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Two-sided estimates for the transition densities of symmetric Markov processes dominated by stable-like processes in $C^{1,\eta}$ open sets

Kyung-Youn Kim[a], Panki Kim[a,b,*]

[a] *Department of Mathematical Sciences, Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu, Seoul 151-747, Republic of Korea*

[b] *Research Institute of Mathematics, Seoul National University, Building 27, 1 Gwanak-ro, Gwanak-gu, Seoul 151-747, Republic of Korea*

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Abstract

In this paper, we study sharp Dirichlet heat kernel estimates for a large class of symmetric Markov processes in $C^{1,\eta}$ open sets. The processes are symmetric pure jump Markov processes with jumping intensity $\kappa(x, y)\psi_1(|x-y|)^{-1}|x-y|^{-d-\alpha}$, where $\alpha \in (0, 2)$. Here, ψ_1 is an increasing function on $[0, \infty)$, with $\psi_1(r) = 1$ on $0 < r \le 1$ and $c_1 e^{c_2 r^\beta} \le \psi_1(r) \le c_3 e^{c_4 r^\beta}$ on $r > 1$ for $\beta \in [0, \infty]$, and $\kappa(x, y)$ is a symmetric function confined between two positive constants, with $|\kappa(x, y) - \kappa(x, x)| \le c_5|x - y|^\rho$ for $|x - y| < 1$ and $\rho > \alpha/2$. We establish two-sided estimates for the transition densities of such processes in $C^{1,\eta}$ open sets when $\eta \in (\alpha/2, 1]$. In particular, our result includes (relativistic) symmetric stable processes and finite-range stable processes in $C^{1,\eta}$ open sets when $\eta \in (\alpha/2, 1]$.

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Keywords: Dirichlet form; Jump process; Jumping kernel; Markov process; Heat kernel; Dirichlet heat kernel; Transition density; Lévy system

* Corresponding author at: Department of Mathematical Sciences, Seoul National University Building 27, 1 Gwanak-ro, Gwanak-gu Seoul 151-747, Republic of Korea. Tel.: +82 2 880 4077.

E-mail addresses: eunicekim@snu.ac.kr (K.-Y. Kim), pkim@snu.ac.kr (P. Kim).

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1. Introduction

Discontinuous Markov processes form a large class of stochastic processes containing stable-like processes and relativistic stable-like processes. Recently, discontinuous Markov processes have often been used to simulate physical and economic systems that cannot be modeled by Gaussian processes (see [32,24,34–36]). Because of such importance in both theory and practice, there has been intense interest in studying discontinuous Markov processes.

Throughout this paper we assume that $\beta \in [0, \infty]$, $\alpha \in (0, 2)$, and $d \geq 1$. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and dx be the d-dimensional Lebesgue measure in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x with radius r. The Euclidean distance between x and y will be denoted by $|x - y|$. For two nonnegative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g. We will use the symbol ":=", which is read as "is defined to be". For any Borel set $A \subset \mathbb{R}^d$, we will use $\mathrm{diam}(A)$ to denote its diameter and $|A|$ to denote its Lebesgue measure.

The infinitesimal generator $\mathcal{L}$ of a discontinuous symmetric Markov process $Y = (Y_t, \mathbb{P}_x)_{t \geq 0,\, x \in \mathbb{R}^d}$ is a symmetric integro-differential operator, and under some mild assumptions the distribution $\mathbb{P}_x(Y_t \in dy)$ is absolutely continuous, for every $x \in \mathbb{R}^d$ and $t > 0$, with respect to Lebesgue measure in $\mathbb{R}^d$. We will use $p(t, x, y)$ to denote the transition density of Y so that $\mathbb{P}_x(Y_t \in A) = \int_A p(t, x, y) dy$. For any open subset $D \subset \mathbb{R}^d$, we denote by Y^D the subprocess of Y killed upon leaving D, and we use $p_D(t, x, y)$ to denote the transition density of Y^D.

The transition density $p_D(t, x, y)$ describes the distribution of the process Y^D. Conversely, from an analytic viewpoint, $p_D(t, x, y)$ is also called a Dirichlet heat kernel of the operator $\mathcal{L}$ on D, because it is a fundamental solution of $\partial_t u = \mathcal{L} u$ and $u = 0$ on D^c. Thus, obtaining sharp two-sided estimates of $p_D(t, x, y)$ is a fundamental problem in both analysis and probability theory. However, it is not easy to obtain two-sided estimates of $p_D(t, x, y)$, especially near the boundary. For Dirichlet heat kernel estimates for killed diffusions, see [18–20] for the upper bound and [40] for the lower bound on bounded $C^{1,1}$ connected open sets.

A prototype of discontinuous Markov processes is a (rotationally) symmetric α-stable Lévy process where $\alpha \in (0, 2)$. The infinitesimal generator of a symmetric α-stable Lévy process is a fractional Laplacian $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$ that is a nonlocal operator. Recall that $\Delta^{\alpha/2}$ can be defined as

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(d, -\alpha) \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{R}^d : |y - x| > \varepsilon\}} (u(y) - u(x)) \frac{dy}{|x - y|^{d+\alpha}}, \tag{1.1}$$

where Γ is the Gamma function and $\mathcal{A}(d, -\alpha) = \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma\left(\frac{d+\alpha}{2}\right) \Gamma(1 - \alpha/2)^{-1}$. Thus, it is a pure jump process and has a Lévy density $y \to \mathcal{A}(d, -\alpha)|y|^{-d-\alpha}$. Chen et al. [10] obtained the Dirichlet heat kernel estimates for the symmetric α-stable process X in $C^{1,1}$ open sets.

Another example of discontinuous Markov processes is a relativistic α-stable process X^m with mass $m > 0$, which is a Lévy process with a characteristic function given by

$$\mathbb{E}_x\left[e^{i\xi \cdot (X_t^m - X_0^m)}\right] = \exp\left(-t\left(\left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m\right)\right) \quad \text{for every } x, \xi \in \mathbb{R}^d.$$

The corresponding infinitesimal generator is $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$. In particular, for $\alpha = 1$ the operator $m - \sqrt{m^2 - \Delta}$ is called the free Hamiltonian corresponding to the quantization of the kinetic energy for a relativistic particle of mass m (e.g., see [5,33]). The Lévy density of X^m is

$$J^m(y) = \mathcal{A}(d, -\alpha)|y|^{-d-\alpha}\psi(m^{1/\alpha}|y|) \quad \text{where } \psi(r) := \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds.$$

ψ is decreasing and is a smooth function of r^2 satisfying $\psi(r) \le 1$ and $\psi(r) \asymp e^{-r}(1 + r^{(d+\alpha-1)/2})$ on $[1, \infty)$ (see [16, pp. 276–277] for details). Thus, $J^m(y)$ is dominated by the Lévy density of the symmetric α-stable process. The approach developed in [10] provides a guideline for establishing sharp two-sided heat kernel estimates for other discontinuous Lévy processes in open subsets of $\mathbb{R}^d$. For example, two-sided Dirichlet heat kernel estimates for X^m are discussed in [12]. Very recently two-sided Dirichlet heat kernel estimates were extended to a large class of symmetric Lévy processes in [13,14].

In this paper, motivated by [8,10,12] we consider a large class of symmetric Markov processes (not necessarily Lévy processes) whose jumping kernels are dominated by the kernel of the fractional Laplacian. We establish the two-sided estimates for Dirichlet heat kernels of the generators of such Markov processes in (possibly unbounded) $C^{1,\eta}$ open sets D. When D is $\mathbb{R}^d$, such a problem has been discussed in [25,38,39]. Our result extends the main results in [10,12] and provides far more.

Let us now describe our assumptions and fix the notation simultaneously. Let ψ_1 be an increasing function on $[0, \infty)$ with $\psi_1(r) = 1$ for $0 < r \le 1$, and let there be constants $\gamma_1, \gamma_2 > 0$ and $\beta \in [0, \infty]$ so that

$$L_1 e^{\gamma_1 r^\beta} \le \psi_1(r) \le L_2 e^{\gamma_2 r^\beta} \quad \text{for every } 1 < r < \infty, \tag{1.2}$$

for some constants $L_1, L_2 > 0$. We define

$$j(r) = \frac{1}{r^{d+\alpha}\psi_1(r)} \quad r > 0. \tag{1.3}$$

We assume that $\kappa(x, y)$ is a positive symmetric function with

$$L_3^{-1} \le \kappa(x, y) \le L_3, \quad x, y \in \mathbb{R}^d, \tag{1.4}$$

and

$$|\kappa(x, y) - \kappa(x, x)|\mathbf{1}_{\{|x-y|<1\}} \le L_4|x - y|^\rho, \quad x, y \in \mathbb{R}^d, \tag{1.5}$$

where $\rho > \alpha/2$ and L_3, L_4 are positive constants. Let J be a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$ such that

$$\begin{aligned} J(x, y) &:= \kappa(x, y) j(|x - y|) \\ &= \begin{cases} \kappa(x, y)|x - y|^{-d-\alpha}\psi_1(|x - y|)^{-1} & \text{if } \beta \in [0, \infty), \\ \kappa(x, y)|x - y|^{-d-\alpha}\mathbf{1}_{\{|x-y|\le 1\}} & \text{if } \beta = \infty. \end{cases} \end{aligned} \tag{1.6}$$

For $u \in L^2(\mathbb{R}^d, dx)$, we define $\mathcal{E}(u, u) := 2^{-1}\int_{\mathbb{R}^d\times\mathbb{R}^d}(u(x) - u(y))^2 J(x, y)dxdy$. Let $C_c(\mathbb{R}^d)$ denote the space of continuous functions with compact support in $\mathbb{R}^d$ and equipped with uniform topology. We define

$$\mathcal{D}(\mathcal{E}) := \{f \in C_c(\mathbb{R}^d) : \mathcal{E}(f, f) < \infty\}. \tag{1.7}$$

By [15, Proposition 2.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}^d, dx)$, where $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_{\mathbb{R}^d} u(x)^2 dx$ and $\mathcal{F} := \overline{\mathcal{D}(\mathcal{E})}^{\mathcal{E}_1}$. Hence, there is a Hunt process on $\mathbb{R}^d$ associated with this Dirichlet form (see [21]).

