal{C} = \mathcal{OL}$ as well as the implication (1.4) together with a partial converse to that implication. The equivalency (1.2) for $\mathcal{C} = \mathcal{OL}$ is crucial for our proof of (1.2) for $\mathcal{C} = \mathcal{L}(\alpha)$ in Section 6.

In Section 6 we prove (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$. The results from Section 3 are crucial to verify the hypothesis of these results.

In Section 7 we give applications to the process classes GZ, CGMY and GH.

In the companion article Albin and Sundén [2] we study the tail behaviour of *superexponential* Lévy processes with lighter than exponential tails. This rich class of processes includes many processes for which the limit H in (1.3) does not exist.

2. Subexponential and exponential distributions

In this section we review classes of probability distributions that feature in our work.

2.1. Subexponential distributions

Here we discuss distributions with tails that are heavier than exponential ones.

The following classes of distributions $\mathcal{L}$ and $\mathcal{S}$ are well-known from the literature:

Definition 2.1. A cumulative distribution function (CDF) F belongs to the class of *long-tailed* distributions $\mathcal{L}$ if

$$\lim_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} = 1 \quad \text{for } x \in \mathbb{R}.$$

A CDF F belongs to the class of *subexponential* distributions $\mathcal{S}$ if

$$\lim_{u\to\infty} \frac{1-F\star F(u)}{1-F(u)} = 2.$$

In Definition 2.1 $\star$ means convolution, that is, $F \star F(u) = \int_{\mathbb{R}} F(u-x)\mathrm{d}F(x)$.

It turns out that $\mathcal{S} \subsetneq \mathcal{L}$ (see Embrechts and Goldie [16], Section 3). It is easy to see that $F \in \mathcal{S}$ if $1-F$ is regularly varying at infinity with a non-positive index:

Definition 2.2. A measurable function $g > 0$ is *regularly varying* at infinity with *index* $\varrho \in \mathbb{R}$, $g \in \mathcal{R}(\varrho)$, if

$$\lim_{u\to\infty} \frac{g(ux)}{g(u)} = x^{\varrho} \quad \text{for } x \in (0,\infty).$$

A measurable function $g > 0$ is *$\mathcal{O}$-regularly varying* at infinity, $g \in \mathcal{OR}$, if

$$0 < \liminf_{u\to\infty} \frac{g(ux)}{g(u)} \leq \limsup_{u\to\infty} \frac{g(ux)}{g(u)} < \infty \quad \text{for } x \in (0, \infty).$$

Example 2.3. Given constants (parameters) $x_0, \varrho > 0$, the Pareto distribution $F(x) = 1 - (x/x_0)^{-\varrho}$ for $x \geq x_0$ satisfies $1 - F \in \mathcal{R}(-\varrho)$, so that $F \in \mathcal{S} \subseteq \mathcal{L}$.

For the class $\mathcal{L}$ we will need the following lemma, the proof of which is elementary:

Lemma 2.4. *A CDF F belongs to the class $\mathcal{L}$ if and only if*

$$\liminf_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} \geq 1 \quad \textit{for some } x > 0.$$

2.2. Exponential distributions

Here we discuss distributions with exponential tails.

The following classes $\mathcal{L}(\alpha)$ and $\mathcal{S}(\alpha)$ are well-known from the literature. The class $\mathcal{OL}$ of Bengtsson [8] and Shimura and Watanabe [37] is an exponential analogue of $\mathcal{OR}$.

Definition 2.5. Given a constant $\alpha \geq 0$, a CDF F belongs to the class $\mathcal{L}(\alpha)$ if

$$\lim_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} = \mathrm{e}^{-\alpha x} \quad \text{for } x \in \mathbb{R}. \tag{2.1}$$

A CDF F belongs to the *exponential* class $\mathcal{S}(\alpha)$ if $F \in \mathcal{L}(\alpha)$ and

$$\lim_{u\to\infty} \frac{1 - F \star F(u)}{1 - F(u)} \quad \text{exists (and is finite).} \tag{2.2}$$

A CDF F belongs to the class of *$\mathcal{O}$-exponential* distributions $\mathcal{OL}$ if

$$0 < \liminf_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} \leq \limsup_{u\to\infty} \frac{1 - F(u + x)}{1 - F(u)} < \infty \quad \text{for } x \in \mathbb{R}.$$

Pitman [31], p. 338, argued that the class $\mathcal{L}$ should be called subexponential rather than $\mathcal{S}$. By his logic the class $\mathcal{L}(\alpha)$ should be called exponential instead of $\mathcal{S}(\alpha)$. In fact, the exponential distribution itself belongs to $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$ by Example 2.11. This is the reason that we talk about exponential distributions when dealing with $\mathcal{L}(\alpha)$.

Note that $\mathcal{L}(0) = \mathcal{L}, \mathcal{S}(0) = \mathcal{S}$ [as the limit (2.2) is 2 for $\mathcal{S}(0)$, see Embrechts and Goldie [17], Section 2], and $\mathcal{S}(\alpha) \subseteq \mathcal{L}(\alpha) \subseteq \mathcal{OL}$. To illustrate how $\mathcal{OL}$ differs from $\cup_{\alpha\geq 0} \mathcal{L}(\alpha)$ we give the following simple result which is proved in Appendix A.1:

Proposition 2.6. *An absolutely continuous CDF F belongs to $\mathcal{L}(\alpha)$ if and only if*

$$F(u) = 1 - \exp\left\{-\int_{-\infty}^{u} (a(x) + b(x))\mathrm{d}x\right\} \quad \textit{for } u \in \mathbb{R}, \tag{2.3}$$

for some measurable functions a and b with $a + b \geq 0$ such that

$$\lim_{x\to\infty} a(x) = \alpha, \qquad \lim_{u\to\infty} \int_{-\infty}^{u} a(x)\mathrm{d}x = \infty \quad \textit{and} \quad \lim_{u\to\infty} \int_{-\infty}^{u} b(x)\mathrm{d}x \quad \textit{exists.} \tag{2.4}$$

An absolutely continuous CDF F belongs to $\mathcal{OL}$ if and only if (2.3) holds for some measurable functions a and b with $a+b \geq 0$ such that

$$\limsup_{x\to\infty} |a(x)| < \infty, \qquad \liminf_{u\to\infty} \int_{-\infty}^{u} a(x)\mathrm{d}x = \infty \quad \textit{and}$$
$$\limsup_{u\to\infty} \left| \int_{-\infty}^{u} b(x)\mathrm{d}x \right| < \infty. \tag{2.5}$$

Example 2.7. Let $a(x) = \alpha\, \mathbf{1}_{[0,\infty)}(x)$ and $b(x) = \beta \cos(\mathrm{e}^x - 1)\mathbf{1}_{[0,\infty)}(x)$ in (2.3) where $|\beta| \leq \alpha$ and $\alpha > 0$ are constants. Then we have $F \in \mathcal{L}(\alpha)$ since $\lim_{u\to\infty} \int_0^u \cos(\mathrm{e}^x - 1)\mathrm{d}x$ exists. Instead, if we take $b(x) = \beta \cos(x)\, \mathbf{1}_{[0,\infty)}(x)$, then (2.5) holds but (2.4) does not unless $\beta = 0$, so that $F \in \mathcal{OL} \setminus \mathcal{L}(\alpha)$ for $\beta \neq 0$.

The following elementary result for the class $\mathcal{OL}$ corresponds to Lemma 2.4 for $\mathcal{L}$:

Lemma 2.8. *A CDF F belongs to the class $\mathcal{OL}$ if and only if*

$$\liminf_{u\to\infty} \frac{1-F(u+x)}{1-F(u)} > 0 \quad \textit{for some } x > 0.$$

2.3. Distributions with semi-heavy tails

In mathematical finance log increments of asset prices are often modelled to have *semi-heavy tails*, see, e.g., Barndorff-Nielsen [6] and Schoutens [34].

The following simple corollary to Proposition 2.6 is proved in Appendix A.2:

Corollary 2.9. *A CDF F is semi-heavy tailed satisfying (1.1) if and only if*

$$F(u) = 1 - \exp\left\{ -\int_{-\infty}^{u} c(x)\mathrm{d}x \right\} \quad \textit{for } u \in \mathbb{R}, \tag{2.6}$$

for some measurable function $c \geq 0$ that satisfies

$$\lim_{x\to\infty} c(x) = \alpha \quad \textit{and}$$
$$\lim_{u\to\infty} \int_{-\infty}^{u} \left(c(x) - \alpha\, \mathbf{1}_{[0,\infty)}(x) + \frac{\rho}{x}\, \mathbf{1}_{[1,\infty)}(x) \right) \mathrm{d}x = \ln\left(\frac{C}{\alpha}\right). \tag{2.7}$$

If any of these equivalent conditions hold so that both of them hold, then $F \in \mathcal{L}(\alpha)$.

Corollary 2.9 shows which distributions in Example 2.7 are semi-heavy:

Example 2.10. The distributions in Example 2.7 are semi-heavy only if $\beta = 0$ as they have $c(x) = \alpha + \beta \cos(\mathrm{e}^x - 1)$ and $c(x) = \alpha + \beta \cos(x)$ for $x \geq 0$ in (2.6).

By Corollary 2.9 semi-heavy-tailed CDF's belong to $\mathcal{L}(\alpha)$. However, by the following example a semi-heavy-tailed CDF belongs to $\mathcal{S}(\alpha)$ only if $\rho < -1$ in (1.1).

