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Mirror and synchronous couplings of geometric Brownian motions

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Abstract

The paper studies the question of whether the classical mirror and synchronous couplings of two Brownian motions minimise and maximise, respectively, the coupling time of the corresponding geometric Brownian motions. We establish a characterisation of the optimality of the two couplings over any finite time horizon and show that, unlike in the case of Brownian motion, the optimality fails in general even if the geometric Brownian motions are martingales. On the other hand, we prove that in the cases of the ergodic average and the infinite time horizon criteria, the mirror coupling and the synchronous coupling are always optimal for general (possibly non-martingale) geometric Brownian motions. We show that the two couplings are efficient if and only if they are optimal over a finite time horizon and give a conjectural answer for the efficient couplings when they are suboptimal.

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Keywords: Mirror and synchronous coupling; Coupling time; Geometric Brownian motions; Efficient coupling; Optimal coupling; Bellman's principle

1. Introduction

Let the process $B = (B_t)_{t \geq 0}$ be a fixed standard Brownian motion and consider a standard Brownian motion $V = (V_t)_{t \geq 0}$ on the same probability space. For any starting points $x, y \in \mathbb{R}$, define the *coupling time* $\tau(V)$ to be the first time the processes $x + B$ and $y + V$ meet. It is obvious that the *synchronous coupling* $V = B$ maximises the coupling time as it makes it infinite almost surely (assuming $x \neq y$). Note further that the coupling time $\tau(V)$ for any Brownian motion

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V cannot be smaller than the first time one of the processes $x + B$ and $y + V$ reaches level $(x + y)/2$. In the case of the *mirror coupling* $V = -B$, this random time actually equals $\tau(V)$ and the coupling inequality becomes an equality. In particular, for any fixed $T \geq 0$, the extremal Brownian motion in the optimisation problem,

$$\text{minimise (resp. maximise)} \quad \mathbb{P}(\tau(V) > T) \quad \text{over all Brownian motions } V, \tag{1}$$

is given by the mirror (resp. synchronous) coupling, uniformly over all finite time horizons.

It is natural to investigate the following closely related problem for geometric Brownian motion: minimise the coupling time of the processes $\mathrm{d}X_t = \sigma_1 X_t \,\mathrm{d}B_t$ and $\mathrm{d}Y_t(V) = \sigma_2 Y_t(V)\,\mathrm{d}V_t$ over all Brownian motions V on a given filtered probability space. The aim here is to maximise the probability of the event that X and $Y(V)$ couple before a given fixed time T. Since the processes X and $Y(V)$ are, at any time t, given by explicit deterministic functions of B_t and V_t respectively, the discussion above might suggest that mirror coupling of B and V should be optimal. Furthermore, since X and $Y(V)$ are martingales, the Dambis–Dubins–Schwarz representation of the difference $X - Y(V)$ intuitively suggests that the two processes will meet as early as possible if the coupling of the Brownian motions B and V is chosen so that the instantaneous volatility of $X - Y(V)$ is as large as possible. Equivalently put, the minimal coupling time should be achieved by the Brownian motion V which maximises (at every moment in time) the instantaneous quadratic variation $\mathrm{d}[X - Y(V)]_s = ((\sigma_1 X_s)^2 + (\sigma_2 Y_s(V))^2)\mathrm{d}s - 2\sigma_1 X_s \sigma_2 Y_s(V)\,\mathrm{d}[B, V]_s$. Since the (random) Lebesgue density of the covariation measure $\mathrm{d}[B, V]_s$ on $[0, \infty)$ is always between -1 and 1, it follows that the mirror coupling $V = -B$ should be optimal. However, as we shall see, both of these intuitive arguments turn out to be false in general.

This paper investigates the problems of minimising and maximising the coupling time of two general (i.e. possibly non-martingale) geometric Brownian motions (GBMs) using a finite time, infinite time and ergodic average criteria. In the finite time horizon case we study the analogue of Problem (1) for GBMs and give a necessary and sufficient condition on the value function for the mirror (resp. synchronous) coupling to be optimal. This leads to an if-and-only-if condition on the parameters of the GBMs, which characterises the suboptimality (and hence optimality) of the mirror (resp. synchronous) coupling for any finite time horizon. In contrast to the intuitive arguments given above, this condition implies that mirror (resp. synchronous) coupling can be suboptimal in Problem (1) for GBMs even if the geometric Brownian motions are martingales. This raises a natural question: is the exponential tail of the mirror (resp. synchronous) coupling optimal or, put differently, is the coupling efficient in the sense of [4]? We show that the mirror (resp. synchronous) coupling is efficient if and only if it is optimal, and hence may be inefficient. In the case where the coupling is suboptimal, the proof of the aforementioned equivalence suggests the conjecture that the synchronous (resp. mirror) coupling is efficient in the minimisation (resp. maximisation) problem.

The *stationary* and *infinite time horizon* (for some "discount" rate $q > 0$) problems are given as the analogues of Problem (1) with $\mathbb{P}(\tau(V) > T)$ replaced by

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V) > t)\,\mathrm{d}t \qquad \text{and} \qquad \int_0^\infty \mathrm{e}^{-qt}\,\mathbb{P}(\tau(V) > t)\,\mathrm{d}t,$$

respectively. It is clear that in the case of Brownian motion, the mirror (resp. synchronous) coupling is optimal according to both of these criteria. In this paper we prove that, unlike in the finite time horizon case, the same holds for all (possibly non-martingale) geometric Brownian motions. In particular this implies that the mirror coupling, which may be inefficient (i.e. has a thicker exponential tail than the optimal coupling), nevertheless minimises both the Laplace transform of the tail probability for any "discount" rate q and its ergodic average. Our proofs are based on Bellman's principle.

An application in mathematical finance of the coupling problems considered in the present paper can be described as follows. Assume that the performance of a portfolio manager is assessed at some fixed future time (e.g. one year from now) with respect to a benchmark security (e.g. some equity index), which evolves as a geometric Brownian motion X. Put differently, the remuneration of the manager depends on whether her portfolio, which evolves as $Y(V)$, exceeds the benchmark X in normalised terms. Assume also that the manager's mandate stipulates that, over the same time horizon, her portfolio may not exceed a pre-specified amount of realised variance. Both of these assumptions are realistic and are used extensively in practice, since the investor wants to beat the index but cannot tolerate arbitrary amounts of volatility in the mean time (e.g. investors like pension funds routinely stipulate such realised variance conditions). Imagine now a situation where the manager has a given amount of time, say T, before the evaluation of her performance, but is behind the benchmark by a certain amount. The question of how to trade in such a way (a) to minimise the probability of not catching up with the benchmark before T and (b) to achieve this without taking unnecessary bets which would increase the realised volatility of the portfolio, is precisely the question of the stochastic minimisation of the coupling time between X and $Y(V)$ (recall that the expected quadratic variation of $Y(V)$, i.e. the realised variance of the manager's portfolio, does not depend on the choice of Brownian motion V).

