

Extremes of the standardized Gaussian noise

Zakhar Kabluchko*

Institute of Stochastics, Ulm University, Helmholtzstr. 18, 89069 Ulm, Germany

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Abstract

Let $\{\xi_k, k \in \mathbb{Z}^d\}$ be a d -dimensional array of independent standard Gaussian random variables. For a finite set $A \subset \mathbb{Z}^d$ define $\mathbb{S}(A) = \sum_{k \in A} \xi_k$. Let $|A|$ be the number of elements in A . We prove that the appropriately normalized maximum of $\mathbb{S}(A)/\sqrt{|A|}$, where A ranges over all discrete cubes or rectangles contained in $\{1, \dots, n\}^d$, converges in law to the Gumbel extreme-value distribution as $n \rightarrow \infty$. We also prove a continuous-time counterpart of this result.

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1. Introduction and statement of results

Let $\{\xi_k, k \in \mathbb{N}\}$ be independent standard Gaussian random variables. Denote by $S_k = \xi_1 + \dots + \xi_k$ the corresponding random walk and let

$$L_n = \max_{0 \leq i < j \leq n} \frac{S_j - S_i}{\sqrt{j - i}}. \quad (1)$$

It has been shown by Siegmund and Venkatraman [24] that for every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[L_n \leq \sqrt{2 \log n} + \frac{\frac{1}{2} \log \log n + \log \frac{H}{2\sqrt{\pi}} + \tau}{\sqrt{2 \log n}} \right] = e^{-e^{-\tau}}, \quad (2)$$

* Tel.: +49 731 5023527.

E-mail addresses: zakhar.kabluchko@uni-ulm.de, kabluch@math.uni-goettingen.de.

where $H \in (0, \infty)$ is some constant. A different proof of the same result has been given in [11] where also the following continuous-time counterpart of (2) can be found. Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. For $n > 1$ define

$$M_n = \sup_{\substack{x, y \in [0, n] \\ y-x \geq 1}} \frac{B(y) - B(x)}{\sqrt{y-x}}. \quad (3)$$

Then, for every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[M_n \leq \sqrt{2 \log n} + \frac{\frac{3}{2} \log \log n - \log(2\sqrt{\pi}) + \tau}{\sqrt{2 \log n}} \right] = e^{-e^{-\tau}}. \quad (4)$$

Almost sure laws of large numbers for L_n , M_n and related quantities have been obtained in [23,26,12,13]. An analogue of L_n in the case of heavy-tailed random variables has been studied in [16].

Our aim here is to prove multidimensional counterparts of (2) and (4). We shall be interested in the maximum of discrete-time or continuous-time d -dimensional Gaussian noise standardized by the square root of its variance. The maximum is taken over some family of d -dimensional subsets. Here, we shall consider two families of subsets, rectangles and cubes, in discrete and continuous settings. Both families are multidimensional generalizations of the collection of one-dimensional intervals.

Let us state our discrete-time results first. Let $\{\xi_k, k \in \mathbb{Z}^d\}$ be a d -dimensional array of independent standard Gaussian random variables. Given a finite set $A \subset \mathbb{Z}^d$ we define

$$\mathbb{S}(A) = \sum_{k \in A} \xi_k. \quad (5)$$

Note that $\text{Var} \mathbb{S}(A) = |A|$, where $|A|$ is the number of elements in the set A .

A set of the form $\{x_1, \dots, x_1 + h\} \times \dots \times \{x_d, \dots, x_d + h\}$, where $x_1, \dots, x_d \in \mathbb{Z}$ and $h \in \mathbb{N} \cup \{0\}$, is called a d -dimensional discrete cube. Denote by \mathfrak{C}^d the set of all discrete d -dimensional cubes and let \mathfrak{C}_n^d be the set of all discrete d -dimensional cubes contained in $\{1, \dots, n\}^d$, where $n \in \mathbb{N}$. Define

$$u_n(\tau) = \sqrt{2d \log n} + \frac{\frac{1}{2} \log(d \log n) + \log \frac{(2d)^d J_d}{\sqrt{\pi}} + \tau}{\sqrt{2d \log n}}, \quad \tau \in \mathbb{R}, \quad (6)$$

where $J_d \in (0, \infty)$ is a constant defined in Lemma 4.1 below.

Theorem 1.1. For every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n(\tau) \right] = e^{-e^{-\tau}}.$$

A set of the form $\{x_1, \dots, y_1\} \times \dots \times \{x_d, \dots, y_d\}$, where $x_i, y_i \in \mathbb{Z}$ and $x_i \leq y_i$ for all $1 \leq i \leq d$, is called a d -dimensional discrete rectangle. Note that a discrete cube is a discrete rectangle whose sides have equal lengths. Denote by \mathfrak{R}^d the collection of all discrete d -dimensional rectangles and let \mathfrak{R}_n^d be the set of all discrete d -dimensional rectangles contained

in $\{1, \dots, n\}^d$, where $n \in \mathbb{N}$. Define

$$u_n(\tau) = \sqrt{2d \log n} + \frac{\left(d - \frac{1}{2}\right) \log(d \log n) + \log \frac{2^{2d-1} d^d G_d^d}{\sqrt{\pi}} + \tau}{\sqrt{2d \log n}}, \quad \tau \in \mathbb{R}, \quad (7)$$

where $G_d \in (0, \infty)$ is a constant defined in Lemma 4.1 below.

Theorem 1.2. For every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{A}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n(\tau) \right] = e^{-e^{-\tau}}.$$

Remark 1.1. The following laws of large numbers hold (see [13]):

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2d \log n}} \max_{A \in \mathfrak{A}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2d \log n}} \max_{A \in \mathfrak{A}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} = 1 \quad \text{a.s.}$$

Remark 1.2. In dimension $d = 1$, both Theorems 1.1 and 1.2 reduce to (2).

We also prove the following continuous-time counterparts of Theorems 1.1 and 1.2. Let $\{\mathbb{W}(A), A \in \mathcal{B}_b(\mathbb{R}^d)\}$ be a Gaussian white noise on \mathbb{R}^d whose intensity is the Lebesgue measure. This means that we are given a zero-mean Gaussian process \mathbb{W} indexed by the collection $\mathcal{B}_b(\mathbb{R}^d)$ of all Borel subsets of \mathbb{R}^d with finite Lebesgue measure such that for every $A_1, A_2 \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\text{Cov}(\mathbb{W}(A_1), \mathbb{W}(A_2)) = \lambda(A_1 \cap A_2),$$

where $\lambda(A)$ denotes the d -dimensional Lebesgue measure of a set $A \in \mathcal{B}_b(\mathbb{R}^d)$.

A set of the form $[x_1, x_1 + h] \times \dots \times [x_d, x_d + h]$, where $x_1, \dots, x_d \in \mathbb{R}$ and $h > 0$, is called a d -dimensional cube. Let \mathcal{C}^d be the collection of all d -dimensional cubes and denote by \mathcal{C}_n^d the set of all cubes contained in $[0, n]^d$, where $n > 0$ need not be integer. Endow \mathcal{C}^d with its natural topology inherited from the identification $\mathcal{C}^d = \mathbb{R}^d \times (0, \infty)$. It is well-known that the process $\{\mathbb{W}(A), A \in \mathcal{C}^d\}$ has a version with a.s. continuous paths. This may be deduced from the continuity of the Brownian sheet process. In the sequel, we shall always deal with such a continuous version. It is not difficult to see that the supremum of the standardized white noise $\mathbb{W}(A)/\sqrt{\lambda(A)}$ taken over all $A \in \mathcal{C}_n^d$ does not exist due to the singularity emerging as the volume of A approaches 0. To avoid the singularity, we fix some $a > 0$ and define $\mathcal{C}_n^d(a)$ to be the set of all cubes from \mathcal{C}_n^d whose side length h satisfies $h \geq a$. With a constant E_d to be specified below (see (30)), define

$$u_n(\tau) = \sqrt{2d \log n} + \frac{\left(d + \frac{1}{2}\right) \log(d \log n) + \log \frac{2^d E_d}{d a^d \sqrt{\pi}} + \tau}{\sqrt{2d \log n}}, \quad \tau \in \mathbb{R}. \quad (8)$$

Theorem 1.3. For every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a)} \frac{\mathbb{W}(A)}{\sqrt{\lambda(A)}} \leq u_n(\tau) \right] = e^{-e^{-\tau}}.$$