Example 2.11. For a semi-heavy-tailed CDF F with $\rho < -1$, Pakes [29], Corollary 2.1 ii and Lemma 2.3 (see also [30]), show that the limit (2.2) exists with value $2\int_{\mathbb{R}} \mathrm{e}^{\alpha x}\mathrm{d}F(x) < \infty$, so that $F \in \mathcal{S}(\alpha)$. For a semi-heavy-tailed CDF F with $\rho \geq -1$ we have $\int_1^\infty \mathrm{e}^{\alpha x}\mathrm{d}F(x) = \infty$. Hence Pakes [29], p. 411 (see also [30]), shows that the ratio in (2.2) goes to infinity as $u \to \infty$, so that $F \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$.

3. Tauberian theorems for $\mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha)$

In this section we first introduce some standard notation for Lévy processes that will be used repeatedly from here on. Then we state and prove the Tauberian results mentioned in the Introduction that are of fundamental importance in Sections 6 and 7.

An *infinitely divisible* CDF F is characterized by a *characteristic triple* (ν, m, s^2) as

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x} \mathrm{d}F(x) = \exp\left\{\mathrm{i}\theta m + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta \mathbf{1}_{(-1,1)}(x)\, x\right) \mathrm{d}\nu(x) - \frac{\theta^2 s^2}{2}\right\}$$
$$\text{for } \theta \in \mathbb{R}. \tag{3.1}$$

Here ν is the *Lévy* (Borel) *measure* on $\mathbb{R}$ that satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge |x|^2) \mathrm{d}\nu(x) < \infty$ while $m \in \mathbb{R}$ and $s^2 \geq 0$ are constants. This triple is unique.

A *Lévy process* is a continuous in probability and càdlàg stochastic process $\{\xi(t)\}_{t\geq 0}$ with $\xi(0) = 0$ that has stationary and independent increments. As the marginal distribution of $\xi(t)$ is infinitely divisible with characteristic triple $(\nu t, mt, s^2 t)$ where (ν, m, s^2) is the characteristic triple of $\xi(1)$, the latter triple is called the *characteristic triple* of ξ.

The Tauberian results Theorem 3.3, Corollary 3.5 and Theorem 3.6 establish that if ξ is a Lévy process with Lévy measure ν that satisfies the condition (3.3), possibly together with additional conditions, then we have the behaviour (3.4) the tails of the Lévy process. The claim (3.4) is a key condition in Sections 6 and 7.

Our first Tauberian result is derived from Braverman [12], Lemma 5, together with the following two results from the literature that are stated here for easy reference:

Theorem 3.1 (*Embrechts, Goldie and Veraverbeke [18], Sgibnev [36]*). *Given a constant $\alpha \geq 0$ and a Lévy process ξ with Lévy measure ν, we have*

$$\frac{\nu([1,\infty) \cap \cdot)}{\nu([1,\infty))} \in \mathcal{S}(\alpha) \Leftrightarrow \xi(t) \in \mathcal{S}(\alpha) \quad \textit{for some } t > 0 \Leftrightarrow \xi(t) \in \mathcal{S}(\alpha) \textit{ for } t > 0.$$

Moreover, if any of these conditions holds so that all of them hold, then we have

$$\lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\nu((u,\infty))} = t\, \mathbf{E}\{\mathrm{e}^{\alpha\xi(t)}\} \quad \textit{for } t > 0.$$

Theorem 3.2 (*Albin [1], Pakes [29,30]*). *Let $\alpha \geq 0$ be a constant and ξ a Lévy process with characteristic triple (ν, m, s^2). Write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with characteristic triples $(\nu(\cdot \cap [1,\infty)), 0, 0)$ and $(\nu(\cdot \cap (-\infty, 1)), m, s^2)$, respectively. We have*

$$\frac{\nu([1,\infty) \cap \cdot)}{\nu([1,\infty))} \in \mathcal{L}(\alpha) \Rightarrow \xi_1(t) \in \mathcal{L}(\alpha) \quad \textit{for } t > 0 \Rightarrow \xi(t) \in \mathcal{L}(\alpha) \textit{ for } t > 0.$$

Moreover, if $\xi_1(t) \in \mathcal{L}(\alpha)$ for $t > 0$, then we have

$$\mathbf{P}\{\xi(t) > u\} \sim \mathbf{P}\{\xi_1(t) > u\} \int_{\mathbb{R}} \mathrm{e}^{\alpha x} \mathrm{d}F_{\xi_2(t)}(x) \quad \textit{as } u \to \infty \textit{ for } t > 0. \tag{3.2}$$

Here is our first Tauberian result. This result is due to Braverman [13], Theorem 1, but our proof is very much shorter. See also Section 8 on priority:

Theorem 3.3 (*Braverman [13]*). *Let ξ be a Lévy process that satisfies*

$$F(\cdot) \equiv \frac{\nu([1 \vee \cdot, \infty))}{\nu([1,\infty))} \in \mathcal{L}(\alpha) \quad \textit{for some } \alpha > 0 \quad \textit{and} \quad \lim_{u\to\infty} \frac{1 - F(u)}{1 - F \star F(u)} = 0. \quad (3.3)$$

Then it holds that

$$\xi(t) \in \mathcal{L}(\alpha) \setminus \mathcal{S}(\alpha) \quad \textit{for } t > 0 \quad \textit{and} \quad \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(s) > u\}}{\mathbf{P}\{\xi(t) > u\}} = 0 \quad \textit{for } 0 < s < t. \quad (3.4)$$

Proof. Let ξ have triple (ν, m, s^2) and write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triples $(\nu(\cdot \cap [1,\infty)), 0, 0)$ and $(\nu(\cdot \cap (-\infty, 1)), m, s^2)$, respectively. From Theorem 3.2 we have $\xi_1(t), \xi(t) \in \mathcal{L}(\alpha)$. As ξ_1 is a compound Poisson process with jump CDF F, and as the second requirement of (3.3) means that F is light-tailed in the sense of Braverman [12], Definition 1, it follows from Braverman [12], Lemma 5, that ξ_1 satisfies the second requirement of (3.4). As tail probabilities for ξ_1 are proportional to those of ξ by (3.2) we see that the second requirement of (3.4) holds also for ξ. Hence $\xi(t) \notin \mathcal{S}(\alpha)$ by Theorem 3.1, which finishes the proof of all claims of the theorem. □

In Corollary 3.5 and Theorem 3.6 below we specialize Theorem 3.3 to semi-heavy-tailed Lévy measures with $\rho > -1$ and $\rho = -1$, respectively (see Example 2.11). We also express the tail probability $\mathbf{P}\{\xi(t) > u\}$ as $u \to \infty$ in terms of the characteristic triple, which makes possibly more explicit results in our applications in Section 7.

For the statement of Corollary 3.5 and Theorem 3.6, consider the moment generating function (MGF) of a Lévy process ξ given by

$$\phi(t,\lambda) = \mathbf{E}\{\mathrm{e}^{-\lambda\xi(t)}\} = \left(\mathbf{E}\{\mathrm{e}^{-\lambda\xi(1)}\}\right)^t = \phi(1,\lambda)^t \equiv \phi(\lambda)^t \quad \text{for } t > 0 \text{ and } \lambda \in \mathbb{R}. \quad (3.5)$$

Writing (ν, m, s^2) for the characteristic triple of ξ, Sato [33], Theorem 25.17, shows that

$$\begin{aligned} &\phi(t,\lambda) < \infty \quad \text{for some } t > 0 \Leftrightarrow \phi(t,\lambda) < \infty \quad \text{for } t > 0 \\ &\quad \Leftrightarrow \int_{\mathbb{R}\setminus(-1,1)} \mathrm{e}^{-\lambda x} \mathrm{d}\nu(x) < \infty. \end{aligned} \quad (3.6)$$

We will need the following functions μ and V given by [cf. (3.1)]

$$\begin{cases} \mu(\lambda) = -\dfrac{\phi'(\lambda)}{\phi(\lambda)} = \displaystyle\int_{\mathbb{R}} \left(x\,\mathrm{e}^{-\lambda x} - x\,\mathbf{1}_{(-1,1)}(x)\right) \mathrm{d}\nu(x) + m - \lambda s^2 \\ V(\lambda) = -\mu'(\lambda) = \displaystyle\int_{\mathbb{R}} x^2 \mathrm{e}^{-\lambda x} \mathrm{d}\nu(x) + s^2 \end{cases} \quad (3.7)$$

for $\lambda \in \mathbb{R}$ such that the definition makes sense. We will also need the inverse function $\mu^{\leftarrow}(u)$ of μ which will be well-defined in all cases we encounter for $u \in \mathbb{R}$ sufficiently large as we will have $\lim_{\lambda\downarrow-\alpha} \mu(\lambda) = \infty$ with $\mu'(\lambda) = -V(\lambda) < 0$ (see below).

We will need the following well-known Tauberian result for compound Poisson processes with semi-heavy Lévy measures. Our version of the result is stated slightly differently than in the literature to better suit our purposes, but it is easy to see that it is equivalent to the results in the literature.