The mirror coupling and the synchronous coupling of Brownian motions and related processes have attracted much attention in the literature. For example the classical book [7] and paper [8] introduce the mirror couplings of Brownian motions and diffusion processes (see also book [10] for the general theory of coupling). In [5] it is established that the mirror coupling is not the only maximal coupling, although it is the unique maximal coupling in the family of Markovian (also known as immersed) couplings. In [2] it is proved that the tracking error of two driftless diffusions is minimised by the synchronous coupling of the driving Brownian motions. In [6] generalised mirror coupling and generalised synchronous coupling of Brownian motions are introduced; the former minimises the coupling time and maximises the tracking error of two regime-switching martingales, whereas the latter does the opposite. Articles [1], [4], and [9] discuss various applications of the mirror coupling of reflected Brownian motions and other processes. In particular in [4], the notion of efficiency of a Markovian coupling, also used in the present paper, is studied in the context of the spectral gap of the generator of a Markov process.

The remainder of the paper is organised as follows. Section 2 describes the setting and basic notation, which is used throughout. Section 3 establishes the optimality of the mirror and synchronous couplings in the infinite time horizon (Section 3.1, Theorem 1) and stationary (Section 3.2, Proposition 5) problems. In Section 4 we characterise the optimality of the mirror and synchronous couplings over a finite time horizon (Section 4.1, Theorem 8) and analyse the efficiency of the two couplings (Section 4.2, Theorem 9). Appendix A contains a well-known lemma from stochastic analysis, which enables us to apply Bellman's principle.

2. Setting and notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space which is rich enough to support a standard $(\mathcal{F}_t)$-Brownian motion $B = (B_t)_{t\geq 0}$. Let

$$\mathcal{V} := \{V = (V_t)_{t\geq 0};\ V \text{ is an } (\mathcal{F}_t)\text{-Brownian motion with } V_0 = 0\} \tag{2}$$

be the set of all (standard) $(\mathcal{F}_t)$-Brownian motions on this probability space.

Let $X = (X_t)_{t\geq 0}$ and $Y(V) = (Y_t(V))_{t\geq 0}$ be geometric Brownian motions, satisfying stochastic differential equations

$$X_t = x + \int_0^t X_s\,(\sigma_1\,\mathrm{d}B_s + a_1\,\mathrm{d}s) \quad \text{and} \quad Y_t(V) = y + \int_0^t Y_s(V)\,(\sigma_2\,\mathrm{d}V_s + a_2\,\mathrm{d}s)\,. \tag{3}$$

The Brownian motion B is fixed throughout and V is any element of the set $\mathcal{V}$, defined in (2). We assume throughout the paper that

$$x, y > 0, \quad a_1, a_2 \in \mathbb{R} \quad \text{and} \quad \sigma_1, \sigma_2 \in \mathbb{R}, \quad \text{such that} \quad \sigma_1\sigma_2 > 0, \tag{4}$$

and define the following constants

$$\mu := a_2 - a_1 + \sigma_1^2/2 - \sigma_2^2/2 \quad \text{and} \quad \sigma_\pm := \sigma_2 \pm \sigma_1. \tag{5}$$

Note that (4) implies $|\sigma_+| > |\sigma_-|$. The symbol $\pm$ denotes either $+$ or $-$. If $\pm$ and $\mp$ appear in the same expression, then they simultaneously denote either $+$ and $-$, or $-$ and $+$.

Define the *coupling time* of the two processes in (3) as

$$\tau(V) := \inf\{t \geq 0;\ X_t = Y_t(V)\} \quad (\inf\emptyset := \infty).$$

The random variable $\tau(V)$ is zero when the two processes start at the same point and positive $\mathbb{P}$-a.s. otherwise. Under mild assumptions (e.g. if the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous or if all the paths of Brownian motions V and B are continuous), $\tau(V)$ is $\mathbb{P}$-a.s. equal to an $(\mathcal{F}_t)$-stopping time. Furthermore, since X and $Y(V)$ are geometric Brownian motions, this coupling time can be expressed as the coupling time of a Brownian motion and a Brownian motion with drift: $\tau(V) = \inf\{t \geq 0;\ V_t = (\sigma_1 B_t - \mu t + \log(x/y))/\sigma_2\}$.

3. Stationary and infinite time horizon problems

3.1. Infinite time horizon problems

For any $q > 0$, we consider the following two problems: find $V^{\inf} \in \mathcal{V}$ and $V^{\sup} \in \mathcal{V}$ (if they exist) such that

$$\inf_{V\in\mathcal{V}} \int_0^\infty \mathrm{e}^{-qt}\,\mathbb{P}\big(\tau(V) > t\big)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt}\,\mathbb{P}\big(\tau\big(V^{\inf}\big) > t\big)\,\mathrm{d}t \tag{qInf}$$

and

$$\sup_{V\in\mathcal{V}} \int_0^\infty \mathrm{e}^{-qt}\,\mathbb{P}\big(\tau(V) > t\big)\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-qt}\,\mathbb{P}\big(\tau(V^{\sup}) > t\big)\,\mathrm{d}t. \tag{qSup}$$

A simple integration by parts yields $\int_0^\infty \mathrm{e}^{-rt}\,\mathbb{P}(\tau > t)\,\mathrm{d}t = (1-\mathbb{E}(\mathrm{e}^{-r\tau}))/r$ for any nonnegative random variable τ and $r > 0$. Therefore Problems (qInf) and (qSup) are equivalent to finding $V^{(+)} \in \mathcal{V}$ and $V^{(-)} \in \mathcal{V}$ respectively, such that

$$\sup_{V\in\mathcal{V}} \pm\mathbb{E}\left(\mathrm{e}^{-q\tau(V)}\right) = \pm\mathbb{E}\left(\mathrm{e}^{-q\tau\left(V^{(\pm)}\right)}\right). \tag{q$\pm$}$$

Note also that if e_q is an exponential random variable with $\mathbb{E}(e_q) = 1/q$, independent of the filtration $(\mathcal{F}_t)_{t\geq 0}$, then Problems (qInf) and (qSup) are equivalent to minimising and maximising $\mathbb{P}(\tau(V) > e_q)$ over $V \in \mathcal{V}$, respectively.

The following theorem holds.

Theorem 1. *A solution to Problem (q±) is (for any $q > 0$) given by*

$$V^{(\pm)} = \mp B.$$

Remark 1. (i) Observe that by Theorem 1, the mirror coupling $V^{(+)} = -B$ solves Problem (qInf) and the synchronous coupling $V^{(-)} = +B$ is the solution to Problem (qSup).
(ii) Note that the solution depends neither on the parameters in (4) nor on the discount rate q.

