



Localization of Wiener functionals of fractional regularity and applications[☆]

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Abstract

In this paper we localize some of Watanabe's results on Wiener functionals of fractional regularity, and use them to give a precise estimate of the difference between two Donsker's delta functionals even with fractional differentiability. As an application, the convergence rate of the density of the Euler scheme for non-Markovian stochastic differential equations is obtained.

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1. Introduction

During the last years the analysis on Wiener functionals with fractional smoothness in the sense of Malliavin calculus has drawn increasing attention. There are two ways to define fractional order Sobolev spaces as intermediate spaces between the Sobolev spaces with integer differential index. One way is the complex interpolation method. It makes use of fractional

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powers of the Ornstein–Uhlenbeck operator, and we denote these fractional order Sobolev spaces by \mathbb{D}_α^p , $p > 1$, $\alpha \in \mathbb{R}$. The spaces \mathbb{D}_α^p are natural and typical ones which corresponds to Bessel potential spaces in classical analysis. However, it is not easy to see that the space \mathbb{D}_α^p with $0 < \alpha < 1$ is invariant under the composition with Lipschitz functions. To circumvent this difficulty, Watanabe [19] introduced real interpolation fractional order Sobolev spaces on Wiener spaces by using the trace method. Then an equivalent method, the K -method, was used in Airault, Malliavin and Ren [2] to study the smoothness of stopping times of diffusion processes. The advantage of the K -method is that it describes explicitly how well one can approximate a fractionally smooth Wiener functional by a sequence of smooth functionals. For the equivalence between the trace method and the K -method, we refer to [18, Chap. 1]. Also it should be mentioned that later Hirsch [8] proved that the complex interpolation fractional order Sobolev spaces are in fact invariant under the composition with Lipschitz functions, too.

The aims of the present paper are, roughly speaking, to study local versions of some of the results of [19] and to investigate their applications in the Euler scheme of non-Markovian stochastic differential equations. In particular, we establish a precise estimate of the difference between two Donsker's delta functionals in terms of fractional order Sobolev norms, and, as a consequence, we then dominate the difference between two conditional expectations in the sense of Hölder norms.

The Euler scheme is a useful tool in the numerical simulation of solutions of stochastic differential equations, and has theoretical value as well. Let $(X(\cdot))$ be the unique solution to

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \quad (1)$$

where b, σ are respectively the Lipschitz continuous mapping from \mathbb{R}^d to \mathbb{R}^d and $\mathbb{R}^d \otimes \mathbb{R}^m$, and $(W(\cdot))$ is an m -dimensional Brownian motion. Let $T > 0$ be a fixed time horizon, and T/n represent the discretization step. Set $X_n(0) = x$, and for $kT/n < t \leq (k+1)T/n$, the Euler scheme is defined by

$$\begin{aligned} X_n(t) = & X_n\left(\frac{kT}{n}\right) + b\left(\frac{kT}{n}, X_n\left(\frac{kT}{n}\right)\right)\left(t - \frac{kT}{n}\right) \\ & + \sigma\left(\frac{kT}{n}, X_n\left(\frac{kT}{n}\right)\right)\left(W(t) - W\left(\frac{kT}{n}\right)\right). \end{aligned}$$

There are two kinds of weak approximations. The first one concerns

$$\xi_1(x, T, n) := E[f(X(T))] - E[f(X_n(T))],$$

where f is a suitable class of test function. The second one is the approximation of the density $p_{X(T)}$ of the law of $X(T)$, i.e.,

$$\xi_2(x, T, n) := p_{X(T)} - p_{X_n(T)}.$$

When studying these two kinds of quantities, people's interest focuses on the convergence rate or an error expansion of $\xi_1(x, T, n)$ and $\xi_2(x, T, n)$ in terms of T/n , due to the fact that analysis of these two kinds of quantities turns out to be more important for applications, for instance in finance and biology. There have been a lot of progresses in this area. Suppose that the test function f and the coefficients b and σ are sufficiently smooth and f has polynomial growth. Without any additional assumption on the generator, Talay and Tubaro [17] derive an error expansion of order 1 for $\xi_1(x, T, n)$. Bally and Talay [5] also obtain the same kind of result for bounded Borel functions f under the hypoellipticity assumption on the coefficients. These

authors also extend their results to $\xi_2(x, T, n)$ for a slightly modified Euler scheme in [6]. It is also worth noting that Kohatsu-Higa and Pettersson in [11] introduce another way to prove weak error expansion of $\xi_1(x, T, n)$ and $\xi_2(x, T, n)$ which is based on the integration by parts formula of the Malliavin calculus. On the other hand, by using the fractional calculus in the Malliavin calculus, Watanabe and the second author of the present paper [15] obtained the convergence of $\xi_2(x, T, n)$ in fractional order Sobolev spaces. All these works are confined in the context of Markovian SDEs.

Establishing the estimate of the difference between two Donsker's delta functionals enables us to study the Euler scheme of non-Markovian stochastic differential equations. In other words, we allow the coefficients in stochastic differential equations to look into the past. More precisely, we consider the solution to the equation of the form:

$$X(t) = x + \int_0^t b(s, X(\cdot))ds + \int_0^t \sigma(s, X(\cdot))dW(s), \quad (2)$$

where $\sigma : [0, \infty] \times C([0, \infty]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ and $b : [0, \infty] \times C([0, \infty]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$.

Let us describe our ideas explicitly as follows. Since the coefficients (σ, b) depend not only on the present values of the solution processes, but also on its previous values too, the analysis of convergence rate or error expansion of $\xi_1(x, T, n)$ and $\xi_2(x, T, n)$ for SDE (2) is quite different to that for SDE (1). In particular, it seems difficult to extend the results in [5,6] to stochastic differential equations which are not Markovian SDEs since the approaches used there rely heavily on the Feynman–Kac partial differential equations associated with SDE (1). Here unlike the approaches in [5,6], by using the fact that the heat kernel can be given by the generalized expectation of Donsker's delta functionals, we will establish the convergence rate of $\xi_2(x, T, n)$ for SDE (2). Of course, since the coefficients in (2) may depend on the past trajectories $\{X_s, 0 \leq s \leq t\}$ of the solution, we should modify the Euler scheme for non-Markovian stochastic differential equations which will be carried out in Section 5. In order to derive a convergence rate of $\xi_2(x, T, n)$, the strong approximation of the Euler scheme for non-Markovian stochastic differential equations in Sobolev spaces of appropriate orders in the Malliavin calculus sense is needed.

The paper is organized as follows. In Section 2, we recall some results of the Malliavin calculus that we will use in the sequel. In Section 3, we study some properties of the fractional order Sobolev spaces under local assumptions. In Section 4, we give a precise estimate of the difference between two Donsker's functionals. In Section 5, we are devoted to the proof of the convergence rate of $\xi_2(x, T, n)$.

2. Recalls on the Malliavin calculus

We first recall briefly some basic ingredients in the Malliavin calculus and the reader is referred, e.g., to [10,13,14] for more details. Let (B, \mathbb{H}, μ) be an abstract Wiener space. We will denote the gradient operator (or Shigekawa's \mathbb{H} -derivative) by D , its dual divergence operator (or the Skorohod operator) by D^* and the Ornstein–Uhlenbeck operator by $L := -D^*D$. Let E be a real separable Hilbert space. The Sobolev spaces $\mathbb{D}_\alpha^p(E)$, $1 < p < \infty$, $\alpha \in \mathbb{R}$, of E -valued Wiener and generalized Wiener functionals are defined by

$$\mathbb{D}_\alpha^p(E) = (1 - L)^{-\alpha/2}(\mathbb{L}^p(E))$$

with the norm

$$\|F\|_{\alpha,p} = \|(1 - L)^{\alpha/2}F\|_{\mathbb{L}^p(E)},$$

where $\mathbb{L}^p(E)$ is the usual $\mathbb{L}^p(E)$ -space. We denote $\mathbb{L}^{\infty-}(E) = \bigcap_{1 < p < \infty} \mathbb{L}^p(E)$, $\mathbb{D}_\alpha^{\infty-}(E) = \bigcap_{1 < p < \infty} \mathbb{D}_\alpha^p(E)$, and $\mathbb{D}_\infty^{\infty-}(E) = \bigcap_{\alpha > 0} \bigcap_{1 < p < \infty} \mathbb{D}_\alpha^p(E)$. If $E = \mathbb{R}$, we simply write \mathbb{L}^p , \mathbb{D}_α^p , $\mathbb{L}^{\infty-}$, $\mathbb{D}_\alpha^{\infty-}$ and $\mathbb{D}_\infty^{\infty-}$.

We denote by $C_p^\infty(\mathbb{R}^d)$ the set of all infinitely continuously differential functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that f and all of its partial derivatives have polynomial growth, and we also denote by $C_b^\infty(\mathbb{R}^d)$ the set of all infinitely continuously differential functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that f and all of its partial derivatives are bounded. Let $C^\beta(\mathbb{R}^d)$, $\beta \geq 0$, be the Banach space of $[\beta]$ -times continuously differentiable functions on \mathbb{R}^d whose $[\beta]$ -th derivatives are uniformly $\{\beta\}$ -Hölder continuous with the norm $([\beta]$ and $\{\beta\} = \beta - [\beta]$ denote the integer and fractional part of β , respectively)

$$\|f\|_{C^\beta(\mathbb{R}^d)} = \sum_{n; |n| \leq [\beta]} |\partial^n f|_\infty + \sum_{n; |n| = [\beta]} \sup_{x \neq y} |\partial^n f(x) - \partial^n f(y)|/|x - y|^{\{\beta\}},$$

with the notation $n = (n_1, \dots, n_d)$, $|n| = \sum_{i=1}^d n_i$ and $\partial^n = \partial_1^{n_1} \dots \partial_d^{n_d}$, where $\partial_i = \partial/\partial x_i$. Note that $(1 - \Delta)^{-\beta/2}(C^0(\mathbb{R}^d)) \subset C^\beta(\mathbb{R}^d)$ (see [16]). Let $S(\mathbb{R}^d)$ be the real space of rapidly decreasing C^∞ -functions.

In the Malliavin calculus, a key role is played by the Malliavin covariance matrix which is defined as follows.

Definition 2.1. Suppose that $F = (F^1, \dots, F^d)$ is a random vector whose components belong to the space $\mathbb{D}_1^{\infty-}$. We associate to F the following random symmetric nonnegative definite matrix:

$$\Sigma_F(\omega) = (\sigma_F^{ij}(\omega))_{1 \leq i, j \leq d} := ((DF^i, DF^j)_\mathbb{H})_{1 \leq i, j \leq d}.$$

The matrix $\Sigma_F(\omega)$ will be called the Malliavin covariance matrix of the random vector F . We will say that a random vector $F = (F^1, \dots, F^d)$ whose components are in $\mathbb{D}_\infty^{\infty-}$ is nondegenerate if its Malliavin covariance matrix $\Sigma_F(\omega)$ is invertible a.s. and

$$\Gamma_F(\omega) = (\gamma_F^{ij}(\omega))_{1 \leq i, j \leq d} := (\Sigma_F(\omega))^{-1} \in \mathbb{L}^{\infty-}(\mathbb{R}^d \otimes \mathbb{R}^d).$$

Set $d\mu_G := G \cdot d\mu$ and

$$p_{F,G}(y) = \frac{\mu_G(F \in dy)}{dy}. \quad (3)$$

Denote by E_G the integral w.r.t μ_G . The following proposition is taken from [3, Proposition 23] or [4, Lemma 2.1].

Proposition 2.2. Let $F \in \mathbb{D}_2^{\infty-}(\mathbb{R}^d)$ and let $G \in \mathbb{D}_1^{\infty-}$ take values on $[0, 1]$ with

$$1 + E_G[\|D \ln G\|_\mathbb{H}^p] < \infty \quad \text{for every } p \geq 1.$$

Assume that A is a measurable set such that $G1_A = 0$ and for any $p > 1$,

$$E[|\det(\Sigma_F)|^{-p} 1_{A^c}] < \infty.$$

Then the law of F under μ_G is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d . Moreover, for every $p > d$ there exist some universal constants C and $q > 1$ depending on d and p such that the density $p_{F,G}$ satisfies

$$p_{F,G}(y) \leq C(1 + E_G[|\det(\Sigma_F)|^{-p}])^q (1 + \|F\|_{2,p,G} + \|LF\|_{p,G})^q (1 + E_G[\|D \ln G\|_\mathbb{H}^p])^q.$$

In the rest of this article, we will adopt the following notations. Let $\Psi : [0, \infty) \mapsto \mathbb{R}$ be a C_b^∞ function (Ψ and all of its partial derivatives are bounded) such that

$$1_{[0, \frac{1}{8}]} \leq \Psi \leq 1_{[0, \frac{1}{4}]},$$

and $\Psi_1 : [0, \infty) \mapsto \mathbb{R}$ be a C_b^∞ function such that

$$1_{[0, \frac{1}{4}]} \leq \Psi_1 \leq 1_{[0, \frac{1}{2}]}$$

and

$$\sup_x |(\ln \Psi_1(x))'|^p \Psi_1(x) < \infty \quad \text{for every } p \geq 1. \quad (4)$$

For $F_1, F_2 \in \mathbb{D}_\infty^{--}(\mathbb{R}^d)$, we define R_{F_1, F_2} by

$$R_{F_1, F_2} = \frac{\|D(F_1 - F_2)\|_{\mathbb{H}}^2 (1 + \|\Sigma_{F_1}\|_2^2)^{(d-1)/2}}{\det(\Sigma_{F_1})},$$

where $\|DF_i\|_{\mathbb{H}}^2 = \sum_{j=1}^d \|DF_i^j\|_{\mathbb{H}}^2$, $i = 1, 2$ and $\|\Sigma_{F_1}\|_2$ is the Hilbert–Schmidt norm of the Malliavin covariance matrix Σ_{F_1} . It is obvious that $\Psi_1(R_{F_1, F_2}) = 1$ on the set $\{\Psi(R_{F_1, F_2}) \neq 0\}$. For the set $\bigcup_{k=0}^\infty \{D^k(\Psi(R_{F_1, F_2})) \neq 0\}$ and $\bigcup_{k=0}^\infty \{D^k(\Psi_1(R_{F_1, F_2})) \neq 0\}$, we have the following result:

$$\bigcup_{k=0}^\infty \{D^k(\Psi(R_{F_1, F_2})) \neq 0\} \subset \left\{ \det(\Sigma_{F_2+t(F_1-F_2)}) \geq 4^{-d} \frac{(\det(\Sigma_{F_1}))^d}{\|\Sigma_{F_1}\|^{d(d-1)}} \right\} \quad \text{a.s.}, \quad (5)$$

and

$$\bigcup_{k=0}^\infty \{D^k(\Psi_1(R_{F_1, F_2})) \neq 0\} \subset \left\{ \det(\Sigma_{F_2+t(F_1-F_2)}) \geq \left(1 - \frac{\sqrt{2}}{2}\right)^{2d} \frac{(\det(\Sigma_{F_1}))^d}{\|\Sigma_{F_1}\|^{d(d-1)}} \right\} \quad \text{a.s.}, \quad (6)$$

where $\|\Sigma_{F_1}\|$ is the operator norm. The proof of this result can be found in [7, Remark 14].

Watanabe [19] has introduced the fractional order Sobolev spaces on the Wiener space and studied their applications to the solutions of stochastic differential equations. An equivalent approach to these spaces using the K -method then appeared in [2,9]. Now let us recall results in this respect that will be needed in the sequel.

Definition 2.3. For any $0 < \alpha < 1$ and any $1 < p < \infty$, we define

$$\mathcal{E}_\alpha^p = (\mathbb{L}^p, \mathbb{D}_1^p)_{\alpha, p},$$

where (\cdot, \cdot) denotes the real interpolation space as in [18].

There are several equivalent norms in \mathcal{E}_α^p (cf. [18]). The one we shall use is given by Peetre's K -method:

$$\|F\|_{\mathcal{E}_\alpha^p} := \left[\int_0^1 [\epsilon^{-\alpha} K(\epsilon, p, F)]^p \frac{d\epsilon}{\epsilon} \right]^{\frac{1}{p}},$$

where

$$K(\epsilon, p, F) := \inf\{\|F_1\|_p + \epsilon\|F_2\|_{1,p}, F_1 + F_2 = F, F_1, F_2 \in \mathbb{L}^p\}.$$

Remark 2.4. Let $0 < \alpha < 1$, $1 < p < \infty$, then $F \in \mathcal{E}_\alpha^p$ if and only if

$$\sum_{n=1}^{\infty} 2^{np\alpha} K(2^{-n}, p, F)^p < \infty.$$

For the relationship between \mathbb{D}_α^p and \mathcal{E}_α^p , the following theorem is proved in [19, Theorem 1.1] directly using specific properties of Wiener functionals.

Theorem 2.5. For every $0 < \alpha < 1$, $1 < p < \infty$ and $\varepsilon > 0$, we have

$$\mathcal{E}_{\alpha+\varepsilon}^p \subset \mathbb{D}_\alpha^p \subset \mathcal{E}_{\alpha-\varepsilon}^p.$$

By the above theorem we deduce immediately that for every $\alpha > 0$, $\mathbb{D}_{\alpha-}^p := \bigcap_{0 < \beta < \alpha} \mathbb{D}_\beta^p = \bigcap_{0 < \beta < \alpha} \mathcal{E}_\beta^p =: \mathcal{E}_{\alpha-}^p$ and $\mathbb{D}_{\alpha-}^{\infty} := \bigcap_{0 < \beta < \alpha} \mathbb{D}_\beta^{\infty} = \bigcap_{0 < \beta < \alpha} \mathcal{E}_\beta^{\infty} =: \mathcal{E}_{\alpha-}^{\infty}$.

3. Properties of the fractional order Sobolev spaces under local assumptions

Now we will study local properties of the space \mathcal{E}_α^p under local assumptions. In what follows we denote by C a generic constant which can be different from one formula to another.