Theorem 3.4 (*Embrechts, Jensen, Maejima and Teugels [19], Homble and McCormick [23, 24], Jensen [25]*). *For a compound Poisson process ξ with Lévy measure ν that is absolutely continuous sufficiently far out to the right with*

$$\frac{\mathrm{d}\nu(u)}{\mathrm{d}u} \sim C\, u^{\rho} \mathrm{e}^{-\alpha u} \quad \textit{as } u \to \infty \textit{ for some constants } C, \alpha > 0 \textit{ and } \rho > -1, \tag{3.8}$$

we have, with the notation (3.5) and (3.7),

$$\mathbf{P}\{\xi(t) > u\} \sim \frac{\mathrm{e}^{u\mu^{\leftarrow}(u/t)} \phi(\mu^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V(\mu^{\leftarrow}(u/t))}} \quad \textit{as } u \to \infty \textit{ for } t > 0.$$

The following corollary to Theorems 3.3 and 3.4 addresses processes which have semi-heavy-tailed Lévy measures with $\rho > -1$:

Corollary 3.5. *Let ξ be a Lévy process with characteristic triple (ν, m, s^2) such that ν is absolutely continuous sufficiently far out to the right and satisfies (3.8). Write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triplets $(\nu(\cdot \cap [1, \infty)), 0, 0)$ and $(\nu(\cdot \cap (-\infty, 1)), m, s^2)$, respectively, and let ϕ_1, μ_1, $\mu_1^{\leftarrow}$ and V_1 denote the quantities ϕ, μ, $\mu^{\leftarrow}$ and V in (3.5) and (3.7) calculated for the process ξ_1 instead of ξ. Eq. (3.4) holds and*

$$\mathbf{P}\{\xi(t) > u\} \sim \mathbf{E}\{\mathrm{e}^{\alpha\xi_2(t)}\} \frac{\mathrm{e}^{u\mu_1^{\leftarrow}(u/t)} \phi_1(\mu_1^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V_1(\mu_1^{\leftarrow}(u/t))}} \quad \textit{as } u \to \infty \textit{ for } t > 0. \tag{3.9}$$

Proof. As (3.3) holds by (3.8) and Example 2.11 we get (3.4) from Theorem 3.3. As ξ_1 is a compound Poisson process that satisfies (3.8) Theorem 3.4 shows that

$$\mathbf{P}\{\xi_1(t) > u\} \sim \frac{\mathrm{e}^{u\mu_1^{\leftarrow}(u/t)} \phi_1(\mu_1^{\leftarrow}(u/t))^t}{\alpha\sqrt{2\pi t\, V_1(\mu_1^{\leftarrow}(u/t))}} \quad \text{as } u \to \infty \text{ for } t > 0.$$

Hence (3.9) follows from (3.2) [as $\xi_1(t) \in \mathcal{L}(\alpha)$ by the proof of Theorem 3.3]. □

By further calculations one can phrase (3.9) in terms of ϕ, μ, $\mu^{\leftarrow}$ and V instead of ϕ_1, μ_1, $\mu_1^{\leftarrow}$ and V_1 in special cases: See the proof of (7.8) below for an example on this!

In Theorem 3.6 we address processes which have semi-heavy-tailed Lévy measures with $\rho = -1$. Now the Tauberian arguments have to be developed in detail from scratch as there are no suitable results in the literature to take off from. However, the idea of the proof to use Esscher transforms to find tail probabilities is well-known and is also used in the companion paper Albin and Sundén [2], where we give a bibliography.

Theorem 3.6. *If ξ is a Lévy process with Lévy measure ν that is absolutely continuous sufficiently far out to the right and satisfies (3.8) with $\rho = -1$, then (3.4) holds. If in addition $Ct > 1$ and*

$$\liminf_{x \downarrow 0} \frac{x}{\ln(1/x)} \frac{\mathrm{d}\nu(x)}{\mathrm{d}x} = \infty, \tag{3.10}$$

then we have

$$\mathbf{P}\{\xi(t) > u\} \sim \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\alpha\, \Gamma(Ct+1)} \frac{\mathrm{e}^{u\mu^{\leftarrow}(u/t)} \phi(\mu^{\leftarrow}(u/t))^t}{\sigma(\mu^{\leftarrow}(u/t))} \quad \textit{as } u \to \infty. \tag{3.11}$$

Proof. We get (3.4) from Theorem 3.3 as (3.3) holds by Example 2.11.

To prove (3.11) consider the Esscher transform of $\xi(t)$, which with the notation (3.5) and (3.7) is a random variable $Z_{t,\lambda}$ with CDF defined by the change of measure

$$\mathrm{d}F_{Z_{t,\lambda}}(x) = \frac{\mathrm{e}^{-\lambda x}\mathrm{d}F_{\xi(t)}(x)}{\phi(\lambda)^t} \quad \text{for } t > 0 \text{ and } \lambda \in \mathbb{R} \text{ such that } \phi(\lambda) < \infty. \tag{3.12}$$

By elementary calculations we have $\mathbf{E}\{Z_{t,\lambda}\} = t\mu(\lambda)$ and $\mathbf{Var}\{Z_{t,\lambda}\} = tV(\lambda) \equiv t\sigma(\lambda)^2$. Further, (3.1) and (3.7) show that $(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)$ has CHF $g_{t,\lambda}$ given by

$$\begin{aligned} g_{t,\lambda}(\theta) &= \frac{\phi(\lambda - \mathrm{i}\theta/\sigma(\lambda))^t}{\phi(\lambda)^t}\,\mathrm{e}^{-\mathrm{i}t\mu(\lambda)\theta/\sigma(\lambda)} \\ &= \exp\left\{t\int_{\mathbb{R}}\left(\cos\left(\frac{\theta x}{\sigma(\lambda)}\right) - 1\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\right. \\ &\quad \left. + \mathrm{i}t\int_{\mathbb{R}}\left(\sin\left(\frac{\theta x}{\sigma(\lambda)}\right) - \frac{\theta x}{\sigma(\lambda)}\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) - \frac{\theta^2 s^2 t}{2V(\lambda)}\right\}. \end{aligned} \tag{3.13}$$

By (3.7), (3.8) and elementary calculations we have

$$\mu(\lambda) \sim \frac{C}{\alpha + \lambda} \quad \text{and} \quad V(\lambda) \sim \frac{C}{(\alpha+\lambda)^2} \quad \text{as } \lambda \downarrow -\alpha. \tag{3.14}$$

Let Γ_{Ct} denote a gamma distributed random variable with PDF $f_{\Gamma_{Ct}}(x) = x^{Ct-1}\mathrm{e}^{-x}/\Gamma(Ct)$ for $x > 0$. By (3.8), (3.14) and the change of variable $x = \sigma(\lambda)\,y$ in (3.13) we have

$$\begin{aligned} g_{t,\lambda}(\theta) &\to \exp\left\{Ct\int_0^\infty \frac{\cos(\theta y) - 1}{y}\,\mathrm{e}^{-\sqrt{C}y}\mathrm{d}y + \mathrm{i}Ct\int_0^\infty \frac{\sin(\theta y) - \theta y}{y}\,\mathrm{e}^{-\sqrt{C}y}\mathrm{d}y\right\} \\ &= \exp\left\{-\frac{Ct\ln(1+\theta^2/C)}{2} + \mathrm{i}Ct\left(\arctan\left(\frac{\theta}{\sqrt{C}}\right) - \frac{\theta}{\sqrt{C}}\right)\right\} \\ &= \frac{1}{(1 - \mathrm{i}\theta/\sqrt{C})^{Ct}}\mathrm{e}^{-\mathrm{i}\sqrt{C}\,t\theta} \\ &= \mathbf{E}\left\{\exp\left[\mathrm{i}\theta(\Gamma_{Ct} - Ct)/\sqrt{C}\right]\right\} \equiv g_t(\theta) \quad \text{as } \lambda \downarrow -\alpha, \end{aligned} \tag{3.15}$$

cf. Erdélyi, Magnus, Oberhettinger and Tricomi [21], Equations 4.2.1, 4.7.59 and 4.7.82.

A key step in the proof is to prove that

$$\limsup_{\lambda \downarrow -\alpha} \int_{|\theta| > K} \left|g_{t,\lambda}(\theta)\right| \mathrm{d}\theta \to 0 \quad \text{as } K \to \infty, \tag{3.16}$$

as this shows that $g_{t,\lambda}$ is integrable for $\lambda > -\alpha$ small enough (since $g_{t,\lambda}$ is bounded by 1), so that $(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)$ has a continuous PDF $f_{(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)}$, which by (3.15) and (3.16) satisfies (using bounded convergence)

$$\begin{aligned} &\limsup_{\lambda \downarrow -\alpha} \sup_{x \in \mathbb{R}} \left| f_{(Z_{t,\lambda} - t\mu(\lambda))/\sigma(\lambda)}(x) - f_{(\Gamma_{Ct} - Ct)/\sqrt{C}}(x)\right| \\ &\quad \le \limsup_{K \to \infty} \limsup_{\lambda \downarrow -\alpha} \left(\int_{|\theta| \le K} \left|g_{t,\lambda}(\theta) - g_t(\theta)\right| \mathrm{d}\theta + \int_{|\theta| > K} \left(|g_{t,\lambda}(\theta)| + |g_t(\theta)|\right) \mathrm{d}\theta\right) \\ &\quad = 0. \end{aligned} \tag{3.17}$$

From (3.12) and (3.17) together with (3.14), we get

$$\begin{aligned} f_{\xi(t)}(t\mu(\lambda)-x/\lambda) &= \frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\, f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}\left(-\frac{x}{\lambda\,\sigma(\lambda)}\right) \\ &\sim \frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\, f_{(\Gamma_{Ct}-Ct)/\sqrt{C}}(0) \\ &= \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)} \quad \text{as } \lambda\downarrow-\alpha. \end{aligned} \tag{3.18}$$

Note that $\sup_{x\in\mathbb{R}} f_{(Z_{t,\lambda}-t\mu(\lambda))/\sigma(\lambda)}(x)$ is bounded for $\lambda > -\alpha$ small enough by (3.17) [as $Ct > 1$], so that we may integrate (3.18) using bounded convergence to obtain

$$\begin{aligned} \mathbf{P}\{\xi(t) > t\mu(\lambda)\} &= \int_0^\infty \frac{f_{\xi(t)}(t\mu(\lambda)-x/\lambda)}{-\lambda}\mathrm{d}x \\ &\sim \int_0^\infty \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{(-\lambda)\,\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)-x}\phi(\lambda)^t}{\sigma(\lambda)}\mathrm{d}x \\ &\sim \frac{\sqrt{C}(Ct)^{Ct}\mathrm{e}^{-Ct}}{\alpha\,\Gamma(Ct+1)}\frac{\mathrm{e}^{t\lambda\mu(\lambda)}\phi(\lambda)^t}{\sigma(\lambda)} \quad \text{as } \lambda\downarrow-\alpha. \end{aligned}$$

From this in turn we get (3.11) by a change of variable in the limit.