We say **UJS** holds for a jumping kernel $\widetilde{J}$ if for a.e. $x, y \in \mathbb{R}^d$,

$$\widetilde{J}(x,y) \le \frac{c}{r^d} \int_{B(x,r)} \widetilde{J}(z,y) dz \quad \text{whenever } r \le \frac{1}{2}|x-y|. \tag{\textbf{UJS}}$$

Note that, since j is decreasing and $J(x,y) \asymp j(|x-y|)$, we have

$$\begin{aligned} \int_{B(x,r)} J(z,y) dz &\ge \int_{B(x,r)\cap\{|z-y|\le|x-y|\}} c_1 j(|z-y|) dz \\ &\ge c_2 r^d j(|x-y|) \ge c_3 r^d J(x,y) \end{aligned}$$

for every $r \le |x-y|/2$. Thus **UJS** holds for our J. Moreover, our conditions (1.2)–(1.4) imply that [8, (1.6)] holds with $\phi(r) = r^\alpha \psi_1(r)$. Thus, the Hunt process Y associated with $(\mathcal{E}, \mathcal{F})$ belongs to a subclass of the processes considered in [8]. Therefore, Y is conservative and it has a Hölder continuous transition density $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with respect to the Lebesgue measure.

The function J is called the jumping intensity kernel of Y, because it gives rise to a Lévy system for Y describing the jumps of the process Y. For any $x \in \mathbb{R}^d$, stopping time S (with respect to the filtration of Y), and nonnegative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y) = 0$ for all $y \in \mathbb{R}^d$ and $s \ge 0$ we have

$$\mathbb{E}_x \left[\sum_{s \le S} f(s, Y_{s-}, Y_s) \right] = \mathbb{E}_x \left[\int_0^S \left(\int_{\mathbb{R}^d} f(s, Y_s, y) J(Y_s, y) dy \right) ds \right] \tag{1.8}$$

(e.g., see [15, Appendix A]).

We first consider the estimate for the transition density $p(t,x,y)$ of Y in $\mathbb{R}^d$. Hereinafter, for $a, b \in \mathbb{R}$, we have $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. For each $a, T, \gamma > 0$, we define a function $h_{a,\gamma,T}(t,r)$ on $(t,r) \in (0,T] \times [0,\infty)$ as

$$h_{a,\gamma,T}(t,r) := \begin{cases} t^{-d/\alpha} \wedge t r^{-d-\alpha} e^{-\gamma r^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge t r^{-d-\alpha} & \text{if } \beta \in (1,\infty] \text{ with } r < 1, \\ t \exp\left\{ -a \left(r \left(\log \frac{Tr}{t} \right)^{\frac{\beta-1}{\beta}} \wedge r^\beta \right) \right\} & \text{if } \beta \in (1,\infty) \text{ with } r \ge 1, \\ (t/(Tr))^{ar} & \text{if } \beta = \infty \text{ with } r \ge 1. \end{cases} \tag{1.9}$$

Even though in [15, Theorem 1.2] and [8, Theorems 1.2 and 1.4] two-sided estimates for $p(t,x,y)$ are stated separately for the cases $0 < t \le 1$ and $t > 1$ (not for the cases $0 < t \le T$ and $t > T$ for some fixed constant $T > 0$), the same proofs work for the cases $0 < t \le T$ and $t > T$. We state two-sided estimates for $p(t,x,y)$ for the case $0 < t \le T$, which we will use.

Theorem 1.1. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in (1.6). Then, the process Y has a continuous transition density function $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. For each positive constant T, there are positive constants C_1, c_1, and $c_2 \ge 1$ which depend on $\alpha, \beta, d, L_3, \psi_1, T$ such that for every $t \in (0,T]$ the* *function $p(t,x,y)$ has the following estimates:*

$$c_2^{-1} h_{c_1,\gamma_2,T}(t,|x-y|) \le p(t,x,y) \le c_2\, h_{C_1,\gamma_1,T}(t,|x-y|). \tag{1.10}$$

Note that, unlike those in [8, Theorem 1.2], the exponents γ_1 and γ_2 in Theorem 1.1 are explicit. When $\beta \in [0,1]$, the upper bound in (1.10) comes from [25, Theorem 2, Proposition 1]. We omit the proof of the upper bound in (1.10) for $\beta \in [1,\infty]$, since the proof is the same, as mentioned above. However, in Section 3, the lower bounds in (1.10) will be proved as a special case of the preliminary lower bound on the heat kernel of the killed process.

The goal of this paper is to obtain the sharp two-sided Dirichlet heat kernel estimates for Y on $C^{1,\eta}$ open sets for $\eta \in (\alpha/2, 1]$. For any open set D, we use τ_D to denote the first exit time from D by the process Y, and we use Y^D to denote the process obtained by killing the process Y upon exiting D. By the strong Markov property, it can easily be verified that $p_D(t,x,y) := p(t,x,y) - \mathbb{E}_x[p(t-\tau_D, Y_{\tau_D}, y); t > \tau_D]$ is the transition density of Y^D. Using the continuity and estimate of p, it is routine to show that $p_D(t,x,y)$ is symmetric and continuous (e.g., see the proof of Theorem 2.4 in [17]).

Recall that an open set D in $\mathbb{R}^d$ (when $d \ge 2$) is said to be $C^{1,\eta}$ with $\eta \in (0,1]$ if there exist a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,\eta}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0,\dots,0)$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(x) - \nabla\phi(w)| \le \Lambda|x-w|^\eta$ and an orthonormal coordinate system CS_z of $z = (z_1,\dots, z_{d-1}, z_d) := (\widetilde{z}, z_d)$ with origin at z such that $B(z,R) \cap D = \{y = (\tilde{y}, y_d) \in B(z,R) \text{ in } CS_z : y_d > \phi(\widetilde{y})\}$. The pair (R,Λ) will be called the $C^{1,\eta}$ characteristics of the open set D. Note that a $C^{1,\eta}$ open set D with characteristics (R,Λ) can be unbounded and disconnected, and the distance between two distinct components of D is at least R. By a $C^{1,\eta}$ open set in $\mathbb{R}$ we mean an open set that can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

When $\beta \in (1,\infty]$, we need to make an assumption for D in order to obtain the lower bound of $p_D(t,x,y)$. We say that *the path distance in each connected component of D is comparable to the Euclidean distance with characteristic* λ_1 if for every x and y in a same component of D there is a rectifiable curve l in D which connects x to y such that the length of l is less than or equal to $\lambda_1|x-y|$. Clearly, such a property holds for all bounded $C^{1,\eta}$ open sets, $C^{1,\eta}$ open sets with compact complements, and connected open sets above graphs of $C^{1,\eta}$ functions.