Let $F \in \mathbb{D}_{\alpha-}^{\infty}$ such that $F > 0$ a.s. and $1/F \in \mathbb{L}^{\infty-}$. It is well known that if α is a positive integer, then $1/F \in \mathbb{D}_{\alpha-}^{\infty}$. When $\alpha > 0$ is not an integer, it is proved in [19] that $1/F$ only belongs to $\mathbb{D}_{\alpha-}^{\infty}$. Our first result is a local version of this result whose proof is based on the K -method.

Theorem 3.1. Let $\alpha = k + \sigma$, $k \in \mathbb{N}$, $0 < \sigma \leq 1$. Suppose that $F \in \mathcal{E}_\alpha^{\infty-}$ or $\mathbb{D}_{\alpha-}^{\infty}$ is a nonnegative Wiener functional, $G \in \mathbb{D}_{k+1}^{\infty-}$, and A is a measurable set such that $G1_A = 0$ and

$$E \left[\left| \frac{1}{F} \right|^p 1_{A^c} \right] < \infty, \quad \text{for every } p > 1.$$

Then we have

$$\frac{1}{F} \cdot G \in \mathbb{D}_{\alpha-}^{\infty}.$$

Furthermore, if α is a positive integer, $F, G \in \mathbb{D}_{\alpha-}^{\infty}$, then we have

$$\frac{1}{F} \cdot G \in \mathbb{D}_{\alpha-}^{\infty}.$$

Remark 3.2. In fact, as can be seen from the proof of Theorem 3.1, for every $\Psi \in C_b^\infty$, we have $\Psi(1/F) \cdot G \in \mathbb{D}_{\alpha-}^{\infty}$, and it is also easy to obtain that for any $\beta < \alpha$, $p < p'$, we have

$$\left\| \Psi \left(\frac{1}{F} \right) \cdot G \right\|_{\beta, p} \leq C \|G\|_{k+1, p'}. \quad (7)$$

Furthermore, if there exists $G_1 \in \mathbb{D}_{k+1}^{\infty-}$ such that $G_1 = 1$ on the set $\{G \neq 0\}$ and $G_1 1_A = 0$, then we have $G = G \cdot G_1$ and by $\Psi(1/F) \cdot G_1 \in \mathbb{D}_{\alpha-}^{\infty-}$,

$$\left\| \Psi\left(\frac{1}{F}\right) \cdot G \right\|_{\beta,p} = \left\| \Psi\left(\frac{1}{F}\right) \cdot G_1 \cdot G \right\|_{\beta,p} \leq C \|G\|_{\beta,p'}. \quad (8)$$

In order to prove [Theorem 3.1](#), we need the following lemma.

Lemma 3.3. *Let $0 < \alpha < 1$, $4 < p < \infty$. Suppose that $F \in \mathbb{L}^p$ is a nonnegative Wiener functional and $F_n \in \mathbb{L}^p$ such that*

(i)

$$\|F_n - F\|_p \leq C 2^{-n\alpha};$$

(ii)

$$E \left[\left| \frac{1}{F} \right|^p 1_{A^c} \right] < \infty.$$

Set

$$\widetilde{F}_n := F_n \vee 0 + 2^{-n\alpha}.$$

Then we have

$$\sup_n E \left[\left| \frac{1}{\widetilde{F}_n} \right|^{p/4} 1_{A^c} \right] < \infty. \quad (9)$$

Proof. Obviously

$$\|\widetilde{F}_n - F\|_p \leq C 2^{-n\alpha}, \quad (10)$$

$$E \left[\left| \frac{1}{\widetilde{F}_n} \right|^q \right] \leq 2^{nq\alpha}, \quad \forall q > 1. \quad (11)$$

We define

$$r_n = \left| \frac{\widetilde{F}_n - F}{F} \right| 1_{A^c}.$$

Therefore we split (9) into two parts:

$$\begin{aligned} E \left[\left| \frac{1}{\widetilde{F}_n} \right|^{p/4} 1_{A^c} \right] &= E \left[\left| \frac{1}{\widetilde{F}_n} \right|^{p/4} 1_{A^c} 1_{\{r_n \leq \frac{1}{2}\}} \right] + E \left[\left| \frac{1}{\widetilde{F}_n} \right|^{p/4} 1_{A^c} 1_{\{r_n > \frac{1}{2}\}} \right] \\ &=: \Xi_1 + \Xi_2. \end{aligned}$$

Since on A^c ,

$$\frac{2}{3F} 1_{\{r_n \leq \frac{1}{2}\}} \leq \frac{1}{\widetilde{F}_n} 1_{\{r_n \leq \frac{1}{2}\}} \leq \frac{2}{F} 1_{\{r_n \leq \frac{1}{2}\}},$$

by Assumption (ii) we have

$$\sup_n \Xi_1 \leq E \left[\left| \frac{2}{F} \right|^{p/4} 1_{A^c} 1_{\{r_n \leq \frac{1}{2}\}} \right] < \infty.$$

Next let us deal with the term Ξ_2 . By (10) and Assumption (ii), we have

$$\begin{aligned} P\left(A^c \left\{r_n > \frac{1}{2}\right\}\right) &\leq 2^{p/2} E\left[\left(\frac{\widetilde{F}_n - F}{F}\right)^{p/2} 1_{A^c}\right] \\ &\leq 2^{p/2} \|\widetilde{F}_n - F\|_p^{p/2} \cdot \left\|\frac{1}{F} 1_{A^c}\right\|_p^{p/2} \\ &\leq C 2^{-np\alpha/2}. \end{aligned}$$

Consequently, by (11) we have

$$\begin{aligned} \sup_n \Xi_2 &\leq E\left[\left|\frac{1}{\widetilde{F}_n}\right|^p\right]^{1/4} \cdot P\left(A^c \left\{r_n > \frac{1}{2}\right\}\right)^{1/2} \\ &\leq \sup_n C 2^{np\alpha/4} \cdot 2^{-np\alpha/4} < \infty, \end{aligned}$$

and hence we complete the proof. \square

Proof of Theorem 3.1. Since there is no essential difference, we assume $0 < \alpha < 1$ and $F \in \mathcal{E}_\alpha^{\infty-}$ for simplicity. For every $p' > 1$, let $p = 18p'$, $r = 2p' = p/9$ and $r' = p/4$. By Remark 2.4, for each n , we can find $F_n \in \mathbb{D}_1^p$ such that

$$\|F_n - F\|_p \leq C 2^{-n\alpha}, \quad \|F_n\|_{1,p} \leq C 2^{n(1-\alpha)}.$$

Set

$$\widetilde{F}_n := F_n \vee 0 + 2^{-n\alpha}.$$

By the previous lemma, we have

$$\sup_n \left\|\frac{1}{\widetilde{F}_n} 1_{A^c}\right\|_{r'} < \infty.$$

Hence by the Hölder inequality, we have

$$\left\|D\left(\frac{1}{\widetilde{F}_n}\right) 1_{A^c}\right\|_r \leq C \|D\widetilde{F}_n\|_p \cdot \left\|\frac{1}{\widetilde{F}_n} 1_{A^c}\right\|_{r'}^2 \leq C 2^{n(1-\alpha)}.$$

Therefore, by the Meyer equivalence and $\{DG \neq 0\} \subset \{G \neq 0\} \subset A^c$ we have

$$\left\|\frac{1}{\widetilde{F}_n} \cdot G\right\|_{1,p'} \leq CE \left[\left(\left|\frac{1}{\widetilde{F}_n}\right|^r + \left\|D\left(\frac{1}{\widetilde{F}_n}\right)\right\|_{\mathbb{H}}^r\right) 1_{A^c}\right]^{\frac{1}{r}} \cdot \|G\|_{1,r} \leq C 2^{n(1-\alpha)}.$$

A similar calculus gives

$$\left\|\frac{1}{\widetilde{F}_n} \cdot G - \frac{1}{F} \cdot G\right\|_{p'} \leq \|G\|_r \cdot \|F - \widetilde{F}_n\|_p \cdot \left\|\frac{1}{\widetilde{F}_n} 1_{A^c}\right\|_{r'} \cdot \left\|\frac{1}{F} 1_{A^c}\right\|_{r'} \leq C 2^{-n\alpha}.$$

By Remark 2.4, we obtain

$$\left\|\frac{1}{F} \cdot G\right\|_{\mathcal{E}_\beta^{p'}} < \infty, \quad \forall \beta < \alpha,$$

and this completes the proof. \square

If the assumption $G \in \mathbb{D}_{k+1}^{\infty-}$ in [Theorem 3.1](#) is replaced by $G \in \mathcal{E}_\alpha^{\infty-}$, we have the following theorem.

Theorem 3.4. *Let $\alpha = k + \sigma$, $k \in \mathbb{N}$, $0 < \sigma \leq 1$. Suppose that $F \in \mathcal{E}_\alpha^{\infty-}$ or $\mathbb{D}_\alpha^{\infty-}$ is a nonnegative Wiener functional, $G \in \mathcal{E}_\alpha^{\infty-}$ or $\mathbb{D}_\alpha^{\infty-}$, and A is a measurable set such that $G1_A = 0$ and*

$$E \left[\left| \frac{1}{F} \right|^p 1_{A^c} \right] < \infty, \quad \text{for every } p > 1.$$

Then we have

$$\frac{1}{F} \cdot G \in \mathbb{D}_{\left(k + \frac{\sigma}{2^{k+1}}\right)-}^{\infty-}.$$

Proof. Without loss of generality, we assume $0 < \alpha < 1$, $F \in \mathcal{E}_\alpha^{\infty-}$ and $G \in \mathcal{E}_\alpha^{\infty-}$ for simplicity. For every $p' > 1$, let p, r, r' be as in the proof of [Theorem 3.1](#). By [Remark 2.4](#), for each n , we can find $F_n \in \mathbb{D}_1^p$, $G_n \in \mathbb{D}_1^{r'}$ such that

$$\|F_n - F\|_p \leq C2^{-n\alpha}, \quad \|F_n\|_{1,p} \leq C2^{n(1-\alpha)}$$

and

$$\|G_n - G\|_r \leq C2^{-n\alpha}, \quad \|G_n\|_{1,r} \leq C2^{n(1-\alpha)}.$$

Let

$$\widetilde{F}_n := F_n \vee 0 + 2^{-n\alpha/2}.$$

Then \widetilde{F}_n satisfies

$$\|\widetilde{F}_n - F\|_p \leq C2^{-n\alpha/2}, \quad \|\widetilde{F}_n\|_{1,p} \leq C2^{n(1-\alpha/2)}.$$

Therefore we have

$$\begin{aligned} \left\| \frac{1}{\widetilde{F}_n} \cdot G_n - \frac{1}{F} \cdot G \right\|_{p'} &\leq \left\| \frac{1}{\widetilde{F}_n} \cdot G - \frac{1}{F} \cdot G \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n} (G_n - G) \right\|_{p'} \\ &=: A_1 + A_2. \end{aligned}$$

By the same method as in the proof of [Theorem 3.1](#), we have

$$A_1 \leq C2^{-n\alpha/2}.$$

By $1/\widetilde{F}_n \leq 2^{n\alpha/2}$ and $\|G_n - G\|_r \leq C2^{-n\alpha}$, we also have

$$A_2 \leq \|G_n - G\|_r \cdot \left\| \frac{1}{\widetilde{F}_n} \right\|_r \leq C2^{-n\alpha/2}.$$

Consequently we have

$$\left\| \frac{1}{\widetilde{F}_n} \cdot G_n - \frac{1}{F} \cdot G \right\|_{p'} \leq C2^{-n\alpha/2}.$$

Similarly, by Meyer equivalence of norms, we have

$$\left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{1,p'} \leq C \left\{ \left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{p'} + \left\| D \left(\frac{1}{\widetilde{F}_n} \cdot G_n \right) \right\|_{p'} \right\}$$

$$\begin{aligned}
&= C \left\{ \left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n^2} \cdot D\widetilde{F}_n \cdot G_n \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n} \cdot DG_n \right\|_{p'} \right\} \\
&\leq C \left\{ \left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n^2} \cdot D\widetilde{F}_n \cdot (G_n - G) \right\|_{p'} \right. \\
&\quad \left. + \left\| \frac{1}{\widetilde{F}_n^2} \cdot D\widetilde{F}_n \cdot G \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n} \cdot DG_n \right\|_{p'} \right\}.
\end{aligned}$$

Similar to the proof of [Theorem 3.1](#), we have

$$\left\| \frac{1}{\widetilde{F}_n^2} \cdot D\widetilde{F}_n \cdot G \right\|_{p'} \leq C2^{n(1-\alpha/2)}.$$

By $1/\widetilde{F}_n \leq 2^{n\alpha/2}$, $\|G_n - G\|_r \leq C2^{-n\alpha}$ and $\|D\widetilde{F}_n\|_p \leq C2^{n(1-\alpha/2)}$, we have

$$\begin{aligned}
\left\| \frac{1}{\widetilde{F}_n^2} \cdot D\widetilde{F}_n \cdot (G_n - G) \right\|_{p'} &\leq \left\| \frac{1}{\widetilde{F}_n} \right\|_{r'}^2 \cdot \|D\widetilde{F}_n\|_p \cdot \|G_n - G\|_r \\
&\leq C2^{n(1-\alpha/2)}.
\end{aligned}$$

By $1/\widetilde{F}_n \leq 2^{n\alpha/2}$, $\|G_n - G\|_r \leq C2^{-n\alpha}$ and $\|G_n\|_{1,r} \leq 2^{n(1-\alpha)}$, we also have

$$\begin{aligned}
\left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{p'} &\leq \left\| \frac{1}{\widetilde{F}_n} \cdot (G_n - G) \right\|_{p'} + \left\| \frac{1}{\widetilde{F}_n} \cdot G \right\|_{p'} \leq C, \\
\left\| \frac{1}{\widetilde{F}_n} \cdot DG_n \right\|_{p'} &\leq C2^{n(1-\alpha/2)}.
\end{aligned}$$

Consequently, we have

$$\left\| \frac{1}{\widetilde{F}_n} \cdot G_n \right\|_{1,p'} \leq C2^{n(1-\alpha/2)}.$$

By [Remark 2.4](#), we obtain

$$\left\| \frac{1}{F} \cdot G \right\|_{\mathcal{E}_\beta^{p'}} < \infty, \quad \forall \beta < \alpha/2,$$

and this completes the proof. \square

The following theorem is a local version of Lemma 2.1 in [\[19\]](#) which will play an essential role in Section 4. Before proceeding, we introduce some notions and notations concerning the Sobolev space on \mathbb{R}^d (cf. [\[1,18\]](#)). Define the family of Bessel potential spaces by

$$\mathbb{L}_\alpha^p(\mathbb{R}^d) = (1 - \Delta)^{-\alpha/2}(\mathbb{L}^p(\mathbb{R}^d)), \quad 1 < p < \infty, \alpha \in \mathbb{R}$$

equipped with the norm

$$\|g\|_{\alpha,p} = \|(1 - \Delta)^{\alpha/2} g\|_p.$$

For $1 < p < \infty$, $\alpha \in \mathbb{R}$, we also define fractional order Sobolev spaces on \mathbb{R}^d as follows:

$$\mathcal{E}_\alpha^p(\mathbb{R}^d) = (\mathbb{L}_k^p(\mathbb{R}^d), \mathbb{L}_{k+1}^p(\mathbb{R}^d))_{\sigma,p} \quad \text{if } k+1 \text{ be the smallest integer larger than } \alpha$$

equipped with the norm

$$\|g\|_{\mathcal{E}_\alpha^p(\mathbb{R}^d)} = \left[\int_0^1 |\epsilon^{-\sigma} K(\epsilon, p, g)|^p \frac{d\epsilon}{\epsilon} \right]^{1/p},$$

where $\sigma := \alpha - k$ and

$$K(\epsilon, p, g) := \inf\{\|g_1\|_{k,p} + \epsilon \|g_2\|_{k+1,p}, g_1 + g_2 = g, g_1, g_2 \in \mathbb{L}_k^p(\mathbb{R}^d)\}.$$

Theorem 3.5. Suppose that $F \in \mathbb{D}_1^{\infty-}(\mathbb{R}^d)$, $G \in \mathbb{D}_1^{\infty-}$ taking values on $[0, 1]$ with

$$1 + E_G[\|D \ln G\|_{\mathbb{H}}^p] < \infty, \quad \text{for every } p \geq 1,$$

and the density $p_{F,G}$ is bounded. Then, for every $1 < p < p' < \infty$ and $0 \leq \alpha < \alpha' \leq 1$, there exists a positive constant $C = C(\alpha, \alpha', p, p')$ such that

$$\|g \circ F \cdot G\|_{\mathcal{E}_\alpha^p} \leq C \|g\|_{\mathcal{E}_{\alpha'}^{p'}(\mathbb{R}^d)} \quad \text{for every } g \in S(\mathbb{R}^d).$$

Proof. Since $p_{F,G}$ is bounded, for every $p' > p$, we can find $C = C(p, p') > 0$ such that

$$\begin{aligned} \|g \circ F \cdot G\|_p &\leq \|g \circ F \cdot G^{1/p'}\|_{p'} \cdot \|G^{1-1/p'}\|_{pp'/(p'-p)} \\ &= \left[\int_{\mathbb{R}^d} |g(x)|^{p'} p_{F,G}(x) dx \right]^{1/p'} \cdot \|G^{1-1/p'}\|_{pp'/(p'-p)} \\ &\leq C \|g\|_{p'}. \end{aligned} \tag{12}$$

Also, by the Meyer equivalence of norms, for every $p' > p$, we have

$$\begin{aligned} \|g \circ F \cdot G\|_{1,p} &= \|(1-L)^{1/2}(g \circ F \cdot G)\|_p \\ &\leq C\{\|g \circ F \cdot G\|_p + \|D(g \circ F \cdot G)\|_p\} \\ &\leq C \left\{ \|g \circ F \cdot G\|_p + \left\| \sum_{i=1}^d (\partial_i g) \circ F \cdot D F_i \cdot G \right\|_p + \|g \circ F \cdot D G\|_p \right\} \\ &\leq C \left\{ \|g \circ F \cdot G\|_p + \left\| \sum_{i=1}^d (\partial_i g) \circ F \cdot D F_i \cdot G \right\|_p + \|g \circ F \cdot D(\ln G) \cdot G\|_p \right\} \\ &\leq C \left\{ \|g \circ F \cdot G\|_p + \sum_{i=1}^d \|(\partial_i g) \circ F \cdot G\|_{p'} \cdot \|D F_i\|_{pp'/(p'-p)} \right. \\ &\quad \left. + \|g \circ F \cdot G^{1-((p'-p)/pp')}\|_{p'} \cdot \|D(\ln G) \cdot G^{(p'-p)/pp'}\|_{pp'/(p'-p)} \right\}. \end{aligned}$$

Then similarly as (12), for every $p' > p$, we can find $C = C(p, p')$ such that

$$\|g \circ F \cdot G\|_{1,p} \leq C \|g\|_{1,p'}. \tag{13}$$

By (12) and (13), we get the desired estimate if $\alpha = 0$ or $\alpha = 1$. If $0 < \alpha < 1$, we proceed as follows. Let $1 < p < p' < \infty$ and $0 < \alpha < \alpha' < 1$ be fixed. For $g \in \mathcal{E}_{\alpha'}^{p'}(\mathbb{R}^d)$, by (12) and (13), we have

$$K(\epsilon, p, g \circ F \cdot G) \leq CK(\epsilon, p', g).$$

Hence by the Hölder inequality we obtain that

$$\begin{aligned} \|g \circ F \cdot G\|_{\mathcal{E}_{\alpha}^p} &= \left[\int_0^1 [\epsilon^{-\alpha} K(\epsilon, p, g \circ F \cdot G)]^p \frac{d\epsilon}{\epsilon} \right]^{\frac{1}{p}} \\ &\leq C \left[\int_0^1 [\epsilon^{-\alpha} K(\epsilon, p', g)]^p \frac{d\epsilon}{\epsilon} \right]^{\frac{1}{p}} \\ &= C \left[\int_0^1 \left[\epsilon^{-\alpha' p - \frac{p}{p'}} K(\epsilon, p', g)^p \right] \left[\epsilon^{(\alpha' - \alpha)p - \frac{p' - p}{p'}} d\epsilon \right]^{\frac{1}{p}} \right] \\ &\leq C \left[\int_0^1 [\epsilon^{-\alpha'} K(\epsilon, p', g)]^{p'} \frac{d\epsilon}{\epsilon} \right]^{\frac{1}{p'}} \left[\int_0^1 \epsilon^{\frac{(\alpha' - \alpha)p' p}{p' - p}} \frac{d\epsilon}{\epsilon} \right]^{\frac{p' - p}{p' p}} \\ &\leq C \|g\|_{\mathcal{E}_{\alpha'}^{p'}(\mathbb{R}^d)} \end{aligned}$$

as desired. \square

We shall need the following local version of the integration by parts formula.