A set of the form $[x_1, y_1] \times \cdots \times [x_d, y_d]$, where $x_i, y_i \in \mathbb{R}$ and $x_i < y_i$ for all $1 \leq i \leq d$, is called a d -dimensional rectangle. Note that a d -dimensional cube is a d -dimensional rectangle with equal side lengths. Let \mathcal{R}^d be the collection of all d -dimensional rectangles. We denote by \mathcal{R}_n^d the set of all rectangles contained in $[0, n]^d$, where $n > 0$ need not be integer. The set \mathcal{R}^d is endowed with the topology inherited from the natural embedding $\mathcal{R}^d \subset \mathbb{R}^{2d}$. Let $\{\mathbb{W}(A), A \in \mathcal{B}_b(\mathbb{R}^d)\}$ be a white noise on \mathbb{R}^d . We shall always deal with an a.s. continuous version of the random field $\{\mathbb{W}(A), A \in \mathcal{R}^d\}$. The existence of such a version is a consequence of the continuity of the Brownian sheet process. Given $a > 0$ we define $\mathcal{R}_n^d(a)$ to be the set of all rectangles $[x_1, y_1] \times \cdots \times [x_d, y_d]$ contained in $[0, n]^d$ such that $y_i - x_i \geq a$ for all $1 \leq i \leq d$. We set

$$u_n(\tau) = \sqrt{2d \log n} + \frac{\left(2d - \frac{1}{2}\right) \log(d \log n) - \log(2a^d \sqrt{\pi}) + \tau}{\sqrt{2d \log n}}, \quad \tau \in \mathbb{R}. \quad (9)$$

Theorem 1.4. For every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a)} \frac{\mathbb{W}(A)}{\sqrt{\lambda(A)}} \leq u_n(\tau) \right] = e^{-e^{-\tau}}.$$

Remark 1.3. Using methods similar to that of [13] it is possible to prove the following laws of large numbers: for every $a > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2d \log n}} \sup_{A \in \mathcal{C}_n^d(a)} \frac{\mathbb{W}(A)}{\sqrt{\lambda(A)}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2d \log n}} \sup_{A \in \mathcal{R}_n^d(a)} \frac{\mathbb{W}(A)}{\sqrt{\lambda(A)}} = 1 \quad \text{a.s.}$$

Remark 1.4. For dimension $d = 1$, both Theorems 1.3 and 1.4 reduce to (4).

The maximum of the standardized i.i.d. noise over the set of discrete rectangles has been studied in [25], where, in particular, our Proposition 4.3 can be found. The arguments of [25] are somewhat heuristical; we use a different method.

2. The asymptotic extreme-value rate

It is well-known that the maximum of a large number of dependent random variables behaves in the same way as the maximum of the same number of independent random variables provided the dependence between the variables is weak enough; see [14, Ch. 3,4] and [7, Ch. 9]. It should be stressed that the results of Section 1 do not fall into this category. To compare the behavior of the maxima of the standardized Gaussian noise to the behavior of independent Gaussian random variables, we introduce a notion of asymptotic extreme-value rate which is of independent interest. To begin with, recall the well-known fact (see [14]) that if $\{\xi_k, k \in \mathbb{N}\}$ are independent standard Gaussian random variables, then for every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{k=1, \dots, n} \xi_k \leq u_n(\tau) \right] = e^{-e^{-\tau}}, \quad (10)$$

where $u_n(\tau)$ is given by

$$u_n(\tau) = \sqrt{2 \log n} + \frac{-\frac{1}{2} \log \log n - \log 2\sqrt{\pi} + \tau}{\sqrt{2 \log n}}, \quad \tau \in \mathbb{R}. \quad (11)$$

Definition 2.1. For every $n \in \mathbb{N}$ let a zero-mean, unit-variance Gaussian field $\mathbb{X}_n = \{\mathbb{X}_n(t), t \in T_n\}$ defined on some parameter space T_n be given. Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be some function satisfying $\lim_{n \rightarrow \infty} f(n) = +\infty$. We say that the sequence of random fields \mathbb{X}_n has extreme-value rate f if for every $\tau \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{t \in T_n} \mathbb{X}_n(t) \leq u_{f(n)}(\tau) \right] = e^{-e^{-\tau}}, \quad (12)$$

where $u_n(\tau)$ is defined as in (11).

Roughly speaking, condition (12) states that the supremum of \mathbb{X}_n has the same asymptotic behavior as the supremum of $f(n)$ i.i.d. standard Gaussian variables. The next elementary lemma is useful for computing extreme-value rates.

Lemma 2.1. Let $u_n(\tau)$ be given by (11) and let $f(n) = \alpha n^\beta (\log n)^\gamma$ for some $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$. Then, as $n \rightarrow \infty$,

$$u_{f(n)}(\tau) = \sqrt{2\beta \log n} + \frac{(\gamma - \frac{1}{2}) \log(\beta \log n) + \log \frac{\alpha}{2\beta^\gamma \sqrt{\pi}} + \tau + o(1)}{\sqrt{2\beta \log n}}. \quad (13)$$

The extreme-value rates of the standardized Gaussian noise over various collections of subsets can be now evaluated by comparing the results of Section 1 with Lemma 2.1. By the convergence to types lemma, the $o(1)$ -term on the right-hand side of (13) can be discarded while computing the extreme-value rates.

Collection of subsets T_n	Extreme-value rate $f(n)$
$T_n = \mathfrak{C}_n^d$, discrete cubes in $\{1, \dots, n\}^d$	$(2d)^{d+1} J_d n^d \log n$
$T_n = \mathfrak{R}_n^d$, discrete rectangles in $\{1, \dots, n\}^d$	$(2d)^{2d} G_d^d n^d (\log n)^d$
$T_n = \mathcal{C}_n^d(a)$, cubes in $[0, n]^d$ with side $\geq a$	$2E_d(\frac{2d}{a})^d n^d (\log n)^{d+1}$
$T_n = \mathcal{R}_n^d(a)$, rectangles in $[0, n]^d$ with sides $\geq a$	$\frac{d^{2d}}{a^d} n^d (\log n)^{2d}$

The rest of the paper is organized as follows. In Section 3 we recall the definition of locally self-similar Gaussian fields. Applications of this notion will be given in Section 4. In Section 5 we recall a Poisson limit theorem for finite-range dependent events. The proofs of our results are given in Sections 6 and 7.

Throughout the paper we use the following notation. We denote d -dimensional vectors by $\mathbf{x} = (x_1, \dots, x_d), \mathbf{y} = (y_1, \dots, y_d)$ etc. We write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for all $1 \leq i \leq d$. Given some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ with $\mathbf{x} \leq \mathbf{y}$, we denote by $[\mathbf{x}, \mathbf{y}]$ the d -dimensional rectangle $[x_1, y_1] \times \dots \times [x_d, y_d]$. Given $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ with $\mathbf{x} \leq \mathbf{y}$, we denote by $[\mathbf{x}, \mathbf{y}]_{\mathbb{Z}^d}$ the discrete d -dimensional rectangle consisting of all $\mathbf{z} \in \mathbb{Z}^d$ such that $x_i \leq z_i < y_i$ for all $1 \leq i \leq d$. We denote by C a large positive constant whose value may change from line to line.

3. Locally self-similar Gaussian fields

The main tool in our proofs is the extreme-value theory of continuous-time Gaussian processes initiated by Pickands [17,18]. Important results in this direction are due to Berman [4–7]. Let $\{\mathbb{X}(t), t \in \mathbb{R}\}$ be a stationary, a.s. continuous zero-mean Gaussian process whose covariance function $r(t) = \mathbb{E}[\mathbb{X}(0)\mathbb{X}(t)]$ satisfies $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ as $t \rightarrow 0$ for some $C > 0$, $\alpha \in (0, 2]$. Suppose that $r(t) = 1$ holds only for $t = 0$. Under these conditions, Pickands [17] proved the asymptotic equality

$$\mathbb{P} \left[\sup_{t \in [0, l]} \mathbb{X}(t) > u \right] \sim \frac{l H_\alpha C^{1/\alpha}}{\sqrt{2\pi}} u^{2/\alpha-1} e^{-u^2/2}, \quad u \rightarrow +\infty, \quad (14)$$

for all $l > 0$, where $H_\alpha \in (0, \infty)$ is the so-called Pickands constant. Only the values $H_1 = 1$ and $H_2 = \pi^{-1/2}$ are known explicitly. Neither the assumption of stationarity nor the one-dimensionality of the parameter set of the process is relevant for Pickands' approach. His result has been extended in various directions by Qualls and Watanabe [21,22], Bickel and Rosenblatt [8], Hüsler [10], Albin [1] (who has presented numerous subsequent works on this topic), Piterbarg and Fatalov [20], Mikhaleva and Piterbarg [15] and Chan and Lai [9]; see also the monographs [19], [14, Ch. 12], [7]. A general non-rigorous approach, called the Poisson clumping heuristics, has been suggested by Aldous [2].