3.1.1. Proof of Theorem 1

Observe that, due to the symmetry in Problem (q±), we may assume without loss of generality that the starting points x, y in (3)–(4) satisfy $(x, y) \in D$, where the set $D \subset \mathbb{R}^2$ is given by

$$D := \{(a, b);\ a \geq b > 0\}. \tag{6}$$

Fix $q > 0$ and define the following function, closely related to the right-hand side in Problem (q±):

$$\Psi^{(\pm)}(x, y) := \mathbb{E}_{x,y}\left(\mathrm{e}^{-q\tau(\mp B)}\right), \quad (x, y) \in D. \tag{7}$$

The proof of Theorem 1 is in two steps: we first establish sufficient conditions for a function $\Psi : D \to \mathbb{R}_+$ implying that $\pm\Psi$ is equal to the right-hand side in Problem (q±) (Lemmas 2 and 3), and then prove that $\Psi^{(\pm)}$ in (7) satisfies these conditions (Lemma 4). Throughout the paper we denote $\mathbb{R}_+ := [0, \infty)$.

For any measurable function $\Psi : D \to \mathbb{R}_+$ and Brownian motion $V \in \mathcal{V}$, consider the process $U(V, \Psi) = (U_t(V, \Psi))_{t\in[0,\infty)}$ defined by

$$U_t(V, \Psi) := \mathrm{e}^{-q(t\wedge\tau(V))}\,\Psi(X_{t\wedge\tau(V)}, Y_{t\wedge\tau(V)}(V)) \tag{8}$$

(here and in the rest of the paper we denote $s \wedge t := \min(s, t)$). Then the following lemma (a suitable version of Bellman's principle) holds.

Lemma 2. *Let $\Psi : D \to \mathbb{R}_+$ be a bounded continuous function satisfying $\Psi(x, x) = 1$ for all $x > 0$. If, for every $(x, y) \in D$, the process $\pm U(V, \Psi)$ is a $\mathbb{P}_{x,y}$-supermartingale for all $V \in \mathcal{V}$ and $U(\mp B, \Psi)$ is a $\mathbb{P}_{x,y}$-martingale, then $V^{(\pm)} = \mp B$ solves Problem (q±).*

Proof. Since $X_{\tau(V)} = Y_{\tau(V)}(V)$ $\mathbb{P}_{x,y}$-a.s. on the event $\{\tau(V) < \infty\}$ for any $V \in \mathcal{V}$, Ψ is continuous and bounded, $\Psi(x, x) = 1$ holds for any $x > 0$ and $q > 0$, the supermartingale property and the Dominated Convergence Theorem imply

$$\begin{aligned}\pm\mathbb{E}_{x,y}\left(\mathrm{e}^{-q\tau(V)}\right) &= \mathbb{E}_{x,y}\left(\pm U_{\tau(V)}(V, \Psi)\mathbb{I}_{\{\tau(V)<\infty\}}\right) \leq \mathbb{E}_{x,y}\left(\pm U_0(V, \Psi)\right) \\ &= \pm\Psi(x, y), \quad (x, y) \in D,\end{aligned}$$

for all $V \in \mathcal{V}$ ($\mathbb{I}_{\{\cdot\}}$ denotes the indicator of the event $\{\cdot\}$). Since $U(\mp B, \Psi)$ is a martingale, for $V^{(\pm)} = \mp B$ this inequality becomes an equality and the lemma follows. □

Our next task is to establish a verification lemma for Problem (q±). Let D° be the interior (in $\mathbb{R}^2$) of the set D defined in (6). For any twice differentiable function $f \in \mathcal{C}^{2,2}(D^\circ)$ we define the function $\mathcal{L}^{(\pm)} f$ by the formula

$$\begin{aligned}&\left(\mathcal{L}^{(\pm)} f\right)(x, y) \\ &:= \left(a_1 x f_x + a_2 y f_y + \frac{1}{2}\sigma_1^2 x^2 f_{xx} + \frac{1}{2}\sigma_2^2 y^2 f_{yy} \mp \sigma_1\sigma_2 x y f_{xy} - q f\right)(x, y), \qquad (9)\end{aligned}$$

where $(x, y) \in D^\circ$ and f_x, f_y, f_{xx}, f_{yy} and f_{xy} denote the partial derivatives of f. For any function $\Psi : D \to \mathbb{R}_+$, such that $\Psi \in \mathcal{C}^{2,2}(D^\circ)$, and Brownian motion $V \in \mathcal{V}$, the local martingale $M(V, \Psi) = (M_t(V, \Psi))_{t\in[0,\infty)}$, given by

$$\begin{aligned}&M_t(V, \Psi) \\ &:= \int_0^{t\wedge\tau(V)} \mathrm{e}^{-qs}\big(\sigma_1 X_s \Psi_x(X_s, Y_s(V))\,\mathrm{d}B_s + \sigma_2 Y_s(V)\Psi_y(X_s, Y_s(V))\,\mathrm{d}V_s\big), \qquad (10)\end{aligned}$$

is well-defined.

Lemma 3. *Assume the following hold:* (I) $\Psi : D \to \mathbb{R}_+$ *is a bounded continuous function with* $\Psi(x, x) = 1$ *for all* $x > 0$*;* (II) $\Psi \in \mathcal{C}^{2,2}(D^\circ)$ *and, in the interior* D°*,* $\Psi_{xy} \le 0$ *and* $\mathcal{L}^{(\pm)}\Psi = 0$*;* (III) $M(V, \Psi)$ *is a* $\mathbb{P}_{x,y}$*-martingale for all* $(x, y) \in D$ *and* $V \in \mathcal{V}$*. Then for any* $(x, y) \in D$*,* $V \in \mathcal{V}$*, the process* $\pm U(V, \Psi)$*, defined in* (8)*, is a* $\mathbb{P}_{x,y}$*-supermartingale and* $U(\mp B, \Psi)$ *is a* $\mathbb{P}_{x,y}$*-martingale.*

Proof. The definition of X and $Y(V)$ in (3) and Lemma 10 in the Appendix imply $\mathrm{d}[X, Y(V)]_t = C_t\sigma_1 X_t \sigma_2 Y_t(V)\,\mathrm{d}t$, where $C = (C_t)_{t\in[0,\infty)}$ is $(\mathcal{F}_t)$-adapted and $\mathbb{P}(C_t \in [-1, 1]) = 1$ for all $t \in [0, \infty)$. Itô's lemma, the assumptions in Lemma 3 and definition (8) of $U(V, \Psi)$ yield

$$\begin{aligned}\pm U_t(V, \Psi) &= \pm\Psi(x, y) \pm M_t(V, \Psi) \\ &\quad + \int_0^{t\wedge\tau(V)} \mathrm{e}^{-qs}\sigma_1\sigma_2(1 \pm C_s)X_s Y_s(V)\Psi_{xy}(X_s, Y_s(V))\,\mathrm{d}s\end{aligned}$$

for all $(x, y) \in D$ and $V \in \mathcal{V}$. Since X, $Y(V)$ and $1 \pm C$ are non-negative processes and, by assumption (4), we have $\sigma_1\sigma_2 > 0$, the integrand in the representation of $\pm U(V, \Psi)$ is non-positive, making $\pm U(V, \Psi)$ a $\mathbb{P}_{x,y}$-supermartingale. This representation, together with assumption (III), implies that $U(\mp B, \Psi)$ is a $\mathbb{P}_{x,y}$-martingale. □

Note the following equivalence:

$$\mathbb{P}_{x,y}(\tau(\mp B) = \infty) = 1 \quad \text{for all } (x, y) \in D^\circ \iff \mp = +, \sigma_2 = \sigma_1, a_2 \le a_1. \qquad (11)$$

It is clear that under condition (11) Theorem 1 holds. Lemmas 2 and 3 imply that in order to establish Theorem 1 in general, it is sufficient to prove that, when (11) fails, the function $\Psi^{(\pm)} : D \to \mathbb{R}_+$ in (7) satisfies the assumptions of Lemma 3. More precisely, the following lemma holds.