Lemma 3.6. *Let $k \in \mathbb{N}$. Suppose*

- (i) $F \in \mathbb{D}_{k+2}^{\infty-}(\mathbb{R}^d)$;
- (ii) $G \in \mathbb{D}_{k+1}^{\infty-}$, A is a measurable set such that $G1_A = 0$ and for any $p > 1$,

$$E[|\det(\Sigma_F)|^{-p} 1_{A^c}] < \infty.$$

Then for all $g \in C_p^\infty(\mathbb{R}^d)$ and all multiindex $\alpha \in \{1, \dots, d\}^{k+1}$, we have

$$E \left[\left(\frac{\partial^{|\alpha|} g}{\partial y^\alpha} \right) \circ F \cdot G \right] = E[g \circ F \cdot H_\alpha(F, G)] \quad (14)$$

where the elements $H_\alpha(F, G) \in \mathbb{D}_{k+1-|\alpha|}^{\infty-}$ are recursively given by

$$\begin{aligned} H_{(i)}(F, G) &= H_i(F, G) \\ &:= \sum_{j=1}^d D^*(\gamma_F^{ij} \cdot G \cdot DF_j) \\ &= - \sum_{j=1}^d \{ \gamma_F^{ij} \cdot LF_j \cdot G + (D\gamma_F^{ij}, DF_j)_{\mathbb{H}} \cdot G + \gamma_F^{ij} \cdot (DG, DF_j)_{\mathbb{H}} \}, \\ H_\alpha(F, G) &= H_{(i_1, \dots, i_k)}(F, G) \\ &:= H_{i_k}(F, H_{(i_1, \dots, i_{k-1})}(F, G)), \end{aligned}$$

which satisfy $H_\alpha(F, G)1_A = 0$ and

$$\|H_\alpha(F, G)\|_{k+1-|\alpha|, p} \leq C \|G\|_{k+1, p'} \quad (15)$$

for every $p < p'$.

Proof. Since

$$(\det(\Sigma_F))^{-1} 1_{A^c} \in \mathbb{L}^{\infty-} \quad \text{and} \quad \det(\Sigma_F) \in \mathbb{D}_{k+1}^{\infty-},$$

by Theorem 3.1, we have

$$(\det(\Sigma_F))^{-1} G \in \mathbb{D}_{k+1}^{\infty-},$$

which in turn yields that

$$\gamma_F^{ij} \cdot G \in \mathbb{D}_{k+1}^{\infty-}.$$

By the chain rule we have

$$(\partial_i g) \circ F \cdot 1_{A^c} = \sum_{j=1}^d (D(g \circ F), DF_j)_{\mathbb{H}} \cdot \gamma_F^{ij} \cdot 1_{A^c},$$

so we obtain

$$(\partial_i g) \circ F \cdot G = \left(D(g \circ F), \sum_{j=1}^d \gamma_F^{ij} \cdot G \cdot DF_j \right)_{\mathbb{H}}.$$

Hence by the duality relationship between the derivative and the divergence operators, we get

$$E[(\partial_i g) \circ F \cdot G] = E[g \circ F \cdot H_i(F, G)],$$

where

$$\begin{aligned} H_i(F, G) &= \sum_{j=1}^d D^*(\gamma_F^{ij} \cdot G \cdot DF_j) \\ &= - \sum_{j=1}^d \{ \gamma_F^{ij} \cdot LF_j \cdot G + (D\gamma_F^{ij}, DF_j)_{\mathbb{H}} \cdot G + \gamma_F^{ij} \cdot (DG, DF_j)_{\mathbb{H}} \}. \end{aligned}$$

It follows by the localness of D that $H_i(F, G)1_A = 0$, and by (7) and the fact that $D^* : \mathbb{D}_{1+\alpha}^p(\mathbb{H}) \rightarrow \mathbb{D}_\alpha^p$ is continuous for every p and every α , we see that for every $p < p'$,

$$\|H_i(F, G)\|_{k, p} \leq C \sum_{j=1}^d \|\gamma_F^{ij} \cdot G \cdot DF_j\|_{k+1, p} \leq C \|G\|_{k+1, p'}.$$

Thus we have

$$E[(\partial_{i_1} \partial_{i_2} g) \circ F \cdot G] = E[g \circ F \cdot H_{i_2}(F, H_{i_1}(F, G))].$$

Obviously we still have $H_{i_2}(F, H_{i_1}(F, G))1_A = 0$ and consequently we can go ahead further. Moreover, it is easy to see that for every $p < p' < p''$,

$$\|H_{i_2}(F, H_{i_1}(F, G))\|_{k-1, p} \leq C \|H_{i_1}(F, G)\|_{k, p'} \leq C \|G\|_{k+1, p''}.$$

By induction we prove the desired results. \square

Remark 3.7. In applications one usually takes $A = \{G = 0\}$. Note that by the localness of the derivative operator one then has $\{D^\beta G \neq 0\} \subset A^c$ for any $\beta \in \mathbb{N}$.

Remark 3.8. Under the conditions of Lemma 3.6, suppose that there exists $G_1 \in \mathbb{D}_k^{\infty-}$ such that $G_1 = 1$ on the set $\{G \neq 0\}$ and $G_1 1_A = 0$, then for any $G' \in \mathbb{D}_{k+1}^{\infty-}$ and $\varepsilon \geq 0$, we have $H_\alpha(F, G \cdot G') = H_\alpha(F, G \cdot G') \cdot G_1$,

$$\|H_i(F, G \cdot G')\|_{k-\varepsilon, p} \leq C \sum_{j=1}^d \|\gamma_F^{ij} \cdot G \cdot G' \cdot DF_j\|_{k+1-\varepsilon, p} \leq C \|G'\|_{k+1-\varepsilon, p'}$$

for every $p < p'$.

Since $\gamma_F^{ij} \cdot G_1 \in \mathbb{D}_k^{\infty-}$,

$$\begin{aligned} \|H_{(i_1, i_2)}(F, G \cdot G')\|_{k-1-\varepsilon, p} &\leq C \sum_{j=1}^d \|\gamma_F^{i_2 j} \cdot H_{i_1}(F, G \cdot G') \cdot G_1 \cdot DF_j\|_{k-\varepsilon, p} \\ &\leq C \|H_{i_1}(F, G \cdot G')\|_{k-\varepsilon, p'} \leq C \|G'\|_{k+1-\varepsilon, p''} \\ &\text{for every } p < p' < p''. \end{aligned}$$

Therefore by induction we have

$$\|H_\alpha(F, G \cdot G')\|_{k+1-|\alpha|-\varepsilon, p} \leq C \|G'\|_{k+1-\varepsilon, p'} \quad (16)$$

for every $p < p'$.

4. Estimate of the difference between two Donsker's delta functions

In this section, we establish an estimate of the difference between two Donsker's delta functionals. In what follows we also denote by C a generic constant which can be different from one formula to another.

Theorem 4.1. Let $\delta = k + \sigma$, $k \in \mathbb{N}$, $0 < \sigma \leq 1$. Suppose that $H, H_1, H_2 : B \rightarrow [0, 1]$ and $F_1, F_2 : B \rightarrow \mathbb{R}^d$ are Wiener functionals such that

- (i) $F_1, F_2 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$;
- (ii) $H, H_1, H_2 \in \mathbb{D}_{k+2}^{\infty-}$ with

$$1 + E_{H_2}[\|D \ln H_2\|_{\mathbb{H}}^p] < \infty \quad \text{for every } p \geq 1,$$

such that $H_1 = 1$ on the set $\{H \neq 0\}$ and $H_2 = 1$ on the set $\{H_1 \neq 0\}$;

- (iii) there is a measurable set A such that $H_2 1_A = 0$ and F_1, F_2 are nondegenerate a.s. on the set A^c .

Then for every p, p', p'', r_1, r_2 and r_3 satisfying $1 < p < p' < p'' < \infty$, $r_1 > pp'/(p' - p)$ and $r_2 > r_3 = p'p''/(p'' - p')$, and for

$$0 < \beta < \alpha \wedge (1 + \delta) - 1 - \frac{d(p-1)}{p}, \quad \beta + \frac{d(p''-1)}{p''} < \delta' < \delta, \quad \delta' \leq \alpha - 1,$$

we can find a positive constant C which may depend on $F_1, F_2, \alpha, \beta, \delta, \delta', p, p', p'', r_1$ and r_2 such that

$$\begin{aligned} &\|(1 - \Delta)^{\beta/2} \delta_x \circ F_1 \cdot H - (1 - \Delta)^{\beta/2} \delta_x \circ F_2 \cdot H\|_{-\alpha, p} \\ &\leq C \|F_1 - F_2\|_{2+\delta', r_1} + C \|F_1 - F_2\|_{1, r_2} + C \|F_1 - F_2\|_{\delta', r_3}. \end{aligned}$$

Remark 4.2. The expression $(1 - \Delta)^{\beta/2} \delta_x \circ F$ is well-defined. We refer the reader to [19, Definition 2.1] for more details.

Remark 4.3. Since $p_{F,G}(x) = E[1_{\{F > x\}} H_{(1,\dots,d)}(F, G)]$ (see [14, Proposition 2.1.5]), by (6), if $k \geq d - 1$ then the conditions (iii), $F_1, F_2 \in \mathbb{D}_{k+2}^{\infty-}(\mathbb{R}^d)$, $H_2 \in \mathbb{D}_{k+1}^{\infty-}$ and $H_2 1_A = 0$ imply that

1. for $i = 1, 2$, the densities p_{F_i, H_2} of the laws of F_i under μ_{H_2} are bounded;
2. for every $1 < p < \infty$, the density $p_{F_2 + t(F_1 - F_2), H_2} \cdot \Psi_1(R_{F_1, F_2})$ is bounded, uniformly in $t \in [0, 1]$.

Recently, Bally and Caramellino [3] have proved that any non-degenerated functional which is twice differentiable in Malliavin sense has a bounded density (see Proposition 2.2). Using this result, the conditions (1) and (2) are automatically satisfied under the conditions of Theorem 4.1 (see [4, Example 2.3] for Ψ_1 satisfying the condition (4)). This fact will play a very important role in the proof of Theorem 4.1.

Now we state the following corollary which is more refined than [4, Proposition 2.2].

Corollary 4.4. *In the circumstance of Theorem 4.1 we have*

$$\|p_{F_1, H} - p_{F_2, H}\|_{C^\beta(\mathbb{R}^d)} \leq C \|F_1 - F_2\|_{2+\delta', r_1 \vee r_2}.$$

Proof. Since $p_{F_i, H}(x) = E_H[1 \cdot \delta_x \circ F_i]$, we have

$$\|(1 - \Delta)^{\beta/2}(p_{F_1, H}(x) - p_{F_2, H}(x))\|_{C^0(\mathbb{R}^d)} \leq C \|F_1 - F_2\|_{2+\delta', r_1 \vee r_2}.$$

Then the conclusion follows from the fact that $(1 - \Delta)^{-\beta/2}(C^0(\mathbb{R}^d)) \subset C^\beta(\mathbb{R}^d)$. \square

To prove the theorem we need some preparations. In what follows for the simplicity of notations sometimes we shall drop the notation of \sum if there is no confusion. We begin with the following lemma which can be found in Watanabe [19].

Lemma 4.5. *If $\alpha > d(p - 1)/p$, then the map $x \in \mathbb{R}^d \rightarrow (1 - \Delta)^{-\alpha/2} \delta_x \in \mathbb{L}^p(\mathbb{R}^d)$ is bounded and continuous.*

The following lemma is a local version of Lemma 2.2 in [19].

Lemma 4.6. *Let $\delta = k + \sigma$, $k \in \mathbb{N}$, $0 < \sigma \leq 1$. Suppose that $G : B \rightarrow \mathbb{R}$ and $F : B \rightarrow \mathbb{R}^d$ be Wiener functionals such that*

- (i) $F \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$;
- (ii) $G \in \mathbb{D}_{k+1}^{\infty-}$, $G_1 \in \mathbb{D}_{k+1}^{\infty-}$ taking values on $[0, 1]$ with

$$1 + E_{G_1}[\|D \ln G_1\|_{\mathbb{H}}^p] < \infty \quad \text{for every } p \geq 1,$$

such that $G_1 = 1$ on the set $\{G \neq 0\}$;

- (iii) *there is a measurable set A such that $G_1 1_A = 0$ and F is nondegenerate a.s. on the set A^c .*

Then, for every $1 < p < p' < \infty$ and $0 < \delta' < \tilde{\delta} \leq \delta$, we can find a positive constant $C = C(\tilde{\delta}, \delta', p, p')$ such that

$$\|g \circ F \cdot G\|_{-\tilde{\delta}, p} \leq C \|g\|_{-\delta', p'} \quad \text{for every } g \in S(\mathbb{R}^d).$$

Proof. By Lemma 3.6, we obtain for $G' \in \mathbb{D}_{k+1}^{\infty-}$,

$$E[(\partial_{i_1} \cdots \partial_{i_m} g) \circ F \cdot G \cdot G'] = E[g \circ F \cdot H_{(i_1, \dots, i_m)}(F, G \cdot G')].$$

Taking $\varepsilon = k + 1 - \tilde{\delta}$, by Remark 3.8, we can find, for every $1 < q' < q$, $\tilde{\delta} > 0$ and $m = 0, 1, \dots, k + 1$, a positive constant $C = C(q, q', \tilde{\delta}, m; F, G, G_1, G')$ such that

$$\|H_{(i_1, \dots, i_m)}(F, G \cdot G')\|_{\tilde{\delta}-m, q'} \leq C \|G'\|_{\tilde{\delta}, q}.$$

Consequently, if $1/p + 1/q = 1$ and $1/p' + 1/q' = 1$, we have

$$\begin{aligned} \|(\partial_{i_1} \cdots \partial_{i_m} g) \circ F \cdot G\|_{-\tilde{\delta}, p} &= \sup\{|E[(\partial_{i_1} \cdots \partial_{i_m} g) \circ F \cdot G \cdot G']| : \|G'\|_{\tilde{\delta}, q} \leq 1\} \\ &\leq \sup\{|E[g \circ F \cdot H_{(i_1, \dots, i_m)}(F, G \cdot G')]| : \|H_{(i_1, \dots, i_m)}(F, G \cdot G')\|_{\tilde{\delta}-m, q'} \leq C\} \\ &= \sup\{|E[g \circ F \cdot H_{(i_1, \dots, i_m)}(F, G \cdot G') \cdot G_1]| : \|H_{(i_1, \dots, i_m)}(F, G \cdot G')\|_{\tilde{\delta}-m, q'} \leq C\} \\ &\leq C \|g \circ F \cdot G_1\|_{m-\tilde{\delta}, p'} \quad \text{for } m = 0, \dots, k + 1. \end{aligned} \quad (17)$$

The rest of the proof is the same as that of [19] and we give it for reader's convenience. If $k = 2l - 1$ is odd, any $g \in S(\mathbb{R}^d)$ can be written as

$$g = \sum_{n=0}^l \Sigma' \pm \partial_{i_1} \cdots \partial_{i_{2n}} (1 - \Delta)^{-l} g,$$

where Σ' is a certain sum over indices (i_1, \dots, i_{2n}) . Hence, by (17), we have

$$\begin{aligned} \|g \circ F \cdot G\|_{-\tilde{\delta}, p} &\leq C \sum_{n=0}^l \|(1 - \Delta)^{-l} g \circ F \cdot G_1\|_{2n-\tilde{\delta}, p'} \\ &\leq C \|(1 - \Delta)^{-(k+1)/2} g \circ F \cdot G_1\|_{k+1-\tilde{\delta}, p'}. \end{aligned}$$

Since

$$0 \leq 1 - \sigma = k + 1 - \tilde{\delta} < 1,$$

using Remark 4.3 (since p_{F, G_1} is bounded) and Theorem 3.5 we have for every $0 < \delta' < \tilde{\delta}$ and $p'' > p'$ with $k + 1 - \delta' < 1$,

$$\|g \circ F \cdot G\|_{-\tilde{\delta}, p} \leq C \|(1 - \Delta)^{-(k+1)/2} g\|_{k+1-\delta', p''} = C \|g\|_{-\delta', p''}.$$

This yields the desired estimate, since δ' and p'' can be chosen arbitrarily close to $\tilde{\delta}$ and p . Similar arguments applies for even k and we refer to [19, Lemma 2.2] for details. The proof is completed. \square

We also need the following technical lemma.