In order to finish the proof of the theorem it remains to prove (3.16). To that end pick constants $\varepsilon\in(0,1)$ and $A\geq 1$ such that $(1-\varepsilon)^5 Ct \geq 1$ and $\varepsilon\, t\mathrm{d}\nu(x)/\mathrm{d}x \geq 16\ln(1/x)/x$ for $x\in(0,1/A)$, cf. (3.10). As $1-\cos(x)\geq \frac{1}{4}x^2$ for $|x|\leq 1$ we have [as $\mathrm{e}^{-\lambda x}\geq 1$ for $x\geq 0$]

$$\begin{aligned} \varepsilon\, t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) &\geq \varepsilon\, t\int_0^{\sigma(\lambda)/|\theta|}\frac{x^2\theta^2}{4\,V(\lambda)}\mathrm{d}\nu(x) \\ &\geq \int_0^{\sigma(\lambda)/|\theta|}\frac{4\,\theta^2}{V(\lambda)}\,x\ln(1/x)\mathrm{d}x \\ &= 1+\ln\left(\frac{\theta^2}{V(\lambda)}\right) \quad \text{for } |\theta|\geq A\sigma(\lambda) \end{aligned}$$

and $\lambda > -\alpha$ small enough. As (3.8) shows that $\mathrm{d}\nu(x)/\mathrm{d}x \geq (1-\varepsilon)\,C\,\mathrm{e}^{-\alpha x}/x$ for $x\geq B$, for some constant $B>0$ large enough to make $B+2\pi/A\leq B/(1-\varepsilon)$, we further have

$$\begin{aligned} &(1-\varepsilon)\,t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \\ &\geq (1-\varepsilon)^2 Ct\sum_{k=0}^\infty\int_{B/\sigma(\lambda)+2\pi k/|\theta|}^{B/\sigma(\lambda)+2\pi(k+1)/|\theta|}(1-\cos(\theta x))\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)x}}{x}\mathrm{d}x \\ &\geq (1-\varepsilon)^2 Ct\sum_{k=0}^\infty\frac{2\pi}{|\theta|}\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)(B/\sigma(\lambda)+2\pi(k+1)/|\theta|)}}{B/\sigma(\lambda)+2\pi(k+1)/|\theta|} \\ &\geq (1-\varepsilon)^2 Ct\int_{B/\sigma(\lambda)}^\infty\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)(x+2\pi/(A\sigma(\lambda)))}}{x+2\pi/(A\sigma(\lambda))}\mathrm{d}x \\ &\geq (1-\varepsilon)^2 Ct\,\mathrm{e}^{-2\pi(\alpha+\lambda)/A}\frac{B}{B+2\pi/A}\int_{B/\sigma(\lambda)}^\infty\frac{\mathrm{e}^{-(\alpha+\lambda)\sigma(\lambda)x}}{x}\mathrm{d}x \end{aligned}$$

$$\geq (1-\varepsilon)^4 Ct \int_{B(\alpha+\lambda)}^{\infty} \frac{\mathrm{e}^{-x}}{x} \mathrm{d}x$$
$$\geq (1-\varepsilon)^5 Ct \ \ln\left(\frac{1}{B(\alpha+\lambda)}\right) \quad \text{for } |\theta| \geq A\sigma(\lambda)$$

and $\lambda > -\alpha$ small enough, where the last inequality is an elementary calculation.

By the estimates of the previous paragraph together with (3.13) and (3.14) we obtain

$$\begin{aligned}\int_{|\theta|\geq A\sigma(\lambda)} |g_{t,\lambda}(\theta)|\mathrm{d}\theta &= \int_{|\theta|\geq A\sigma(\lambda)} \exp\left\{-t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x)\right\}\mathrm{d}\theta \\ &\leq \int_{|\theta|\geq A\sigma(\lambda)} \mathrm{e}^{(1-\varepsilon)^5 Ct\gamma}(B(\alpha+\lambda))^{(1-\varepsilon)^5 Ct}\frac{V(\lambda)}{\theta^2}\mathrm{d}\theta \\ &= \mathrm{e}^{(1-\varepsilon)^5 Ct\gamma}\frac{2(B(\alpha+\lambda))^{(1-\varepsilon)^5 Ct}\sigma(\lambda)}{A} \to 0 \quad \text{as } \lambda\downarrow -\alpha. \quad (3.19)\end{aligned}$$

Moreover, we have, using Erdélyi, Magnus, Oberhettinger and Tricomi [21], Eq. 4.7.59, together with the inequality $1-\cos(x) \leq x^2/2$ and (3.14),

$$\begin{aligned}&t\int_{\mathbb{R}}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\mathrm{e}^{-\lambda x}\mathrm{d}\nu(x) \\ &\geq (1-\varepsilon)\,Ct\int_B^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}\mathrm{d}x \\ &\geq (1-\varepsilon)\,Ct\int_0^{\infty}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{\mathrm{e}^{-(\alpha+\lambda)x}}{x}\mathrm{d}x - Ct\int_0^{B}\left(1-\cos\left(\frac{\theta x}{\sigma(\lambda)}\right)\right)\frac{1}{x}\mathrm{d}x \\ &\geq \frac{(1-\varepsilon)\,Ct}{2}\ln\left(1+\frac{\theta^2}{(\alpha+\lambda)^2 V(\lambda)}\right) - Ct\int_0^B\frac{\theta^2 x}{2\,V(\lambda)}\mathrm{d}x \\ &\geq \frac{(1-\varepsilon)\,Ct}{2}\ln\left(\frac{\theta^2}{2C}\right) - Ct\frac{A^2B^2}{4} \\ &\quad \text{for } |\theta|\leq A\sigma(\lambda) \text{ and } \lambda > -\alpha \text{ small enough.}\end{aligned}$$

Hence we have (for $\lambda > -\alpha$ small enough)

$$\begin{aligned}\int_{K\leq|\theta|\leq A\sigma(\lambda)} |g_{t,\lambda}(\theta)|\mathrm{d}\theta &\leq 2\int_K^{\infty}\exp\left\{Ct\frac{A^2B^2}{4} - \frac{(1-\varepsilon)\,Ct}{2}\ln\left(\frac{\theta^2}{2C}\right)\right\}\mathrm{d}\theta \\ &= 2\exp\left\{Ct\frac{A^2B^2}{4}\right\}\frac{(2C)^{(1-\varepsilon)Ct/2}}{(1-\varepsilon)Ct-1}K^{1-(1-\varepsilon)Ct},\end{aligned}$$

which goes to 0 as $K\to\infty$ since $(1-\varepsilon)\,Ct > 1$. Recalling (3.19) this gives (3.16). □

4. Subexponential Lévy processes

The following result we will later extend from long-tailed processes to exponential ones.

Theorem 4.1 (*Berman [9], Marcus [28], Willekens [39]*). *For a Lévy process ξ we have*

$$\xi(h)\in\mathcal{L} \Leftrightarrow \sup_{t\in[0,h]}\xi(t)\in\mathcal{L}.$$

Moreover, if any of these memberships holds so that both of them hold, then

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} = 1. \tag{4.1}$$

The following simple converse to Theorem 4.1 is new and will be used in Section 6 to prove that $H > 1$ in (1.3) for $\mathcal{C} = \mathcal{S}(\alpha)$.

Theorem 4.2. *For a Lévy process ξ that satisfies (4.1), but is not a subordinator, one of the following two conditions holds:*

1. $\xi(h) \in \mathcal{L}$;
2. $$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = 0 \quad \text{for all } t \in (0, h).$$

Proof. Let (4.1) hold and assume that the liminf in Condition 2 takes a value $\ell(t) > 0$ for some $t \in (0, h)$. To show that Condition 1 holds note that

$$\begin{aligned} 0 &= \lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \left(\mathbf{P}\left\{\sup_{s\in[0,h]} \xi(s) > u\right\} - \mathbf{P}\{\xi(h) > u\}\right) \\ &\geq \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(h) \leq u, \xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} \\ &\geq \mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \\ &= \ell(t)\, \mathbf{P}\{\xi(h-t) \leq -\varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{u < \xi(t) \leq u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} \quad \text{for } \varepsilon > 0. \end{aligned} \tag{4.2}$$

As ξ is not a subordinator we have $\mathbf{P}\{\xi(h-t) \leq -\varepsilon\} > 0$ for ε small enough, see, e.g., Sato [33], Theorem 24.7. Therefore (4.2) shows that

$$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u+\varepsilon\}}{\mathbf{P}\{\xi(t) > u\}} = 1.$$

Hence $\xi(t) \in \mathcal{L}$ by Lemma 2.4, so that $\xi(h) \in \mathcal{L}$ by the existence of the limit $\ell(t)$. $\square$

While subordination of ξ implies (4.1), as does Condition 1 of Theorem 4.2 by Theorem 4.1, Condition 2 does not imply (4.1) as is exemplified by Brownian motion.

5. $\mathcal{O}$-exponential Lévy processes

In this section we extend Theorems 4.1 and 4.2 from the class $\mathcal{L}$ to $\mathcal{OL}$. The extension of Theorem 4.1 will be used in Section 6 in the proof of (1.2) and (1.3) for $\mathcal{C} = \mathcal{L}(\alpha)$.