Lemma 4. *Assumptions* (I)–(III) *of Lemma 3 hold for the function* $\Psi^{(\pm)} : D \to \mathbb{R}_+$ *in* (7)*, if for some* $(x, y) \in D^\circ$ *we have* $\mathbb{P}_{x,y}(\tau(\mp B) = \infty) < 1$.

Proof. Under the assumption of the lemma, the following representation holds:

$$\Psi^{(\pm)}(x, y) = \left(\frac{y}{x}\right)^{k_\pm} \quad \text{for } (x, y) \in D, \qquad (12)$$

where

$$k_\pm := \begin{cases} -\mu/\sigma_\pm^2 + \sqrt{(\mu/\sigma_\pm^2)^2 + 2q/\sigma_\pm^2}, & \text{if } \sigma_\pm \neq 0, \\ q/\mu, & \text{if } \sigma_\pm = 0, \end{cases}$$

and $\sigma_\pm$ and μ are defined in (5). Since, by assumption, the condition on the right-hand side in (11) is not satisfied, the equality $\sigma_\pm = 0$ implies $\mu > 0$, making $k_\pm$ a well-defined real number. Formula (12) follows from the fact that $\tau(\mp B)$ has the same law as the first-passage time of the Brownian motion with drift $(\sigma_\pm B_t + \mu t)_{t\in[0,\infty)}$ over the level $\log(x/y)$. The Laplace transform of this random time is given in [3, p. 295] and amounts to the right-hand side of (12).

Assumption (I) in Lemma 3 follows from (12). Furthermore it is clear that $\Psi^{(\pm)} \in \mathcal{C}^{2,2}(D^\circ)$. The formula in (12) and some simple calculations imply that for $(x, y) \in D^\circ$ the following holds:

$$\Psi_x^{(\pm)}(x, y) = -\frac{k_\pm}{x}\, \Psi^{(\pm)}(x, y), \qquad \Psi_y^{(\pm)}(x, y) = \frac{k_\pm}{y}\, \Psi^{(\pm)}(x, y), \tag{13}$$

and

$$\Psi_{xy}^{(\pm)}(x, y) = -\frac{k_\pm^2}{xy}\, \Psi^{(\pm)}(x, y) \le 0, \qquad \left(\mathcal{L}^{(\pm)}\, \Psi^{(\pm)}\right)(x, y) = 0.$$

Hence assumption (II) of Lemma 3 is also satisfied. The equalities in (13) and the definition in (10) of the local martingale $M(V, \Psi^{(\pm)})$ imply that the integrands in the stochastic integrals are bounded processes and therefore square integrable. Hence $M(V, \Psi^{(\pm)})$ is a $\mathbb{P}_{x,y}$-martingale for all $(x, y) \in D$ and $V \in \mathcal{V}$ and assumption (III) of Lemma 3 also holds. $\square$

3.2. Stationary problems

Note first that Fubini's theorem and the Dominated Convergence Theorem imply the existence of the limit:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V) > t)\, dt = \lim_{T\to\infty} \mathbb{E}((\tau(V)/T) \wedge 1) = \mathbb{P}(\tau(V) = \infty). \tag{14}$$

Hence the stationary problems from the introduction can be rephrased as: find $V^{\inf} \in \mathcal{V}$ and $V^{\sup} \in \mathcal{V}$ such that

$$\inf_{V\in\mathcal{V}} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V) > t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V^{\inf}) > t)\, dt \tag{SInf}$$

and

$$\sup_{V\in\mathcal{V}} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V) > t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{P}(\tau(V^{\sup}) > t)\, dt. \tag{SSup}$$

A solution to these problems, independent of the values of the parameters of the geometric Brownian motions in (3), is given in the following proposition. Note in particular that, unlike in the finite time horizon case, no new phenomena arise when the ergodic average criterion is used (i.e. the solution is completely analogous to the infinite time horizon case).

Proposition 5. *The Brownian motions $V^{\inf} = -B$ and $V^{\sup} = B$ solve Problems (SInf) and (SSup) respectively.*

Proof. As in Section 3.1.1 we may assume that, due to symmetry, the starting points of X and $Y(V)$ satisfy $(x, y) \in D$ (see (6)). By (3) and the definition of $\tau(V)$ in Section 2 we have

$$\tau(V) = \inf\{t \geq 0;\ \sigma_2 V_t - \sigma_1 B_t + \mu t = \log(x/y)\}, \tag{15}$$

where μ is defined in (5) and the convention $\inf \emptyset = \infty$ is used. If $x = y$ we have $\tau(V) = 0$ for all $V \in \mathcal{V}$ and Proposition 5 follows. So we can assume $(x, y) \in D^\circ$ in the rest of the proof.

We first analyse the case $\mu > 0$. By (14), Problems (SInf) and (SSup) are equivalent to finding $V^{(\pm)} \in \mathcal{V}$ such that

$$\inf_{V \in \mathcal{V}} \pm\mathbb{P}(\tau(V) = \infty) = \pm\mathbb{P}(\tau(V^{(\pm)}) = \infty). \tag{S$\pm$}$$

The strong law of large numbers for Brownian motion (e.g. [3, p. 53]), representation (15) and $\log(x/y) > 0$ imply the equality $\mathbb{P}_{x,y}(\tau(V) = \infty) = 0$ for every $V \in \mathcal{V}$ and Proposition 5 follows.

In the case $\mu \leq 0$ we return to the formulation of Problems (SInf) and (SSup) above. Observe that Theorem 8b below yields the optimal couplings that minimise and maximise the probability $\mathbb{P}(\tau(V) > t)$ for every $t \geq 0$. Since the couplings are independent of t, they also minimise and maximise the stationary criteria in Problems (SInf) and (SSup), which concludes the proof. $\square$

Remark 2. The proof of Proposition 5 relies in an obvious way on Theorem 8b below. We would like to stress that there is no circularity in this argument since Proposition 5 is not used in Section 4. Stationary problems are considered in Section 3 rather than later on in the paper, because the structure of the solution is the same as that of the infinite time horizon problems.