Lemma 4.7. Let $H, H_1 \in B \rightarrow [0, 1]$ and $F_1, F_2 \in B \rightarrow \mathbb{R}^d$ satisfy all the conditions (i)–(iii) of Theorem 4.1. Then, for every δ', r, r', r'' satisfying $0 < \delta' < \delta$, $r' > r > 1$ and $r'' > r'r/(r' - r)$, we can find a positive constant $C = C(F_1, F_2, \delta', r, r', r'')$ such that

$$\|H_i(F_1, H \cdot G) - H_i(F_2, H \cdot G)\|_{\delta', r} \leq C \|G\|_{1+\delta', r'} \cdot \|F_1 - F_2\|_{2+\delta', r''},$$

where $G \in \mathbb{D}_{1+\delta}^{\infty-}$.

Proof. In what follows, $r, m, n, m', m'^c, r', r'^c, r'', r''^c, r''^{cc}$ satisfy

$$\begin{aligned} \frac{1}{m} + \frac{1}{n} &= \frac{1}{r}, & \frac{1}{m'} + \frac{1}{m'^c} &= \frac{1}{m}, \\ \frac{1}{r'} + \frac{1}{r'^c} &= \frac{1}{n}, & \frac{1}{r''^c} + \frac{1}{r''^{cc}} &= \frac{1}{m'^c}, \quad m' < r''. \end{aligned}$$

Since the Malliavin covariance matrix Σ_{F_1} satisfies that $\sigma_{F_1}^{ij} \in \mathbb{D}_{1+\delta}^{\infty-}$, and the space $\mathbb{D}_{1+\delta}^{\infty-}$ is an algebra, so $\det(\Sigma_{F_1}) \in \mathbb{D}_{1+\delta}^{\infty-}$, then by Theorem 3.1, we have

$$(\det(\Sigma_{F_1}))^{-1} \cdot H \in \mathbb{D}_{(1+\delta)-}^{\infty-},$$

and we deduce easily from this that $\gamma_{F_1}^{ij} \cdot H \in \mathbb{D}_{(1+\delta)-}^{\infty-}$. Since

$$H_i(F_1, H \cdot G) = \sum_{j=1}^d D^*(\gamma_{F_1}^{ij} \cdot H \cdot G \cdot DF_1^j)$$

and

$$H_i(F_2, H \cdot G) = \sum_{j=1}^d D^*(\gamma_{F_2}^{ij} \cdot H \cdot G \cdot DF_2^j),$$

using the fact that $D^* : \mathbb{D}_{1+\alpha}^p(\mathbb{H}) \rightarrow \mathbb{D}_\alpha^p$ is continuous for every p and α , we have

$$\begin{aligned} & \|H_i(F_1, H \cdot G) - H_i(F_2, H \cdot G)\|_{\delta', r} \\ & \leq C \|\gamma_{F_1}^{ij} \cdot H \cdot G \cdot DF_1^j - \gamma_{F_2}^{ij} \cdot H \cdot G \cdot DF_2^j\|_{1+\delta', r} \\ & \leq C \|\gamma_{F_1}^{ij} \cdot H \cdot G \cdot DF_1^j - \gamma_{F_2}^{ij} \cdot H \cdot G \cdot DF_1^j\|_{1+\delta', r} \\ & \quad + C \|\gamma_{F_2}^{ij} \cdot H \cdot G \cdot DF_1^j - \gamma_{F_2}^{ij} \cdot H \cdot G \cdot DF_2^j\|_{1+\delta', r} \\ & =: A_1 + A_2. \end{aligned}$$

We first observe that

$$(\Sigma_{F_2})^{-1} - (\Sigma_{F_1})^{-1} = (\Sigma_{F_1})^{-1}(\Sigma_{F_1} - \Sigma_{F_2})(\Sigma_{F_2})^{-1}.$$

Therefore we have

$$\gamma_{F_2}^{ij} - \gamma_{F_1}^{ij} = \sum_{k,l=1}^d \gamma_{F_1}^{ik}(\sigma_{F_1}^{kl} - \sigma_{F_2}^{kl})\gamma_{F_2}^{lj}.$$

In view of $\gamma_{F_1}^{ij} \cdot H \in \mathbb{D}_{(1+\delta)-}^{\infty-}$, $\gamma_{F_2}^{ij} \cdot H_1 \in \mathbb{D}_{(1+\delta)-}^{\infty-}$, $F_1 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$, $F_2 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$, we obtain

$$\begin{aligned} A_1 & \leq C \|G \cdot DF_1^j\|_{1+\delta', n} \cdot \|\gamma_{F_2}^{ij} \cdot H - \gamma_{F_1}^{ij} \cdot H\|_{1+\delta', m} \\ & \leq C \|G\|_{1+\delta', r'} \cdot \|DF_1^j\|_{1+\delta', r'^c} \cdot \|\sigma_{F_1}^{kl} - \sigma_{F_2}^{kl}\|_{1+\delta', m'} \\ & \quad \cdot \|\gamma_{F_1}^{ik} \cdot H\|_{1+\delta', r''} \cdot \|\gamma_{F_2}^{lj} \cdot H_1\|_{1+\delta', r''c} \\ & \leq C \|G\|_{1+\delta', r'} \cdot \|\sigma_{F_1}^{kl} - \sigma_{F_2}^{kl}\|_{1+\delta', m'} \\ & \leq C \|G\|_{1+\delta', r'} \cdot \|F_1 - F_2\|_{2+\delta', r''}. \end{aligned} \tag{18}$$

For the second term, in view of $\gamma_{F_2}^{ij} \cdot H \in \mathbb{D}_{(1+\delta)-}^{\infty-}$, we obtain

$$\begin{aligned} A_2 & \leq C \|\gamma_{F_2}^{ij} \cdot H \cdot G\|_{1+\delta', n} \cdot \|DF_1^j - DF_2^j\|_{1+\delta', m} \\ & \leq C \|\gamma_{F_2}^{ij} \cdot H\|_{1+\delta', r'^c} \cdot \|G\|_{1+\delta', r'} \cdot \|DF_1^j - DF_2^j\|_{1+\delta', m} \\ & \leq C \|G\|_{1+\delta', r'} \cdot \|F_1 - F_2\|_{2+\delta', m}. \end{aligned} \tag{19}$$

Hence combining (18) and (19), we have

$$\|H_i(F_1, H \cdot G) - H_i(F_2, H \cdot G)\|_{\delta', r} \leq C \|G\|_{1+\delta', r'} \cdot \|F_1 - F_2\|_{2+\delta', r''}.$$

Thus the proof is completed. \square

With the above preparation we can establish the following results which will play a crucial role in the proof of [Theorem 4.1](#).

Lemma 4.8. *Let $H, H_1, H_2 \in B \rightarrow [0, 1]$ and $F_1, F_2 \in B \rightarrow \mathbb{R}^d$ satisfy all the conditions (i)–(iii) of [Theorem 4.1](#). Then, for every $0 < \delta'' < \delta' < \delta$ and $p, p', p'', p''', r_1, r_2$ satisfying $1 < p < p' < p'' < p''' < \infty$, $r_1 > pp'/(p' - p)$ and $r_2 > r_3 = p'p''/(p'' - p')$, we can find a positive constant C such that*

$$\begin{aligned} & \|g \circ F_1 \cdot H - g \circ F_2 \cdot H\|_{-(1+\delta'), p} \\ & \leq C \|g\|_{-(1+\delta''), p''} \cdot \|F_1 - F_2\|_{2+\delta', r_1} + C \|g\|_{-\delta', p'''} \cdot \|F_1 - F_2\|_{1, r_2} \\ & \quad + C \|g\|_{-\delta'', p'''} \cdot \|F_1 - F_2\|_{\delta', r_3} \end{aligned}$$

for every $g \in S(\mathbb{R}^d)$. The positive constant C may depend on $F_1, F_2, \delta', \delta'', p, p', p'', p''', r_1, r_2$.

Proof. In what follows, $q = p/(p - 1)$, $q' = p'/(p' - 1)$, $q'' = p''/(p'' - 1)$. First using the integration by parts formula, we have for $G \in \mathbb{D}_{1+\delta}^{\infty-}$,

$$\begin{aligned} & |E[(\partial_i g) \circ F_1 \cdot H \cdot G - (\partial_i g) \circ F_2 \cdot H \cdot G]| \\ & = |E[g \circ F_1 \cdot H_1 \cdot H_i(F_1, H \cdot G) - g \circ F_2 \cdot H_1 \cdot H_i(F_2, H \cdot G)]| \\ & \leq |E[g \circ F_1 \cdot H_1 \cdot H_i(F_1, H \cdot G) - g \circ F_2 \cdot H_1 \cdot H_i(F_1, H \cdot G)]| \\ & \quad + |E[g \circ F_2 \cdot H_1 \cdot H_i(F_1, H \cdot G) - g \circ F_2 \cdot H_1 \cdot H_i(F_2, H \cdot G)]| \\ & \leq \|g \circ F_1 \cdot H_1 - g \circ F_2 \cdot H_1\|_{-\delta', p'} \cdot \|H_i(F_1, H \cdot G)\|_{\delta', q'} \\ & \quad + \|g \circ F_2 \cdot H_1\|_{-\delta', p'} \cdot \|H_i(F_1, H \cdot G) - H_i(F_2, H \cdot G)\|_{\delta', q'}. \end{aligned}$$

Since

$$H_i(F_1, H \cdot G) = \sum_{j=1}^d D^*(\gamma_{F_1}^{ij} \cdot H \cdot G \cdot DF_1^j),$$

by $\gamma_{F_1}^{ij} \cdot H \in \mathbb{D}_{(1+\delta)-}^{\infty-}$ and $D^*: \mathbb{D}_{1+\alpha}^p(\mathbb{H}) \rightarrow \mathbb{D}_\alpha^p$ is continuous for every p and every α , the map $G \rightarrow H_i(F_1, H \cdot G)$ can be extended to a continuous operator $\mathbb{D}_{(1+\delta)-}^q \rightarrow \mathbb{D}_{\delta-}^{q-} := \bigcap_{q' < q} \mathbb{D}_{\delta-}^{q'}$ for every q and every δ , that is, we can find, for every $1 < q' < q$, $0 < \delta' < \delta$, a positive constant $C = C(q, q', \delta'; F_1)$ such that

$$\|H_i(F_1, H \cdot G)\|_{\delta', q'} \leq C \|G\|_{1+\delta', q}.$$

Then by [Lemma 4.7](#), we have

$$\begin{aligned} & \|(\partial_i g) \circ F_1 \cdot H - (\partial_i g) \circ F_2 \cdot H\|_{-(1+\delta'), p} \\ & = \sup\{|E[(\partial_i g) \circ F_1 \cdot H \cdot G - (\partial_i g) \circ F_2 \cdot H \cdot G]| : \|G\|_{1+\delta', q} \leq 1\} \\ & \leq \sup\{\|g \circ F_1 \cdot H_1 - g \circ F_2 \cdot H_1\|_{-\delta', p'} \\ & \quad \cdot \|H_i(F_1, H \cdot G)\|_{\delta', q'} : \|H_i(F_1, H \cdot G)\|_{\delta', q'} \leq C\} \\ & \quad + \sup\{\|g \circ F_2 \cdot H_1\|_{-\delta', p'} \cdot \|H_i(F_1, H \cdot G) - H_i(F_2, H \cdot G)\|_{\delta', q'}\} \end{aligned}$$

$$\begin{aligned}
& -H_i(F_2, H \cdot G)\|_{\delta', q'} : \|G\|_{1+\delta', q} \leq 1\} \\
& \leq C\|g \circ F_1 \cdot H_1 - g \circ F_2 \cdot H_1\|_{-\delta', p'} + C\|g \circ F_2 \cdot H_1\|_{-\delta', p'} \cdot \|F_1 - F_2\|_{2+\delta', r_1}. \quad (20)
\end{aligned}$$

Since any $g \in \mathcal{S}(\mathbb{R}^d)$ can be written in the form

$$g = \sum_{n=0}^1 \Sigma' \pm \partial_{i_1} \cdots \partial_{i_{2n}} (1 - \Delta)^{-1} g,$$

where Σ' is a certain sum over indices (i_1, \dots, i_{2n}) , by (20), we have

$$\begin{aligned}
& \|g \circ F_1 \cdot H - g \circ F_2 \cdot H\|_{-(1+\delta'), p} = \left\| \sum_{n=0}^1 \Sigma' \pm \partial_{i_1} \cdots \partial_{i_{2n}} (1 - \Delta)^{-1} g \circ F_1 \cdot H \right. \\
& \quad \left. - \sum_{n=0}^1 \Sigma' \pm \partial_{i_1} \cdots \partial_{i_{2n}} (1 - \Delta)^{-1} g \circ F_2 \cdot H \right\|_{-(1+\delta'), p} \\
& \leq C \Sigma' \|\partial_{i_2} (1 - \Delta)^{-1} g \circ F_1 \cdot H_1 - \partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1\|_{-\delta', p'} \\
& \quad + C \Sigma' \|\partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1\|_{-\delta', p'} \cdot \|F_1 - F_2\|_{2+\delta', r_1} \\
& =: \Xi_1 + \Xi_2.
\end{aligned}$$

By Lemma 4.6 and the $\mathbb{L}_\alpha^p(\mathbb{R}^d)$ boundedness of $\partial_{i_2} (1 - \Delta)^{-\frac{1}{2}}$, we have

$$\|\partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1\|_{-\delta', p'} \leq C \|\partial_{i_2} (1 - \Delta)^{-1} g\|_{-\delta'', p''} \leq C \|g\|_{-(1+\delta''), p''}.$$

Then we obtain

$$\Xi_2 \leq C \|g\|_{-(1+\delta''), p''} \cdot \|F_1 - F_2\|_{2+\delta', r_1}. \quad (21)$$

Now we deal with the first term Ξ_1 . We split Ξ_1 into two parts

$$\begin{aligned}
& \|\partial_{i_2} (1 - \Delta)^{-1} g \circ F_1 \cdot H_1 - \partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1\|_{-\delta', p'} \\
& = \sup\{|E[\partial_{i_2} (1 - \Delta)^{-1} g \circ F_1 \cdot H_1 \cdot G \\
& \quad - \partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1 \cdot G]| : \|G\|_{\delta', q'} \leq 1\} \\
& \leq \sup\{|E[(\partial_{i_2} (1 - \Delta)^{-1} g \circ F_1 \cdot H_1 \cdot G \\
& \quad - \partial_{i_2} (1 - \Delta)^{-1} g \circ F_2 \cdot H_1 \cdot G) \cdot (1 - \Psi(R_{F_1, F_2}))]| : \|G\|_{\delta', q'} \leq 1\} \\
& \quad + \sup\{|E[(\partial_{i_2} (1 - \Delta)^{-1} g \circ F_1 \cdot H_1 \cdot G - \partial_{i_2} (1 - \Delta)^{-1} g \\
& \quad \circ F_2 \cdot H_1 \cdot G) \cdot \Psi(R_{F_1, F_2})]| : \|G\|_{\delta', q'} \leq 1\} \\
& =: \Xi_3 + \Xi_4,
\end{aligned}$$

where Ψ and R_{F_1, F_2} are defined in Section 2. First we consider the term Ξ_3 . In view of $(\det(\Sigma_{F_1}))^{-1} \cdot 1_{\{H_2 \neq 0\}} \in \mathbb{L}^{\infty-}$, $F_1 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$, we have

$$\begin{aligned}
& P(\{\Psi(R_{F_1, F_2}) \neq 1\} \cap \{H_2 \neq 0\}) \\
& \leq P\left(\left\{((\det(\Sigma_{F_1}))^{-1})^{1/2} (1 + \|\Sigma_{F_1}\|_2^2)^{(d-1)/4} \|D(F_1 - F_2)\|_{\mathbb{H}} > \frac{1}{2\sqrt{2}}\right\} \cap \{H_2 \neq 0\}\right) \\
& \leq (2\sqrt{2})^{r'_3} E[(((\det(\Sigma_{F_1}))^{-1})^{1/2} (1 + \|\Sigma_{F_1}\|_2^2)^{(d-1)/4})^{r'_3} \|D(F_1 - F_2)\|_{\mathbb{H}}^{r'_3} \cdot 1_{\{H_2 \neq 0\}}] \\
& \leq CE[\|DF_1 - DF_2\|_{\mathbb{H}}^{r'_3/r_2}],
\end{aligned}$$

where $r_3 < r'_3 < r_2$. Then by Remark 3.7, we obtain

$$\begin{aligned}
 & |E[(\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1 \cdot G - \partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1 \cdot G) \cdot (1 - \Psi(R_{F_1, F_2}))]| \\
 & \leq |E[\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1 \cdot H_2 \cdot G \cdot (1 - \Psi(R_{F_1, F_2}))]| \\
 & \quad + |E[\partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1 \cdot H_2 \cdot G \cdot (1 - \Psi(R_{F_1, F_2}))]| \\
 & \leq C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1\|_{-k', p''} \cdot \|H_2 \cdot G \cdot (1 - \Psi(R_{F_1, F_2}))\|_{k', q''} \\
 & \quad + C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1\|_{-k', p''} \cdot \|H_2 \cdot G \cdot (1 - \Psi(R_{F_1, F_2}))\|_{k', q''} \\
 & \leq C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \cdot \|H_2 \cdot (1 - \Psi(R_{F_1, F_2}))\|_{k', r_3} \\
 & \quad + C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \cdot \|H_2 \cdot (1 - \Psi(R_{F_1, F_2}))\|_{k', r_3} \\
 & \leq C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \cdot P(\{\Psi(R_{F_1, F_2}) \neq 1\} \cap \{H_2 \neq 0\})^{1/r'_3} \\
 & \quad + C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \\
 & \quad \cdot P(\{\Psi(R_{F_1, F_2}) \neq 1\} \cap \{H_2 \neq 0\})^{1/r'_3} \\
 & \leq C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \cdot \|F_1 - F_2\|_{1, r_2} \\
 & \quad + C\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1\|_{-k', p''} \cdot \|G\|_{k', q'} \cdot \|F_1 - F_2\|_{1, r_2},
 \end{aligned}$$

where $k' < \delta' \leq k' + 1$. By Lemma 4.6 and Remark 4.3, it is obvious that

$$\|\partial_{i_2}(1-\Delta)^{-1}g \circ F_i \cdot H_1\|_{-k', p''} \leq C\|\partial_{i_2}(1-\Delta)^{-1}g\|_{-k', p'''} \leq C\|g\|_{-(k'+1), p'''},$$