We recall a result of Chan and Lai [9] (see also [15]) which will play a fundamental role in the sequel. It is an extension of (14) to locally self-similar fields (also called locally stationary fields), a class of Gaussian fields satisfying certain local conditions. A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is called homogeneous of order $\alpha > 0$ if $f(vs) = |v|^\alpha f(s)$ for every $s \in \mathbb{R}^d$ and $v \in \mathbb{R}$. Let $H(\alpha)$ be the linear space of all continuous homogeneous functions of order α endowed with the norm $\|f\| = \sup_{\|t\|_2=1} f(t)$. Here, $\|t\|_2$ is the Euclidean norm of $t \in \mathbb{R}^d$. We may identify $H(\alpha)$ with the Banach space of continuous functions, symmetric with respect to the origin, on the unit sphere in \mathbb{R}^d . Let $H^+(\alpha)$ be the cone of all strictly positive functions in $H(\alpha)$.

Definition 3.1 (see [9]). Let $\{\mathbb{X}(t), t \in D\}$ be an a.s. continuous, zero-mean, unit-variance Gaussian random field defined on some domain $D \subset \mathbb{R}^d$. Let $r(t_1, t_2) = \mathbb{E}[\mathbb{X}(t_1)\mathbb{X}(t_2)]$ be the covariance function of \mathbb{X} and assume that $r(t_1, t_2) < 1$ for $t_1 \neq t_2$. The field \mathbb{X} is called *locally self-similar* with index $\alpha \in (0, 2]$ if for every $t \in D$ there exists a function $C_t \in H^+(\alpha)$ such that the following two conditions hold:

1. We have $\lim_{\|s\|_2 \rightarrow 0} \frac{1-r(t, t+s)}{C_t(s)} = 1$, where the convergence is uniform in $t \in K$ for every compact set $K \subset D$.
2. The map $C_\bullet : D \rightarrow H^+(\alpha)$, sending t to C_t , is continuous.

The collection of homogeneous functions C_t , $t \in D$, is referred to as the *local structure* of the field \mathbb{X} . It can be shown (see [11]) that for every $t \in D$ there exists a finite measure Γ_t , symmetric with respect to the origin, on the unit sphere in \mathbb{R}^d such that the following representation holds:

$$C_t(s) = \int_{\mathbb{S}^{d-1}} |\langle s, x \rangle|^\alpha \Gamma_t(dx).$$

The next theorem, proved by Chan and Lai [9] (see also [15] for a similar result), describes the asymptotic behavior of the high excursion probability of a locally self-similar Gaussian field.

Theorem 3.1 (see [9,15]). Let $\{\mathbb{X}(t), t \in D\}$ be a Gaussian field defined on some domain $D \subset \mathbb{R}^d$. Suppose that \mathbb{X} is locally self-similar of index α with local structure $C_t(s)$. Let $K \subset D$ be a compact set with positive Jordan measure. Then, as $u \rightarrow +\infty$,

$$\mathbb{P} \left[\sup_{t \in K} \mathbb{X}(t) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left(\int_K \Lambda(t) dt \right) u^{\frac{2d}{\alpha}-1} e^{-u^2/2}, \quad (15)$$

where the function $\Lambda : D \rightarrow (0, \infty)$ is defined in (18) below.

The function Λ , which might be called the *high excursion intensity* of \mathbb{X} , is defined as follows. For each $t \in D$, let $\{Y_t(s), s \in \mathbb{R}^d\}$ be an a.s. continuous Gaussian field such that for all $s, s_1, s_2 \in \mathbb{R}^d$,

$$\mathbb{E} Y_t(s) = -C_t(s), \quad (16)$$

$$\text{Cov}(Y_t(s_1), Y_t(s_2)) = C_t(s_1) + C_t(s_2) - C_t(s_1 - s_2). \quad (17)$$

The field Y_t describes the local behavior of the field \mathbb{X} conditioned to reach an extremely high value at t and will be therefore called the *tangent process* of \mathbb{X} in the sequel. Note that $\{Y_t(s) + C_t(s), s \in \mathbb{R}^d\}$ is a zero-mean self-similar Gaussian field with stationary increments, a non-isotropic generalization of the fractional Brownian motion. With the above notation, it has been shown in [9] that the following limit exists in $(0, \infty)$ and is a continuous function of t :

$$\Lambda(t) = \lim_{T \rightarrow \infty} \frac{1}{T^d} \mathbb{E} \left[\exp \left(\sup_{s \in [0, T]^d} Y_t(s) \right) \right]. \quad (18)$$

The following theorem has been obtained in [9] as a by-product of the above Theorem 3.1. It describes the asymptotic behavior of the high excursion probability over a finite grid with mesh size going to 0. For one-dimensional stationary processes it can be found in [14, Lemma 12.2.4].

Theorem 3.2. Suppose that the conditions of Theorem 3.1 are satisfied. Let $u \uparrow \infty$ and $q \downarrow 0$ in such a way that $qu^{2/\alpha} \rightarrow \kappa$ for some constant $\kappa > 0$. Then,

$$\mathbb{P} \left[\max_{t \in K \cap q\mathbb{Z}^d} \mathbb{X}(t) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left(\int_K \Lambda(t; \kappa) dt \right) u^{\frac{2d}{\alpha}-1} e^{-u^2/2}, \quad (19)$$

where

$$\Lambda(t; \kappa) = \lim_{T \rightarrow \infty} \frac{1}{T^d} \mathbb{E} \left[\exp \left(\max_{s \in [0, T]^d \cap \kappa\mathbb{Z}^d} Y_t(s) \right) \right]. \quad (20)$$

Furthermore, $\lim_{\kappa \downarrow 0} \Lambda(t; \kappa) = \Lambda(t)$, where $\Lambda(t)$ is the high excursion intensity of \mathbb{X} .

4. Applications to the standardized Gaussian noise

The next example (see [2, J25], [9]) will play a major role in the sequel. Let $\{B(t), t \in \mathbb{R}\}$ be the standard Brownian motion. Let $\mathcal{D} = \{(x, y) \in \mathbb{R}^2 | x < y\}$ be the space of all intervals in \mathbb{R} . Define a Gaussian field $\{\mathbb{U}(x, y), (x, y) \in \mathcal{D}\}$ by

$$\mathbb{U}(x, y) = \frac{B(y) - B(x)}{\sqrt{y - x}}. \quad (21)$$

It is elementary to verify (see [2, J25], [9,11]) that the field \mathbb{U} is locally self-similar with index $\alpha = 1$ and a local structure given by

$$C_{(x,y)}(p,q) = \frac{1}{2} \cdot \frac{|p| + |q|}{y - x}, \quad (x,y) \in \mathcal{D}, \quad (p,q) \in \mathbb{R}^2. \quad (22)$$

We shall be interested in multidimensional generalizations of this example. Recall that a rectangle in \mathbb{R}^d is a set of the form $[\mathbf{x}, \mathbf{y}] = [x_1, y_1] \times \cdots \times [x_d, y_d]$, where $x_i < y_i$ for all $1 \leq i \leq d$. We can identify the collection \mathcal{R}^d of rectangles with \mathcal{D}^d . Also, we shall sometimes identify a rectangle $[\mathbf{x}, \mathbf{y}] \in \mathcal{R}^d$ with a pair $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2d}$. Denote by $\{\mathbb{W}(A), A \in \mathcal{R}^d\}$ a white noise indexed by rectangles. Recall our standing assumption that the sample paths of \mathbb{W} are continuous. Define a Gaussian random field $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$, called the standardized white noise, by

$$\mathbb{X}(A) = \frac{\mathbb{W}(A)}{\sqrt{\lambda(A)}}. \quad (23)$$

The covariance function $r_{\mathbb{X}}$ of the random field $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ is an n -fold tensor product of the covariance function $r_{\mathbb{U}}$ of the process \mathbb{U} from (21), i.e., if $A = [\mathbf{x}, \mathbf{y}]$ and $A' = [\mathbf{x}', \mathbf{y}']$ are in \mathcal{R}^d , then

$$r_{\mathbb{X}}(A, A') = \prod_{i=1}^d r_{\mathbb{U}}((x_i, y_i), (x'_i, y'_i)). \quad (24)$$

Proposition 4.1. *Let $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ be the standardized white noise defined in (23). Let $\mathcal{I} \subset \mathcal{R}^d$ be a compact subset of the space of rectangles \mathcal{R}^d having a positive Jordan measure. Then, as $u \rightarrow +\infty$,*

$$\mathbb{P} \left[\sup_{A \in \mathcal{I}} \mathbb{X}(A) > u \right] \sim \frac{1}{4^d \sqrt{2\pi}} \left(\int_{\mathcal{I}} \frac{d\mathbf{x} d\mathbf{y}}{\prod_{i=1}^d (y_i - x_i)^2} \right) u^{4d-1} e^{-u^2/2}. \quad (25)$$