4. Finite time horizon problems and the efficiency of the couplings

4.1. Finite time horizon problems

Retain the setting and notation from Section 2. For any $T > 0$, consider the following problems:

$$\text{find } V^{(\pm)} \in \mathcal{V} \text{ such that } \inf_{V \in \mathcal{V}} \pm\mathbb{P}\big(\tau(V) > T\big) = \pm\mathbb{P}\big(\tau\big(V^{(\pm)}\big) > T\big). \tag{T$\pm$}$$

As in Section 3, we can reduce Problem (T±) to the case where diffusions in (3) start at $(x, y) \in D$, where D is given in (6). Define the set $E := D \times [0, T]$ and recall that the *value function* for Problem (T±) is defined by

$$F(x, y, t) := \inf_{V \in \mathcal{V}} \pm\mathbb{P}_{x,y}(\tau(V) > t), \quad (x, y, t) \in E. \tag{16}$$

Based on the results in Section 3, one might expect that $\pm\Phi^{(\pm)}$, where

$$\Phi^{(\pm)}(x, y, t) := \mathbb{P}_{x,y}\left(\tau(\mp B) > t\right), \quad (x, y, t) \in E, \tag{17}$$

would be the value function for Problem (T±). In order to investigate this, we define the function $\mathcal{A}^{(\pm)} f$ for any $f \in \mathcal{C}^{2,2,1}(E^\circ)$ (E° is the interior of E in $\mathbb{R}^3$) by the formula

$$\left(\mathcal{A}^{(\pm)} f\right)(x, y, t) := \left(a_1 x f_x + a_2 y f_y + \frac{1}{2}\sigma_1^2 x^2 f_{xx} + \frac{1}{2}\sigma_2^2 y^2 f_{yy} \mp \sigma_1\sigma_2 xy f_{xy} - f_t\right)(x, y, t),$$

where $(x, y, t) \in E^\circ$ and f_x, f_y, f_t, etc. denote the partial derivatives of f. For any sufficiently smooth function $\Phi : E \to \mathbb{R}_+$ and any Brownian motion $V \in \mathcal{V}$, we define the local martingale $N(V, \Phi) = (N_t(V, \Phi))_{t\in[0,T]}$ by

$$N_t(V, \Phi) := \int_0^{t\wedge\tau(V)} \big(\sigma_1 X_s \Phi_x(X_s, Y_s(V), T - s)\, \mathrm{d}B_s + \sigma_2 Y_s(V) \Phi_y(X_s, Y_s(V), T - s)\, \mathrm{d}V_s\big). \tag{18}$$

The following proposition provides the key ingredient in the proof of Theorem 8 below.

Proposition 6. *Let a bounded function $\Phi : E \to \mathbb{R}_+$ satisfy:* (i) *$\Phi(x, x, t) = 0$ for all $x > 0$ and $t \in [0, T]$, and $\Phi(x, y, 0) = 1$ for all $(x, y) \in D^\circ$;* (ii) *$\Phi \in \mathcal{C}^{2,2,1}(E^\circ)$ and, in the interior E°, the equality $\mathcal{A}^{(\pm)}\Phi = 0$ holds;* (iii) *$N(V, \Phi)$ is a $\mathbb{P}_{x,y}$-martingale for all $(x, y) \in D$ and $V \in \mathcal{V}$. Then the following equivalence holds:*

$$\Phi_{xy} \ge 0 \text{ on } E^\circ \iff V^{(\pm)} = \mp B \text{ solves Problem } (T\pm) \text{ and } \pm\Phi \text{ is its value function.}$$

Proof. ($\Rightarrow$): The proof of this implication is analogous to that of Lemma 2 (Bellman's principle) and Lemma 3 (submartingale property) in Section 3. The process $\pm U(V, \Phi) = (\pm U_t(V, \Phi))_{t\in[0,T]}$,

$$U_t(V, \Phi) := \Phi(X_{t\wedge\tau(V)}, Y_{t\wedge\tau(V)}(V), T - t), \tag{19}$$

is a $\mathbb{P}_{x,y}$-submartingale for any $V \in \mathcal{V}$ and $(x, y) \in D$ (proof as in Lemma 3). For any $t \in [0, T]$, the boundary conditions in assumption (i) imply

$$U_t(V, \Phi) = U_{\tau(V)}(V, \Phi) = 0 \quad \mathbb{P}_{x,y}\text{-a.s. on } \{t \ge \tau(V)\}.$$

Hence, for any $(x, y) \in D$ and $V \in \mathcal{V}$, the submartingale property yields the inequality

$$\begin{aligned}\pm\mathbb{P}_{x,y}(\tau(V) > T) &= \mathbb{E}_{x,y}\big(\pm U_T(V, \Phi)\mathbb{I}_{\{\tau(V)>T\}}\big) = \mathbb{E}_{x,y}(\pm U_T(V, \Phi)) \\ &\ge \pm\mathbb{E}_{x,y} U_0(V, \Phi) = \pm\Phi(x, y, T).\end{aligned}$$

As in Lemma 2, this establishes the implication (note that, unlike Lemma 2, in this case we do not need, and in fact do not have, the continuity of Φ on E).

($\Leftarrow$): Assume that there exists $(x_0, y_0, T_0) \in E^\circ$, such that $\Phi_{xy}(x_0, y_0, T_0) < 0$, and that $\pm\Phi$ is the value function of Problem (T±). Bellman's principle implies that the process $\pm U(V, \Phi)$, defined in (19), is a $\mathbb{P}_{x,y}$-submartingale for any $V \in \mathcal{V}$ and $(x, y) \in D$. Using our assumption, we now construct a Brownian motion $\tilde{V}^{(\pm)} \in \mathcal{V}$, such that $\pm U(\tilde{V}^{(\pm)}, \Phi)$ fails to be a $\mathbb{P}_{x,y}$-submartingale (for any pair $(x, y) \in D^\circ$), which will imply the proposition.

The continuity of Φ_{xy} implies that there exists $r > 0$, such that Φ_{xy} is strictly negative on the set $K_2 := H_2 \times [T_0 - 2r, T_0 + 2r] \subset E^\circ$, where $H_2 := [x_0 - 2r, x_0 + 2r] \times [y_0 - 2r, y_0 + 2r]$. Let $H_1 := [x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r]$ and define the stopping times $\tau_1^{(\pm)}$ and $\tau_2^{(\pm)}$ by:

$$\begin{aligned}\tau_1^{(\pm)} &:= \inf\{t \in [0, T]; (X_t, Y_t(\mp B)) \in H_1\}, \\ \tau_2^{(\pm)} &:= \inf\{t \in [\tau_1, T]; (X_t, Y_t(\pm B)) \notin H_2\}\end{aligned}$$

(where $\inf\emptyset := T$). Note that $\tau_1^{(\pm)} \le \tau_2^{(\pm)} \le T$ $\mathbb{P}_{x,y}$-a.s. and $\mathbb{P}_{x,y}(\tau_1^{(\pm)} < \tau_2^{(\pm)}) > 0$ (there is a slight abuse of notation in the definition of $\tau_2^{(\pm)}$ as it is assumed that the process $Y(\pm B)$, defined in (3), is driven by the Brownian motion $\pm B$ as indicated, but started at the random time $\tau_1^{(\pm)}$ and point $Y_{\tau_1^{(\pm)}}(\mp B)$; ditto for X).