We are now ready to state the main result of this paper. Recall that C_1 is the constant in Theorem 1.1. Let $\delta_D(x)$ be a distance between x and D^c, and let

$$\Psi(t,x) := \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right). \tag{1.11}$$

Theorem 1.2. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in (1.6). Suppose that $\eta \in (\alpha/2, 1]$, $T > 0$, and D is a $C^{1,\eta}$ open set in $\mathbb{R}^d$ with characteristics (R,Λ). Then, the transition density $p_D(t,x,y)$ of Y^D has the following estimates.*

(1) *There is a positive constant $c_1 = c_1(\alpha,\beta,R,\Lambda,T,d,\psi_1,L_3,L_4,\eta)$ such that for all $(t,x,y) \in (0,T] \times D \times D$ we have*

$$p_D(t,x,y) \le c_1\, \Psi(t,x)\Psi(t,y) \begin{cases} h_{C_1\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta \in [0,\infty), \\ h_{C_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta = \infty. \end{cases}$$

(2) *There is a positive constant* $c_2 = c_2(\alpha, \beta, R, \Lambda, T, d, \psi_1, L_3, L_4, \eta)$ *such that for all* $(t, x, y) \in (0, T] \times D \times D$ *we have*

$$p_D(t,x,y) \ge c_2\, \Psi(t,x)\,\Psi(t,y) \times \begin{cases} t^{-d/\alpha} \wedge t|x-y|^{-d-\alpha} e^{-\gamma_2|x-y|^\beta} & \text{if } \beta \in [0,1], \\ t^{-d/\alpha} \wedge t|x-y|^{-d-\alpha} & \text{if } \beta \in (1,\infty) \text{ and } |x-y| < 1, \\ & \text{or } \beta = \infty \text{ and } |x-y| \le 4/5. \end{cases}$$

(3) *Suppose in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic* λ_1. *Then, there are positive constants* $c_i = c_i(\alpha, \beta, R, \Lambda, T, d, \psi_1, L_3, L_4, \eta, \lambda_1)$, $i = 3, 4$, *such that if* x, y *are in a same component of* D *and* $t \in (0, T]$, *we have*

$$p_D(t,x,y) \ge c_3\, \Psi(t,x)\,\Psi(t,y) \times \begin{cases} h_{c_4,\gamma_2,T}(t, |x-y|) & \text{if } \beta \in (1,\infty) \text{ and } |x-y| \ge 1, \\ h_{c_4,\gamma_2,T}(t, 5|x-y|/4) & \text{if } \beta = \infty \text{ and } |x-y| \ge 4/5. \end{cases}$$

(4) *If* $\beta \in (1, \infty)$, *there is a positive constant* $c_5 = c_5(\alpha, \beta, R, \Lambda, T, d, \psi_1, L_3, L_4, \eta)$ *such that for every* x, y *in different components of* D *with* $|x - y| \ge 1$ *and* $t \in (0, T]$ *we have*

$$p_D(t,x,y) \ge c_5\, \Psi(t,x)\,\Psi(t,y) \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma_2 (5|x-y|/4)^\beta}.$$

(5) *Suppose in addition that* D *is bounded and connected. Then, there are positive constants* $c_i = c_i(\alpha, \beta, R, \Lambda, T, d, \psi_1, L_3, L_4, \eta, \mathrm{diam}(D))$, $i = 6, 7$, *such that for all* $(t, x, y) \in [T, \infty) \times D \times D$ *we have*

$$c_6\, e^{-t\lambda^D}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \le p_D(t,x,y) \le c_7\, e^{-t\lambda^D}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

where $-\lambda^D < 0$ *is the largest eigenvalue of the generator of* Y^D.

The cutoff values $5/4$ in Theorem 1.2(3)–(4) (and $4/5$ in Theorem 1.2(2)) are not essential. Further analysis reveals that for any $\varepsilon > 0$ we can choose $1 + \varepsilon$ as the cutoff value. However, it seems that we cannot choose 1 as the cutoff value.

If D is a $C^{1,\eta}$ connected open set and the path distance in D is comparable to the Euclidean distance, then by Theorem 1.2(1)–(4) we can rewrite the two-sided estimates for $p_D(t, x, y)$.

Corollary 1.3. *Suppose that* Y *is the symmetric pure jump Hunt process with the jumping intensity kernel* J *defined in* (1.6). *Suppose further that* D *is a* $C^{1,\eta}$ *connected open set with* $\eta \in (\alpha/2, 1]$ *and that the path distance in* D *is comparable to the Euclidean distance with characteristic* λ_1. *Then, for each* $T > 0$ *there exist* $c_i = c_i(\alpha, \beta, R, \Lambda, T, d, \psi_1, L_3, L_4, \eta, \lambda_1) > 0$, $i = 1, 2$, *such that for every* $t \in (0, T]$ *we have*

$$\begin{aligned} c_1^{-1}\, \Psi(t,x)\,\Psi(t,y) h_{c_2,\gamma_2,T}(t, 5|x-y|/4) &\le p_D(t,x,y) \\ &\le c_1\, \Psi(t,x)\,\Psi(t,y) h_{C_1\wedge\gamma_1,\gamma_1,T}(t, |x-y|/6). \end{aligned}$$

The boundary Harnack principle for classical harmonic functions (for Brownian motion) describes how harmonic functions decay near the boundary of D. This principle is important to studies of not only boundary value problems for partial differential equations but also the potential theory of Markov processes. The boundary Harnack principle has recently been generalized to a large class of discontinuous processes (see [2–4,22,26,27,30,37]).

Unfortunately, the boundary Harnack principle does not hold for our process Y when $\beta > 1$ (see [4,26] for counterexamples). This is one of the main difficulties in obtaining the boundary decay rate of $p_D(t,x,y)$. In this paper, by using Dynkin's formula and the test function method, the key estimates for exit distributions are obtained directly.

Note that when D is bounded, Theorem 1.2 gives the sharp estimates for $p_D(t,x,y)$ for all $t>0$, and the estimate for $p_D(t,x,y)$ has the same form as that obtained for symmetric stable processes in [10]. Thus, by integrating the two-sided heat kernel estimates in Theorem 1.2 with respect to t and following the proof of [10, Corollary 1.2], the estimates for the Green function $G_D(x,y)=\int_0^\infty p_D(t,x,y)dt$ in [10] can be extended to $C^{1,\eta}$ open sets. Since the proof is the same, we omit the proof.

Corollary 1.4. *Suppose that Y is the symmetric pure jump Hunt process with the jumping intensity kernel J defined in (1.6). Suppose further that $\eta\in(\alpha/2,1]$ and D is a bounded $C^{1,\eta}$ open set in $\mathbb{R}^d$. When $\beta=\infty$, we assume that D is connected. Then, on $D\times D$ we have*

$$G_D(x,y)\asymp\begin{cases}\dfrac{1}{|x-y|^{d-\alpha}}\left(1\wedge\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d>\alpha,\\ \log\left(1+\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d=1=\alpha,\\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2}\wedge\dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha.\end{cases}$$

The rest of this paper is organized as follows. In Section 2, we first solve the martingale-type problem for Y, which yields the Dynkin-type formula (2.4). Then, in Theorem 2.6, we give the key estimate for exit distributions. In Sections 3 and 5, we prove the lower bound estimates for $p_D(t,x,y)$. In Section 3, we first consider the case $\delta_D(x)\wedge\delta_D(y)\geq t^{1/\alpha}$; that is, x and y are kept away from the boundary of D. The result and our estimates for the exit distributions are used in Section 5 to prove the lower bound for all $x,y\in D$. Section 4 contains the proof of the upper bound. When $|x-y|<c$, we use Meyer's construction. Then, by using Lemma 4.1 twice, we prove the upper bound of $p_D(t,x,y)$ without using the lower bound of $p(t,x,y)$. This enables us to write the bound of $p_D(t,x,y)$ in a compact form.

Throughout the rest of this paper, the positive constants $C_1, C_*, L_1, L_2, L_3, L_4, \gamma_1, \gamma_2$ can be regarded as fixed. In the statements of results and the proofs, the constants $c_i=c_i(a,b,c,\ldots)$, $i=1,2,3,\ldots$, denote generic constants depending on $a,b,c,\ldots$, whose exact values are unimportant. These are given anew in each statement and each proof. The dependence of the constants on the dimension $d\geq 1$, on $\alpha\in(0,2)$, and on the positive constants L_1, L_2, L_3, L_4, γ_1, γ_2 will not be mentioned explicitly.

2. Estimates for exit distributions

In this section we give some key estimates for exit distributions. First, we introduce an inequality that is used several times in this paper.

Lemma 2.1. *Suppose that $\beta\in[0,\infty)$. For any $r_0>0$, there exists a positive constant $c=c(\beta,r_0)$ such that*

$$j(r)\leq cj(2r)\quad \text{for every } r\in(0,r_0]. \tag{2.1}$$

Moreover, (2.1) holds for $\beta=\infty$ if $r_0\leq 1/4$.