Proof. It follows from Definition 3.1, Eqn. (22) and the tensor product structure (24) that the random field $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ is locally self-similar with index $\alpha = 1$ and its local structure is given by

$$C_{\mathbf{x},\mathbf{y}}(\mathbf{p}, \mathbf{q}) = \frac{1}{2} \sum_{i=1}^d \frac{|p_i| + |q_i|}{y_i - x_i}, \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^d. \quad (26)$$

Let $\{V_i(s), s \in \mathbb{R}\}$ and $\{W_i(s), s \in \mathbb{R}\}$, $1 \leq i \leq d$, be independent standard Brownian motions with drift $-|s|/2$. It follows from (16), (17) and (26) that the tangent process of $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ at $[\mathbf{x}, \mathbf{y}] \in \mathcal{R}^d$ is given (in distribution) by

$$Y_{\mathbf{x},\mathbf{y}}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^d \left(V_i \left(\frac{p_i}{y_i - x_i} \right) + W_i \left(\frac{q_i}{y_i - x_i} \right) \right), \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^d. \quad (27)$$

We compute the high excursion intensity Λ given in (18). Using a simple change of variables and the independence of the processes $V_1, W_1, \dots, V_d, W_d$, we obtain that

$$\begin{aligned}\Lambda(\mathbf{x}, \mathbf{y}) &= \frac{1}{4^d \prod_{i=1}^d (y_i - x_i)^2} \lim_{T \rightarrow \infty} \frac{1}{T^{2d}} \mathbb{E} \left[\exp \sup_{\mathbf{p}, \mathbf{q} \in [0, T]^d} \sum_{i=1}^d (V_i(2p_i) + W_i(2q_i)) \right] \\ &= \frac{1}{4^d \prod_{i=1}^d (y_i - x_i)^2} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\exp \sup_{p_1 \in [0, T]} V_1(2p_1) \right] \right\}^{2d}.\end{aligned}$$

The limit on the right-hand side is the Pickands constant $H_1 = 1$; see [17]. Hence, $\Lambda(\mathbf{x}, \mathbf{y}) = 4^{-d} \prod_{i=1}^d (y_i - x_i)^{-2}$. The proposition follows now by Theorem 3.1. \square

Next we prove a similar result for the maximum of the standardized white noise over a subset of the space of cubes. We identify a cube $[\mathbf{x}, \mathbf{x} + h]$ with the point $(\mathbf{x}, h) \in \mathbb{R}^d \times (0, \infty)$.

Proposition 4.2. *Let $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ be the standardized white noise as in (23). Let $\mathcal{I} \subset \mathcal{C}^d$ be a compact subset of the space of cubes \mathcal{C}^d having a positive Jordan measure. There is a constant $E_d > 0$ such that as $u \rightarrow +\infty$,*

$$\mathbb{P} \left[\sup_{A \in \mathcal{I}} \mathbb{X}(A) > u \right] \sim \frac{E_d}{\sqrt{2\pi}} \cdot \left(\int_{\mathcal{I}} \frac{d\mathbf{x}dh}{h^{d+1}} \right) u^{2d+1} e^{-u^2/2}. \quad (28)$$

Proof. The random field $\{\mathbb{X}(A), A \in \mathcal{C}^d\}$ is locally self-similar with index $\alpha = 1$ since it is a restriction of a locally self-similar field $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ to a linear subspace. Let V_i and W_i , $1 \leq i \leq d$, be drifted Brownian motions as in the previous proof. The tangent process defined in (16) and (17) is given by

$$Y_{\mathbf{x}, h}(\mathbf{p}, g) = \sum_{i=1}^d \left(V_i \left(\frac{p_i}{h} \right) + W_i \left(\frac{p_i + g}{h} \right) \right), \quad \mathbf{p} \in \mathbb{R}^d, g \in \mathbb{R}. \quad (29)$$

The high excursion intensity $\Lambda(\mathbf{x}, h)$ defined in (18) is given by

$$\Lambda(\mathbf{x}, h) = \lim_{T \rightarrow \infty} \frac{1}{T^{d+1}} \mathbb{E} \left[\exp \sup_{\substack{\mathbf{p} \in [0, T]^d \\ g \in [0, T]}} \sum_{i=1}^d \left(V_i \left(\frac{p_i}{h} \right) + W_i \left(\frac{p_i + g}{h} \right) \right) \right].$$

A change of variables shows that we have $\Lambda(\mathbf{x}, h) = h^{-(d+1)} E_d$ where E_d is a constant given by

$$E_d = \lim_{T \rightarrow \infty} \frac{1}{T^{d+1}} \mathbb{E} \left[\exp \sup_{\substack{\mathbf{p} \in [0, T]^d \\ g \in [0, T]}} \sum_{i=1}^d (V_i(p_i) + W_i(p_i + g)) \right]. \quad (30)$$

The proof is completed by applying Theorem 3.1. \square

Next we consider the maximum of the standardized Gaussian noise taken over a set of discrete rectangles. Let $\{B(s), s \geq 0\}$ be a standard Brownian motion. For $h, \kappa > 0$ define

$$G(h; \kappa) = \frac{1}{h^2} F^2 \left(\frac{\kappa}{h} \right), \quad F(\kappa) = \lim_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\exp \max_{s \in [0, T] \cap \kappa \mathbb{Z}} \left(B(s) - \frac{s}{2} \right) \right]. \quad (31)$$

Using the fluctuation theory of random walks it can be shown (see [11]) that with $\bar{\Phi}(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-w^2/2} dw$ denoting the tail of the standard normal law,

$$F(\kappa) = \frac{1}{\kappa} \exp \left\{ -2 \sum_{n=1}^{\infty} \frac{1}{n} \bar{\Phi} \left(\frac{1}{2} \sqrt{\kappa n} \right) \right\}. \quad (32)$$

Proposition 4.3. *Let $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ be the standardized white noise as in (23). Let $\mathcal{I} \subset \mathcal{R}^d$ be a compact subset of the space of rectangles \mathcal{R}^d having a positive Jordan measure. Let $u \uparrow \infty$ and $q \downarrow 0$ in such a way that $qu^2 \rightarrow \kappa$ for some constant $\kappa > 0$. Then,*

$$\mathbb{P} \left[\max_{A \in \mathcal{I} \cap q\mathbb{Z}^{2d}} \mathbb{X}(A) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left(\int_{\mathcal{I}} \prod_{i=1}^d G(y_i - x_i; \kappa) dx dy \right) u^{4d-1} e^{-u^2/2}. \quad (33)$$

Proof. We compute the function $\Lambda(\mathbf{x}, \mathbf{y}; \kappa)$ given in Theorem 3.2. Recall that the tangent process is given by (27). Setting $h_i := y_i - x_i$ and making a change of variables, we obtain that $\Lambda(\mathbf{x}, \mathbf{y}; \kappa)$ is equal to

$$\frac{1}{\prod_{i=1}^d h_i^2} \lim_{T \rightarrow \infty} \frac{1}{T^{2d}} \mathbb{E} \exp \left(\sum_{i=1}^d \max_{\substack{p_i \in [0, T] \\ p_i \in \kappa h_i^{-1} \mathbb{Z}}} V_i(p_i) + \sum_{i=1}^d \max_{\substack{q_i \in [0, T] \\ q_i \in \kappa h_i^{-1} \mathbb{Z}}} W_i(q_i) \right).$$

Consequently, by the independence of $V_1, \dots, V_d, W_1, \dots, W_d$ and (31), we obtain $\Lambda(\mathbf{x}, \mathbf{y}; \kappa) = G(h_1; \kappa) \dots G(h_d; \kappa)$. The proposition follows from Theorem 3.2. \square

Proposition 4.4. *Let $\{\mathbb{X}(A), A \in \mathcal{R}^d\}$ be the standardized white noise defined in (23). Let $\mathcal{I} \subset \mathcal{C}^d$ be a compact subset of the space of cubes \mathcal{C}^d having a positive Jordan measure. Let $u \uparrow \infty$ and $q \downarrow 0$ in such a way that $qu^2 \rightarrow \kappa$ for some constant $\kappa > 0$. Then, with a function $J_d(h; \kappa)$ defined in (35) and (36) below,*

$$\mathbb{P} \left[\max_{A \in \mathcal{I} \cap q\mathbb{Z}^{d+1}} \mathbb{X}(A) > u \right] \sim \frac{1}{\sqrt{2\pi}} \left(\int_{\mathcal{I}} J_d(h; \kappa) dx dh \right) u^{2d+1} e^{-u^2/2}. \quad (34)$$