Define the process $\tilde{V}^{(\pm)} = (\tilde{V}_t^{(\pm)})_{t\in[0,\infty)}$ by the following formula:

$$\tilde{V}_t^{(\pm)} := \int_0^t \left(\mp \mathbb{I}_{\{s<\tau_1^{(\pm)}\}} \pm \mathbb{I}_{\{\tau_1^{(\pm)}\le s<\tau_2^{(\pm)}\}} \mp \mathbb{I}_{\{s\ge\tau_2^{(\pm)}\}}\right) \mathrm{d}B_s,$$

where $\mathbb{I}_{\{\cdot\}}$ is the indicator of the event $\{\cdot\}$. Note that $\tilde{V}^{(\pm)}$ is an $(\mathcal{F}_t)$-Brownian motion by Lévy's characterisation theorem. Itô's formula on the stochastic interval $[\tau_1^{(\pm)}, \tau_2^{(\pm)}]$ and assumptions (i)–(iii) in the proposition imply the following representation:

$$\begin{aligned}
&\mathbb{E}_{x,y}\left[\pm U_{\tau_2^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi)\middle|\mathcal{F}_{\tau_1^{(\pm)}}\right] \\
&\quad = \pm U_{\tau_1^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi) \\
&\quad + \mathbb{E}_{x,y}\left[\int_{\tau_1^{(\pm)}}^{\tau_2^{(\pm)}} 2\sigma_1\sigma_2 X_s Y_s(\tilde{V}^{(\pm)})\,\Phi_{xy}(X_s, Y_s(\tilde{V}^{(\pm)}), T-s)\,\mathrm{d}s\middle|\mathcal{F}_{\tau_1^{(\pm)}}\right].
\end{aligned}$$

The event $\{\tau_1^{(\pm)} \in (T_0 - r, T_0 + r), \tau(\tilde{V}^{(\pm)}) > T_0 + 2r\}$ has strictly positive probability and the integrand under the conditional expectation is strictly negative on this event. We therefore find

$$\begin{aligned}
&\mathbb{E}_{x,y}\left[\pm U_{\tau_2^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi)\middle|\mathcal{F}_{\tau_1^{(\pm)}}\right] < \pm U_{\tau_1^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi) \quad \text{on} \\
&\quad \{\tau_1^{(\pm)} \in (T_0 - r, T_0 + r), \tau(\tilde{V}^{(\pm)}) > T_0 + 2r\}
\end{aligned}$$

$\mathbb{P}_{x,y}$-a.s. This inequality contradicts the $\mathbb{P}_{x,y}$-a.s. inequality

$$\mathbb{E}_{x,y}\left[\pm U_{\tau_2^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi)\middle|\mathcal{F}_{\tau_1^{(\pm)}}\right] \ge \pm U_{\tau_1^{(\pm)}}(\tilde{V}^{(\pm)}, \Phi),$$

which follows from the optional sampling theorem applied to the bounded $\mathbb{P}_{x,y}$-submartingale $U(\tilde{V}^{(\pm)}, \Phi)$. This concludes the proof. □

We will now apply Proposition 6 to study the question of whether $\pm\Phi^{(\pm)}$, defined in (17), is the value function for Problem (T±).

Lemma 7. *Recall that μ and $\sigma_\pm$ are given in (5) and assume $\sigma_\pm \neq 0$. Then, assumptions* (i)–(iii) *of Proposition 6 hold for the function $\Phi^{(\pm)}$ defined in (17). Furthermore, we have*

$$\begin{aligned}
\Phi_{xy}^{(\pm)}(x, y, t) &= \frac{2\log(x/y) - 4\mu t}{xy(|\sigma_\pm|\sqrt{t})^3}\, n\left(\frac{\log(x/y) - \mu t}{|\sigma_\pm|\sqrt{t}}\right) \\
&\quad + \frac{4\mu^2}{xy\sigma_\pm^4}\left(\frac{x}{y}\right)^{2\mu/\sigma_\pm^2} N\left(\frac{-\log(x/y) - \mu t}{|\sigma_\pm|\sqrt{t}}\right)
\end{aligned}$$

for all $(x, y) \in D^\circ$ and $t > 0$, where $N(\cdot)$ is the standard normal distribution function and $n(\cdot)$ is its density.

Proof. The explicit formula for the distribution of the running maximum of a Brownian motion with drift (see e.g. [3, p. 250]) yields the following representation of the function in (17):

$$\Phi^{(\pm)}(x, y, t) = h^{(\pm)}(\log(x/y), t) \quad \text{for } (x, y) \in D, \tag{20}$$

where, for any $z \geq 0$ and $s > 0$, we define

$$h^{(\pm)}(z,s) := N\left(\frac{z-\mu s}{|\sigma_\pm|\sqrt{s}}\right) - \exp\left(\frac{2\mu z}{\sigma_\pm^2}\right) N\left(\frac{-z-\mu s}{|\sigma_\pm|\sqrt{s}}\right). \tag{21}$$

Simple (but tedious) calculations using this representation yield the properties required in assumptions (i)–(iii) of Proposition 6. Indeed, note that the partial derivatives $h_z^{(\pm)}$, $h_{zz}^{(\pm)}$ and $h_s^{(\pm)}$, take the form (recall $n'(x) = -xn(x)$):

$$h_z^{(\pm)}(z,s) = \frac{2}{|\sigma_\pm|\sqrt{s}} n\left(\frac{z-\mu s}{|\sigma_\pm|\sqrt{s}}\right) - \frac{2\mu}{\sigma_\pm^2}\exp\left(\frac{2\mu z}{\sigma_\pm^2}\right) N\left(\frac{-z-\mu s}{|\sigma_\pm|\sqrt{s}}\right);$$

$$h_{zz}^{(\pm)}(z,s) = \frac{4s\mu - 2z}{(|\sigma_\pm|\sqrt{s})^3} n\left(\frac{z-\mu s}{|\sigma_\pm|\sqrt{s}}\right) - \frac{4\mu^2}{\sigma_\pm^4}\exp\left(\frac{2\mu z}{\sigma_\pm^2}\right) N\left(\frac{-z-\mu s}{|\sigma_\pm|\sqrt{s}}\right);$$

$$h_s^{(\pm)}(z,s) = -\frac{z}{|\sigma_\pm| s^{3/2}} n\left(\frac{z-\mu s}{|\sigma_\pm|\sqrt{s}}\right).$$

These formulae and the representation in (20) imply the formula for $\Phi_{xy}^{(\pm)}(x,y,t)$, as well as assumptions (i) and (ii) of Proposition 6. The martingale property of the process in (18) (i.e. assumption (iii) in Proposition 6) follows by Itô's isometry from the fact that both functions

$$x\Phi_x^{(\pm)}(x,y,t) = h_z^{(\pm)}(\log(x/y),t) \quad \text{and} \quad y\Phi_y^{(\pm)}(x,y,t) = -h_z^{(\pm)}(\log(x/y),t)$$

are bounded on E. This completes the proof of the lemma. $\square$

We are now ready to prove that the mirror (resp. synchronous) coupling of the driving Brownian motions in (3) is not necessarily optimal in Problem (T+) (resp. (T−)). In Theorem 8, we give a necessary and sufficient condition for the function $\pm\Phi^{(\pm)}$, defined (17), to be the value function for Problem (T±).