Theorem 5.1. *For a Lévy process ξ we have*

$$\xi(h) \in \mathcal{OL} \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}.$$

Moreover, if any of these memberships holds so that both of them hold, then

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} < \infty. \tag{5.1}$$

Proof. The fact that $\xi(h) \in \mathcal{OL}$ implies $\sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}$ and (5.1) follows as

$$\mathbf{P}\{\xi(h) > x\} \le \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > x\right\} \le \frac{\mathbf{P}\{\xi(h) > x-1\}}{\mathbf{P}\{\inf_{t\in[0,h]} \xi(t) > -1\}} \le \frac{C\,\mathbf{P}\{\xi(h) > x\}}{\mathbf{P}\{\inf_{t\in[0,h]} \xi(t) > -1\}}$$

for x large enough for some constant $C > 0$, where the middle inequality follows from Sato [33], Remark 45.9. Conversely, if $\sup_{t\in[0,h]} \xi(t) \in \mathcal{OL}$, then by the same inequality

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(h) > u+x\}}{\mathbf{P}\{\xi(h) > u\}} \\
&\quad \ge \liminf_{u\to\infty} \mathbf{P}\left\{\inf_{t\in[0,h]} \xi(t) > -1\right\} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u+x+1\right\} / \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} > 0
\end{aligned}$$

for $x > 0$, so that $\xi(h) \in \mathcal{OL}$ by Lemma 2.8. □

Theorem 5.2. *For a Lévy process ξ such that $-\xi$ is not a subordinator and (5.1) holds one of the following two conditions holds:*

1. $\xi(h) \in \mathcal{OL}$;
2. $$\liminf_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} = 0 \quad \text{for } t \in (0, h).$$

Proof. If (5.1) holds and the liminf in Condition 2 equals $\ell(t) > 0$ for a $t \in (0, h)$, then

$$\begin{aligned}
\infty &> \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} \\
&\ge \mathbf{P}\{\xi(h-t) \ge \varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u-\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \\
&\ge \ell(t)\mathbf{P}\{\xi(h-t) \ge \varepsilon\} \limsup_{u\to\infty} \frac{\mathbf{P}\{\xi(h) > u-\varepsilon\}}{\mathbf{P}\{\xi(h) > u\}} \quad \text{for } \varepsilon > 0.
\end{aligned}$$

As $-\xi$ is not a subordinator we get $\mathbf{P}\{\xi(h-t) \ge \varepsilon\} > 0$ for $\varepsilon > 0$ small enough as in the proof of Theorem 4.2. Hence we see that $\xi(h) \in \mathcal{OL}$ using Lemma 2.8. □

6. Exponential Lévy processes

For Lévy processes in $\mathcal{S}(\alpha)$ Braverman and Samorodnitsky [14], Theorem 3.1, proved that

$$\lim_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} = H \quad \text{exists with value } H \in [1, \infty). \tag{6.1}$$

Although Braverman [11], Theorem 2.1, expresses H in terms of the characteristic triple, he also notes that the expression typically cannot be evaluated except for subordinators.

The next theorem extends Theorem 4.1 from $\mathcal{L}$ to $\mathcal{L}(\alpha)$ as well as (6.1) from $\mathcal{S}(\alpha)$ to $\mathcal{L}(\alpha)$ assuming (6.2) given below. It seems that specific processes in $\mathcal{L}(\alpha)$ always satisfy (6.2) (see Sections 3 and 7), making our result very useful in practice, while we are unsure of the real theoretical significance of (6.2). Our proof is quite short and transparent while in the literature already proofs of (6.1) for $\mathcal{S}(\alpha)$ are long and difficult.

Theorem 6.1. *For a constant $\alpha \geq 0$ and a Lévy process ξ such that*

$$L(t) = \lim_{u\to\infty} \frac{\mathbf{P}\{\xi(t) > u\}}{\mathbf{P}\{\xi(h) > u\}} \quad \text{exists for } t \in (0,h) \tag{6.2}$$

we have

$$\xi(h) \in \mathcal{L}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha) \quad \text{for } \alpha \geq 0.$$

Moreover, if any of these memberships holds so that both of them hold, then (6.1) holds. In that case we have $H = 1$ if $L(t) = 0$ for $t \in (0,h)$.

Proof. Assume that $\xi(h) \in \mathcal{L}(\alpha)$. Note that for each $t \in (0,h)$ with $L(t) > 0$ we have $\xi(t) \in \mathcal{L}(\alpha)$ by (6.2). This in turn by inspection of (2.1) means that

$$\lim_{u\to\infty} \mathbf{P}\{\xi(t) - u > x \mid \xi(t) > u\} = \mathrm{e}^{-\alpha x} \quad \text{for } x \geq 0. \tag{6.3}$$

Letting η be an $\exp(\alpha)$ distributed random variable that is independent of ξ (6.3) gives

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} \\
&\geq \limsup_{a\downarrow 0} \liminf_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\max_{k=0,\ldots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} \\
&= \limsup_{a\downarrow 0} \liminf_{u\to\infty} \sum_{k=0}^{\lfloor h/a\rfloor} \frac{\mathbf{P}\{\xi(h-ka) > u\}}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1}\{\xi(h-\ell a) \leq u\} \,\middle|\, \xi(h-ka) > u\right\} \\
&= \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\, \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1}\{\xi((k-\ell)a) + \eta \leq 0\}\right\}
\end{aligned} \tag{6.4}$$

(where $\bigcap_{\ell=0}^{-1}$ is the empty intersection, that is, the whole sample space). For a matching upper bound, note that the strong Markov property gives

$$\begin{aligned}
&\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u + x\right\} \\
&\leq \mathbf{P}\left\{\max_{k=0,\ldots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} + \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u + x\right\} \mathbf{P}\left\{\inf_{t\in[0,a]} \xi(t) < -x\right\} \\
&\quad \text{for } x > 0.
\end{aligned}$$

From this together with (6.4) and the fact that $\xi(h) \in \mathcal{L}(\alpha)$ we get

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\}$$

$$= \limsup_{x\downarrow 0} \limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u+x\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u+x\right\}$$
$$\leq \limsup_{x\downarrow 0} \liminf_{a\downarrow 0} \limsup_{u\to\infty} \frac{\mathrm{e}^{\alpha x}}{\mathbf{P}\{\xi(h) > u\}}$$
$$\times \mathbf{P}\left\{\max_{k=0,\ldots,\lfloor h/a\rfloor} \xi(h-ka) > u\right\} \Big/ \mathbf{P}\left\{\inf_{t\in[0,a]} \xi(t) \geq -x\right\}$$
$$\leq \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\, \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a) + \eta \leq 0\}\right\}. \tag{6.5}$$

From (6.4) together with (6.5) we conclude that (6.1) holds with

$$H = \lim_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\, \mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a) + \eta \leq 0\}\right\}. \tag{6.6}$$

Here $H \geq 1$ by (6.1) with $H = 1$ if $L(t) = 0$ for $t \in (0, h)$ by (6.6), while $H < \infty$ by Theorem 5.1.

Conversely, assume that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$. To finish the proof we have to show that $\xi(h) \in \mathcal{L}(\alpha)$. Assume that $\alpha > 0$ as we are done otherwise by Theorem 4.1. Observe that it is enough to show that given any $x \geq 0$ we have

$$\lim_{n\to\infty} \frac{\mathbf{P}\{\xi(h) > u_n + x\}}{\mathbf{P}\{\xi(h) > u_n\}} = \mathrm{e}^{-\alpha x} \tag{6.7}$$

for any sequence $u_n \to \infty$ as $n \to \infty$ such that the limit (6.7) really exists. This is so because the ratio in (6.7) is bounded so that every subsequence of that ratio has a further subsequence that converges to the limit $\mathrm{e}^{-\alpha x}$. It follows that (2.1) holds for $x \geq 0$, which in turn gives (2.1) in general by an elementary argument.

Consider the distributions supported on $[0, \infty)$ with CDF given by

$$F_n(x) = \mathbf{P}\{\xi(h) \leq u_n + x \mid \xi(h) > u_n\} = 1 - \frac{\mathbf{P}\{\xi(h) > u_n + x\}}{\mathbf{P}\{\xi(h) > u_n\}} \quad \text{for } x \geq 0.$$

For suitable constants $N \in \mathbb{N}$ and $C, \varepsilon > 0$, Theorem 5.1 gives [use (5.1) in the first step]

$$\limsup_{x\to\infty} \sup_{n\geq N} (1 - F_n(x))$$
$$\leq \limsup_{x\to\infty} C \sup_{n\geq N} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + x\right\} \Big/ \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n\right\}$$
$$\leq \limsup_{x\to\infty} C \sup_{n\geq N} \prod_{k=1}^{\lfloor x\rfloor} \frac{\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + k\right\}}{\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_n + k - 1\right\}}$$
$$\leq \limsup_{x\to\infty} C \sup_{n\geq N} \prod_{k=1}^{\lfloor x\rfloor} (1+\varepsilon)\, \mathrm{e}^{-\alpha} = 0.$$

Hence the sequence $\{F_n\}_{n=1}^{\infty}$ is tight in the sense of weak convergence. Therefore Prohorov's theorem shows that there exists a weakly convergent subsequence $F_{n_k} \xrightarrow{\mathrm{d}} F$.