Proof. Recall that the tangent process is given by (29). The high excursion intensity $\Lambda(\mathbf{x}, h; \kappa)$ defined in Theorem 3.2 is given by

$$\Lambda(\mathbf{x}, h; \kappa) = \lim_{T \rightarrow \infty} \frac{1}{T^{d+1}} \mathbb{E} \left[\exp \max_{\substack{p \in [0, T]^d \cap \kappa \mathbb{Z}^d \\ g \in [0, T] \cap \kappa \mathbb{Z}}} \sum_{i=1}^d \left(V_i \left(\frac{p_i}{h} \right) + W_i \left(\frac{p_i + g}{h} \right) \right) \right].$$

By a change of variables, we have $\Lambda(\mathbf{x}, h; \kappa) = J_d(h; \kappa)$, where

$$J_d(h; \kappa) = h^{-(d+1)} E_d(\kappa/h), \quad (35)$$

$$E_d(\kappa) = \lim_{T \rightarrow \infty} \frac{1}{T^{d+1}} \mathbb{E} \left[\exp \max_{\substack{p \in [0, T]^d \cap \kappa \mathbb{Z}^d \\ g \in [0, T] \cap \kappa \mathbb{Z}}} \sum_{i=1}^d (V_i(p_i) + W_i(p_i + g)) \right]. \quad (36)$$

The proposition follows from Theorem 3.2. \square

Lemma 4.1. Define $J_d(h) = J_d(h; 2d)$ by (35), (36) and $G_d(h) = G(h; 2d)$ by (31) with $\kappa = 2d$. Then, $J_d := \int_0^\infty J_d(h)dh < \infty$ and $G_d := \int_0^\infty G_d(h)dh < \infty$.

Proof. We have $\lim_{\kappa \downarrow 0} F(\kappa) = 1/2$ by [14, Ch. 12] and $\lim_{\kappa \downarrow 0} E_d(\kappa) = E_d$ by [9]. It follows that $G_d(h) \sim 1/(4h^2)$ and $J_d(h) \sim E_d/h^{d+1}$ as $h \uparrow +\infty$. Hence, $G_d < \infty$ and $J_d < \infty$. \square

5. A Poisson limit theorem for dependent events

In this section we recall a Poisson limit theorem for finite-range dependent random events. This result will be needed in our proofs. A more general statement can be found in [3, Thm. 1]. For every $N \in \mathbb{N}$ let E_{1N}, \dots, E_{NN} be (in general, dependent) random events such that $\mathbb{P}[E_{iN}] = p_N$ for all $1 \leq i \leq N$, where p_N is a sequence satisfying $\mu := \lim_{N \rightarrow \infty} Np_N \in (0, \infty)$. We assume that the events E_{1N}, \dots, E_{NN} are finite-range dependent in the following sense: there exist B_{1N}, \dots, B_{NN} , subsets of $\{1, \dots, N\}$, such that the following three conditions are satisfied:

1. There is a constant $C > 0$ not depending on N such that $|B_{iN}| < C$ for every $1 \leq i \leq N$ and $N \in \mathbb{N}$.
2. For every $1 \leq i \leq N$, the random event E_{iN} is independent of the collection $\{E_{jN}, j \notin B_{iN}\}$.
3. We have $\lim_{N \rightarrow \infty} \sum_{i=1}^N \sum_{j \in B_{iN} \setminus \{i\}} \mathbb{P}[E_{iN} \cap E_{jN}] = 0$.

Theorem 5.1. Under the above assumptions, the distribution of the random variable $\sum_{i=1}^N \mathbf{1}_{E_{iN}}$ converges as $N \rightarrow \infty$ to the Poisson distribution with mean μ . In particular,

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\bigcap_{i=1}^N E_{iN}^c \right] = e^{-\mu}. \quad (37)$$

6. Proofs in the continuous-time case

6.1. Proof of Theorem 1.3

Let $\mathbb{X}(A) = \mathbb{W}(A)/\sqrt{\lambda(A)}$ be the standardized white noise. Recall that \mathcal{C}^d is the set of all d -dimensional cubes and \mathcal{C}_n^d is the set of all d -dimensional cubes contained in $[0, n]^d$. For $0 < a < b \leq n$ let $\mathcal{C}_n^d(a, b)$ be the set of all cubes $[\mathbf{x}, \mathbf{x} + h] \in \mathcal{C}_n^d$ such that $h \in [a, b]$. In the next lemma we shall evaluate the high excursion probability over the set $\mathcal{C}_n^d(a, b)$.

Lemma 6.1. Fix some $\tau \in \mathbb{R}$, $0 < a < b$, and let $u_n = u_n(\tau)$ be defined by (8). We have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, b)} \mathbb{X}(A) \leq u_n \right] = e^{-e^{-\tau} \left(1 - \left(\frac{a}{b}\right)^d\right)}.$$

Proof. For $\mathbf{k} \in \mathbb{Z}^d$ define $\mathcal{I}_{\mathbf{k}}$ to be the set of cubes of the form $[\mathbf{x}, \mathbf{x} + h] \in \mathcal{C}^d$ such that $h \in [a, b]$ and $x_i \in [k_i, k_i + 1]$ for every $1 \leq i \leq d$. Let $E_{\mathbf{k}, n}$ be the random event $\{\sup_{A \in \mathcal{I}_{\mathbf{k}}} \mathbb{X}(A) > u_n\}$. Note that by the translation invariance, the probability $p_n := \mathbb{P}[E_{\mathbf{k}, n}]$ is independent of $\mathbf{k} \in \mathbb{Z}^d$. By Proposition 4.2 and (8), we have as $n \rightarrow \infty$,

$$p_n \sim \frac{E_d}{\sqrt{2\pi}} u_n^{2d+1} e^{-u_n^2/2} \int_{\mathcal{I}_{\mathbf{k}}} \frac{d\mathbf{x}dh}{h^{d+1}} \sim \frac{da^d e^{-\tau}}{n^d} \int_a^b \frac{dh}{h^{d+1}} \sim \frac{e^{-\tau}}{n^d} \left(1 - \left(\frac{a}{b}\right)^d\right). \quad (38)$$

To prove the lemma, we shall verify the assumptions of Section 5. Define index sets $K'_n = [0, n]^d \cap \mathbb{Z}^d$ and $K''_n = [0, n - b - 1]^d \cap \mathbb{Z}^d$. Clearly, we have

$$\bigcap_{k \in K'_n} E_{k,n}^c \subset \left\{ \sup_{A \in \mathcal{C}_n^d(a,b)} \mathbb{X}(A) \leq u_n \right\} \subset \bigcap_{k \in K''_n} E_{k,n}^c. \quad (39)$$

We shall show that the families of random events $\{E_{k,n}, k \in K'_n\}$ and $\{E_{k,n}, k \in K''_n\}$ satisfy the assumptions of Theorem 5.1. Note that we have $|K'_n| \sim n^d$ and $|K''_n| \sim n^d$ as $n \rightarrow \infty$. Hence, by (38),

$$\lim_{n \rightarrow \infty} p_n |K'_n| = \lim_{n \rightarrow \infty} p_n |K''_n| = e^{-\tau} \left(1 - \left(\frac{a}{b} \right)^d \right).$$

The events $E_{k,n}$ are finite-range dependent, i.e., $E_{k_1,n}$ and $E_{k_2,n}$ are independent provided that $\|k_1 - k_2\|_\infty > b + 1$. Hence, conditions 1 and 2 of Section 5 are satisfied. By Proposition 4.2,

$$\mathbb{P}[E_{k_1,n} \cup E_{k_2,n}] \sim \frac{E_d}{\sqrt{2\pi}} u_n^{2d+1} e^{-u_n^2/2} \int_{\mathcal{I}_{k_1} \cup \mathcal{I}_{k_2}} \frac{dx dh}{h^{d+1}} \sim 2 \frac{e^{-\tau}}{n^d} \left(1 - \left(\frac{a}{b} \right)^d \right). \quad (40)$$

It follows from (38) and (40) that uniformly in $k_1, k_2 \in \mathbb{Z}^d$, we have $\mathbb{P}[E_{k_1,n} \cap E_{k_2,n}] = o(1/n^d)$ as $n \rightarrow \infty$. Together with the finite-range dependence, this implies that condition 3 of Section 5 is satisfied. By Theorem 5.1, we have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\bigcap_{k \in K'_n} E_{k,n}^c \right] = \lim_{n \rightarrow \infty} \mathbb{P} \left[\bigcap_{k \in K''_n} E_{k,n}^c \right] = e^{-e^{-\tau} \left(1 - \left(\frac{a}{b} \right)^d \right)}. \quad (41)$$

The statement of the lemma follows from (39) and (41). \square

To prove Theorem 1.3 we need to take the limit $b \rightarrow +\infty$ in Lemma 6.1. The next lemma estimates the high excursion probability over $\mathcal{C}_n^d(b) = \mathcal{C}_n^d(b, n)$, the set of all cubes $[x, x + h] \in \mathcal{C}_n^d$ such that $h \geq b$.