Theorem 8. *Recall that μ and $\sigma_\pm$ are given in (5). Then the following holds for any positive time horizon and distinct starting points:*

(a) *If $\mu > 0$ and $\sigma_\pm \neq 0$, then $V^{(\pm)} = \mp B$ does NOT solve Problem (T±).*
(b) *If $\mu \leq 0$, then $V^{(\pm)} = \mp B$ solves Problem (T±) with the value function $\pm\Phi^{(\pm)}$ in (17).*

Remark 3. (i) Note that under the assumptions of Theorem 8a, the mirror and synchronous couplings are suboptimal in Problems (T+) and (T−) respectively. Furthermore, if $\pm = +$, then $\sigma_\pm > 0$ and hence the optimality of the mirror coupling can fail even if the laws of X and $Y(V)$ are equivalent (i.e. $\sigma_1 = \sigma_2$) for all $V \in \mathcal{V}$.

(ii) In the case $\mu > 0$ and $\sigma_\pm = 0$ we have $\pm = -$, $\sigma_1 = \sigma_2$ and $\Phi^{(-)}(x,y,t) = \mathbb{I}_{\{t\mu < \log(x/y)\}}$ for all $(x,y) \in D^\circ$, $t \in [0,T]$ (recall (15)), which implies that the synchronous coupling is suboptimal if and only if $T \geq \log(x/y)/\mu$.

Proof. (a) By Proposition 6 it suffices to show that for any fixed $t > 0$, there exists $(x,y) \in D^\circ$ (see (6) for the definition of D) such that $\Phi_{xy}^{(\pm)}(x,y,t) < 0$.

Define $z := \log(x/y)/(|\sigma_\pm|\sqrt{t}) > 0$ and $\alpha := \mu\sqrt{t}/|\sigma_\pm| > 0$. Note that, since we are allowed to choose the point $(x,y) \in D^\circ$ arbitrarily close to the diagonal half-line in the boundary of D, a Taylor expansion of order one of $z \mapsto n(z-\alpha)$ and $z \mapsto N(-z-\alpha)$ around $z=0$, the representation of $\Phi^{(\pm)}_{xy}$ in Lemma 7 and the inequality

$$\alpha N(-\alpha) < n(-\alpha) \tag{22}$$

imply that $\Phi^{(\pm)}_{xy}(x, y, t) < 0$ for some $(x, y) \in D^\circ$. To check (22), note that $un(u) = -n'(u)$ and

$$\alpha N(-\alpha) = \int_\alpha^\infty \alpha n(u)\,\mathrm{d}u < \int_\alpha^\infty un(u)\,\mathrm{d}u = n(-\alpha).$$

(b) Assume first $\sigma_\pm \neq 0$. Then the representation of $\Phi^{(\pm)}_{xy}$ in Lemma 7 and the assumption $\mu \le 0$ imply $\Phi_{x,y} \ge 0$ on E°. Hence Proposition 6 yields the theorem. If $\sigma_\pm = 0$, we have $\pm = -$, $\sigma_1 = \sigma_2$ and, by (15), it follows $\Phi^{(-)}(x, y, t) = 1$ for all $(x, y) \in D^\circ$, $t \in [0, T]$. Hence $-\Phi^{(-)}$ is the value function for Problem (T−) and the theorem follows. □

4.2. *Efficiency of the mirror and synchronous couplings*

In this section we examine further the (lack of) optimality of the mirror and synchronous couplings characterised by the assumptions of Theorem 8a. Since in this case the two couplings do not minimise and maximise (respectively) the coupling times of the geometric Brownian motions in (3) over finite time horizons, but are nonetheless optimal both over the infinite time horizon (Section 3.1) and for the stationary criterion (Section 3.2), it is natural to analyse whether the two couplings are efficient. A coupling $V \in \mathcal{V}$ is *(exponentially) efficient* (for some $(x, y) \in D^\circ$) in Problem (T±) if the rate of the exponential decay of the tail of its coupling time is the same as the exponential decay of the value function F, defined in (16), in the following sense:

$$\pm \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x,y}\left(\tau(V) > t\right) \le \pm \liminf_{t\to\infty} \frac{1}{t} \log\left(\pm F(x, y, t)\right). \tag{23}$$

Note that by (16) the opposite inequality in (23) holds for any coupling V. Hence we could have defined exponential efficiency by requiring equality in (23). Furthermore, if the limits on both sides of (23) exist, the definition of the exponential efficiency in Problem (T±) can be further simplified to

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x,y}\left(\tau(V) > t\right) = \lim_{t\to\infty} \frac{1}{t} \log\left(\pm F(x, y, t)\right).$$

It is clear that if a coupling solves Problem (T±) for all time horizons $T > 0$ and does not depend on T, then it is also efficient according to the definition in (23). Hence, by Theorem 8b, the mirror and synchronous couplings are efficient if $\mu \le 0$. However, the following statement holds.

Theorem 9. *If $\mu > 0$, the mirror and synchronous couplings are NOT efficient (for any $(x, y) \in D^\circ$) in Problems* (T+) *and* (T−) *respectively.*

Remark 4. We thank the referee for noting that Theorem 9 holds without assuming $\sigma_\pm \neq 0$.

Proof. The following bounds hold for the standard normal distribution function $N(\cdot)$,

$$-\frac{z}{1+z^2}\, n(z) \le N(z) \le -z^{-1} n(z) \qquad \text{for any } z < 0, \text{ where } n = N'. \tag{24}$$

The first inequality follows from the identity $\int_r^\infty (1+y^{-2})\mathrm{e}^{-y^2/2}\,\mathrm{d}y = r^{-1}\mathrm{e}^{-r^2/2}$ for all $r > 0$, and the second is given in (22).