Consequently

$$\Xi_3 \leq C\|g\|_{-(k'+1), p'''} \cdot \|F_1 - F_2\|_{1, r_2} \leq C\|g\|_{-\delta', p'''} \cdot \|F_1 - F_2\|_{1, r_2}. \quad (22)$$

Next we consider the second term Ξ_4 . First by (6), we have

$$\{\Psi_1(R_{F_1, F_2}) \neq 0\} \subset \left\{ \det(\Sigma_{(F_2+t(F_1-F_2))}) \geq \left(1 - \frac{\sqrt{2}}{2}\right)^{2d} \frac{(\det(\Sigma_{F_1}))^d}{\|\Sigma_{F_1}\|^{d(d-1)}} \right\} \quad \text{a.s.}$$

Then by $(\det(\Sigma_{F_1}))^{-1} \cdot 1_{\{H_2 \neq 0\}} \in \mathbb{L}^{\infty-}$, $F_1 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$, we obtain that $F_2 + t(F_1 - F_2)$ is nondegenerate a.s. on the set $\{H_2 \neq 0\} \cap \{\Psi_1(R_{F_1, F_2}) \neq 0\}$, uniformly in $t \in [0, 1]$. Besides, by Remark 3.2 it is obvious that $F_2 + t(F_1 - F_2) \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$, $\Psi(R_{F_1, F_2}) \cdot H_1 \in \mathbb{D}_{(1+\delta)-}^{\infty-}$ and $\Psi_1(R_{F_1, F_2}) \cdot H_2 \in \mathbb{D}_{(1+\delta)-}^{\infty-}$, and hence $F_2 + t(F_1 - F_2) \in \mathbb{D}_{2+\delta'}^{\infty-}(\mathbb{R}^d)$, $\Psi(R_{F_1, F_2}) \cdot H_1 \in \mathbb{D}_{k'+1}^{\infty-}$ and $\Psi_1(R_{F_1, F_2}) \cdot H_2 \in \mathbb{D}_{k'+1}^{\infty-}$. Therefore, $F_2 + t(F_1 - F_2)$, $\Psi(R_{F_1, F_2}) \cdot H_1$ and $\Psi_1(R_{F_1, F_2}) \cdot H_2$ satisfy all the conditions in Lemma 4.6 by condition (4). Consequently, applying mean value theorem, Remark 4.3(2) and Lemma 4.6, we have

$$\begin{aligned}
 \Xi_4 & = \|\partial_{i_2}(1-\Delta)^{-1}g \circ F_1 \cdot H_1 \cdot \Psi(R_{F_1, F_2}) \\
 & \quad - \partial_{i_2}(1-\Delta)^{-1}g \circ F_2 \cdot H_1 \cdot \Psi(R_{F_1, F_2})\|_{-\delta', p'} \\
 & \leq \int_0^1 \|\partial_i \partial_{i_2}(1-\Delta)^{-1}g \circ (F_2 + t(F_1 - F_2)) \\
 & \quad \cdot (F_1^i - F_2^i) \cdot H_1 \cdot \Psi(R_{F_1, F_2})\|_{-\delta', p'} dt \\
 & \leq C \int_0^1 \|\partial_i \partial_{i_2}(1-\Delta)^{-1}g \circ (F_2 + t(F_1 - F_2)) \\
 & \quad \cdot H_1 \cdot \Psi(R_{F_1, F_2})\|_{-\delta', p''} \cdot \|F_1^i - F_2^i\|_{\delta', r_3} dt \\
 & \leq C\|\partial_i \partial_{i_2}(1-\Delta)^{-1}g\|_{-\delta'', p'''} \cdot \|F_1 - F_2\|_{\delta', r_3},
 \end{aligned} \quad (23)$$

and, noting the $\mathbb{L}_\alpha^p(\mathbb{R}^d)$ boundedness of $\partial_{i_1}\partial_{i_2}(1-\Delta)^{-1}$, this is further dominated by $C\|g\|_{-\delta'',p'''} \cdot \|F_1 - F_2\|_{\delta',r_3}$. Thus combining (21)–(23), we have

$$\begin{aligned} & \|g \circ F_1 \cdot H - g \circ F_2 \cdot H\|_{-(1+\delta'),p} \\ & \leq C_1\|g\|_{-(1+\delta''),p''} \cdot \|F_1 - F_2\|_{2+\delta',r_1} + C_2\|g\|_{-\delta',p'''} \cdot \|F_1 - F_2\|_{1,r_2} \\ & \quad + C_3\|g\|_{-\delta'',p'''} \cdot \|F_1 - F_2\|_{\delta',r_3}. \end{aligned}$$

This completes the proof of the lemma. \square

Remark 4.9. In the proof of Lemma 4.8, the reason why we make use of the integration by parts formula rather than a direct use of the mean value theorem, is to ensure that $\Psi(R_{F_1,F_2}) \cdot H_1$ satisfies the condition (ii) of Lemma 4.6.

Proof of Theorem 4.1. In what follows, $q = p/(p-1)$, $q' = p'/(p'-1)$, $q'' = p''/(p''-1)$. Since $0 < \beta < \alpha \wedge (1+\delta) - 1 - d/q$, we can choose $1 < q''' < q''$ and $0 < \delta'' < \delta' < \delta$ such that $\delta'' - \beta > d/q'''$. Hence by Lemma 4.5, we have

$$x \rightarrow (1-\Delta)^{\beta/2}\delta_x \in \mathbb{L}_{-\delta''}^{p'''}(\mathbb{R}^d) \quad (24)$$

is bounded and continuous, where $1/p''' + 1/q''' = 1$. Hence by Lemma 4.8 and (24), we can find $C > 0$ such that

$$\begin{aligned} & \|(1-\Delta)^{\beta/2}\delta_x \circ F_1 \cdot H - (1-\Delta)^{\beta/2}\delta_x \circ F_2 \cdot H\|_{-\alpha,p} \\ & \leq \|(1-\Delta)^{\beta/2}\delta_x \circ F_1 \cdot H - (1-\Delta)^{\beta/2}\delta_x \circ F_2 \cdot H\|_{-(1+\delta'),p} \\ & \leq C\|(1-\Delta)^{\beta/2}\delta_x\|_{-(1+\delta''),p''} \cdot \|F_1 - F_2\|_{2+\delta',r_1} \\ & \quad + C\|(1-\Delta)^{\beta/2}\delta_x\|_{-\delta',p'''} \cdot \|F_1 - F_2\|_{1,r_2} \\ & \quad + C\|(1-\Delta)^{\beta/2}\delta_x\|_{-\delta'',p'''} \cdot \|F_1 - F_2\|_{\delta',r_3} \\ & \leq C\|F_1 - F_2\|_{2+\delta',r_1} + C\|F_1 - F_2\|_{1,r_2} + C\|F_1 - F_2\|_{\delta',r_3}. \end{aligned}$$

The proof is thus completed. \square

Similar to the proof of Theorem 4.1, by Theorem 3.4, we have the following theorem which shows that the condition $H, H_1, H_2 \in \mathbb{D}_{k+2}^{\infty-}$ can be improved to $H, H_1, H_2 \in \mathbb{D}_{(1+\delta)-}^{\infty-}$.

Theorem 4.10. Let $\delta = k + \sigma$, $k \in \mathbb{N}$, $0 < \sigma \leq 1$. Suppose that $H, H_1, H_2 : B \rightarrow [0, 1]$ and $F_1, F_2 : B \rightarrow \mathbb{R}^d$ are Wiener functionals such that

- (i) $F_1, F_2 \in \mathbb{D}_{2+\delta}^{\infty-}(\mathbb{R}^d)$;
- (ii) $H, H_1, H_2 \in \mathbb{D}_{(1+\delta)-}^{\infty-}$ with

$$1 + E_{H_2}[|D \ln H_2|_{\mathbb{H}}^p] < \infty \quad \text{for every } p \geq 1,$$

such that $H_1 = 1$ on the set $\{H \neq 0\}$ and $H_2 = 1$ on the set $\{H_1 \neq 0\}$;

- (iii) there is a measurable set A such that $H_2 1_A = 0$ and F_1, F_2 are nondegenerate a.s. on the set A^c .

Then for every p, p', p'', r_1, r_2 and r_3 satisfying $1 < p < p' < p'' < \infty$, $r_1 > pp'/(p' - p)$ and $r_2 > r_3 = p'p''/(p'' - p')$, and for

$$\begin{aligned} 0 < \beta < \alpha \wedge \left(1 + k + \frac{\sigma}{2^{k+2}}\right) - 1 - \frac{d(p-1)}{p}, \\ \beta + \frac{d(p''-1)}{p''} < \delta' < k + \frac{\sigma}{2^{k+2}} < \delta, \quad \delta' \leq \alpha - 1, \end{aligned}$$

we can find a positive constant C which may depend on $F_1, F_2, \alpha, \beta, \delta, \delta', p, p', p'', r_1, r_2$ and r_3 such that

$$\begin{aligned} & \| (1 - \Delta)^{\beta/2} \delta_x \circ F_1 \cdot H - (1 - \Delta)^{\beta/2} \delta_x \circ F_2 \cdot H \|_{-\alpha, p} \\ & \leq C \| F_1 - F_2 \|_{2+\delta', r_1} + C \| F_1 - F_2 \|_{1, r_2} + C \| F_1 - F_2 \|_{\delta', r_3}. \end{aligned}$$

5. Convergence rate of density of the Euler scheme for non-Markovian stochastic differential equations: applications of Theorem 4.1

In this section we study the convergence rate of density of the Euler scheme for non-Markovian stochastic differential equations. Before proceeding, we introduce some notations and notions. Given compact metric spaces M_1, \dots, M_d and real separable Hilbert spaces E_1, \dots, E_d and E , let $k \geq 1$ be an integer, we denote by $C_p^k(C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d); E)$ the class of continuous $F : C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d) \rightarrow E$ such that for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $1 \leq |\alpha| = \alpha_1 + \dots + \alpha_d \leq k$ and any $g_j^i \in C(M_i; E_i)$, $1 \leq i \leq d$, $0 \leq j \leq \alpha_i$, the map

$$(x_1^1, \dots, x_{\alpha_1}^1, \dots, x_1^d, \dots, x_{\alpha_d}^d) \in \mathbb{R}^{|\alpha|} \mapsto F \begin{pmatrix} g_0^1 + \sum_{j=1}^{\alpha_1} x_j^1 g_j^1 \\ \vdots \\ g_0^d + \sum_{j=1}^{\alpha_d} x_j^d g_j^d \end{pmatrix} \text{ is } C^k;$$

there is a continuous

$$F^{(\alpha)} : C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d) \rightarrow \text{Hom}(C(M^\alpha; E^{\otimes \alpha}); E)$$

for which

$$\begin{aligned} & \frac{\partial^\alpha}{\partial x_1^1 \dots \partial x_{\alpha_1}^1 \dots \partial x_1^d \dots \partial x_{\alpha_d}^d} F \begin{pmatrix} g_0^1 + \sum_{j=1}^{\alpha_1} x_j^1 g_j^1 \\ \vdots \\ g_0^d + \sum_{j=1}^{\alpha_d} x_j^d g_j^d \end{pmatrix} \bigg|_{x_1^1 = \dots = x_{\alpha_d}^d = 0} \\ & = F^{(\alpha)} \begin{pmatrix} g_0^1 \\ \vdots \\ g_0^d \end{pmatrix} (g_1^1(*_{1,1}) \otimes \dots \otimes g_{\alpha_1}^1(*_{1,\alpha_1}) \otimes \dots \otimes g_1^d(*_{d,1}) \otimes \dots \otimes g_{\alpha_d}^d(*_{d,\alpha_d})); \quad (25) \end{aligned}$$

and, for any $0 \leq |\alpha| \leq k$, there exist $C_\alpha < \infty$ and $\lambda_\alpha < \infty$ such that

$$\left\| F^{(\alpha)} \begin{pmatrix} g^1 \\ \vdots \\ g^d \end{pmatrix} \right\|_{\text{Hom}(C(M^\alpha; E^{\otimes \alpha}); E)} \leq C_\alpha \left(1 + \sum_{i=0}^d \|g^i\|_{C(M_i; E_i)} \right)^{\lambda_\alpha}$$

for $g^i \in C(M_i; E_i)$, $i = 1, \dots, d$. Similarly, let $k \geq 1$ be an integer, we denote by $C_b^k(C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d); E)$ the class of continuous $F : C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d) \rightarrow E$

such that for any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $1 \leq |\alpha| = \alpha_1 + \dots + \alpha_d \leq k$ and any $g_j^i \in C(M_i; E_i)$, $1 \leq i \leq d$, $0 \leq j \leq \alpha_i$, the map

$$(x_1^1, \dots, x_{\alpha_1}^1, \dots, x_1^d, \dots, x_{\alpha_d}^d) \in \mathbb{R}^{|\alpha|} \mapsto F \begin{pmatrix} g_0^1 + \sum_{j=1}^{\alpha_1} x_j^1 g_j^1 \\ \vdots \\ g_0^d + \sum_{j=1}^{\alpha_d} x_j^d g_j^d \end{pmatrix} \text{ is } C^k;$$

there is a continuous

$$F^{(\alpha)} : C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d) \rightarrow \text{Hom}(C(M^\alpha; E^{\otimes \alpha}); E)$$

for which

$$\begin{aligned} & \frac{\partial^\alpha}{\partial x_1^1 \dots \partial x_{\alpha_1}^1 \dots \partial x_1^d \dots \partial x_{\alpha_d}^d} F \begin{pmatrix} g_0^1 + \sum_{j=1}^{\alpha_1} x_j^1 g_j^1 \\ \vdots \\ g_0^d + \sum_{j=1}^{\alpha_d} x_j^d g_j^d \end{pmatrix} \Big|_{x_1^1 = \dots = x_{\alpha_d}^d = 0} \\ &= F^{(\alpha)} \begin{pmatrix} g_0^1 \\ \vdots \\ g_0^d \end{pmatrix} (g_1^1(*_{1,1}) \otimes \dots \otimes g_{\alpha_1}^1(*_{1,\alpha_1}) \otimes \dots \otimes g_1^d(*_{d,1}) \otimes \dots \otimes g_{\alpha_d}^d(*_{d,\alpha_d})); \end{aligned}$$

and, for any $0 \leq |\alpha| \leq k$, there exists $C_\alpha < \infty$ such that

$$\left\| F^{(\alpha)} \begin{pmatrix} g^1 \\ \vdots \\ g^d \end{pmatrix} \right\|_{\text{Hom}(C(M^\alpha; E^{\otimes \alpha}); E)} \leq C_\alpha$$

for $g^i \in C(M_i; E_i)$, $i = 1, \dots, d$, furnished with the following norm:

$$\|F\|_{C_b^k(C(M_1; E_1) \otimes \dots \otimes C(M_d; E_d); E)} = \sum_{|\alpha|=0}^k \sup_{\substack{g^i \in C(M_i; E_i) \\ i=1, \dots, d}} \left\| F^{(\alpha)} \begin{pmatrix} g^1 \\ \vdots \\ g^d \end{pmatrix} \right\|_{\text{Hom}(C(M^\alpha; E^{\otimes \alpha}); E)}.$$

For any $p > 1$, let \mathcal{G}_0^p be the class of continuous progressively measurable functions: $\xi : [0, T] \times B \rightarrow E$ such that

$$\|\xi\|_{0,p,T;E} := \sup_{0 \leq t \leq T} \|\xi(t)\|_E \|\mathbb{L}^p < \infty.$$

For every integer $k \geq 1$ and any $p > 1$, let \mathcal{G}_k^p be the class of $\xi \in \mathcal{G}_0^p$ such that $\xi(t) \in \mathbb{D}_j^p(E)$ for each $t \in [0, T]$ and $j = 1, \dots, k$, equipped with the following norm

$$\|\xi\|_{k,p,T;E} := \sum_{0 \leq j \leq k} \sup_{0 \leq t \leq T} \|D^j \xi(t)\|_{\mathbb{H}^{\otimes j} \otimes E} \|\mathbb{L}^p < \infty. \quad (26)$$

Finally, we define $\mathcal{G}_\infty^{\infty-}$ by

$$\mathcal{G}_\infty^{\infty-} = \bigcap_{k \geq 0} \bigcap_{1 < p < \infty} \mathcal{G}_k^p.$$

Now we introduced the following fractional order Sobolev space \mathcal{G}_α^p .

Definition 5.1. For every integer $k \geq 0$, any $0 < \alpha < 1$ and any $1 < p < \infty$, we define

$$\mathcal{G}_{k+\alpha}^p = (\mathcal{G}_k^p, \mathcal{G}_{k+1}^p)_{\alpha, p}.$$

We shall use the following norms which was given by Peetre's K -method.

$$\|\xi\|_{k+\alpha, p, T; E} := \left[\int_0^1 [\epsilon^{-\alpha} K(\epsilon, p, \xi)]^p \frac{d\epsilon}{\epsilon} \right]^{\frac{1}{p}}, \quad (27)$$

where

$$K(\epsilon, p, \xi) := \inf\{\|\xi_1\|_{k, p, T; E} + \epsilon \|\xi_2\|_{k+1, p, T; E}, \xi_1 + \xi_2 = \xi, \xi_1, \xi_2 \in \mathcal{G}_k^p\}.$$

The following definition is taken from [12].