Proof. The result follows immediately from $L_2^{-1}e^{-\gamma_2 r^\beta}r^{-d-\alpha}\leq j(r)\leq L_1^{-1}e^{-\gamma_1 r^\beta}r^{-d-\alpha}$. □

For $\varepsilon \in (0, 1/2)$, we define the operators $\mathcal{A}_\varepsilon$ and $\mathcal{A}$ by

$$\mathcal{A}_\varepsilon g(x) := \int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}} (g(y)-g(x))J(y,x)dy \quad \text{and} \quad \mathcal{A}g(x) := \lim_{\varepsilon\downarrow 0} \mathcal{A}_\varepsilon g(x)$$

whenever these exist pointwise. We use $C_c^2(\mathbb{R}^d)$ to denote the space of twice differentiable functions with compact support. For every $g \in C_c^2(\mathbb{R}^d)$ and $r \in (\varepsilon/2, 1]$ we have

$$\begin{aligned}
\mathcal{A}_\varepsilon g(x) &= \left(\int_{\{y\in\mathbb{R}^d:r>|y-x|>\varepsilon\}} + \int_{\{y\in\mathbb{R}^d:|y-x|\geq r\}}\right)(g(y)-g(x))\kappa(x,y)j(|x-y|)dy \\
&= \kappa(x,x)\int_{\{y\in\mathbb{R}^d:r>|y-x|>\varepsilon\}} (g(y)-g(x))j(|x-y|)dy \\
&\quad + \int_{\{y\in\mathbb{R}^d:r>|y-x|>\varepsilon\}} (g(y)-g(x))(\kappa(x,y)-\kappa(x,x))j(|x-y|)dy \\
&\quad + \int_{\{y\in\mathbb{R}^d:|y-x|\geq r\}} (g(y)-g(x))\kappa(x,y)j(|x-y|)dy \\
&= \kappa(x,x)\int_{\{y\in\mathbb{R}^d:r>|y-x|>\varepsilon\}} (g(y)-g(x)-(y-x)\cdot\nabla g(x))\, j(|y-x|)dy \\
&\quad + \int_{\mathbb{R}^d} (g(y)-g(x))j(|x-y|)\big(\mathbf{1}_{\{r>|x-y|>\varepsilon\}}(y)(\kappa(x,y)-\kappa(x,x)) \\
&\quad + \mathbf{1}_{\{|x-y|\geq r\}}(y)\kappa(x,y)\big)dy. \qquad (2.2)
\end{aligned}$$

By (1.2)–(1.5) we have

$$\begin{aligned}
&\left|(g(y)-g(x))j(|x-y|)\left(\mathbf{1}_{\{r>|x-y|>\varepsilon\}}(y)(\kappa(x,y)-\kappa(x,x)) + \mathbf{1}_{\{|x-y|\geq r\}}(y)\kappa(x,y)\right)\right| \\
&\quad \leq L_4\mathbf{1}_{\{r>|x-y|>\varepsilon\}}(y)|g(y)-g(x)||x-y|^{-d-\alpha+\rho} \\
&\qquad + 2L_3\|g\|_\infty \mathbf{1}_{\{|x-y|>r\}}(y)|x-y|^{-d-\alpha}.
\end{aligned}$$

By this and the assumption $\rho > \alpha/2$ which is greater than $\alpha - 1$, we see that $\mathcal{A}g$ is well defined in $\mathbb{R}^d$ and that $\mathcal{A}_\varepsilon g$ converges to $\mathcal{A}g$ locally uniformly in $\mathbb{R}^d$ as $\varepsilon \to 0$. Furthermore, for every $r \in (0, 1]$ we have

$$\begin{aligned}
\mathcal{A}g(x) &= \kappa(x,x)\int_{\{y\in\mathbb{R}^d:r>|y-x|\}} (g(y)-g(x)-(y-x)\cdot\nabla g(x))\, j(|y-x|)dy \\
&\quad + \int_{\mathbb{R}^d} (g(y)-g(x))j(|x-y|)\big(\mathbf{1}_{\{r>|x-y|\}}(y)(\kappa(x,y)-\kappa(x,x)) \\
&\quad + \mathbf{1}_{\{|x-y|\geq r\}}(y)\kappa(x,y)\big)dy, \qquad (2.3)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{A}g\|_\infty &\leq c_1\int_{\mathbb{R}^d}\Big(\mathbf{1}_{\{1>|y|\}}(y)(|y|^{-d-\alpha+2} \\
&\qquad + |y|^{-d-\alpha+\rho+1}) + \mathbf{1}_{\{|y|\geq 1\}}(y)|y|^{-d-\alpha}\Big)\, dy < \infty.
\end{aligned}$$

Next, we solve the martingale-type problem for the operator $\mathcal{A}$ on $C_c^2(\mathbb{R}^d)$ and show that the Dynkin-type formula in terms of $\mathcal{A}$ is valid for every $f \in C_c^2(\mathbb{R}^d)$ (cf. [23, Section 6]).

Proposition 2.2. *For each $f \in C_c^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, there exists a $\mathbb{P}_x$-martingale M_t^f with respect to the filtration of Y such that $M_t^f = f(Y_t) - f(Y_0) - \int_0^t \mathcal{A}f(Y_s)ds$ $\mathbb{P}_x$-a.s. In particular, for every $f \in C_c^2(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$ we have*

$$\mathbb{E}_x \int_0^{\tau_U} \mathcal{A}f(Y_t)dt = \mathbb{E}_x[f(Y_{\tau_U})] - f(x). \tag{2.4}$$

Proof. We fix $f \in C_c^2(\mathbb{R}^d)$ and assume that the support of f is a subset of $B(0, R/2)$. We use a strict version of Fukushima's decomposition [21, Theorem 5.2.5]. First, it is clear from (1.7) that $f \in \mathcal{F}$. The energy measure $\mu_{\langle f \rangle}$ of f has the density $\Gamma(f)(x) = \int_{\mathbb{R}^d} (f(x) - f(y))^2 J(x, y)dy$.

Now, by Fubini's theorem and the dominated convergence theorem, for any $g \in C_c^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\mathcal{E}(f, g) &= \frac{1}{2} \lim_{\varepsilon \downarrow 0} \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d,\ |y-x|>\varepsilon\}} (g(y) - g(x))(f(y) - f(x))J(y, x)\, dx\, dy \\ &= -\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} g(x) \left(\int_{\{y\in\mathbb{R}^d:|y-x|>\varepsilon\}} (f(y) - f(x))J(y, x)dy \right) dx \\ &= -\int_{\mathbb{R}^d} g(x)\mathcal{A}f(x)\, dx.\end{aligned}$$

We recall from [21] that S_0 is the collection of positive Radon measures of finite energy integrals and

$$S_{00} := \left\{ \mu \in S_0 : \mu(\mathbb{R}^d) < \infty,\ \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-t} p(t, x, y)dt\mu(dy) < \infty \right\}.$$

Let $\nu := \nu_+ - \nu_-$, where $\nu_+(dx) := -\mathbf{1}_{\{\mathcal{A}f(x)<0\}}\mathcal{A}f(x)dx$ and $\nu_-(dx) := \mathbf{1}_{\{\mathcal{A}f(x)\ge0\}} \mathcal{A}f(x)dx$, so that $\mathcal{E}(f, g) = \int_{\mathbb{R}^d} g(x)\nu(dx)$. Note that $\|\mathcal{A}f\|_\infty < \infty$ and that $|\mathcal{A}f(x)| \le c_1|x|^{-d-\alpha}$ for $x \in B(0, R)^c$. Thus, $|\nu|(\mathbb{R}^d) < \infty$. Moreover, clearly

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty e^{-t} p(t, x, y)dt|\nu|(dy) \le \|\mathcal{A}f\|_\infty \int_0^\infty e^{-t}dt < \infty.$$

Thus, ν_+ and ν_- are in S_{00}. Similarly, since $\|\Gamma(f)\|_\infty < \infty$ and $|\Gamma(f)(x)| \le c_2|x|^{-d-\alpha}$ for $x \in B(0, R)^c$, we have $\mu_{\langle f \rangle}(\mathbb{R}^d) < \infty$ and so $\mu_{\langle f \rangle}$ is also in S_{00}.

Since $-\int_0^t \mathbf{1}_{\{\mathcal{A}f(X_s)<0\}}\mathcal{A}f(X_s)ds$ and $\int_0^t \mathbf{1}_{\{\mathcal{A}f(X_s)\ge0\}}\mathcal{A}f(X_s)ds$ are positive continuous additive functionals in the strict sense with Revuz measures ν_+ and ν_-, respectively, upon applying [21, Theorem 5.2.5] we conclude that for every $x \in \mathbb{R}^d$ we have

$$\begin{aligned}f(Y_t) - f(Y_0) &= M_t^f + \int_0^t \mathbf{1}_{\{\mathcal{A}f(x)\ge0\}}\mathcal{A}f(Y_s)ds + \int_0^t \mathbf{1}_{\{\mathcal{A}f(x)<0\}}\mathcal{A}f(Y_s)ds \\ &= M_t^f + \int_0^t \mathcal{A}f(Y_s)ds,\end{aligned}$$

where M_t^f is a $\mathbb{P}_x$-martingale additive functional in the strict sense with Revuz measure $\mu_{\langle f \rangle}$. □

Using (2.4), we prove the following lemma, which is used several times in Section 4. The proof of the next result is well-known (e.g., see [28, Lemma 4.15]).