Assume now that $\sigma_\pm \neq 0$. Let

$$Z(t) := \frac{\log(x/y) - \mu t}{|\sigma_\pm|\sqrt{t}} \qquad \text{and} \qquad \widehat{Z}(t) := \frac{-\log(x/y) - \mu t}{|\sigma_\pm|\sqrt{t}},$$

and note that for all large $t > 0$ we have $\widehat{Z}(t) < Z(t) < 0$ and the equality

$$n\left(Z(t)\right) = \left(\frac{x}{y}\right)^{2\mu/\sigma_\pm^2} n\left(\widehat{Z}(t)\right) \tag{25}$$

holds. The representations in (20) and (21) imply

$$\Phi^{(\pm)}(x, y, t) = N(Z(t))\left[1 - \left(\frac{x}{y}\right)^{2\mu/\sigma_\pm^2} \frac{N(\widehat{Z}(t))}{N(Z(t))}\right]. \tag{26}$$

The inequalities in (24) imply the limit

$$\lim_{t\to\infty} \frac{1}{t}\log N(Z(t)) = -\frac{\mu^2}{2\sigma_\pm^2}. \tag{27}$$

In order to deal with the second factor on the right-hand side of (26), we note the following inequalities,

$$1 - \left(\frac{x}{y}\right)^{2\mu/\sigma_\pm^2} \frac{N(\widehat{Z}(t))}{N(Z(t))} \geq 1 + (1 + Z^2(t))\frac{N(\widehat{Z}(t))}{n(\widehat{Z}(t))Z(t)} \geq 1 - \frac{1 + Z^2(t)}{\widehat{Z}(t)Z(t)},$$

which are a consequence of two applications of the second inequality in (24) and identity (25). Let the assumption

$$\log(x/y) > \frac{\sigma_\pm^2}{2\mu} \tag{28}$$

hold. Then the following inequality is satisfied

$$1 - \frac{1 + Z^2(t)}{\widehat{Z}(t)Z(t)} = \frac{t^{-1}(2\mu\log(x/y) - \sigma_\pm^2) - t^{-2}2\log^2(x/y)}{\mu^2 - t^{-2}\log^2(x/y)} > 0 \quad \text{for all large } t > 0,$$

and the limit holds:

$$\lim_{t\to\infty} \frac{1}{t}\log\left(1 - \frac{1 + Z^2(t)}{\widehat{Z}(t)Z(t)}\right) = 0. \tag{29}$$

By (26) we have

$$N(Z(t))\left[1 - \frac{1 + Z^2(t)}{\widehat{Z}(t)Z(t)}\right] \leq \Phi^{(\pm)}(x, y, t) \leq N(Z(t)).$$

If the starting points x, y satisfy (28), then (27), (29), the inequalities in the line above and the fact that log is increasing imply the limit: $\lim_{t\to\infty} \frac{1}{t}\log \Phi^{(\pm)}(x, y, t) = -\frac{\mu^2}{2\sigma_\pm^2}$.

In order to see that this limit holds without assumption (28), i.e. for $(x, y) \in D$ such that $\log(x/y) \in (0, \sigma_\pm^2/(2\mu)]$, define a Brownian motion with drift W, started from 0, and its first-passage time $T(z)$:

$$W_t := \mp\sigma_\pm B_t + \mu t, \quad t \geq 0, \qquad \text{and} \qquad T(z) := \inf\{t \geq 0 : W_t = z\}, \quad z \in \mathbb{R},$$

and note that $\mathbb{P}_{x,y}(\tau(\mp B) > t) = \mathbb{P}(T(\log(x/y)) > t)$ holds for any $(x, y) \in D$ (cf. (15)). Fix $(x, y) \in D$ that violates assumption (28) and pick $\alpha_0 < 0$ and $(x_0, y_0) \in D^\circ$ such that the following holds:

$$\log(x_0/y_0) = \log(x/y) - \alpha_0 > \frac{\sigma_\pm^2}{2\mu}.$$

Denote the constant $q := \mathbb{P}(W_1 < \alpha_0, T(\log(x/y)) > 1)$, which clearly satisfies $q \in (0, 1)$. The Markov property of W at time 1 yields the following inequalities for all $t > 1$:

$$\begin{aligned}\mathbb{P}_{x,y}(\tau(\mp B) > t) &= \mathbb{P}(T(\log(x/y)) > t)\\ &\geq q\mathbb{P}(T(\log(x/y) - \alpha_0) > t - 1)\\ &> q\mathbb{P}(T(\log(x/y) - \alpha_0) > t) = q\mathbb{P}_{x_0,y_0}(\tau(\mp B) > t).\end{aligned}$$

Since (26) implies the bound $\mathbb{P}_{x,y}(\tau(\mp B) > t) \leq N(Z(t))$ for any $(x, y) \in D^\circ$, the following limits hold

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x,y}(\tau(\mp B) > t) = \lim_{t\to\infty} \frac{1}{t} \log \Phi^{(\pm)}(x, y, t) = -\frac{\mu^2}{2\sigma_\pm^2}, \tag{30}$$

by the inequality above, our analysis under assumption (28) and the limit in (27).

Definition (5) and assumption $\sigma_\pm \neq 0$ imply $|\sigma_+| > |\sigma_-| > 0$ and hence $\mu/(2\sigma_+^2) < \mu/(2\sigma_-^2)$. The mirror coupling is therefore not efficient for Problem (T+) since it has a strictly thicker exponential tail than the synchronous coupling. Likewise, the synchronous coupling is not efficient for Problem (T−), which requires the thickest possible exponential tail among all couplings, since it has a thinner tail than the mirror coupling.

In the case $\sigma_\pm = 0$ we have $\sigma_1 = \sigma_2$ and, by (15), $\tau(B) = \log(x/y)/\mu$. Hence $\mathbb{P}_{x,y}(\tau(B) > t) = 0$ for all $t \geq \log(x/y)/\mu$. Since the limit in (30) still holds for $\pm = +$ (note that $|\sigma_+| > 0$), we obtain the inequality:

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x,y}(\tau(B) > t) = -\infty < -\frac{\mu^2}{2\sigma_+^2} = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{x,y}(\tau(-B) > t).$$

This inequality and definition (23) imply that the mirror (resp. synchronous) coupling is not efficient for Problem (T+) (resp. (T−)). □

Remark 5. It is the presence of the positive drift $\mu > 0$ that makes the mirror coupling suboptimal in Problem (T+) (see Theorem 8). The proof of Theorem 9 suggests that if the drift μ is strictly positive and the time horizon T is large, it is in fact better (according to the exponential tail criterion) to use synchronous coupling. This naturally leads to the following conjecture: *if* $\mu > 0$, *the synchronous (resp. mirror) coupling is efficient in Problem* (T+) *(resp.* (T−)*).*

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Appendix. Family of Brownian motions on a filtered probability space

Recall that $\mathcal{V}$ is defined in (2). See e.g. [6, Lemma 2.1] for the proof of Lemma 10.

Lemma 10. *For any Brownian motion $V \in \mathcal{V}$, there exists an $(\mathcal{F}_t)$-Brownian motion $W \in \mathcal{V}$ and a process $C = (C_t)_{t \geq 0}$, such that B and W are independent, C is progressively measurable with $-1 \leq C_t \leq 1$ for all $t \geq 0$ $\mathbb{P}$-a.s., and the following representation holds:*

$$V_t = \int_0^t C_s \, \mathrm{d}B_s + \int_0^t \sqrt{1 - C_s^2} \, \mathrm{d}W_s.$$

Remark 6. The proof of this lemma requires the existence of a Brownian motion $B^{\perp} \in \mathcal{V}$ that is independent of B. If our probability space did not support such a Brownian motion, we could enlarge it, which would only increase the set $\mathcal{V}$. Since the optimal Brownian motions in Theorems 1 and 8b are constructed from B alone, they would also have to be optimal in the original problem. Therefore we can assume that $B^{\perp}$ exists.

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