Lemma 6.2. *There is a constant C such that for every $0 < b < n$ and $u > 1$,*

$$\mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(b,n)} \mathbb{X}(A) > u \right] \leq C b^{-d} u^{2d+1} e^{-u^2/2} n^d.$$

Proof. For $k \in \mathbb{Z}^d$ and $l \in \mathbb{Z}$ denote by $\mathcal{I}_{k,l}$ the set of all cubes $[x, x + h] \in \mathcal{C}^d$ such that $h \in [2^l, 2^{l+1}]$ and $x_i \in [2^l k_i, 2^l(k_i + 1)]$ for all $1 \leq i \leq d$. By Proposition 4.2, for every $u > 1$,

$$\mathbb{P} \left[\sup_{A \in \mathcal{I}_{k,l}} \mathbb{X}(A) > u \right] \leq C u^{2d+1} e^{-u^2/2} \int_{\mathcal{I}_{k,l}} \frac{dx dh}{h^{d+1}} \leq C u^{2d+1} e^{-u^2/2}. \quad (42)$$

Note that the constant C is independent of k, l since the left-hand side does not depend on k, l by the affine invariance. Without restriction of generality, we may assume that $n = 2^L$ and $b = 2^{L_b}$ for some $L, L_b \in \mathbb{Z}$. Otherwise, we may replace n by $2^{\lceil \log_2 n \rceil}$ and b by $2^{\lceil \log_2 b \rceil}$. The set $\mathcal{C}_n^d(b, n)$ can be written as a union of sets of the form $\mathcal{C}_n^d(2^l, 2^{l+1})$, $l = L_b, \dots, L - 1$. Now, the set $\mathcal{C}_n^d(2^l, 2^{l+1})$ can be covered by $n^d/2^{ld}$ sets of the form $\mathcal{I}_{k,l}$. Hence, we can cover the set $\mathcal{C}_n^d(b, n)$ by at most $\sum_{l=L_b}^{L-1} (n^d/2^{ld}) \leq 2n^d/b^d$ sets of the form $\mathcal{I}_{k,l}$. The statement of the lemma follows by applying to each of these sets (42). \square

We are now in position to complete the proof of [Theorem 1.3](#). Fix $a > 0$ and let $u_n = u_n(\tau)$ be chosen as in [\(8\)](#). We have $\mathcal{C}_n^d(a, b) \subset \mathcal{C}_n^d(a, n)$ for every $a < b \leq n$. By [Lemma 6.1](#) we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, n)} \mathbb{X}(A) \leq u_n \right] &\leq \lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, b)} \mathbb{X}(A) \leq u_n \right] \\ &= e^{-e^{-\tau} \left(1 - \left(\frac{a}{b}\right)^d\right)}. \end{aligned} \quad (43)$$

Let us prove a converse inequality. By [\(8\)](#), we have $u_n^{2d+1} e^{-u_n^2/2} \leq C n^{-d}$. Note that $\mathcal{C}_n^d(a, n) \setminus \mathcal{C}_n^d(a, b) = \mathcal{C}_n^d(b, n)$. It follows from [Lemma 6.2](#) that

$$\mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, n) \setminus \mathcal{C}_n^d(a, b)} \mathbb{X}(A) > u_n \right] \leq \frac{C}{b^d}.$$

Consequently, by [Lemma 6.1](#) we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, n)} \mathbb{X}(A) \leq u_n \right] &\geq \lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{C}_n^d(a, b)} \mathbb{X}(A) \leq u_n \right] - \frac{C}{b^d} \\ &= e^{-e^{-\tau} \left(1 - \left(\frac{a}{b}\right)^d\right)} - \frac{C}{b^d}. \end{aligned} \quad (44)$$

The proof is completed by letting $b \rightarrow +\infty$ in [\(43\)](#) and [\(44\)](#).

6.2. Proof of [Theorem 1.4](#)

Let $\mathbb{X}(A) = \mathbb{W}(A)/\sqrt{\lambda(A)}$ be the standardized white noise. Recall that \mathcal{R}^d is the set of all d -dimensional rectangles and \mathcal{R}_n^d is the set of all d -dimensional rectangles contained in $[0, n]^d$. For $0 < a < b \leq n$ let $\mathcal{R}_n^d(a, b)$ be the set of all rectangles $[x, x + h] \in \mathcal{R}_n^d$ such that $h_i \in [a, b]$ for all $1 \leq i \leq d$.

Lemma 6.3. Fix some $\tau \in \mathbb{R}$, $0 < a < b$, and let $u_n = u_n(\tau)$ be defined by [\(9\)](#). Then,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a, b)} \mathbb{X}(A) \leq u_n \right] = e^{-e^{-\tau} \left(1 - \left(\frac{a}{b}\right)^d\right)}.$$

Proof. For $k \in \mathbb{Z}^d$ let \mathcal{I}_k be the set of all rectangles of the form $[x, x + h] \in \mathcal{R}^d$, such that $x_i \in [k_i, k_i + 1]$ and $h_i \in [a, b]$ for every $1 \leq i \leq d$. Let $E_{k,n}$ be the random event $\{\sup_{A \in \mathcal{I}_k} \mathbb{X}(A) > u_n\}$. The probability $p_n := \mathbb{P}[E_{k,n}]$ does not depend on k by translation invariance. By [Proposition 4.1](#) and [\(9\)](#),

$$p_n \sim \frac{1}{4^d \sqrt{2\pi}} u_n^{4d-1} e^{-u_n^2/2} \int_{\mathcal{I}_k} \frac{dx dy}{\prod_{i=1}^d (y_i - x_i)^2} \sim \frac{e^{-\tau}}{n^d} \left(1 - \frac{a}{b}\right)^d, \quad n \rightarrow \infty. \quad (45)$$

The same argument as in the proof of [Lemma 6.1](#) shows that the conditions of [Section 5](#) are satisfied. The statement of the lemma follows from [Theorem 5.1](#) applied to the events $E_{k,n}$. \square

In the next lemma we shall estimate the high excursion probability over the set $\mathcal{P}_n(a, b) := \mathcal{R}_n^d(a, n) \setminus \mathcal{R}_n^d(a, b)$.

Lemma 6.4. *There is a constant C such that for every $0 < a \leq b < n$ and $u > 1$,*

$$\mathbb{P} \left[\sup_{A \in \mathcal{P}_n(a, b)} \mathbb{X}(A) > u \right] \leq C b^{-1} a^{-(d-1)} u^{4d-1} e^{-u^2/2} n^d.$$

Proof. We may write $\mathcal{P}_n(a, b) = \bigcup_{m=1}^d \mathcal{P}_{n,m}(a, b)$, where $\mathcal{P}_{n,m}(a, b)$, $1 \leq m \leq d$, is the set of all rectangles $[x, x+h] \in \mathcal{R}_n^d$ such that $h_m > b$ and $h_i \geq a$ for all $1 \leq i \leq d$. It suffices to estimate the high excursion probability over the set $\mathcal{P}_{n,1}(a, b)$. For $k \in \mathbb{Z}^d$ and $l \in \mathbb{Z}^d$ denote by $\mathcal{I}_{k,l}$ the set of all rectangles $[x, x+h] \in \mathcal{R}_n^d$ such that $x_i \in [2^l k_i, 2^l(k_i+1)]$ and $h_i \in [2^l, 2^{l+1}]$ for all $1 \leq i \leq d$. By Proposition 4.1, we have for every $u > 1$,

$$\mathbb{P} \left[\sup_{A \in \mathcal{I}_{k,l}} \mathbb{X}(A) > u \right] \leq C u^{4d-1} e^{-u^2/2} \int_{\mathcal{I}_{k,l}} \frac{dx dy}{\prod_{i=1}^d (y_i - x_i)^2} < C u^{4d-1} e^{-u^2/2}.$$