Lemma 2.3. *For every $a \in (0, 1]$, there exists a positive constant $c = c(a)$ such that, for any $\beta \in [0, \infty]$, any $r \in (0, 1]$, and any open sets U and D with $B(0, ar) \cap D \subset U \subset D$, we have*

$$\mathbb{P}_x\left(Y_{\tau_U} \in D\right) \le c\, r^{-\alpha}\, \mathbb{E}_x[\tau_U], \quad x \in D \cap B(0, ar/2).$$

Proof. For fixed $a \in (0, 1]$, we take a sequence of radial functions $(\phi_m)_{m \ge 1}$ in $C_c^\infty(\mathbb{R}^d)$ such that $0 \le \phi_m \le 1$, with

$$\phi_m(y) = \begin{cases} 0, & \text{if } |y| < a/2 \text{ or } |y| > m+2, \\ 1, & \text{if } a \le |y| \le m+1, \end{cases}$$

and

$$\sup_{m \ge 1} \left(\sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} \phi_m \right\|_\infty + \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} \phi_m \right\|_\infty \right) < c_1 = c_1(a) < \infty.$$

For any $r \in (0, 1]$, define $\phi_{m,r}(y) = \phi_m(\frac{y}{r})$ so that $0 \le \phi_{m,r} \le 1$,

$$\phi_{m,r}(y) = \begin{cases} 0, & \text{if } |y| < ar/2 \text{ or } |y| > r(m+2) \\ 1, & \text{if } ar \le |y| \le r(m+1), \end{cases} \tag{2.5}$$

and

$$\sup_{m \ge 1} \sum_{i=1}^d \left\| \frac{\partial}{\partial y_i} \phi_{m,r} \right\|_\infty < c_1 r^{-1} \quad \text{and} \quad \sup_{m \ge 1} \sum_{i,j=1}^d \left\| \frac{\partial^2}{\partial y_i \partial y_j} \phi_{m,r} \right\|_\infty < c_1 r^{-2}. \tag{2.6}$$

Using (1.4), (1.5), (2.6), and the assumption that $\rho > \alpha/2$, for every $x \in \mathbb{R}^d$, $r \in (0, 1]$, and $m \ge 1$ we have

$$\begin{aligned} &\left| \kappa(x, x) \int_{\{y \in \mathbb{R}^d : |x-y| < r\}} \left(\phi_{m,r}(y) - \phi_{m,r}(x) - (y - x) \cdot \nabla \phi_{m,r}(x)\right) j(|y - x|) dy \right| \\ &\quad + \int_{\mathbb{R}^d} |\phi_{m,r}(y) - \phi_{m,r}(x)| j(|x - y|) \\ &\quad \times \left(\mathbf{1}_{\{|x-y|<r\}}(y) |\kappa(x, y) - \kappa(x, x)| + \mathbf{1}_{\{|x-y| \ge r\}}(y) \kappa(x, y)\right) dy \\ &\le \frac{c_2}{r^2} \int_{\{|x-y|<r\}} |x - y|^{-d-\alpha+2} dy + \frac{c_2}{r} \int_{\{|x-y|<r\}} |x - y|^{-d-\alpha+1+\rho} dy \\ &\quad + c_2 \int_{\{|x-y| \ge r\}} |x - y|^{-d-\alpha} dy \\ &\le c_3(r^{-\alpha} + r^{-\alpha+\rho}) \le 2c_3 r^{-\alpha} \end{aligned} \tag{2.7}$$

for some $c_3 = c_3(a) > 0$. Now, by combining (2.3)–(2.5) and (2.7), we find that for any $x \in D \cap B(0, ar/2)$ we have

$$\begin{aligned} &\mathbb{P}_x\left(Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right) \\ &= \mathbb{E}_x\left[\phi_{m,r}\left(Y_{\tau_U}\right) : Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right] \\ &\le \mathbb{E}_x\left[\phi_{m,r}\left(Y_{\tau_U}\right)\right] = \mathbb{E}_x\left[\int_0^{\tau_U} \mathcal{A}\phi_{m,r}(Y_t) dt\right] \le \|\mathcal{A}\phi_{m,r}\|_\infty \mathbb{E}_x[\tau_U] \le 2c_3 r^{-\alpha} \mathbb{E}_x[\tau_U]. \end{aligned}$$

Therefore, since $B(0, ar) \cap D \subset U$, we obtain

$$\mathbb{P}_x\left(Y_{\tau_U} \in D\right) = \lim_{m \to \infty} \mathbb{P}_x\left(Y_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right) \le 2c_3\, r^{-\alpha} \mathbb{E}_x[\tau_U]. \quad \square$$

For the remainder of this section we assume that $\eta \in (\alpha/2, 1]$ and that D is a $C^{1,\eta}$ open set with $C^{1,\eta}$ characteristics (R, Λ). Without loss of generality, we assume that $R \le 1$ and $\Lambda \ge 1$. For each fixed $Q \in \partial D$ and for every $r \le R$ we define

$$h_{Q,r}(y) := \delta_D(y)^{\alpha/2} \mathbf{1}_{D \cap B(Q,r)}(y). \tag{2.8}$$

We next establish two lemmas that are used to obtain the key estimates for exit distribution. The next lemma and its proof are similar to [9, Lemma 2.3] and [26, Lemma 3.7] and their proofs. We provide the proof here for completeness. Recall that $\Delta^{\alpha/2}$ is defined in (1.1).

Lemma 2.4. *There exists a positive constant $c = c(\eta, R, \Lambda)$ independent of $Q \in \partial D$ such that $\Delta^{\alpha/2} h_{Q,R/2}$ is well defined in $D \cap B(Q, R/8)$ and*

$$|\Delta^{\alpha/2} h_{Q,R/2}(x)| \le c \quad \text{for all } x \in D \cap B(Q, R/8).$$

Proof. Since the case of $d = 1$ is easier, we give the proof only for $d \ge 2$.

Let $h(\cdot) := h_{Q,R/2}(\cdot)$. Fix $x \in D \cap B(Q, R/8)$ and let $z_x \in \partial D$ such that $\delta_D(x) = |x - z_x|$. Let ϕ be a $C^{1,\eta}$ function and $CS = CS_{z_x}$ be an orthonormal coordinate system with z_x chosen so that $x = (\widetilde{0}, x_d)$, $B(0, R) \cap D = \{y = (\widetilde{y}, y_d) \text{ in } CS : y \in B(0, R), y_d > \phi(\widetilde{y})\}$, $\phi(\widetilde{0}) = 0$, $\nabla\phi(\widetilde{0}) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda$, and $|\nabla\phi(\widetilde{y}) - \nabla\phi_Q(\widetilde{z})| \le \Lambda|\widetilde{y} - \widetilde{z}|^\eta$. We fix the function ϕ and the coordinate system CS, and we define a function $h_x(y) = \delta_{H^+}(y)^{\alpha/2}$, where $H^+ = \{y = (\widetilde{y}, y_d) \text{ in } CS : y_d > 0\}$ is the half space in CS.

We define $\widehat{\phi} : B(\widetilde{0}, R) \to \mathbb{R}$ by $\widehat{\phi}(\widetilde{y}) := 2\Lambda|\widetilde{y}|^{\eta+1}$. Since $\nabla\phi(0) = 0$, by the mean value theorem we have $-\widehat{\phi}(\widetilde{y}) \le \phi(\widetilde{y}) \le \widehat{\phi}(\widetilde{y})$ for any $y \in D \cap B(x, R/8)$. Since $\Delta^{\alpha/2} h_x(y) = 0$ for any $y \in H^+$ (see, [22, (6.6)]), it is enough to show that $\Delta^{\alpha/2}(h - h_x)(x)$ is well defined and that there exists a constant $c_1 = c_1(\eta, R, \Lambda) > 0$ independent of $x \in D \cap B(Q, R/8)$ and $Q \in \partial D$ such that

$$\int_{D \cup H^+} \frac{|h(y) - h_x(y)|}{|y - x|^{d+\alpha}} dy \le c_1 < \infty. \tag{2.9}$$

Let $A := \{y \in (D \cup H^+) \cap B(x, R/8) : -\widehat{\phi}(\widetilde{y}) \le y_d \le \widehat{\phi}(\widetilde{y})\}$ and $E := \{y \in B(x, R/8) : y_d > \widehat{\phi}(\widetilde{y})\}$. We prove (2.9) by showing that $I + II + III \le c_1$, where

$$I := \int_{B(x,R/8)^c} \frac{h(y) + h_x(y)}{|y - x|^{d+\alpha}} dy, \qquad II := \int_A \frac{h(y) + h_x(y)}{|y - x|^{d+\alpha}} dy,$$
$$III := \int_E \frac{|h(y) - h_x(y)|}{|y - x|^{d+\alpha}} dy.$$

For I, since $h = 0$ on $B(Q, R/2)^c$ and $\delta_{H^+}(y) = y_d \le |y - z| + z_d \le 2|y - z|$ for $0 < z_d < R/8$ and $y \in B(z, \frac{R}{8})^c \cap H^+$, we have

$$\begin{aligned} I &\le (R/2)^{\alpha/2} \int_{B\left(x, \frac{R}{8}\right)^c} \frac{1}{|y - x|^{d+\alpha}} dy + \sup_{\{z \in \mathbb{R}^d : 0 < z_d < R/8\}} \int_{B\left(z, \frac{R}{8}\right)^c \cap H^+} \frac{\delta_{H^+}(y)^{\alpha/2}}{|y - z|^{d+\alpha}} dy \\ &\le (R/2)^{\alpha/2} \int_{B\left(0, \frac{R}{8}\right)^c} \frac{1}{|y|^{d+\alpha}} dy + \int_{B\left(0, \frac{R}{8}\right)^c} \frac{1}{|y|^{d+\alpha/2}} dy < \infty. \end{aligned}$$

For II, we first note that for any $y \in A$, $h(y) + h_x(y) \le c_2|\widetilde{y}|^{(1+\eta)\alpha/2}$ and $m_{d-1}(\{y : |\widetilde{y}| = r, -\widehat{\phi}(\widetilde{y}) \le y_d \le \widehat{\phi}(\widetilde{y})\}) \le c_3 r^{d+\eta-1}$ for $r \le R/8$, where $m_{d-1}(dy)$ is the surface measure.