Letting η be a random variable with CDF F that is independent of ξ (6.4) gives

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \\
&\quad \geq \limsup_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a)+\eta \leq 0\}\right\}.
\end{aligned}$$

Using that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$ we further get the following version of (6.5):

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \\
&\quad = \limsup_{x\downarrow 0} \limsup_{k\to\infty} \frac{\mathrm{e}^{\alpha x}}{\mathbf{P}\{\xi(h) > u_{n_k}\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}+x\right\} \\
&\quad \leq \liminf_{a\downarrow 0} \sum_{k=0}^{\lfloor h/a\rfloor} L(h-ka)\,\mathbf{P}\left\{\bigcap_{\ell=0}^{k-1} \{\xi((k-\ell)a)+\eta \leq 0\}\right\}.
\end{aligned}$$

With the notation (6.6) we thus obtain the following version of (6.1):

$$\mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u_{n_k}\right\} \sim H\,\mathbf{P}\{\xi(h) > u_{n_k}\} \quad \text{as } k\to\infty$$

with $H \in [1,\infty)$ as before. This gives the required (6.7) since $\sup_{t\in[0,h]} \xi(t) \in \mathcal{L}(\alpha)$. □

We immediately get the following powerful corollary to Theorem 6.1. See also Braverman [13], Theorem 2, and Section 8 below on priority issues:

Corollary 6.2. *For a Lévy process ξ satisfying (3.4) we have (6.1) with $H = 1$.*

We now complete the result (6.1) of Braverman [11] and Braverman and Samorodnitsky [14] for $\mathcal{S}(\alpha)$ to a result in the fashion of Theorem 4.1. We also show that $H > 1$ unless ξ is a subordinator which does not follow from the mentioned literature.

Corollary 6.3. *For a constant $\alpha \geq 0$ and a Lévy process ξ we have*

$$\frac{\nu([1,\infty)\cap\cdot)}{\nu([1,\infty))} \in \mathcal{S}(\alpha) \Leftrightarrow \xi(h) \in \mathcal{S}(\alpha) \Leftrightarrow \sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha) \quad \text{and (6.2).}$$

Moreover, if any of these memberships holds so that all of them hold, then (6.1) holds. In that case we have $H > 1$ unless $\alpha = 0$ or ξ is a subordinator.

Proof. The left equivalency in the corollary follows from Theorem 3.1, as does the fact that $\xi(h) \in \mathcal{S}(\alpha)$ implies (6.2). Hence Theorem 6.1 shows that $\xi(h) \in \mathcal{S}(\alpha)$ also implies $\sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha)$ as (6.1) holds. Conversely, the fact that $\sup_{t\in[0,h]} \xi(t) \in \mathcal{S}(\alpha)$ and (6.2) imply $\xi(h) \in \mathcal{S}(\alpha)$ follows from Theorem 6.1 alone, again as (6.1) holds.

As any of the equivalent statements in the corollary implies that (6.2) holds with $\xi(h) \in \mathcal{L}(\alpha)$, Theorem 6.1 shows that they also imply (6.1).

If $\alpha > 0$, then Condition 1 of Theorem 4.2 fails. As $\xi(h) \in \mathcal{S}(\alpha)$ implies that the limit in (6.2) is strictly positive, by Theorem 3.1, also Condition 2 of Theorem 4.2 fails. Since ξ is not a subordinator Theorem 4.2 thus shows that

$$\limsup_{u\to\infty} \frac{1}{\mathbf{P}\{\xi(h) > u\}} \mathbf{P}\left\{\sup_{t\in[0,h]} \xi(t) > u\right\} > 1,$$

which combines with (6.1) to show that $H > 1$. $\square$

7. Applications

We now consider applications of our results to GZ, CGMY and GH processes.

7.1. GZ processes

The GZ process was introduced by Grigelionis [22], thereby generalizing the z processes of Prentice [32]. The GZ process is a Lévy process with characteristic triple

$$\begin{aligned}&\left(\frac{\mathrm{d}\nu(x)}{\mathrm{d}x}, m, s^2\right)\\ &= \left(\frac{2\delta}{|x|(1-\mathrm{e}^{-2\pi|x|/\zeta})}\left(\mathrm{e}^{2\pi\beta_1 x}\mathbf{1}_{(-\infty,0)}(x) + \mathrm{e}^{-2\pi\beta_2 x}\mathbf{1}_{(0,\infty)}(x)\right), m, 0\right),\end{aligned} \tag{7.1}$$

where $\beta_1, \beta_2, \delta, \zeta > 0$ and $m \in \mathbb{R}$ are parameters.

Theorem 7.1. *For a GZ Lévy process (6.1) holds with $H = 1$. If in addition $h > 1/(2\delta)$, then we have*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{\mathrm{e}^{(A-2\delta\gamma)h}}{2\pi\beta_2\Gamma(2\delta h)} u^{2\delta h-1}\mathrm{e}^{-2\pi\beta_2 u} \quad \text{as } u \to \infty, \tag{7.2}$$

where γ is Euler's constant and

$$\begin{aligned}A = 2\pi\beta_2 m &+ \int_{\mathbb{R}} \left(\mathrm{e}^{2\pi\beta_2 x}\mathbf{1}_{(-\infty,1)}(x) - 1 - 2\pi\beta_2\mathbf{1}_{(-1,1)}(x)\,x\right)\mathrm{d}\nu(x)\\ &+ \int_1^\infty \left(\mathrm{e}^{2\pi\beta_2 x}\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} - \frac{2\delta}{x}\right)\mathrm{d}x.\end{aligned}$$

Proof. By inspection of (7.1) we see that (3.8) holds with $C = 2\delta$, $\rho = -1$ and $\alpha = 2\pi\beta_2$. Hence (3.4) holds by Theorem 3.6, so that Corollary 6.2 gives (6.1) with $H = 1$.

If $h > 1/(2\delta)$, then the hypothesis of the second part of Theorem 3.6 holds, as (3.10) holds by inspection of (7.1). Hence (3.11) applies. By (3.14) we have $\mu^{\leftarrow}(u) + 2\pi\beta_2 \sim 2\delta/u$ as $u \to \infty$, so that by dominated convergence [note that $\mu^{\leftarrow}(u/h) + 2\pi\beta_2 \geq 0$ and $(\mu^{\leftarrow}(u/h) + 2\pi\beta_2)\,u/(2h\delta) \geq 1/2$ for u large enough] and a change of variable

$$\begin{aligned}&\int_1^\infty \mathrm{e}^{-\mu^{\leftarrow}(u/h)\,x}\mathrm{d}\nu(x)\\ &= \int_1^\infty \mathrm{e}^{-(\mu^{\leftarrow}(u/h)+2\pi\beta_2)\,x}\left(\mathrm{e}^{2\pi\beta_2 x}\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} - \frac{2\delta}{x}\right)\mathrm{d}x + 2\delta\int_1^\infty \mathrm{e}^{-(\mu^{\leftarrow}(u/h)+2\pi\beta_2)\,x}\frac{\mathrm{d}x}{x}\\ &= \int_1^\infty \left(\mathrm{e}^{2\pi\beta_2 x}\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} - \frac{2\delta}{x}\right)\mathrm{d}x + 2\delta\int_{2h\delta/u}^\infty \frac{\mathrm{e}^{-x}}{x}\mathrm{d}x + o(1) \quad \text{as } u \to \infty.\end{aligned} \tag{7.3}$$

Using (3.1) together with (3.11), (3.14) and (7.3), elementary calculations give

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{(2\delta h)^{2\delta h} e^{Ah}}{2\pi\beta_2 \Gamma(2\delta h)} \frac{e^{-2\pi\beta_2 u}}{u} \exp\left\{2\delta h \int_{2\delta h/u}^{\infty} \frac{e^{-x}}{x} dx\right\}$$
$$\sim \frac{(2\delta h)^{2\delta h} e^{Ah}}{2\pi\beta_2 \Gamma(2\delta h)} \frac{e^{-2\pi\beta_2 u}}{u} \left(\frac{e^{-\gamma} u}{2\delta h}\right)^{2\delta h} \quad \text{as } u \to \infty,$$

which establishes (7.2): Here the last asymptotic relation follows from, e.g., Erdélyi, Magnus, Oberhettinger and Tricomi [20], Equations 6.9.25 and 6.7.13. □

Example 7.2. The Meixner process of Schoutens and Teugels [35] is a GZ process with $\beta_1, \beta_2 > 0$ and $\beta_1 + \beta_2 = 1$. Thus it satisfies (6.1) with $H = 1$. Further, (7.2) holds when $h > 1/(2\delta)$.

Remark 7.3. According to Barndorff-Nielsen, Kent and Sørensen [7], Theorem 5.2, if for a constant $\alpha > 0$ and a PDF f the function

$$F_k(x) = \int_0^x y^k e^{\alpha y} f(y) dy, \quad x > 0,$$

has an ultimately monotone derivative with a Laplace transform that satisfies

$$\int_0^\infty x^k e^{(\alpha+s)x} f(x) dx \sim C\, \Gamma(k-\rho)(-s)^{-(k+\rho+1)} \quad \text{as } s \uparrow 0, \tag{7.4}$$

for some constants $C > 0$, $k \in \mathbb{N}$ and $\rho > -k-1$, then f satisfies (1.1). However, in general one cannot tell whether F_k has an ultimately monotone derivative just by inspection of the Laplace transform. And should such additional information on f be available the Tauberian result should typically not be needed anyway.

For example, Grigelionis [22], Corollary 1, deduces that

$$f_{\xi(t)}(u) \sim \left(\frac{2\pi\, \Gamma(\beta_1+\beta_2)}{\zeta\, \Gamma(\beta_1)\, \Gamma(\beta_2)}\right)^{2\delta t} \frac{u^{2\delta t-1}}{\Gamma(2\delta t)} \exp\left\{-\frac{2\pi\beta_2(u-mt)}{\zeta}\right\} \quad \text{as } u \to \infty$$

for GZ processes from information like (7.4) only, with the property that F_k has a monotone derivative waived as "standard calculations": We find this argument incomplete!