Definition 5.2. Let k be an integer and E_1 and E_2 be real separable Hilbert spaces. We say that a function $F : [0, T] \times C([0, T]; E_1) \rightarrow E_2$ satisfies (A_{k, E_1, E_2}) if

(i) F is measurable, and for each $t \in [0, T]$, there is an

$$F(t) \in C_p^k(C([0, t]; E_1); E_2)$$

such that $F(t, \psi) = F(t)(\psi|_{[0, t]})$ for all $\psi \in C([0, T]; E_1)$;

(ii) for each integer $0 \leq n \leq k$, there exist $C_n < \infty$ and γ_n , where $\gamma_n = 1$ for $n = 0$ and $\gamma_n = 0$ for $n \geq 1$, such that

$$\|F(t)^{(n)}(\psi)\|_{Hom(C([0, t]^n; E_1^{\otimes n}); E_2)} \leq C_n(1 + \|\psi\|_{C([0, t]; E_1)})^{\gamma_n} \quad (28)$$

for all $t \in [0, T]$ and $\psi \in C([0, t]; E_1)$;

(iii) for each integer $0 \leq n \leq k$, there exists $C_n < \infty$ such that

$$\begin{aligned} & \|F(t)^{(n)}(\psi)(g|_{[0, t]^n}) - F(s)^{(n)}(\psi)(g|_{[0, s]^n})\|_{E_2} \\ & \leq C_n |t - s|^{1/2} (1 + \|\psi\|_{C([0, T]; E_1)}) \end{aligned} \quad (29)$$

for all $t, s \in [0, T]$, $\psi \in C([0, T]; E_1)$ and $g \in C([0, T]^n; E_1^{\otimes n})$.

We also need the following definition.

Definition 5.3. Let $\delta = k + \theta$, $k \in \mathbb{N}$, $0 < \theta < 1$, $p > 1$ and E_1 and E_2 be real separable Hilbert spaces. We say that a function $F : [0, T] \times C([0, T]; E_1) \rightarrow E_2$ satisfies $(A_{1+\delta, E_1, E_2})$ (or $(A_{2+\delta, E_1, E_2})$) if

(i) F is measurable, and for each $t \in [0, T]$, there is an

$$\begin{aligned} & F(t) \in (C_b^{1+k}(C([0, t]; E_1); E_2), C_b^{2+k}(C([0, t]; E_1); E_2))_{\theta, p} \\ & \text{(or } F(t) \in (C_b^{2+k}(C([0, t]; E_1); E_2), C_b^{3+k}(C([0, t]; E_1); E_2))_{\theta, p}) \end{aligned}$$

such that $F(t, \psi) = F(t)(\psi|_{[0,t]})$ for all $\psi \in C([0, T]; E_1)$, where the norm

$$\|F(t)\|_{(C_b^{1+k}(C([0,t]; E_1); E_2), C_b^{2+k}(C([0,t]; E_1); E_2))_{\theta,p}} \\ \text{(or } \|F(t)\|_{(C_b^{2+k}(C([0,t]; E_1); E_2), C_b^{3+k}(C([0,t]; E_1); E_2))_{\theta,p}})$$

is controlled by a constant which is independent of t ;

(ii) for each integer $0 \leq n \leq 1 + k$ (or $2 + k$), there exists $C_n < \infty$ such that

$$\|F(t)^{(n)}(\psi)(g|_{[0,t]^n}) - F(s)^{(n)}(\psi)(g|_{[0,s]^n})\|_{E_2} \leq C_n |t - s|^{1/2} (1 + \|\psi\|_{C([0,T]; E_1)})$$

for all $t, s \in [0, T]$, $\psi \in C([0, T]; E_1)$ and $g \in C([0, T]^n; E_1^{\otimes n})$.

Now we introduce the Euler scheme for SDE (2). Let $T > 0$ be a fixed time horizon, and let T/n be the discretization step for every integer $n > 0$. Set $X_n(0) = x$ and for $kT/n < t \leq (k+1)T/n$, $X_n(t)$ is inductively defined by

$$X_n(t) = X_n\left(\frac{kT}{n}\right) + \int_{\frac{kT}{n}}^t b\left(\frac{kT}{n}, X_n\left(\cdot \wedge \frac{kT}{n}\right)\right) ds \\ + \int_{\frac{kT}{n}}^t \sigma\left(\frac{kT}{n}, X_n\left(\cdot \wedge \frac{kT}{n}\right)\right) dW(s).$$

First we need the following results. We shall make use of the following assumptions.

- (A.I) $\sigma : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ satisfies the condition $(A_{1+k, \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m})$ and $b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ satisfies the condition $(A_{1+k, \mathbb{R}^d, \mathbb{R}^d})$;
- (A.II) σ is bounded and uniformly nondegenerate, i.e., there exists a constant $c > 0$ such that $\sigma(t, \varphi) \cdot \sigma^*(t, \varphi) \geq c \cdot I$, where $(\sigma(t, \varphi) \cdot \sigma^*(t, \varphi))^{ij} = \sum_{k=1}^r \sigma_k^i(t, \varphi) \sigma_k^j(t, \varphi)$, $i, j = 1, \dots, d$, for all $t \in [0, T]$ and $\varphi \in C([0, T]; \mathbb{R}^d)$.

Theorem 5.4. Suppose that the coefficients (σ, b) of SDE (2) satisfy (A.I) and (A.II). Then for any $p > 1$, we have

$$\|X(\cdot, x)\|_{k,p,T;\mathbb{R}^d} \vee \sup_{n \geq 1} \|X_n(\cdot, x)\|_{k,p,T;\mathbb{R}^d} < \infty \quad (30)$$

and

$$\|X_n(\cdot, x) - X(\cdot, x)\|_{k,p,T;\mathbb{R}^d} = O(n^{-1/2}), \quad (31)$$

where the norm $\|\cdot\|_{k,p,T;\mathbb{R}^d}$ is defined in (26). Furthermore, we also have

$$\|(\det \Sigma_{X(t,x)})^{-1}\|_p < \infty \quad (32)$$

and

$$\sup_{n \geq 1} \|(\det \Sigma_{X_n(t,x)})^{-1}\|_p < \infty \quad (33)$$

for every $1 < p < \infty$, $t \in [0, T]$.

Now we give a proof of Theorem 5.4. Before proceeding, we prepare some propositions. In what follows we denote by C a generic constant which can be different from one formula to another. For convenience we set $k_n(t) = k$ if $kT/n \leq t < (k+1)T/n$, and we denote $\eta_n(t) = k_n(t)T/n$.

Proposition 5.5. Suppose that the coefficients (σ, b) of SDE (2) satisfy the following conditions: there exists $C_0 < \infty$ such that

$$\|\sigma(t, \phi) - \sigma(t, \psi)\|_{HS(\mathbb{R}^d; \mathbb{R}^m)} \vee \|b(t, \phi) - b(t, \psi)\|_{\mathbb{R}^d} \leq C_0 \|\phi - \psi\|_{C([0, t]; \mathbb{R}^d)}, \quad (34)$$

$$\begin{aligned} \|\sigma(t, \phi) - \sigma(s, \phi)\|_{HS(\mathbb{R}^d; \mathbb{R}^m)} \vee \|b(t, \phi) - b(s, \phi)\|_{\mathbb{R}^d} \\ \leq C_0 |t - s|^{1/2} (1 + \|\phi\|_{C([0, T]; \mathbb{R}^d)}) \end{aligned} \quad (35)$$

and

$$\|\sigma(t, \phi)\|_{HS(\mathbb{R}^d; \mathbb{R}^m)} \vee \|b(t, \phi)\|_{\mathbb{R}^d} \leq C_0 (1 + \|\phi\|_{C([0, t]; \mathbb{R}^d)}) \quad (36)$$

for every $t, s \in [0, T]$ and $\phi, \psi \in C([0, T]; \mathbb{R}^d)$. Then we have

$$\sup_{n \geq 1} E \left[\sup_{0 \leq s \leq t} \|X_n(s)\|^{2p} \right] < \infty$$

for all $p > 1$ and $t \in [0, T]$. Moreover, for every $p > 1$ there exists a constant $C = C(p) < \infty$ such that

$$\sup_{0 \leq k \leq n-1} E \left[\sup_{\frac{kT}{n} \leq s < \frac{(k+1)T}{n}} \|X_n(s) - X_n(\eta_n(s))\|^{2p} \right] \leq C n^{-p}$$

for all n .

Proof. It is well known that the conditions (34) and (36) ensure the existence and uniqueness of the solution of SDE (2). Since

$$\begin{aligned} X_n(t) &= X_n(\eta_n(t)) + \int_{\eta_n(t)}^t b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) ds \\ &\quad + \int_{\eta_n(t)}^t \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) dW(s), \end{aligned}$$

we have

$$X_n(t) = x + \int_0^t b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) ds + \int_0^t \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) dW(s).$$

Consequently

$$\begin{aligned} E \left[\sup_{0 \leq s \leq t} \|X_n(s)\|^{2p} \right] &\leq 2^{2p-1} E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s b(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) du \right\|^{2p} \right] \\ &\quad + 2^{2p-1} E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s \sigma(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) dW(u) \right\|^{2p} \right] \\ &\leq 2^{2p-1} t^{2p-1} C_{2p} C_0 \int_0^t E \left[\left(1 + \sup_{0 \leq u \leq s} \|X_n(u)\| \right)^{2p} \right] ds \\ &\quad + 2^{2p-1} t^{p-1} C_{2p} C_0 \int_0^t E \left[\left(1 + \sup_{0 \leq u \leq s} \|X_n(u)\| \right)^{2p} \right] ds. \end{aligned}$$

Thus by Gronwall's lemma, the proof of the first conclusion is completed. For the second conclusion, we proceed as follows. Since

$$\begin{aligned} X_n(t) - X_n(\eta_n(t)) &= \int_{\eta_n(t)}^t b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) ds \\ &\quad + \int_{\eta_n(t)}^t \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) dW(s), \end{aligned}$$

for fixed k , where $k = 0, 1, \dots, n-1$, using BDG's inequality and (36), we have

$$\begin{aligned} &E \left[\sup_{\frac{kT}{n} \leq s < \frac{(k+1)T}{n}} \|X_n(s) - X_n(\eta_n(s))\|^{2p} \right] \\ &\leq 2^{2p-1} E \left[\sup_{\frac{kT}{n} \leq s < \frac{(k+1)T}{n}} \left\| \int_{\frac{kT}{n}}^s b(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) du \right\|^{2p} \right] \\ &\quad + 2^{2p-1} E \left[\sup_{\frac{kT}{n} \leq s < \frac{(k+1)T}{n}} \left\| \int_{\frac{kT}{n}}^s \sigma(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) dW(u) \right\|^{2p} \right] \\ &\leq 2^{2p-1} n^{-2p+1} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} E[\|b(\eta_n(s), X_n(\cdot \wedge \eta_n(s)))\|^{2p}] ds \\ &\quad + 2^{2p-1} n^{-p+1} C_{2p} \int_{\frac{kT}{n}}^{\frac{(k+1)T}{n}} E[\|\sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s)))\|^{2p}] ds \\ &\leq C_1 n^{-2p} + C_2 n^{-p}, \end{aligned}$$

and the proof is completed. \square

Proposition 5.6. Suppose that the coefficients (σ, b) of SDE (2) satisfy the conditions of Proposition 5.5. Then we have

$$(E[\|X_n - X\|_{C([0,t]; \mathbb{R}^d)}^p])^{1/p} = O(n^{-1/2})$$

for all $p > 1$ and $t \in [0, T]$.

Proof.

$$\begin{aligned} X_n(t) - X(t) &= X_n(\eta_n(t)) - X(\eta_n(t)) + \int_{\eta_n(t)}^t b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) ds \\ &\quad - \int_{\eta_n(t)}^t b(s, X(\cdot)) ds \\ &\quad + \int_{\eta_n(t)}^t \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) dW(s) - \int_{\eta_n(t)}^t \sigma(s, X(\cdot)) dW(s) \\ &= X_n(\eta_n(t)) - X(\eta_n(t)) + \int_{\eta_n(t)}^t (b(s, X_n(\cdot)) - b(s, X(\cdot))) ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma(s, X_n(\cdot)) - \sigma(s, X(\cdot))) dW(s) + A_{\eta_n(t), t}, \end{aligned}$$

where

$$\begin{aligned} A_{\eta_n(t),t} &:= \int_{\eta_n(t)}^t (b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) - b(s, X_n(\cdot \wedge \eta_n(s))))ds \\ &\quad + \int_{\eta_n(t)}^t (b(s, X_n(\cdot \wedge \eta_n(s))) - b(s, X_n(\cdot)))ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) - \sigma(s, X_n(\cdot \wedge \eta_n(s))))dW(s) \\ &\quad + \int_{\eta_n(t)}^t (\sigma(s, X_n(\cdot \wedge \eta_n(s))) - \sigma(s, X_n(\cdot)))dW(s). \end{aligned}$$

Repeatedly using the above formula, we obtain

$$\begin{aligned} X_n(t) - X(t) &= \int_0^t (b(s, X_n(\cdot)) - b(s, X(\cdot)))ds \\ &\quad + \int_0^t (\sigma(s, X_n(\cdot)) - \sigma(s, X(\cdot)))dW(s) + \sum_{k=0}^{k_n(t)} A_{\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t}. \end{aligned}$$

Then using BDG's inequality and (34), we have

$$\begin{aligned} E[\|X_n - X\|_{C([0,t];\mathbb{R}^d)}^{2p}] &\leq C_1 \int_0^t E[\|X_n - X\|_{C([0,s];\mathbb{R}^d)}^{2p}]ds \\ &\quad + C_2 E \left[\sup_{0 \leq s \leq t} \left\| \sum_{k=0}^{k_n(s)} A_{\frac{kT}{n}, \frac{(k+1)T}{n} \wedge s} \right\|^{2p} \right]. \end{aligned}$$

For the last term we proceed as follows. First we have

$$E \left[\sup_{0 \leq s \leq t} \left\| \sum_{k=0}^{k_n(s)} A_{\frac{kT}{n}, \frac{(k+1)T}{n} \wedge s} \right\|^{2p} \right] \leq C(A_1 + A_2 + A_3 + A_4),$$

where

$$\begin{aligned} A_1 &= E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s (b(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) - b(u, X_n(\cdot \wedge \eta_n(u))))du \right\|^{2p} \right], \\ A_2 &= E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s (b(u, X_n(\cdot \wedge \eta_n(u))) - b(u, X_n(\cdot)))du \right\|^{2p} \right], \\ A_3 &= E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s (\sigma(\eta_n(u), X_n(\cdot \wedge \eta_n(u))) - \sigma(u, X_n(\cdot \wedge \eta_n(u))))dW(u) \right\|^{2p} \right], \\ A_4 &= E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s (\sigma(u, X_n(\cdot \wedge \eta_n(u))) - \sigma(u, X_n(\cdot)))dW(u) \right\|^{2p} \right]. \end{aligned}$$

In view of BDG's inequality and (35), the terms A_1 and A_3 are dominated by Cn^{-p} . Next we estimate the term A_4 . In view of BDG's inequality and (34), we have

$$A_4 = E \left[\sup_{0 \leq s \leq t} \left\| \int_0^s (\sigma(u, X_n(\cdot \wedge \eta_n(u))) - \sigma(u, X_n(\cdot)))dW(u) \right\|^{2p} \right]$$

$$\begin{aligned} &\leq C_{2p} t^{p-1} \int_0^t E[\|\sigma(s, X_n(\cdot \wedge \eta_n(s))) - \sigma(s, X_n(\cdot))\|^{2p}] ds \\ &\leq C_{2p} C_0 t^{p-1} \int_0^t E[\sup_{0 \leq u \leq s} \|X_n(u \wedge \eta_n(s)) - X_n(u)\|^{2p}] ds. \end{aligned}$$

By Proposition 5.5, the above expression is dominated by $C' n^{-p}$, where $C' = C_{2p} C_0 C t^p$. The term A_2 can be estimated similarly. Thus combining the above estimates, we have

$$E[\|X_n - X\|_{C([0,t];\mathbb{R}^d)}^{2p}] \leq C_1 \int_0^t E[\|X_n - X\|_{C([0,s];\mathbb{R}^d)}^{2p}] ds + C_2 n^{-p} + C_3 n^{-2p}.$$

Then the proof is completed by Gronwall's lemma. \square

We denote by $D_t^j F$, $t \in [0, T]$, $j = 1, \dots, m$, the derivative of a random variable F as an element of $\mathbb{L}^2([0, T] \times B; \mathbb{R}^m) \cong \mathbb{L}^2(B; \mathbb{H})$. Here the separable Hilbert space \mathbb{H} is an \mathbb{L}^2 space of the form $\mathbb{H} = \mathbb{L}^2([0, T], \mathbb{R}^m)$. Similarly we denote by $D_{t_1, \dots, t_n}^{j_1, \dots, j_n} F$ the n -th derivative of F . Before the proof of our main proposition, we need the following proposition.