The constant C does not depend on k, l since the left-hand side does not depend on k, l by the affine invariance. To complete the proof we shall show that the set $\mathcal{P}_{n,1}(a, b)$ can be covered by at most $C b^{-1} a^{-(d-1)} n^d$ sets of the form $\mathcal{I}_{k,l}$. Without restriction of generality we may assume that $n = 2^L$, $a = 2^{L_a}$, $b = 2^{L_b}$ for some $L, L_a, L_b \in \mathbb{Z}$. For $l \in \mathbb{Z}^d$ denote by $\mathcal{Q}_n(l)$ the set of all rectangles $[x, x+h] \in \mathcal{R}_n^d$ such that $h_i \in [2^l, 2^{l+1}]$ for all $1 \leq i \leq d$. Clearly, for every fixed $l \in \mathbb{Z}^d$ the set $\mathcal{Q}_n(l)$ can be covered by $n^d / \prod_{i=1}^d 2^{l_i}$ sets of the form $\mathcal{I}_{k,l}$ with varying k . We have

$$\mathcal{P}_{n,1}(a, b) \subset \bigcup_{l_1=L_b}^{L-1} \bigcup_{l_2=L_a}^{L-1} \dots \bigcup_{l_d=L_a}^{L-1} \mathcal{Q}_n(l).$$

Hence, the set $\mathcal{P}_{n,1}(a, b)$ can be covered by at most

$$n^d \sum_{l_1=L_b}^{L-1} \sum_{l_2=L_a}^{L-1} \dots \sum_{l_d=L_a}^{L-1} \prod_{i=1}^d 2^{-l_i} \leq n^d \left(\sum_{l=L_b}^{\infty} 2^{-l} \right) \left(\sum_{l=L_a}^{\infty} 2^{-l} \right)^{d-1} \leq C b^{-1} a^{-(d-1)} n^d$$

sets of the form $\mathcal{I}_{k,l}$. This completes the proof of the lemma. \square

The proof of Theorem 1.4 can be completed as follows. Fix $a > 0$ and choose $u_n = u_n(\tau)$ as in (9). Since $\mathcal{R}_n^d(a, b) \subset \mathcal{R}_n^d(a, n)$ for every $a < b \leq n$, it follows from Lemma 6.3 that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a, n)} \mathbb{X}(A) \leq u_n \right] \leq e^{-e^{-\tau} (1 - \frac{a}{b})^d}. \quad (46)$$

The converse inequality can be proven as follows. By (9), we have $u_n^{4d-1} e^{-u_n^2/2} \leq C n^{-d}$. It follows from Lemma 6.4 that

$$\mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a, n) \setminus \mathcal{R}_n^d(a, b)} \mathbb{X}(A) > u_n \right] \leq C b^{-1}.$$

Consequently, by Lemma 6.3 we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a,n)} \mathbb{X}(A) \leq u_n \right] &\geq \lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathcal{R}_n^d(a,b)} \mathbb{X}(A) \leq u_n \right] - Cb^{-1} \\ &= e^{-e^{-\tau}(1-\frac{a}{b})^d} - Cb^{-1}. \end{aligned} \quad (47)$$

The proof of Theorem 1.4 is completed by letting $b \rightarrow +\infty$ in (46) and (47).

7. Proofs in the discrete-time case

7.1. Proof of Theorem 1.1

Recall that \mathfrak{C}^d is the set of all discrete d -dimensional cubes and \mathfrak{C}_n^d is the set of all discrete d -dimensional cubes contained in $\{1, \dots, n\}^d$. For $0 < a < b \leq n$ let $\mathfrak{C}_n^d(a, b)$ be the set of all discrete cubes $[\mathbf{x}, \mathbf{x} + h]_{\mathbb{Z}^d} \in \mathfrak{C}_n^d$ such that $h \in [a, b]$. We write $l_n = \lfloor \log n \rfloor$ and $q_n = 1/\lfloor \log n \rfloor$.

Lemma 7.1. Fix some $\tau \in \mathbb{R}$, $0 < a < b$, and let $u_n = u_n(\tau)$ be given by (6). Then,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d(al_n, bl_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] = e^{-e^{-\tau} \frac{1}{J_d} \int_a^b J_d(h) dh}.$$

Proof. For $\mathbf{k} \in \mathbb{Z}^d$ define $\mathcal{I}_{\mathbf{k},n}$ to be the set of all discrete cubes $[\mathbf{x}, \mathbf{x} + h]_{\mathbb{Z}^d}$ such that $h \in [al_n, bl_n]$ and $x_i \in [k_i l_n, (k_i + 1)l_n]$ for all $1 \leq i \leq d$. Let also \mathcal{I} be the set of all (non-discrete) cubes $[\mathbf{x}, \mathbf{x} + h]$ such that $h \in [a, b]$ and $x_i \in [0, 1]$ for all $1 \leq i \leq d$. Let $E_{\mathbf{k},n}$ be the random event $\{\max_{A \in \mathcal{I}_{\mathbf{k},n}} \mathbb{S}(A)/\sqrt{|A|} > u_n\}$. Recall that \mathbb{X} is the standardized white noise. Using the affine invariance, we obtain

$$p_n := \mathbb{P}[E_{\mathbf{k},n}] = \mathbb{P} \left[\max_{A \in q_n \mathbb{Z}^{d+1} \cap \mathcal{I}} \mathbb{X}(A) > u_n \right]. \quad (48)$$

Note that $\kappa := \lim_{n \rightarrow \infty} q_n u_n^2 = 2d$. Proposition 4.4 and (6) imply that

$$p_n \sim \frac{1}{\sqrt{2\pi}} u_n^{2d+1} e^{-u_n^2/2} \left(\int_{\mathcal{I}} J_d(h) d\mathbf{x} dh \right) \sim e^{-\tau} \frac{l_n^d}{n^d} \frac{1}{J_d} \int_a^b J_d(h) dh, \quad n \rightarrow \infty.$$

The set $\mathfrak{C}_n^d(al_n, bl_n)$ can be covered by approximately n^d/l_n^d sets of the form $\mathcal{I}_{\mathbf{k},n}$. The statement of the lemma follows by applying Theorem 5.1. Its conditions can be verified in the same way as in the proof of Lemma 6.1. \square

The proof of Theorem 1.1 can be completed as follows. Since $\mathfrak{C}_n^d(al_n, bl_n) \subset \mathfrak{C}_n^d$ for every $0 < a < b$ and n large enough, it follows from Lemma 7.1 that

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] \leq e^{-e^{-\tau} \frac{1}{J_d} \int_a^b J_d(h) dh}. \quad (49)$$

Let us prove a converse inequality. The number of elements in the finite set $\mathfrak{C}_n^d(0, al_n)$ does not exceed $al_n n^d$. Recall that the Gaussian tail probability $\bar{\Phi}$ satisfies $\bar{\Phi}(u) \leq Cu^{-1}e^{-u^2/2}$, $u > 0$. We obtain

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d(0, al_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq \limsup_{n \rightarrow \infty} C(al_n n^d) \cdot (u_n^{-1} e^{-u_n^2/2}) \leq Ca, \quad (50)$$

where the last inequality is a consequence of (6). Also, it follows from Lemma 6.2 that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d(bl_n, n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] &\leq \limsup_{n \rightarrow \infty} \mathbb{P} \left[\sup_{A \in \mathfrak{C}_n^d(bl_n, n)} \mathbb{X}(A) > u_n \right] \\ &\leq \limsup_{n \rightarrow \infty} C(bl_n)^{-d} u_n^{2d+1} e^{-u_n^2/2} n^d \\ &\leq Cb^{-d}, \end{aligned} \quad (51)$$

where the last step follows from (6). It follows from (50), (51) and Lemma 7.1 that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{C}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] \geq e^{-e^{-\tau} \frac{1}{J_d} \int_a^b J_d(h) dh} - C(a + b^{-d}). \quad (52)$$

By Lemma 4.1, $J_d = \int_0^\infty J_d(h) dh$ is finite. Hence, $\lim_{a \downarrow 0} \lim_{b \uparrow \infty} \int_a^b J_d(h) dh = J_d$. Letting $a \downarrow 0$ and $b \uparrow \infty$ in (49) and (52), we obtain the statement of Theorem 1.1.

7.2. Proof of Theorem 1.2

Recall that \mathfrak{R}^d is the set of all discrete d -dimensional rectangles and \mathfrak{R}_n^d is the set of all discrete d -dimensional rectangles contained in $\{1, \dots, n\}^d$. For $0 < a < b \leq n$ let $\mathfrak{R}_n^d(a, b)$ be the set of all discrete rectangles $[\mathbf{x}, \mathbf{x} + \mathbf{h}]_{\mathbb{Z}^d} \in \mathfrak{R}_n^d$ such that $h_i \in [a, b]$ for all $1 \leq i \leq d$. Recall that we write $l_n = \lfloor \log n \rfloor$ and $q_n = 1/\lfloor \log n \rfloor$.