7.2. CGMY processes

The CGMY Lévy process of Carr, Geman, Madan and Yor [15] has characteristic triple

$$\left(\frac{d\nu(x)}{dx}, m, s^2\right)$$
$$= \left(C_-(-x)^{-1-Y_-} e^{Gx} \mathbf{1}_{(-\infty,0)}(x) + C_+ x^{-1-Y_+} e^{-Mx} \mathbf{1}_{(0,\infty)}(x), m, 0\right), \tag{7.5}$$

where $C_-, C_+, G, M > 0$, $Y_-, Y_+ < 2$ and $m \in \mathbb{R}$ are parameters.

Theorem 7.4. *For a CGMY process (6.1) holds. Further, we have $H > 1$ and*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{C_+ h}{M} \exp\left\{hMm + h\int_{\mathbb{R}} \left(e^{Mx} - 1 - M\mathbf{1}_{(-1,1)}(x)\, x\right) d\nu(x)\right\} \frac{e^{-Mu}}{u^{1+Y_+}} \tag{7.6}$$

as $u \to \infty$ *for* $Y_+ > 0$, *while* $H = 1$ *and*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{M^{C_+ h-1} \mathrm{e}^{Bh}}{\Gamma(C_+ h)} u^{C_+ h-1} \mathrm{e}^{-Mu} \tag{7.7}$$

as $u \to \infty$ *for* $Y_+ = 0$, *where* γ *is Euler's constant and*

$$B = Mm + \int_{\mathbb{R}} \left(\mathrm{e}^{Mx} \mathbf{1}_{(-\infty,0)}(x) - 1 - M \mathbf{1}_{(-1,1)}(x)\, x \right) \mathrm{d}\nu(x),$$

and $H = 1$ *and*

$$\mathbf{P}\{\xi(h) > u\} \sim \frac{(C_+ h \Gamma(1-Y_+))^{-1/(2(1-Y_+))}}{M\sqrt{2\pi(1-Y_+)}\, u^{(1-Y_+/2)/(1-Y_+)}} \times \exp\left\{ -Mu + \frac{(1+1/\Gamma(1-Y_+))\, u^{-Y_+/(1-Y_+)}}{(C_+ h \Gamma(1-Y_+))^{-1/(1-Y_+)}} + Bh \right\} \tag{7.8}$$

as $u \to \infty$ *for* $Y_+ < 0$.

Proof. For $Y_+ > 0$, (7.5) and Example 2.11 show that $\nu([1,\infty) \cap \cdot)/\nu([1,\infty)) \in \mathcal{S}(M)$, so that Corollary 6.3 gives $\xi(h) \in \mathcal{S}(M)$ and (6.1) with $H > 1$ (as ξ is not a subordinator). Further, we get (7.6) by insertion of (7.5) in the second part of Theorem 3.1.

For $Y_+ = 0$, (7.5) shows that (3.8) holds with $\rho = -1$, so that Theorem 3.6 gives (3.4). Hence (6.1) holds with $H = 1$ by Corollary 6.2. To prove (7.7), write $\xi = \xi_1 + \xi_2$ where ξ_1 and ξ_2 are independent Lévy processes with triplets $(\nu((0,\infty) \cap \cdot), C_+(1-\mathrm{e}^{-M})/M, 0)$ and $(\nu((-\infty,0) \cap \cdot), m - C_+(1-\mathrm{e}^{-M})/M, 0)$, respectively. Then ξ_1 is a gamma subordinator, see, e.g., Schoutens [34], Section 5.3.3, and satisfies

$$\mathbf{P}\{\xi_1(h) > u\} \sim \frac{M^{C_+ h-1}}{\Gamma(C_+ h)} u^{C_+ h-1} \mathrm{e}^{-Mu} \quad \text{as } u \to \infty, \tag{7.9}$$

so that $\xi_1(h) \in \mathcal{L}(M)$. As $\mathbf{E}\{\mathrm{e}^{\beta \xi_2(h)}\} < \infty$ for $\beta \geq 0$ [recall (3.6)] it follows from Pakes [29], Lemma 2.1 (see also [30]), that [cf. (3.2)]

$$\mathbf{P}\{\xi(h) > u\} \sim \mathbf{P}\{\xi_1(h) > u\} \mathbf{E}\{\mathrm{e}^{M\xi_2(h)}\} \quad \text{as } u \to \infty. \tag{7.10}$$

Inserting (7.9) and calculating the MGF in (7.10) using (3.1) and (7.5) we get (7.7).

For $Y_+ < 0$, (7.5) shows that (3.8) holds with $\rho = -1 - Y_+ > -1$, $\alpha = M$ and $C = C_+$, so that Corollary 3.5 gives (3.4). Hence (6.1) holds with $H = 1$ by Corollary 6.2, with the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (3.9), by Corollary 3.5. To evaluate (3.9) we note that (3.7) and (7.5) give

$$\mu_1(\lambda) = C_+ \int_1^\infty x^{-Y_+} \mathrm{e}^{-(M+\lambda)x} \mathrm{d}x = C_+ \Gamma(1-Y_+)(M+\lambda)^{Y_+-1} - \frac{C_+}{1-Y_+} + o(1)$$

$$V_1(\lambda) = C_+ \int_1^\infty x^{1-Y_+} \mathrm{e}^{-(M+\lambda)x} \mathrm{d}x \sim C_+ \Gamma(2-Y_+)(M+\lambda)^{Y_+-2}$$

as $\lambda \downarrow -M$. It follows that

$$\mu_1^{\leftarrow}(u) = -M + \left(\frac{u - C_+/(1-Y_+)}{C_+ \Gamma(1-Y_+)} \right)^{1/(Y_+-1)} + o\left(u^{1/(Y_+-1)-1}\right) \quad \text{as } u \to \infty,$$

so that

$$\begin{aligned} u\mu_1^{\leftarrow}(u/h) &\sim -Mu + (C_+ h\,\Gamma(1-Y_+))^{1/(1-Y_+)} u^{Y_+/(Y_+-1)} \\ V_1(\mu_1^{\leftarrow}(u/h)) &\sim C_+\Gamma(2-Y_+)(C_+ h\,\Gamma(1-Y_+))^{(Y_+-2)/(1-Y_+)} u^{(Y_+-2)/(Y_+-1)} \\ &= (1-Y_+)\,h^{-1}(C_+ h\,\Gamma(1-Y_+))^{-1/(1-Y_+)} u^{(Y_+-2)/(Y_+-1)} \end{aligned}$$

as $u \to \infty$. Moreover, we have

$$\begin{aligned} \ln\left(\mathbf{E}\{e^{M\xi_2(h)}\}\phi_1(\lambda)^h\right) &\sim h\int_0^\infty x^{-1-Y_+} e^{-(M+\lambda)x} dx + Bh \\ &= C_+ h\,\Gamma(-Y_+)(M+\lambda)^{Y_+} + Bh \end{aligned}$$

as $\lambda \downarrow -M$, so that

$$\begin{aligned} &\ln\left(\mathbf{E}\{e^{M\xi_2(h)}\}\phi_1(\mu_1^{\leftarrow}(u/h))^h\right) \sim C_+ h(C_+ h\,\Gamma(1-Y_+))^{Y_+/(1-Y_+)} u^{Y_+/(Y_+-1)} + Bh \\ &\quad \sim (C_+ h\,\Gamma(1-Y_+))^{1/(1-Y_+)}\Gamma(1-Y_+)^{-1} u^{Y_+/(Y_+-1)} + Bh \end{aligned}$$

as $u \to \infty$. Inserting the above findings into (3.9) we readily obtain (7.8). □

Braverman [11] and Braverman and Samorodnitsky [14] apply to CGMY for $Y_+ > 0$.

CGMY processes with $Y_-, Y_+ < 0$ are compound Poisson and light tailed in the sense of Braverman [12], Definition 1, by Example 2.11. Hence Braverman's Theorem 1 applies if the process has non-negative drift while his Theorem 3 applies if $0 < M < 1$ and the drift is negative, in both cases to yield (6.1) with $H = 1$.

Example 7.5. As the variance gamma processes of Madan and Seneta [27] are CGMY processes with $C_- = C_+$ and $Y_- = Y_+ = 0$ they satisfy (6.1) with $H = 1$ and the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (7.7).

Example 7.6. The Kou [26] jump-diffusion Lévy process has characteristic triple

$$\left(\frac{d\nu(x)}{dx}, m, s^2\right) = \left((1-p)\lambda e^{\lambda_- x}\mathbf{1}_{(-\infty,0)}(x) + p\lambda e^{-\lambda_+ x}\mathbf{1}_{(0,\infty)}(x), m,\ s^2\right), \tag{7.11}$$

where $p \in (0,1)$, $m \in \mathbb{R}$ and $\lambda, \lambda_-, \lambda_+, s^2 > 0$ are parameters.

Besides the Gaussian component s^2, (7.11) is (7.5) with $Y_- = Y_+ = -1$. Hence the proof of Theorem 7.4 carries over to show that (6.1) holds with $H = 1$ and the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) given by (3.9). The evaluation of (3.9) in the proof of Theorem 7.4 can be modified in a straightforward manner to include the Gaussian component, but we omit the details.