Hence, for $\alpha/2 < \eta$ we have

$$II \le c_2 \int_0^{R/8} \int_{|\widetilde{y}|=r} \mathbf{1}_A(y)|\widetilde{y}|^{(1+\eta)\alpha/2}|\widetilde{y}|^{-d-\alpha} m_{d-1}(dy)dr$$
$$\le c_4 \int_0^{R/8} r^{-\alpha/2+\eta-1+\eta\alpha/2} dr < \infty.$$

Next we estimate III. If $0 < y_d = \delta_{H^+}(y) \le \delta_D(y)$, then $\delta_D(y) - y_d \le 4\Lambda|\widetilde{y}|^{1+\eta}$ and

$$h(y) - h_x(y) \le (y_d + 4\Lambda|\widetilde{y}|^{1+\eta})^{\alpha/2} - y_d^{\alpha/2} \le 2\alpha\Lambda|\widetilde{y}|^{1+\eta}|y_d|^{\frac{\alpha}{2}-1}.$$

If $y_d = \delta_{H^+}(y) > \delta_D(y)$, then $\delta_D(y) \ge y_d - 2\Lambda|\widetilde{y}|^{\eta+1}$ and

$$h_x(y) - h(y) \le y_d^{\alpha/2} - (y_d - 2\Lambda|\widetilde{y}|^{\eta+1})^{\alpha/2} \le \alpha\Lambda|\widetilde{y}|^{1+\eta}(y_d - 2\Lambda|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}.$$

Since $|y_d|^{\frac{\alpha}{2}-1} \le (y_d - 2\Lambda|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}$, we only need to consider the second case. Thus, using $E \subset \{|\widetilde{y}| < R/4, \widehat{\phi}(\widetilde{y}) < y_d < \widehat{\phi}(\widetilde{y}) + R/4\}$ and the change of variable $s = y_d - \widehat{\phi}(r)$, we have

$$III \le c_5 \int_E \frac{|\widetilde{y}|^{1+\eta}(y_d - 2\Lambda|\widetilde{y}|^{\eta+1})^{\frac{\alpha}{2}-1}}{(|\widetilde{y}| + |x_d - y_d|)^{d+\alpha}} dy$$
$$\le c_6 \int_0^{R/4} \int_{\widehat{\phi}(r)}^{\widehat{\phi}(r)+R/4} \frac{(y_d - \widehat{\phi}(r))^{\frac{\alpha}{2}-1}}{(r + |x_d - y_d|)^{\alpha+1-\eta}} dy_d dr$$
$$= c_6 \int_0^{R/4} \int_0^{R/4} \frac{s^{\frac{\alpha}{2}-1}}{(r + |x_d - (s + \widehat{\phi}(r))|)^{\alpha+1-\eta}} ds dr.$$

Then, we use [31, Lemma 4.4], which is a consequence of the rearrangement inequality, and obtain

$$III \le 2c_6 \int_0^{R/2} \left(\int_0^u t^{\frac{\alpha}{2}-1} dt \right) u^{-\alpha-1+\eta} du = \frac{4c_6}{\alpha} \int_0^{R/2} u^{-\frac{\alpha}{2}-1+\eta} du < \infty. \quad \square$$

Recall that $h_{Q,r}(y)$ is defined in (2.8) for each $Q \in \partial D$ and $r \le R$.

Lemma 2.5. *For any $k > 0$, let $B_k := \left\{ y \in D \cap B(Q, \frac{r}{8}) : \delta_{D\cap B(Q,\frac{r}{8})}(y) \ge 2^{-k} \right\}$. Then, for every $|z| < 2^{-k}$,*

$$\widehat{\mathcal{A}}_z h_{Q,r/2}(w) := \lim_{\varepsilon\to 0} \int_{|(w-z)-y|>\varepsilon} \left(h_{Q,r/2}(y) - h_{Q,r/2}(w-z) \right) J(w, z+y)\, dy \tag{2.10}$$

is well defined in B_k. Moreover, there exists $C_* = C_*(\eta, R, \Lambda, \rho) > 0$ independent of Q, k, and $r \le R$ such that

$$|\widehat{\mathcal{A}}_z h_{Q,r/2}(w)| \le C_* r^{-\alpha/2} \quad \text{for all } w \in B_k,\ |z| < 2^{-k}.$$

Proof. For $x \in D \cap B(Q, \frac{r}{8})$, let

$$I = I(x) := \int_{\mathbb{R}^d} \left| h_{Q,r/2}(y) - h_{Q,r/2}(x) \right| \frac{|x-y|^\rho \wedge 1}{|x-y|^{d+\alpha}} dy$$

and

$$II_\varepsilon = II_\varepsilon(x) := \int_{|y-x|>\varepsilon} \left(h_{Q,r/2}(y) - h_{Q,r/2}(x) \right) \frac{dy}{|x-y|^{d+\alpha}}.$$

For $r \le R$, let $x^r = r^{-1}x$, $Q^r = r^{-1}Q$, and $D^r = r^{-1}D$. The D^r are $C^{1,\eta}$ open sets with the same $C^{1,\eta}$ characteristics $(1, \Lambda)$ for all $r \le R$, and

$$II_\varepsilon = r^{-\alpha/2} \int_{|v-x^r|>\varepsilon r^{-1}} \left(\delta_{D^r}(v)^{\alpha/2} \mathbf{1}_{D^r \cap B(Q^r,1/2)}(v) - \delta_{D^r}(x^r)^{\alpha/2}\right) \frac{dv}{|x^r - v|^{d+\alpha}}.$$

Thus, by Lemma 2.4, $\lim_{\varepsilon \to 0} II_\varepsilon$ exists and satisfies $|\lim_{\varepsilon\to 0} II_\varepsilon| \le c_1 r^{-\alpha/2}$.

Similarly, we obtain

$$I = r^{-\alpha/2} \int_{\mathbb{R}^d} \left|\delta_{D^r}(v)^{\alpha/2} \mathbf{1}_{D^r \cap B(Q^r,1/2)}(v) - \delta_{D^r}(x^r)^{\alpha/2}\right| \frac{r^\rho |x^r - v|^\rho \wedge 1}{|x^r - v|^{d+\alpha}} dv.$$

Since $|\delta_{D^r}(v)^{\alpha/2} - \delta_{D^r}(x^r)^{\alpha/2}| \le |\delta_{D^r}(v) - \delta_{D^r}(x^r)|^{\alpha/2} \le |v - x^r|^{\alpha/2}$, for $r \le 1$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\delta_{D^r}(v)^{\alpha/2} \mathbf{1}_{D^r \cap B(Q^r,1/2)}(v) - \delta_{D^r}(x^r)^{\alpha/2}\right| \frac{r^\rho |x^r - v|^\rho \wedge 1}{|x^r - v|^{d+\alpha}} dv \\
&\le \int_{D^r \cap B(Q^r,1/2)} \frac{|x^r - v|^{\rho+\alpha/2}}{|x^r - v|^{d+\alpha}} dv + c_2 \int_{(D^r \cap B(Q^r,1/2))^c} \frac{1}{|x^r - v|^{d+\alpha}} dv \\
&\le c_3 \int_{\mathbb{R}^d} \frac{|u|^{\rho+\alpha/2} \wedge 1}{|u|^{d+\alpha}} du < \infty.
\end{aligned}$$

In the last inequality above we used the assumption that $\rho > \alpha/2$.