Proposition 5.7. Suppose that the coefficients (σ, b) of SDE (2) satisfy the Assumption (A.I), then for any $p > 1$, we have

$$\|X(\cdot, x)\|_{k,p,T;\mathbb{R}^d} \vee \sup_{n \geq 1} \|X_n(\cdot, x)\|_{k,p,T;\mathbb{R}^d} < \infty.$$

Proof. We only give a proof of the case of the first order derivative, and the cases of higher order derivatives can be given in a similar way. By results in Kusuoka and Stroock [12], we have

$$\begin{aligned} D_r^j X_n^i(t) &= D_r^j X_n^i(\eta_n(t)) + \sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(t), t]}(r) \\ &\quad + \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) dW(s)^l, \end{aligned}$$

where $(b^i)^{\alpha_k}$ and $(\sigma_l^i)^{\alpha_k}$ are defined in (25), $\alpha_k = (0, \dots, 0, 1, 0, \dots, 0)$. Repeatedly using the above formula, we have

$$\begin{aligned} D_r^j X_n^i(t) &= \sum_{k=0}^{k_n(t)} \sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right]}(r) \\ &\quad + \int_0^t (b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) ds \\ &\quad + \int_0^t (\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) dW(s)^l. \end{aligned}$$

Thus using BDG's inequality and (28), we obtain

$$\sup_{n \geq 1} \sup_{0 \leq r \leq T} E[\sup_{0 \leq s \leq t} \|D_r^j X_n(s)\|^{2p}] < \infty$$

for all $p > 1$, $t \in [0, T]$ and $j = 1, \dots, m$. Similarly, we also have

$$\sup_{0 \leq r \leq T} E[\sup_{0 \leq s \leq t} \|D_r^j X(s)\|^{2p}] < \infty$$

for all $p > 1$, $t \in [0, T]$ and $j = 1, \dots, m$, and the proof is thus established. \square

Proposition 5.8. Suppose that the coefficients (σ, b) of SDE (2) satisfy the Assumption (A.I), then for any $p > 1$, we have

$$\|X_n(\cdot, x) - X(\cdot, x)\|_{k,p,T;\mathbb{R}^d} = O(n^{-1/2}).$$

Proof. First by results in Kusuoka and Stroock [12], we have

$$\begin{aligned} D_r^j X^i(t) &= D_r^j X^i(\eta_n(t)) + \sigma_j^i(r, X(\cdot)) 1_{(\eta_n(t), t]}(r) \\ &\quad + \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(s, X(\cdot)) D_r^j X^k(*) ds + \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot)) D_r^j X^k(*) dW(s)^l \end{aligned}$$

and

$$\begin{aligned} D_r^j X_n^i(t) &= D_r^j X_n^i(\eta_n(t)) + \sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(t), t]}(r) \\ &\quad + \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(*) \wedge \eta_n(s) ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(*) \wedge \eta_n(s) dW(s)^l. \end{aligned}$$

Thus we have

$$\begin{aligned} D_r^j X_n^i(t) - D_r^j X^i(t) &= D_r^j X_n^i(\eta_n(t)) - D_r^j X^i(\eta_n(t)) \\ &\quad + \sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(t), t]}(r) - \sigma_j^i(r, X(\cdot)) 1_{(\eta_n(t), t]}(r) \\ &\quad + \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(*) \wedge \eta_n(s) ds \\ &\quad - \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(s, X(\cdot)) D_r^j X^k(*) ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(*) \wedge \eta_n(s) dW(s)^l \\ &\quad - \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot)) D_r^j X^k(*) dW(s)^l \\ &= D_r^j X_n^i(\eta_n(t)) - D_r^j X^i(\eta_n(t)) \\ &\quad + \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(s, X(\cdot)) (D_r^j X_n^k(*) - D_r^j X^k(*) ds \\ &\quad + \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot)) (D_r^j X_n^k(*) - D_r^j X^k(*) dW(s)^l \\ &\quad + \Lambda_{\eta_n(t), t}(r), \end{aligned}$$

where $\Lambda_{\eta_n(t), t}(r)$ denotes the remaining terms. Repeatedly using the above formula, we obtain

$$\begin{aligned} D_r^j X_n^i(t) - D_r^j X^i(t) &= \int_0^t (b^i)^{\alpha_k}(s, X(\cdot)) (D_r^j X_n^k(*) - D_r^j X^k(*) ds \\ &\quad + \int_0^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot)) (D_r^j X_n^k(*) - D_r^j X^k(*) dW(s)^l + \sum_{k=0}^{k_n(t)} \Lambda_{\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t}(r) \end{aligned}$$

$$\begin{aligned}
&= \int_0^t (b^i)^{\alpha_k}(s, X(\cdot))(D_r^j X_n^k(*) - D_r^j X^k(*))ds \\
&\quad + \int_0^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot))(D_r^j X_n^k(*) - D_r^j X^k(*))dW(s)^l + \sum_{m=1}^9 \Lambda_m(t, r),
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1(t, r) &= \sum_{k=0}^{k_n(t)} \left[\sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right)}(r) \right. \\
&\quad \left. - \sigma_j^i(r, X(\cdot)) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right)}(r) \right], \\
\Lambda_2(t, r) &= \int_0^t ((b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) \\
&\quad - (b^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s)))) D_r^j X_n^k(* \wedge \eta_n(s)) ds, \\
\Lambda_3(t, r) &= \int_0^t (b^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s)))(D_r^j X_n^k(* \wedge \eta_n(s)) - D_r^j X_n^k(*)) ds, \\
\Lambda_4(t, r) &= \int_0^t ((b^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s))) - (b^i)^{\alpha_k}(s, X_n(\cdot))) D_r^j X_n^k(*) ds, \\
\Lambda_5(t, r) &= \int_0^t ((b^i)^{\alpha_k}(s, X_n(\cdot)) - (b^i)^{\alpha_k}(s, X(\cdot))) D_r^j X_n^k(*) ds, \\
\Lambda_6(t, r) &= \int_0^t ((\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) \\
&\quad - (\sigma_l^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s)))) D_r^j X_n^k(* \wedge \eta_n(s)) dW(s)^l, \\
\Lambda_7(t, r) &= \int_0^t (\sigma_l^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s)))(D_r^j X_n^k(* \wedge \eta_n(s)) - D_r^j X_n^k(*)) dW(s)^l, \\
\Lambda_8(t, r) &= \int_0^t ((\sigma_l^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s))) - (\sigma_l^i)^{\alpha_k}(s, X_n(\cdot))) D_r^j X_n^k(*) dW(s)^l, \\
\Lambda_9(t, r) &= \int_0^t ((\sigma_l^i)^{\alpha_k}(s, X_n(\cdot)) - (\sigma_l^i)^{\alpha_k}(s, X(\cdot))) D_r^j X_n^k(*) dW(s)^l.
\end{aligned}$$

Then by BDG's inequality for Hilbert space valued stochastic integrals (cf. [12, Lemma 2.1]) and Assumption (A.I), we have

$$\begin{aligned}
E \left[\sup_{0 \leq t \leq T} \|DX_n^i(t) - DX^i(t)\|_{\mathbb{H}}^2 \right] &= E \left[\sup_{0 \leq t \leq T} \left(\int_0^t \|D_r X_n^i(t) - D_r X^i(t)\|^2 dr \right)^p \right] \\
&\leq 3^{2p-1} E \left[\sup_{0 \leq t \leq T} \left(\int_0^t \left(\int_0^t (b^i)^{\alpha_k}(s, X(\cdot))(D_r^j X_n^k(*) - D_r^j X^k(*)) ds \right)^2 dr \right)^p \right] \\
&\quad + 3^{2p-1} E \left[\sup_{0 \leq t \leq T} \left(\int_0^t \left(\int_0^t (\sigma_l^i)^{\alpha_k}(s, X(\cdot))(D_r^j X_n^k(*) \right. \right. \right. \\
&\quad \left. \left. \left. - D_r^j X^k(*)) dW(s)^l \right)^2 dr \right)^p \right]
\end{aligned}$$

$$\begin{aligned}
& + 3^{2p-1} E \left[\sup_{0 \leq t \leq T} \left(\int_0^T \left(\sum_{m=1}^9 \Lambda_m(t, r) \right)^2 dr \right)^p \right] \\
& \leq C_1 \int_0^T E \left[\sup_{0 \leq s \leq t} \|DX_n(s) - DX(s)\|_{\mathbb{H} \otimes \mathbb{R}^d}^{2p} \right] dt \\
& + C_2 \sum_{m=1}^9 E \left[\sup_{0 \leq t \leq T} \left(\int_0^T (\Lambda_m(t, r))^2 dr \right)^p \right].
\end{aligned}$$

Let

$$\Xi_m := E \left[\sup_{0 \leq t \leq T} \left(\int_0^T (\Lambda_m(t, r))^2 dr \right)^p \right], \quad m = 1, \dots, 9.$$

Then to complete the proof, we just have to show each of nine terms Ξ_m are dominated by Cn^{-p} . We first observe that

$$\begin{aligned}
& \int_0^T \left(\sum_{k=0}^{k_n(t)} \left[\sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right]}(r) \right. \right. \\
& \quad \left. \left. - \sigma_j^i(r, X(\cdot)) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right]}(r) \right] \right)^2 dr \\
& = \int_0^t (\sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) - \sigma_j^i(r, X(\cdot)))^2 dr \\
& \leq 3 \int_0^t (\sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) - \sigma_j^i(r, X_n(\cdot \wedge \eta_n(r))))^2 dr \\
& \quad + 3 \int_0^t (\sigma_j^i(r, X_n(\cdot \wedge \eta_n(r))) - \sigma_j^i(r, X_n(\cdot)))^2 dr \\
& \quad + 3 \int_0^t (\sigma_j^i(r, X_n(\cdot)) - \sigma_j^i(r, X(\cdot)))^2 dr \\
& \leq 3TC_0^2(1 + \|X_n(\cdot)\|_{C([0, T]; \mathbb{R}^d)})^2 n^{-1} + 3C_0^2 \int_0^t \sup_{0 \leq u \leq r} |X_n(u \wedge \eta_n(r)) - X_n(u)|^2 dr \\
& \quad + 3C_0^2 \int_0^t \sup_{0 \leq u \leq r} |X_n(u) - X(u)|^2 dr.
\end{aligned}$$

Then in view of [Propositions 5.5](#) and [5.6](#), we have

$$\begin{aligned}
\Xi_1 & = E \left[\sup_{0 \leq t \leq T} \left(\int_0^T \left(\sum_{k=0}^{k_n(t)} \left[\sigma_j^i(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right]}(r) \right. \right. \right. \right. \\
& \quad \left. \left. \left. - \sigma_j^i(r, X(\cdot)) 1_{\left(\frac{kT}{n}, \frac{(k+1)T}{n} \wedge t\right]}(r) \right] \right)^2 dr \right)^p \right] \\
& \leq 3^{p-1} 3^p T^p C_0^{2p} E[(1 + \|X_n(\cdot)\|_{C([0, T]; \mathbb{R}^d)})^{2p}] n^{-p} \\
& \quad + 3^{p-1} 3^p T^{p-1} C_0^{2p} \int_0^T E \left[\sup_{0 \leq u \leq r} |X_n(u \wedge \eta_n(r)) - X_n(u)|^{2p} \right] dr
\end{aligned}$$

$$\begin{aligned}
& + 3^{p-1} 3^p T^{p-1} C_0^{2p} \int_0^T E \left[\sup_{0 \leq u \leq r} |X_n(u) - X(u)|^{2p} \right] dr \\
& \leq 3^{p-1} 3^p T^p C_0^{2p} C n^{-p} + 3^{p-1} 3^p T^p C_0^{2p} C n^{-p} + 3^{p-1} 3^p T^p C_0^{2p} C n^{-p} \\
& \leq C n^{-p}.
\end{aligned}$$

Next we estimate the terms Ξ_2 , Ξ_4 , Ξ_5 , Ξ_6 , Ξ_8 and Ξ_9 . Using BDG's inequality for Hilbert space valued stochastic integrals, (28), Propositions 5.5 and 5.7, we obtain

$$\begin{aligned}
\Xi_8 &= E \left[\sup_{0 \leq t \leq T} \left(\int_0^T \left(\int_0^t ((\sigma_l^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s))) \right. \right. \right. \\
&\quad \left. \left. \left. - (\sigma_l^i)^{\alpha_k}(s, X_n(\cdot)) \right) D_r^j X_n^k(*) dW(s)^l \right)^2 dr \right)^p \right] \\
&\leq C_{2p} T^{p-1} \int_0^T E \left[\left(\int_0^T (((\sigma_l^i)^{\alpha_k}(t, X_n(\cdot \wedge \eta_n(t))) \right. \right. \right. \\
&\quad \left. \left. \left. - (\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) \right) D_r^j X_n^k(*)^2 dr \right)^p \right] dt \\
&\leq C_{2p} T^{2p-2} \int_0^T \int_0^T E [(((\sigma_l^i)^{\alpha_k}(t, X_n(\cdot \wedge \eta_n(t))) \\
&\quad - (\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) D_r^j X_n^k(*)^2)^p] dr dt \\
&\leq C_{2p} T^{2p-1} \int_0^T E [\| (\sigma_l^i)^{\alpha_k}(t, X_n(\cdot \wedge \eta_n(t))) \\
&\quad - (\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) \|_{Hom(C([0,t]; \mathbb{R}^d); \mathbb{R})}^{2p}] dt \\
&\leq C_{2p} C_1 T^{2p-1} \int_0^T E \left[\sup_{0 \leq s \leq t} \| X_n(s \wedge \eta_n(t)) - X_n(s) \|^2 \right] dt \\
&\leq C_{2p} C_1 C' T^{2p} n^{-p}.
\end{aligned}$$

The term Ξ_4 can be estimated similarly. Using BDG's inequality for Hilbert space valued stochastic integrals, (28), Propositions 5.6 and 5.7, we have

$$\begin{aligned}
\Xi_9 &= E \left[\sup_{0 \leq t \leq T} \left(\int_0^T \left(\int_0^t ((\sigma_l^i)^{\alpha_k}(s, X_n(\cdot)) \right. \right. \right. \\
&\quad \left. \left. \left. - (\sigma_l^i)^{\alpha_k}(s, X(\cdot)) \right) D_r^j X_n^k(*) dW(s)^l \right)^2 dr \right)^p \right] \\
&\leq C_{2p} T^{p-1} \int_0^T E \left[\left(\int_0^T (((\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) - (\sigma_l^i)^{\alpha_k}(t, X(\cdot))) D_r^j X_n^k(*)^2 dr \right)^p \right] dt \\
&\leq C_{2p} T^{2p-2} \int_0^T \int_0^T E [(((\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) - (\sigma_l^i)^{\alpha_k}(t, X(\cdot))) D_r^j X_n^k(*)^2)^p] dr dt
\end{aligned}$$

$$\begin{aligned}
&\leq C_{2p} T^{2p-1} \int_0^T E[\|(\sigma_l^i)^{\alpha_k}(t, X_n(\cdot)) - (\sigma_l^i)^{\alpha_k}(t, X(\cdot))\|_{Hom(C([0,t];\mathbb{R}^d);\mathbb{R})}^{2p}] dt \\
&\leq C_{2p} C_1 T^{2p-1} \int_0^T E[\sup_{0 \leq s \leq t} \|X_n(s) - X(s)\|^{2p}] dt \\
&\leq C_{2p} C_1 C T^{2p} n^{-p}.
\end{aligned}$$

We can apply the similar way to estimate the term Ξ_5 . In view of BDG's inequality for Hilbert space valued stochastic integrals and (29), we can also obtain that the terms Ξ_2 and Ξ_6 are dominated by Cn^{-p} . Finally we deal with the terms Ξ_3 and Ξ_7 . Since

$$\begin{aligned}
D_r^j X_n^i(t) - D_r^j X_n^i(\eta_n(t)) &= \sigma_f^j(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(t), t]}(r) \\
&+ \int_{\eta_n(t)}^t (b^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) ds \\
&+ \int_{\eta_n(t)}^t (\sigma_l^i)^{\alpha_k}(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) D_r^j X_n^k(* \wedge \eta_n(s)) dW(s)^l,
\end{aligned}$$

using BDG's inequality for Hilbert space valued stochastic integrals and (28), we have

$$\begin{aligned}
\Xi_7 &= E \left[\sup_{0 \leq t \leq T} \left(\int_0^T \left(\int_0^t (\sigma_l^i)^{\alpha_k}(s, X_n(\cdot \wedge \eta_n(s))) \right. \right. \right. \\
&\quad \left. \left. \left(D_r^j X_n^k(* \wedge \eta_n(s)) - D_r^j X_n^k(*) \right) dW(s)^l \right)^2 dr \right)^p \Big] \\
&\leq C_{2p} C_1 T^{p-1} \int_0^T E \left[\left(\int_0^T ((\sigma_l^i)^{\alpha_k}(t, X_n(\cdot \wedge \eta_n(t))) \right. \right. \\
&\quad \left. \left. (D_r^j X_n^k(* \wedge \eta_n(t)) - D_r^j X_n^k(*)) \right)^2 dr \right)^p \Big] dt \\
&\leq C_{2p} C_1 T^{p-1} \int_0^T E \left[\left(\int_0^T \|D_r^j X_n^k(* \wedge \eta_n(t)) - D_r^j X_n^k(*)\|_{C([\eta_n(t), t]; \mathbb{R})}^2 dr \right)^p \right] dt \\
&\leq 3^{2p-1} C_{2p} C_1 T^{p-1} \int_0^T E \left[\left(\int_0^T \sup_{\eta_n(t) \leq s \leq t} |\sigma_f^j(\eta_n(r), \right. \right. \\
&\quad \left. \left. X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(s), s]}(r)|^2 dr \right)^p \right] dt \\
&\quad + 3^{2p-1} C_{2p} C_1 T^{p-1} \int_0^T E \left[\left(\int_0^T \sup_{\eta_n(t) \leq s \leq t} \left| \int_{\eta_n(s)}^s (b^k)^{\alpha_i}(\eta_n(u), \right. \right. \right. \\
&\quad \left. \left. \left. X_n(\cdot \wedge \eta_n(u))) D_r^j X_n^i(* \wedge \eta_n(u)) du \right|^2 dr \right)^p \right] dt
\end{aligned}$$

$$\begin{aligned}
& + 3^{2p-1} C_{2p} C_1 T^{p-1} \int_0^T E \left[\left(\int_0^T \sup_{\eta_n(t) \leq s \leq t} \left| \int_{\eta_n(s)}^s (\sigma_l^k)^{\alpha_i} (\eta_n(u), \right. \right. \right. \\
& \quad \left. \left. \left. X_n(\cdot \wedge \eta_n(u)) \right) D_r^j X_n^i(* \wedge \eta_n(u)) dW(u)^l \right|^2 dr \right)^p \Big] dt \\
& := C(\Xi_{71} + \Xi_{72} + \Xi_{73}).
\end{aligned}$$