7.3. GH processes

The GH Lévy process introduced by Barndorff-Nielsen [3,4] has characteristic triple

$$\begin{aligned} &\left(\frac{d\nu(x)}{dx}, m, s^2\right) \\ &= \left(\frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\varrho^2}\}}{\pi^2 y\left(J_{|\zeta|}(\delta\sqrt{2y})^2 + Y_{|\zeta|}(\delta\sqrt{2y})^2\right)} dy + \zeta^+ e^{-\varrho|x|}\right), m, 0\right), \end{aligned} \tag{7.12}$$

where $\beta, \zeta, m \in \mathbb{R}$, $\delta > 0$ and $\varrho > |\beta|$ are parameters. Here J_ζ denotes the Bessel function and Y_ζ the Bessel function of the second kind, respectively.

Theorem 7.7. *For a GH process (6.1) holds where $H = 1$ if $\zeta \geq 0$ while $H > 1$ if $\zeta < 0$.*

Proof. Using the facts from Watson [38], Equations 3.1.8, 3.51.1, 3.52.3 and 3.54.1-2, about the asymptotic behaviour of $J_\zeta(y)$ and $Y_\zeta(y)$ as $y \downarrow 0$ we readily obtain

$$\int_0^\infty \frac{\exp\{-|x|\sqrt{2y+\varrho^2}\}}{\pi^2 y\left(J_\zeta(\delta\sqrt{2y})^2 + Y_\zeta(\delta\sqrt{2y})^2\right)}\mathrm{d}y \sim \int_0^\infty \frac{\exp\{-x(\varrho + y/\varrho)\}}{\pi^2 y Y_\zeta(\delta\sqrt{2y})^2}\mathrm{d}y$$
$$\sim \begin{cases} \displaystyle\int_0^\infty \frac{\delta^{2\zeta}[\sin(\pi\zeta)\Gamma(1-\zeta)]^2 y^{\zeta-1}\exp\{-x(\varrho+y/\varrho)\}}{\pi^2 2^\zeta}\mathrm{d}y & \text{for } \zeta \in [0,\infty)\setminus\mathbb{N} \\ \displaystyle\int_0^\infty \frac{\delta^{2\zeta} y^{\zeta-1}\exp\{-x(\varrho+y/\varrho)\}}{2^\zeta \Gamma(\zeta)^2}\mathrm{d}y & \text{for } \zeta \in \mathbb{N}\setminus\{0\} \\ \displaystyle\int_0^\infty \frac{\exp\{-x(\varrho+y/\varrho)\}}{y\ln(y/\varrho)^2}\mathrm{d}y & \text{for } \zeta = 0 \end{cases}$$

as $x \to \infty$, so that by insertion in (7.12)

$$\frac{\mathrm{d}\nu(x)}{\mathrm{d}x} \sim \begin{cases} \dfrac{2^\zeta[\sin(\pi\zeta)\Gamma(1+\zeta)]^2\exp\{-(\varrho-\beta)x\}}{\pi^2\delta^{2\zeta}\varrho^\zeta x^{1-\zeta}} & \text{for } \zeta \in (-\infty,0]\setminus\mathbb{N} \\ \dfrac{2^\zeta\exp\{-(\varrho-\beta)x\}}{\delta^{2\zeta}\varrho^\zeta\Gamma(-\zeta)^2 x^{1-\zeta}} & \text{for } \zeta \in (-\mathbb{N})\setminus\{0\} \\ \dfrac{\exp\{-(\varrho-\beta)x\}}{\ln(2)x} & \text{for } \zeta = 0 \\ \dfrac{\zeta\mathrm{e}^{-(\varrho-\beta)x}}{x} & \text{for } \zeta > 0. \end{cases} \tag{7.13}$$

For $\zeta \geq 0$, (7.13) shows that (3.8) holds with $\rho = -1$, so that Theorem 3.6 gives (3.4). Hence Corollary 6.2 shows that (6.1) holds with $H = 1$. For $\zeta < 0$, (7.13) and Example 2.11 show that $\nu([1,\infty)\cap\cdot)/\nu([1,\infty)) \in \mathcal{S}(\varrho-\beta)$, so that Corollary 6.3 gives (6.1) with $H > 1$ (as ξ is not a subordinator). $\square$

Braverman [11] and Braverman and Samorodnitsky [14] apply to GH for $\zeta < 0$.

Example 7.8. The normal inverse Gaussian process introduced by Barndorff-Nielsen [4,5] is a GH processes with $\zeta = -\frac{1}{2}$. Thus it satisfies (6.1) with $H > 1$.

Example 7.9. The hyperbolic process introduced by Barndorff-Nielsen [3] is a GH process with $\zeta = 1$. Thus is satisfies (6.1) with $H = 1$.

Remark 7.10. When $\zeta < 0$ the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ as $u \to \infty$ in (6.1) for GH processes is given by the second part of Theorem 3.1 together with integration of (7.13). When $\zeta \geq 0$ the asymptotic behaviour of $\mathbf{P}\{\xi(h) > u\}$ can be calculated from (3.11) in the fashion of Theorem 7.1 for h large enough. We have omitted these calculations to avoid additional technicalities.

8. Two priority issues

Theorem 3.3 is due to Braverman [13], Theorem 1.

Braverman showed us his result during Fall 2004, so we were aware of his finding when submitting our paper (although, frankly, we had forgotten about it for the final version of our article, so we had to be reminded about it).

Braverman [13], Theorem 2, states that the conclusion of Corollary 6.2 holds under the hypothesis of Theorem 3.3. Thus our Corollary 6.2 is more general than Braverman [13], Theorem 2. However, from a practical point of view, one may argue the importance of the added generality.

Our version of Corollary 6.2 is implicit in Chapter 5 of the thesis of Bengtsson [8] (now named Sundén) that already appeared during Fall 2004, which is also acknowledged by Braverman [13], Remark 1.

It should be noted that there is a more or less complete difference between the approach and methods of proof of Theorem 3.3 and Corollary 6.2 and those of the corresponding results of Braveman.

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Appendix. Technical details of Section 2

Here we prove Proposition 2.6 and Corollary 2.9. We remark that these results are not very far from some standard results on regular variation that can be found in e.g., Bingham, Goldie and Teugels [10], as is indeed indicated by their proofs: The results and proofs are just supplied as a service to the reader not very expert in regular variation.

A.1. Proof of Proposition 2.6

We have $F \in \mathcal{L}(\alpha)$ if and only if $1 - F(\ln(\cdot)) \in \mathcal{R}(-\alpha)$ and $F \in \mathcal{OL}$ if and only if $1 - F(\ln(\cdot)) \in \mathcal{OR}$, see, e.g., Shimura and Watanabe [37], p. 451. By the representation theorems for $\mathcal{R}(-\alpha)$ and $\mathcal{OR}$ (see e.g., Bingham, Goldie and Teugels [10], Theorems 1.3.1 and 2.2.7) a function $1 - F(\ln(\cdot))$ belongs to $\mathcal{R}(-\alpha)$ [$\mathcal{OR}$] if and only if

$$1 - F(\ln(u)) = \hat{c}(u) \exp\left\{-\int_0^u \frac{\hat{a}(x)}{x} \mathrm{d}x\right\} \quad \text{for } u \in \mathbb{R} \text{ large enough,} \tag{A.1}$$

for some measurable functions $\hat{a}$ and $\hat{c}$ such that $\lim_{x\to\infty} \hat{a}(x) = \alpha$ and $\lim_{x\to\infty} \hat{c}(x) > 0$ exists [$\limsup_{x\to\infty} |\hat{a}(x)| < \infty$ and $0 < \liminf_{x\to\infty} \hat{c}(x) \le \limsup_{x\to\infty} \hat{c}(x) < \infty$]. Since F is absolutely continuous with $\lim_{u\to-\infty} F(u) = 0$ and $\lim_{u\to\infty} F(u) = 1$ we can rewrite (A.1) as

$$1 - F(u) = \exp\left\{-\int_0^{e^u} \frac{\hat{a}(x) + \hat{b}(x)}{x} dx\right\}$$
$$= \exp\left\{-\int_{-\infty}^{u} \left(\hat{a}(e^x) + \hat{b}(e^x)\right) dx\right\} \quad \text{for } u \in \mathbb{R},$$

where $a(x) \equiv \hat{a}(e^x)$ and $b(x) \equiv \hat{b}(e^x)$ satisfies (2.4) and (2.5), respectively, depending on whether $F \in \mathcal{L}(\alpha)$ or $F \in \mathcal{OL}$. Finally, as F is non-decreasing we have $a + b \geq 0$. $\square$

A.2. Proof of Corollary 2.9

Integrating (1.1) we readily obtain

$$1 - F(u) = \int_u^{\infty} f(x)dx \sim \frac{C}{\alpha} u^{\rho} e^{-\alpha u} \sim \frac{f(u)}{\alpha} \quad \text{as } u \to \infty. \tag{A.2}$$

In particular we see that $F \in \mathcal{L}(\alpha)$, so that (2.3) holds with $a + b \geq 0$ as in (2.4), where

$$\lim_{u\to\infty} \int_{-\infty}^{u} \left(a(x) + b(x) - \alpha\, \mathbf{1}_{[0,\infty)}(x) + \frac{\rho}{x} \mathbf{1}_{[1,\infty)}(x)\right) dx = \ln\left(\frac{C}{\alpha}\right)$$

by (A.2). Writing $c = a + b$, we thus have (2.6) with $c \geq 0$ as well as the second part of (2.7). Differentiating both sides of (2.6) and using (A.2) we get

$$f(u) = c(u) \exp\left\{-\int_{-\infty}^{u} c(x)dx\right\} = c(u)\ (1 - F(u)) \sim c(u)\frac{f(u)}{\alpha} \quad \text{as } u \to \infty, \tag{A.3}$$

which gives the first part of (2.7).

Conversely, if (2.6) and (2.7) hold, then it is immediate that (A.2) holds so that $F \in \mathcal{L}(\alpha)$, while (1.1) follows from (A.3). $\square$

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