From (1.2)–(1.6) we observe that

$$\begin{aligned}
&\int_{\{y:|(w-z)-y|>\varepsilon\}} \left(h_{Q,r/2}(y) - h_{Q,r/2}(w-z)\right) J(w, z+y)\, dy \\
&= \int_{\{y:|(w-z)-y|>\varepsilon\}} \left(h_{Q,r/2}(y) - h_{Q,r/2}(w-z)\right) \frac{\kappa(w, z+y)}{|w-z-y|^{d+\alpha} \psi_1(|w-z-y|)} dy \\
&= \int_{\{y:|(w-z)-y|>\varepsilon\}} (\kappa(w, z+y) - \kappa(w, w)) \frac{(h_{Q,r/2}(y) - h_{Q,r/2}(w-z))}{\psi_1(|w-z-y|)|w-z-y|^{d+\alpha}} dy \\
&\quad + \kappa(w, w) \int_{\{y:|(w-z)-y|>\varepsilon\}} \frac{(h_{Q,r/2}(y) - h_{Q,r/2}(w-z))}{\psi_1(|w-z-y|)|w-z-y|^{d+\alpha}} dy
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{\{y:|(w-z)-y|>\varepsilon\}} \left(h_{Q,r/2}(y) - h_{Q,r/2}(w-z)\right) J(w, z+y)\, dy\right| \\
&\le c_4 I(w-z) + c_4 II_\varepsilon(w-z).
\end{aligned}$$

Therefore, $\widehat{\mathcal{A}}_z h_{Q,r/2}(w)$ exists on B_k and we have $|\widehat{\mathcal{A}}_z h_{Q,r/2}(w)| \le c_5 r^{-\alpha/2}$ for every $w \in B_k$ and $|z| < 2^{-k}$. □

Using Lemma 2.5, we prove the following theorem which plays a critical role in estimating the exit distribution. The proof of the next theorem is modeled after the proof of [29, Lemma 4.5]. In the next theorem for $x \in D$, we use z_x to denote a point on ∂D such that $|z_x - x| = \delta_D(x)$, and we use the coordinate system CS_{z_x} with a $C^{1,\eta}$ function ϕ such that $\phi(0) = 0$, $\nabla\phi(0) = (0, \ldots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda$, $|\nabla\phi(\widetilde{y}) - \nabla\phi(\widetilde{w})| \le \Lambda|\widetilde{y} - \widetilde{w}|^\eta$, and $B(0, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(z_x, R)$ in $CS_{z_x} : \phi(\widetilde{y}) < y_d\}$. For the next theorem and its proof, we always use this coordinate system CS_{z_x}.

Theorem 2.6. *There are constants $b_1 = b_1(\eta, R, \Lambda, \rho) \in (0, 1/10)$ and $c_1 = c_1(\eta, R, \Lambda) > 1$ such that for any $r \le b_1(R \wedge 1)/2$ and $x \in D$ with $\delta_D(x) < r$ we have*

$$\mathbb{E}_x\left[\tau_{D\cap B(z_x,r)}\right] \le c_1\, r^{\alpha/2}\delta_D(x)^{\alpha/2} \quad \text{where } z_x \in \partial D \text{ with } \delta_D(x) = |x - z_x|, \tag{2.11}$$

and for any $r \le (R \wedge 1)/4$, $\lambda \ge 4$ and $x \in D$ with $\delta_D(x) < \lambda^{-1}r/2$ we have

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(z_x,\lambda^{-1}r)}} \in \{2\Lambda|\widetilde{y}| < y_d, \lambda^{-1}r < |y| < r \text{ in } CS_{z_x}\}\right) \ge c_1^{-1}\frac{\delta_D(x)^{\alpha/2}}{r^{\alpha/2}} \tag{2.12}$$

where $z_x \in \partial D$ and $\delta_D(x) = |x - z_x|$.

Proof. Without loss of generality, we assume that $z_x = 0$ and let $A(a, b) := B(0, b)\backslash B(0, a)$ with $0 < a < b$. Let $r \le (R \wedge 1)/2$ and $h(y) = h_{0,r/2}(y)$ (see (2.8)). Let f be a nonnegative smooth radial function such that $f(y) = 0$ for $|y| > 1$ and $\int_{\mathbb{R}^d} f(y)dy = 1$. For $k \ge 1$, we define $f_k(y) := 2^{kd} f(2^k y)$ and $h^{(k)}(z) := (f_k * h)(z) \in C_c^\infty(\mathbb{R}^d)$, and we let $B_k := \{y \in D \cap B(0, r/8) : \delta_{D\cap B(0,r/8)}(y) \ge 2^{-k}\}$.

By Lemma 2.5, $\widehat{\mathcal{A}}_z h(w)$ exists for $w \in B_k$ and $z \in B(0, 2^{-k})$, with $-C_* r^{-\alpha/2} \le \widehat{\mathcal{A}}_z h(w) \le C_* r^{-\alpha/2}$, where $\widehat{\mathcal{A}}_z h(w)$ is defined in (2.10) and C_* is the constant in Lemma 2.5. Then, by letting $\varepsilon \to 0$ and using the dominated convergence theorem, it follows that $\mathcal{A}h^{(k)}$ is well defined everywhere and for large k and $|z| < 2^{-k}$ we have

$$\begin{aligned} |\mathcal{A}h^{(k)}(w)| &= \left|\int_{|z|<2^{-k}} f_k(z)\widehat{\mathcal{A}}_z h(w)\, dz\right| \le C_* r^{-\alpha/2}\int_{|z|<2^{-k}} f_k(z)\, dz \\ &\le C_* r^{-\alpha/2} \quad \text{on } B_k. \end{aligned} \tag{2.13}$$

Applying (2.4) to $U_\lambda^k := D \cap B(0, \lambda^{-1}r) \cap B_k$ with $\lambda \ge 8$ and $h^{(k)}$ and using (2.13) we have

$$\begin{aligned} \mathbb{E}_x\left[h^{(k)}\left(Y_{\tau_{U_\lambda^k}}\right)\right] - C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{U_\lambda^k}\right] \le h^{(k)}(x) \le \mathbb{E}_x\left[h^{(k)}\left(Y_{\tau_{U_\lambda^k}}\right)\right] \\ + C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{U_\lambda^k}\right], \quad x \in U_\lambda^k. \end{aligned}$$

Since $h^{(k)}$ is in $C_c^\infty(\mathbb{R}^d)$ and by letting $k \to \infty$, for all $\lambda \ge 8$ and $x \in D \cap B(0, \lambda^{-1}r)$ we obtain

$$\delta_D(x)^{\alpha/2} \ge \mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] - C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \tag{2.14}$$

and

$$\delta_D(x)^{\alpha/2} \le \mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] + C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]. \tag{2.15}$$

For any $z \in D \cap B(0, \lambda^{-1}r)$ and $y \in D \cap (B(0, 2^{-1}r)\backslash B(0, \lambda^{-1}r))$, since $2|y| \le r \le 1/2$, we have $j(|y - z|) \ge j(|y| + |z|) \ge j(2|y|) \ge c_1 j(|y|)$. Thus, by (1.8) we obtain

$$\begin{aligned} \mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] &= \mathbb{E}_x \int_{D\cap A(\lambda^{-1}r,2^{-1}r)} \int_0^{\tau_{D\cap B(0,\lambda^{-1}r)}} j(|Y_t - y|)dt\,\delta_D(y)^{\alpha/2}dy \\ &\ge c_1\, \mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]\int_{D\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy. \end{aligned} \tag{2.16}$$

Similarly, with $V := \{2\Lambda|\widetilde{y}| < y_d\}$ we also have

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \geq c_2\,\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]\int_{V\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)dy. \tag{2.17}$$

Clearly,

$$\int_{V\cap A(\lambda^{-1}r,2^{-1}r)} |y|^{-d-a}dy \geq \frac{c_3}{a}r^{-a}\left(\lambda^a - 2^a\right) \geq \frac{c_3}{2a}r^{-a}\left(\lambda^a - 1\right) \quad \text{for every } a > 0. \tag{2.18}$$

Since for every $y \in B(0,R)\cap D$ with $2\Lambda|\widetilde{y}| < y_d$ we have

$$\begin{aligned}\delta_D(y) &\geq (1+\Lambda)^{-1}\left(y_d - \phi(\widetilde{y})\right) \geq (2\Lambda)^{-1}(y_d - \Lambda|\widetilde{y}|)\\ &> (4\Lambda)^{-1}y_d \geq ((2\Lambda)^{-2}+1)^{-1/2}(4\Lambda)^{-1}|y|,\end{aligned}$$

by changing to polar coordinates with $|y| = s$ and using (2.18) we obtain

$$\begin{aligned}\int_{D\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy &\geq \int_{V\cap A(\lambda^{-1}r,2^{-1}r)} j(|y|)\delta_D(y)^{\alpha/2}dy\\ &\geq c_4\int_{\{(\widetilde{y},y_d):2\Lambda|\widetilde{y}|<y_d,\lambda^{-1}r<|y|<2^{-1}r\}} |y|^{-d-\alpha}|y|^{\alpha/2}dy\\ &\geq c_5 r^{-\alpha/2}\left(\lambda^{\alpha/2}-1\right).\end{aligned} \tag{2.19}$$

Then, combining (2.16) and (2.19) yields

$$\mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] \geq c_6 r^{-\alpha/2}\left(\lambda^{\alpha/2}-1\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \tag{2.20}$$

and (2.17) and (2.18) yield

$$\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}R)}} \in V\cap A(\lambda^{-1}r,2^{-1}r)\right) \geq c_6 r^{-\alpha}\left(\lambda^\alpha - 1\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right]. \tag{2.21}$$

Hence, by (2.14) and (2.20), we find that for every $\lambda \geq \lambda_0 := (2+2C_*/c_6)^{2/\alpha} \vee (10)$ and $\delta_D(x) < \lambda^{-1}r$ we have

$$\begin{aligned}\delta_D(x)^{\alpha/2} &\geq \left(c_6\lambda^{\alpha/2} - (c_6+C_*)\right)r^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}r)}]\\ &\geq \frac{c_6}{2}(\lambda^{-1}r)^{-\alpha/2}\mathbb{E}_x[\tau_{D\cap B(0,\lambda^{-1}r)}].\end{aligned} \tag{2.22}$$

Thus, we have proved (2.11) with $b_1 = \lambda_0^{-1}$ and $r = (R\wedge 1)/2$.