Lemma 7.2. Fix some $\tau \in \mathbb{R}$, $0 < a < b$, and let $u_n = u_n(\tau)$ be defined by (7). Then,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d(al_n, bl_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] = e^{-e^{-\tau} \left(\frac{1}{G_d} \int_a^b G_d(h) dh \right)^d}.$$

Proof. For $\mathbf{k} \in \mathbb{Z}^d$ let $\mathcal{I}_{\mathbf{k}, n}$ be the set of discrete rectangles of the form $[\mathbf{x}, \mathbf{x} + \mathbf{h}]_{\mathbb{Z}^d}$ such that $x_i \in [k_i l_n, (k_i + 1) l_n]$ and $h_i \in [al_n, bl_n]$ for every $1 \leq i \leq d$. Let also \mathcal{I} be the set of all (non-discrete) rectangles of the form $[\mathbf{x}, \mathbf{x} + \mathbf{h}]$, where $x_i \in [0, 1]$ and $h_i \in [a, b]$ for every $1 \leq i \leq d$. Denote by $E_{\mathbf{k}, n}$ the random event $\{\max_{A \in \mathcal{I}_{\mathbf{k}, n}} \mathbb{S}(A)/\sqrt{|A|} > u_n\}$. Then, by the affine invariance

$$p_n := \mathbb{P}[E_{\mathbf{k}, n}] = \mathbb{P} \left[\max_{A \in \mathcal{I} \cap q_n \mathbb{Z}^{2d}} \mathbb{X}(A) > u_n \right].$$

Note that $\kappa := \lim_{n \rightarrow \infty} q_n u_n^2 = 2d$. By Proposition 4.3 and (7),

$$\begin{aligned} p_n &\sim \frac{1}{\sqrt{2\pi}} u_n^{4d-1} e^{-u_n^2/2} \int_{\mathcal{I}_{\mathbf{k}}} \left(\prod_{i=1}^d G(y_i - x_i; 2d) \right) d\mathbf{x} d\mathbf{h} \\ &= e^{-\tau} \frac{l_n^d}{n^d} \left(\frac{1}{G_d} \int_a^b G_d(h) dh \right)^d, \quad n \rightarrow \infty. \end{aligned} \quad (53)$$

The set $\mathfrak{R}_n^d(al_n, bl_n)$ can be covered by approximately n^d/l_n^d sets of the form $\mathcal{I}_{\mathbf{k}, n}$. To complete the proof, apply Theorem 5.1 as in the proof of Lemma 6.1. \square

In the next lemma we estimate the high crossing probability over the set of “thin” rectangles. For $a > 0$ let $\mathfrak{P}_n^d(a)$ be the set of all discrete rectangles $[\mathbf{x}, \mathbf{x} + \mathbf{h}]_{\mathbb{Z}^d} \in \mathfrak{R}_n^d$ such that $h_m \leq al_n$ for some $1 \leq m \leq d$.

Lemma 7.3. *Let u_n be a sequence such that $u_n \sim c\sqrt{\log n}$ as $n \rightarrow \infty$ for some $c > 0$. Then,*

$$\mathbb{P} \left[\max_{A \in \mathfrak{P}_n^d(a)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq C a u_n^{2d-1} e^{-u_n^2/2} n^d. \quad (54)$$

Proof. We shall prove (54) by induction over the dimension $d \in \mathbb{N}$. If $d = 1$, then (54) follows from the Gaussian tail estimate $\bar{\Phi}(u_n) \leq C u_n^{-1} e^{-u_n^2/2}$ and the fact that $|\mathfrak{P}_n^1(a)| \leq al_n n$. Before proceeding further, let us show that (54) implies that

$$\mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq C u_n^{2d-1} e^{-u_n^2/2} n^d. \quad (55)$$

Let $\mathfrak{R}_n^d(a, n)$ be the set of all discrete rectangles $[\mathbf{x}, \mathbf{x} + \mathbf{h}]_{\mathbb{Z}^d} \in \mathfrak{R}_n^d$ such that $h_i \geq a$ for all $1 \leq i \leq d$, and recall that $\mathfrak{R}_n^d(a, n)$ is the set of non-discrete rectangles $[\mathbf{x}, \mathbf{x} + \mathbf{h}] \in \mathfrak{R}_n^d$ such that $h_i \geq a$ for all $1 \leq i \leq d$. Taking $a = b = l_n$ in Lemma 6.4, we obtain

$$\mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d(l_n, n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq \mathbb{P} \left[\sup_{A \in \mathfrak{R}_n^d(l_n, n)} \mathbb{X}(A) > u_n \right] \leq C u_n^{2d-1} e^{-u_n^2/2} n^d.$$

Noting that $\mathfrak{R}_n^d = \mathfrak{P}_n^d(1) \cup \mathfrak{R}_n^d(l_n, n)$, we see that (54) implies (55).

Now, assume that (54) and, consequently, (55) have been established in the $(d-1)$ -dimensional setting. We may write $\mathfrak{P}_n^d(a) = \bigcup_{m=1}^d \mathfrak{P}_{n,m}^d(a)$, where $\mathfrak{P}_{n,m}^d(a)$, $1 \leq m \leq d$, is the set of all discrete rectangles $[\mathbf{x}, \mathbf{x} + \mathbf{h}]_{\mathbb{Z}^d} \in \mathfrak{R}_n^d$ such that $h_m \leq al_n$. Consider the set $\mathfrak{Q}_{n,m}^d(x, h)$ of all discrete rectangles $[\mathbf{x}, \mathbf{y}]_{\mathbb{Z}^d} \in \mathfrak{R}_n^d$ such that $x_m = x$ and $h_m = h$. The set $\mathfrak{P}_{n,m}^d(a)$ can be written as a union of at most $al_n n$ sets of the form $\mathfrak{Q}_{n,m}^d(x, h)$. An easy inspection shows that as long as $x + h \leq n$, we have the following equality of laws of random fields:

$$\left\{ \frac{\mathbb{S}(A)}{\sqrt{|A|}}, A \in \mathfrak{Q}_{n,m}^d(x, h) \right\} \stackrel{d}{=} \left\{ \frac{\mathbb{S}(A)}{\sqrt{|A|}}, A \in \mathfrak{R}_n^{d-1} \right\}.$$

By the induction assumption, Eqn. (55) holds in the $(d-1)$ -dimensional setting. Hence,

$$\mathbb{P} \left[\max_{A \in \mathfrak{P}_{n,m}^d(a)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq al_n n \cdot \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^{d-1}} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq C u_n^{2d-1} e^{-u_n^2/2} n^d.$$

Summing over $1 \leq m \leq d$ establishes (54) in the d -dimensional setting and completes the proof. \square

We are now in a position to complete the proof of Theorem 1.2. For every $0 < a < b$, we have by Lemma 7.2,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] &\leq \lim_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d(al_n, bl_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] \\ &= e^{-e^{-\tau} \left(\frac{1}{G_d} \int_a^b G_d(h) dh \right)^d}. \end{aligned} \quad (56)$$

We prove a converse inequality. By Lemma 6.4,

$$\begin{aligned} \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d(al_n, n) \setminus \mathfrak{R}_n^d(al_n, bl_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] &\leq \mathbb{P} \left[\sup_{A \in \mathcal{P}_n(al_n, bl_n)} \mathbb{X}(A) > u_n \right] \\ &\leq C b^{-1} a^{-(d-1)} l_n^{-d} u_n^{4d-1} e^{-u_n^2/2} n^d \\ &\leq C b^{-1} a^{-(d-1)}, \end{aligned} \quad (57)$$

where the last inequality follows from (7). Note that $\mathfrak{R}_n^d \setminus \mathfrak{R}_n^d(al_n, bl_n) = \mathfrak{P}_n^d(a) \cup (\mathfrak{R}_n^d(al_n, n) \setminus \mathfrak{R}_n^d(al_n, bl_n))$. Hence, Lemma 7.3 and (57) imply that

$$\mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d \setminus \mathfrak{R}_n^d(al_n, bl_n)} \frac{\mathbb{S}(A)}{\sqrt{|A|}} > u_n \right] \leq C(b^{-1} a^{-(d-1)} + a). \quad (58)$$

It follows from Lemma 7.2 and (58) that

$$\liminf_{n \rightarrow \infty} \mathbb{P} \left[\max_{A \in \mathfrak{R}_n^d} \frac{\mathbb{S}(A)}{\sqrt{|A|}} \leq u_n \right] \geq e^{-e^{-\tau} \left(\frac{1}{G_d} \int_a^b G_d(h) dh \right)^d} - C(b^{-1} a^{-(d-1)} + a). \quad (59)$$

Letting $a \downarrow 0$ and $b \uparrow \infty$ in (56) and (59) in such a way that $b^{-1} a^{-(d-1)} \rightarrow 0$, we complete the proof of Theorem 1.2.

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