For the term Ξ_{71} , we proceed as follows. We notice that

$$\begin{aligned}
& \int_0^T \sup_{\eta_n(t) \leq s \leq t} |\sigma_j^k(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(s), s]}(r)|^2 dr \\
& = \int_0^T [|\sigma_j^k(\eta_n(r), X_n(\cdot \wedge \eta_n(r))) 1_{(\eta_n(t), t]}(r)|^2] dr \\
& = \int_{\eta_n(t)}^t [|\sigma_j^k(\eta_n(r), X_n(\cdot \wedge \eta_n(r)))|^2] dr \\
& \leq C_0^2 (1 + \sup_{0 \leq u \leq T} |X_n(u \wedge \eta_n(r))|)^2 n^{-1}.
\end{aligned}$$

Then we obtain

$$\Xi_{71} \leq T C_0^{2p} E[(1 + \sup_{0 \leq u \leq T} |X_n(u \wedge \eta_n(r))|)^{2p}] n^{-p} \leq C n^{-p}.$$

Now we consider the term Ξ_{73} . By BDG's inequality and Assumption (A.I), we have

$$\begin{aligned}
\Xi_{73} & \leq T^{p-1} \int_0^T \int_0^T E \left[\sup_{\eta_n(t) \leq s \leq t} \left| \int_{\eta_n(s)}^s (\sigma_l^k)^{\alpha_i} (\eta_n(u), X_n(\cdot \wedge \eta_n(u))) \right. \right. \\
& \quad \left. \left. D_r^j X_n^i(* \wedge \eta_n(u)) dW(u)^l \right|^{2p} \right] dr dt \\
& \leq C_{2p} T^{p-1} n^{-p+1} \int_0^T \int_0^T \int_{\eta_n(t)}^t E[|(\sigma_l^k)^{\alpha_i} (\eta_n(s), X_n(\cdot \wedge \eta_n(s))) \\
& \quad D_r^j X_n^i(* \wedge \eta_n(s))|^{2p}] ds dr dt \\
& \leq C_{2p} T^{p-1} n^{-p+1} \int_0^T \int_0^T \int_{\eta_n(t)}^t E[\sup_{0 \leq u \leq \eta_n(s)} |D_r^j X_n^i(u)|^{2p}] ds dr dt \\
& \leq C n^{-p}.
\end{aligned}$$

A similar estimate holds for the term Ξ_{72} . The term Ξ_3 can be estimated similarly. Thus we have shown

$$\begin{aligned}
& E[\sup_{0 \leq t \leq T} \|DX_n(t) - DX(t)\|_{\mathbb{H} \otimes \mathbb{R}^d}^{2p}] \\
& \leq C_1 \int_0^T E[\sup_{0 \leq s \leq t} \|DX_n(s) - DX(s)\|_{\mathbb{H} \otimes \mathbb{R}^d}^{2p}] dt + C_2 n^{-p}.
\end{aligned}$$

Then by Gronwall's lemma, we have

$$E[\sup_{0 \leq t \leq T} \|DX_n(t) - DX(t)\|_{\mathbb{H} \otimes \mathbb{R}^d}^{2p}] \leq C n^{-p}.$$

For the case of higher order derivatives, we proceed in the same way. The proof is therefore completed. \square

Now we are ready to give the proof of [Theorem 5.4](#).

Proof of Theorem 5.4. The conclusions (30) and (31) have been proved in [Propositions 5.7](#) and [5.8](#), respectively. So it remains only to prove (32) and (33). Since

$$X_n(t) = x + \int_0^t b(\eta_n(s), X_n(\cdot \wedge \eta_n(s)))ds + \int_0^t \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s)))dW(s),$$

we can rewrite it in the following form:

$$X_n(t) = x + \int_0^t b_n(s, X_n(\cdot))ds + \int_0^t \sigma_n(s, X_n(\cdot))dW(s),$$

where

$$b_n(s, X_n(\cdot)) := b(\eta_n(s), X_n(\cdot \wedge \eta_n(s))) \quad \text{and} \quad \sigma_n(s, X_n(\cdot)) := \sigma(\eta_n(s), X_n(\cdot \wedge \eta_n(s))),$$

and the coefficients (σ_n, b_n) also satisfy the uniform elliptic condition. Thus by [[12](#), Theorem 3.5, Corollary 3.9], we obtain the nondegeneracy of $X(T)$ and $X_n(T)$. Therefore we complete the proof. \square

We can now state our main results which extend [Theorem 5.4](#) to the case of interpolation spaces. We shall replace the above Assumption (A.I) with the following Assumption (A.III).

(A.III) $\sigma : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ satisfies the condition $(A_{1+\delta, \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m})$ and $b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ satisfies the condition $(A_{1+\delta, \mathbb{R}^d, \mathbb{R}^d})$.

Theorem 5.9. Let $\delta' = k + \theta'$, $k \in \mathbb{N}$, $0 < \delta' < \delta$ and $k_n = 2^{2n}$. Suppose that the coefficients (σ, b) of SDE (2) satisfy (A.II) and (A.III). Then we have

$$\|X(\cdot, x)\|_{\delta', p, T; \mathbb{R}^d} \vee \sup_{n \geq 1} \|X_n(\cdot, x)\|_{\delta', p, T; \mathbb{R}^d} < \infty \quad (37)$$

and

$$\|X_{k_n}(\cdot, x) - X(\cdot, x)\|_{\delta', p, T; \mathbb{R}^d} = O(2^{-n\theta}), \quad (38)$$

where the norm $\|\cdot\|_{\delta', p, T; \mathbb{R}^d}$ is defined in (27).

Proof. For simplicity of notations we denote X_{k_n} by X_n . We will assume $0 < \delta < 1$. The general case can be treated in the same way by considering the SDE satisfied by the Malliavin–Shigekawa gradient of X and X_n . Since $\sigma : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ satisfies the condition $(A_{1+\delta, \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m})$ and $b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ satisfies the condition $(A_{1+\delta, \mathbb{R}^d, \mathbb{R}^d})$, there are $\sigma(t) \in (C_b^1(C([0, t]; \mathbb{R}^d); \mathbb{R}^d \otimes \mathbb{R}^m), C_b^2(C([0, t]; \mathbb{R}^d); \mathbb{R}^d \otimes \mathbb{R}^m))_{\delta, p}$ and $b(t) \in (C_b^1(C([0, t]; \mathbb{R}^d); \mathbb{R}^d), C_b^2(C([0, t]; \mathbb{R}^d); \mathbb{R}^d))_{\delta, p}$ such that $\sigma(t, \psi) = \sigma(t)(\psi|_{[0, t]})$ and $b(t, \psi) = b(t)(\psi|_{[0, t]})$ for all $\psi \in C([0, T]; \mathbb{R}^d)$. By [Remark 2.4](#), for each m , we can find $\sigma_m(t)$ and $b_m(t)$ such that

$$\begin{aligned} \|\sigma_m(t) - \sigma(t)\|_{C_b^1(C([0, t]; \mathbb{R}^d); \mathbb{R}^d \otimes \mathbb{R}^m)} &\leq C2^{-m\delta}, \\ \|\sigma_m(t)\|_{C_b^2(C([0, t]; \mathbb{R}^d); \mathbb{R}^d \otimes \mathbb{R}^m)} &\leq C2^{m(1-\delta)}, \\ \|b_m(t) - b(t)\|_{C_b^1(C([0, t]; \mathbb{R}^d); \mathbb{R}^d)} &\leq C2^{-m\delta}, \\ \|b_m(t)\|_{C_b^2(C([0, t]; \mathbb{R}^d); \mathbb{R}^d)} &\leq C2^{m(1-\delta)}. \end{aligned} \quad (39)$$

Now we consider the following SDEs:

$$\begin{aligned} X_{0,m}(t) &= x + \int_0^t b_{m+n}(s, X_{0,m}(\cdot))ds + \int_0^t \sigma_{m+n}(s, X_{0,m}(\cdot))dW(s), \\ X_{n,m}(t) &= X_{n,m}\left(\frac{kT}{2^{2n}}\right) + \int_{\frac{kT}{2^{2n}}}^t b_{m+n}\left(\frac{kT}{2^{2n}}, X_{n,m}\left(\cdot \wedge \frac{kT}{2^{2n}}\right)\right)ds \\ &\quad + \int_{\frac{kT}{2^{2n}}}^t \sigma_{m+n}\left(\frac{kT}{2^{2n}}, X_{n,m}\left(\cdot \wedge \frac{kT}{2^{2n}}\right)\right)dW(s). \end{aligned}$$

Now we prove the second conclusion (38). By Assumption (A.III), from the proof of Theorem 5.4, we can obtain

$$\begin{aligned} &\|X_{0,m}(\cdot) - X_{n,m}(\cdot)\|_{1,p,T;\mathbb{R}^d}^p \\ &\leq C(\|b_{m+n}\|_{C_b^2(C([0,t];\mathbb{R}^d);\mathbb{R}^d)}^p \vee \|\sigma_{m+n}\|_{C_b^2(C([0,t];\mathbb{R}^d);\mathbb{R}^d \otimes \mathbb{R}^m)}^p) \\ &\quad \times e^{C(\|b_{m+n}\|_{C_b^1(C([0,t];\mathbb{R}^d);\mathbb{R}^d)} \vee \|\sigma_{m+n}\|_{C_b^1(C([0,t];\mathbb{R}^d);\mathbb{R}^d \otimes \mathbb{R}^m)}^2)2^{-np}}. \end{aligned}$$

Hence by (39) we have

$$\|X_{0,m}(\cdot) - X_{n,m}(\cdot)\|_{1,p,T;\mathbb{R}^d}^p \leq C2^{(m+n)(1-\delta)p}2^{-np}.$$

Since

$$\begin{aligned} X(t) - X_{0,m}(t) &= \int_0^t (b(s, X(\cdot)) - b_{m+n}(s, X_{0,m}(\cdot)))ds \\ &\quad + \int_0^t (\sigma(s, X(\cdot)) - \sigma_{m+n}(s, X_{0,m}(\cdot)))dW(s) \\ &= \int_0^t (b(s, X(\cdot)) - b(s, X_{0,m}(\cdot)))ds + \int_0^t (\sigma(s, X(\cdot)) \\ &\quad - \sigma(s, X_{0,m}(\cdot)))dW(s) + \int_0^t (b(s, X_{0,m}(\cdot)) - b_{m+n}(s, X_{0,m}(\cdot)))ds \\ &\quad + \int_0^t (\sigma(s, X_{0,m}(\cdot)) - \sigma_{m+n}(s, X_{0,m}(\cdot)))dW(s), \end{aligned}$$

by BDG's inequality, Assumption (A.III) and (39), we have

$$E[\|X(\cdot) - X_{0,m}(\cdot)\|_{C([0,t];\mathbb{R})}^{2p}] \leq C_1 \int_0^t E[\|X(\cdot) - X_{0,m}(\cdot)\|_{C([0,s];\mathbb{R})}^{2p}]ds + C_2 2^{-2(m+n)\delta p}.$$

Therefore we deduce by Gronwall's lemma that

$$\|X(\cdot) - X_{0,m}(\cdot)\|_{0,p,T;\mathbb{R}^d}^p \leq C2^{-(m+n)\delta p}.$$

Similarly, we also have

$$\|X_n(\cdot) - X_{n,m}(\cdot)\|_{0,p,T;\mathbb{R}^d}^p \leq C2^{-(m+n)\delta p}.$$

Combining with the above two inequalities we have by Remark 2.4

$$\begin{aligned} \|X_n(\cdot) - X(\cdot)\|_{\delta',p,T;\mathbb{R}^d}^p &\leq C \sum_{m=1}^{\infty} 2^{\delta' mp} (\|X(\cdot) - X_{0,m}(\cdot)\|_{0,p,T;\mathbb{R}^d}^p \\ &\quad + 2^{-mp} \|X_{0,m}(\cdot) - X_{n,m}(\cdot)\|_{1,p,T;\mathbb{R}^d}^p) \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{m=1}^{\infty} 2^{\delta' mp} (2^{-(m+n)\delta p} + 2^{-mp} 2^{(m+n)(1-\delta)p} 2^{-np}) \\
&= C \sum_{m=1}^{\infty} 2^{\delta' mp} (2^{-\delta mp} 2^{-\delta np} + 2^{-\delta mp} 2^{-n\delta p}) \\
&\leq C 2^{-n\delta p}.
\end{aligned}$$

Applying the same procedure as used as above, we can get the first conclusion (37), and we thus complete the proof. \square

Combining this with Theorem 4.1, we can obtain the convergence rate of $\xi_2(x, T, n)$. In order to get this result, we shall make use of the following Assumption (A.IV).

(A.IV) $\sigma : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ satisfies the condition $(A_{2+\delta, \mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^m})$ and $b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ satisfies the condition $(A_{2+\delta, \mathbb{R}^d, \mathbb{R}^d})$.

Theorem 5.10. Let $k_n = 2^{2n}$. Suppose that the coefficients (σ, b) of SDE (2) satisfy (A.II) and (A.IV). If $0 < \beta < \alpha \wedge (1 + \delta) - 1 - d/q$, $1/p + 1/q = 1$, $G \in \mathbb{D}_\alpha^q$ and $t \in [0, T]$, then we have $p_{X_n(t), G} \in C^\beta(\mathbb{R}^d)$, $p_{X(t), G} \in C^\beta(\mathbb{R}^d)$ and

$$\|p_{X_{k_n}(t), G} - p_{X(t), G}\|_{C^\beta(\mathbb{R}^d)} = O(2^{-n\theta}), \quad (40)$$

where $p_{X_{k_n}(t), G}$ and $p_{X(t), G}$ are defined by (3), i.e.,

$$p_{X_{k_n}(t), G}(y) = E[G \cdot \delta_y \circ X_{k_n}(t)] = E(G | X_{k_n}(t) = y) p_{X_{k_n}(t)}(y)$$

and

$$p_{X(t), G}(y) = E[G \cdot \delta_y \circ X(t)] = E(G | X(t) = y) p_{X(t)}(y).$$

In particular, taking $G = \mathbf{1} \in \mathbb{D}_{\infty}^{\infty-}$, we conclude that $p_{X_{k_n}(t)} \in C^\beta(\mathbb{R}^d)$, $p_{X(t)} \in C^\beta(\mathbb{R}^d)$ and

$$\|p_{X_{k_n}(t)} - p_{X(t)}\|_{C^\beta(\mathbb{R}^d)} = O(2^{-n\theta}). \quad (41)$$

Furthermore, if G_n , $n = 1, 2, \dots$ and G are in \mathbb{D}_α^q and

$$\|G_n - G\|_{\alpha, q} = O(2^{-n\lambda}),$$

then we have

$$\|p_{X_{k_n}(t), G_n} - p_{X(t), G}\|_{C^\beta(\mathbb{R}^d)} = O(2^{-n(\lambda \wedge \theta)}). \quad (42)$$

Proof. One only needs to simply combine Theorems 5.9 and 4.1. \square

Remark 5.11. It is worth noting that for some special non-Markovian stochastic differential equations such as stochastic differential delay equations the convergence rate of the density can be improved to a better result. Indeed, Clément, Kohatsu-Higa and Lamberton [7] have obtained the convergence rate of the density for some stochastic differential delay equations is $1/n$.

Examples 5.12. Consider the following stochastic delay differential equations

$$x(t) = x + \int_0^t b(x(s - \tau), x(s)) ds + \int_0^t \sigma(x(s - \tau), x(s)) dW(s),$$

where the coefficients b and σ are mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}$. Let $T > 0$ be a fixed time horizon, and T/n represent the discretization step. Set $x_n(0) = x$, and for $kT/n < t \leq (k+1)T/n$, the

Euler scheme is defined by

$$x_n(t) = x_n\left(\frac{kT}{n}\right) + \int_{\frac{kT}{n}}^t b\left(x_n\left(\frac{kT}{n} - \tau\right), x_n\left(\frac{kT}{n}\right)\right) ds \\ + \int_{\frac{kT}{n}}^t \sigma\left(x_n\left(\frac{kT}{n} - \tau\right), x_n\left(\frac{kT}{n}\right)\right) dW(s).$$

Let $\delta = k + \theta$, $k \in \mathbb{N}$, $0 < \theta \leq 1$ and $p > 1$. Suppose that the coefficients $(b(x), \sigma(x))$ satisfy the following assumptions: $b, \sigma \in (C_b^{2+k}(\mathbb{R}^2; \mathbb{R}), C_b^{3+k}(\mathbb{R}^2; \mathbb{R}))_{\theta, p}$; σ is bounded and uniformly nondegenerate. Then we can use [Theorem 5.10](#) to obtain the convergence rate of density of the Euler scheme for stochastic delay differential equations. Similar results hold of course for more general delay equations.

Remark 5.13. Finally we point out that on a finite dimensional Euclidean space, say \mathbb{R}^m , the space $(C^k(\mathbb{R}^m), C^{k+1}(\mathbb{R}^m))_{\theta, p}$ is very close to $C^{k+\theta}(\mathbb{R}^m)$. In fact, it is known that (see [\[18\]](#)) for every $\epsilon > 0$,

$$C^{k+\theta}(\mathbb{R}^m) = (C^k(\mathbb{R}^m), C^{k+1}(\mathbb{R}^m))_{\theta, \infty} \subset (C^k(\mathbb{R}^m), C^{k+1}(\mathbb{R}^m))_{\theta-\epsilon, p} \\ \subset (C^k(\mathbb{R}^m), C^{k+1}(\mathbb{R}^m))_{\theta-\epsilon, \infty} = C^{k+\theta-\epsilon}(\mathbb{R}^m).$$

On an infinite dimensional Banach space E , this full chain of inclusions does not hold in general, but we still have the following

$$(C^k(E), C^{k+1}(E))_{\theta, p} \subset C^{k+\theta}(E), \quad \forall p \in [1, \infty).$$

Therefore the above example shows that we can obtain the convergence rate of density of the Euler scheme for stochastic delay differential equations when the coefficients are in Hölder spaces.

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