Conversely, since h is zero on D^c and bounded above by $(r/2)^{\alpha/2}$,

$$\mathbb{E}_x\left[h\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}}\right)\right] \leq (r/2)^{\alpha/2}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in D\right).$$

Thus using (2.15) and this, and then using Lemma 2.3 and (2.21), we find that for every $\lambda \geq 8$ and $\delta_D(x) < \lambda^{-1}r/2$ we obtain

$$\begin{aligned}\delta_D(x)^{\alpha/2} &\leq (r/2)^{\alpha/2}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in D\right) + C_* r^{-\alpha/2}\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \\ &\leq r^{-\alpha/2}\left(c_7\lambda^\alpha + C_*\right)\mathbb{E}_x\left[\tau_{D\cap B(0,\lambda^{-1}r)}\right] \\ &\leq r^{\alpha/2}\frac{c_7\lambda^\alpha + C_*}{(\lambda^\alpha - 1)c_6}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \\ &\leq r^{\alpha/2}\frac{(c_7 + C_*)\lambda^\alpha}{(\lambda^\alpha - (\lambda/2)^\alpha)c_6}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right) \\ &= r^{\alpha/2}\frac{c_7 + C_*}{(1 - 2^{-\alpha})c_6}\mathbb{P}_x\left(Y_{\tau_{D\cap B(0,\lambda^{-1}r)}} \in V \cap A(\lambda^{-1}r, 2^{-1}r)\right).\end{aligned}$$

Thus, we have proved (2.12). □

3. Preliminary lower bound estimates

In this section, we discuss a preliminary lower bound for $p_D(t, x, y)$.

Note that conditions in [7] are even weaker than conditions in [8]. Thus Y satisfies conditions imposed in [7]. Using [7, Theorem 1.4 and Lemma 2.5], the proof of the next lemma is the same as that of [12, Lemma 3.1]. Thus, we omit the proof.

Lemma 3.1. *Let T, a, and b be positive constants. For any $\beta \in [0, \infty]$, there exists a constant $c = c(a, b, \beta, T) > 0$ such that for all $\lambda \in (0, T]$ we have*

$$\inf_{\substack{y\in\mathbb{R}^d \\ |y-z|\leq b\lambda^{1/\alpha}}} \mathbb{P}_y\left(\tau_{B(z,2b\lambda^{1/\alpha})} > a\lambda\right) \geq c.$$

Next, we give some preliminary lower bound estimates for $p_D(t, x, y)$ on $\delta_D(x)\wedge\delta_D(y)\wedge T \geq t^{1/\alpha}$, which are used to derive the sharp two-sided estimates for $p_D(t, x, y)$. We first consider D an arbitrary nonempty open set, and we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$. This convention allows us to derive the lower bound of $p(t, x, y)$ simultaneously.

Using [7, Theorem 1.4] and Lemma 3.1, the proof of the next lemma is the same as that of [12, Proposition 3.2]. Thus, we omit the proof.

Proposition 3.2. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$, with $\delta_D(x) \geq at^{1/\alpha} \geq 2|x - y|$. Then, for any $\beta \in [0, \infty]$, there exists a positive constant $c = c(a, \beta, T)$ such that $p_D(t, x, y) \geq c\, t^{-d/\alpha}$.*

Proposition 3.3. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t, x, y) \in (0, T] \times D \times D$, with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$ and $at^{1/\alpha} \leq 2|x - y|$. Then, for any $\beta \in [0, \infty]$, there exists a constant $c = c(a, \beta, T) > 0$ such that $p_D(t, x, y) \geq ctj(|x - y|)$.*

Proof. By Lemma 3.1, starting at $z \in B(y, 4^{-1}at^{1/\alpha})$, with probability at least $c_1 = c_1(a, \beta, T) > 0$ the process Y does not move more than $6^{-1}at^{1/\alpha}$ by time t. Thus, by the strong Markov property

$$\mathbb{P}_x\left(Y_t^D \in B\big(y,\, 2^{-1}at^{1/\alpha}\big)\right) \geq c_1\mathbb{P}_x\left(Y^D \text{ hits the ball } B(y,\, 4^{-1}at^{1/\alpha}) \text{ by time } t\right).$$

Now using this and the Lévy system in (1.8), we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_t^D \in B\big(y,\, 2^{-1}at^{1/\alpha}\big)\right)\\
&\geq c_1\mathbb{P}_x(Y^D_{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}} \in B(y,\, 4^{-1}at^{1/\alpha}) \text{ and } t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})} \text{ is a jumping time})\\
&= c_1\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(y,\,4^{-1}at^{1/\alpha})} J(Y_s,u)du ds\right].
\end{aligned} \tag{3.1}$$

Lemma 3.1 also implies that

$$\mathbb{E}_x\left[t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}\right] \geq t\,\mathbb{P}_x\left(\tau_{B(x,6^{-1}at^{1/\alpha})}\geq t\right) \geq c_2\, t \quad \text{for all } t\in(0,T]. \tag{3.2}$$

We fix the point w on the line connecting $|x-y|$ (i.e., $|x-y| = |x-w| + |w-y|$) such that $|w-y| = 7\cdot 2^{-5}at^{1/\alpha}$, which is possible because $\delta_D(y)\geq at^{1/\alpha}$. Then, $B(w,2^{-5}at^{1/\alpha})\subset B(y,\,4^{-1}at^{1/\alpha})$. Moreover, for every $(z,u)\in B(x,6^{-1}at^{1/\alpha})\times B(w,2^{-5}at^{1/\alpha})$ we have

$$\begin{aligned}
|z-u| &< 6^{-1}at^{1/\alpha} + 2^{-5}at^{1/\alpha} + |x-w|\\
&= |x-y| + (6^{-1}+2^{-5}-7\cdot 2^{-5})at^{1/\alpha} < |x-y|.
\end{aligned}$$

Thus, $B(w,2^{-5}at^{1/\alpha})\subset B(y,\,4^{-1}at^{1/\alpha})\cap\{u : |u-z| < |x-y|\}$. Combining this result with (1.4) and (3.2), we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(y,\,4^{-1}at^{1/\alpha})} J(Y_s,u)du ds\right]\\
&\geq \mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}}\int_{B(w,2^{-5}at^{1/\alpha})} J(Y_s,u)\mathbf{1}_{\{|Y_s-u|<|x-y|\}} du ds\right]\\
&\geq c_3\mathbb{E}_x\left[t\wedge\tau_{B(x,6^{-1}at^{1/\alpha})}\right]|B(w,2^{-5}at^{1/\alpha})|j(|x-y|) > c_4 t^{1+d/\alpha} j(|x-y|).
\end{aligned} \tag{3.3}$$

Then, using the semigroup property along with (3.3) and Proposition 3.2, the proposition follows from the proof of [12, Proposition 3.4]. □

Combining Propositions 3.2 and 3.3 with the definition of j, we obtain a lower bound for $p_D(t,x,y)$ that yields the preliminary lower bound for $p_D(t,x,y)$ and $p(t,x,y)$ for the case $\beta\in[0,1]$ and the case $\beta\in(1,\infty]$ with $|x-y|<1$.

Proposition 3.4. *Let D be an arbitrary open set and let a and T be positive constants. Suppose that $(t,x,y)\in(0,T]\times D\times D$, with $\delta_D(x)\wedge\delta_D(y)\geq at^{1/\alpha}$. Then, for any $\beta\in[0,\infty]$, there exists a positive constant $c = c(a,\beta,T)$ such that*

$$p_D(t,x,y) \geq c\left(t^{-d/\alpha}\wedge tj(|x-y|)\right).$$

We next consider cases $\beta\in(1,\infty]$ with $|x-y|\geq 1$. We will closely follow the proofs of [6, Theorem 3.6] and [8, Theorem 5.5].

For the remainder of this section, we assume that D is an open set with the following property: there exist $\lambda_1\in[1,\infty)$ and $\lambda_2\in(0,1]$ such that for every $r\leq 1$ and x,y in a same component of D with $\delta_D(x)\wedge\delta_D(y)\geq r$ there exists in D a length parameterized rectifiable curve l connecting x to y with the length $|l|$ of l less than or equal to $\lambda_1|x-y|$ and $\delta_D(l(u))\geq\lambda_2 r$ for $u\in[0,|l|]$.

Under this assumption, we prove the preliminary lower bound of $p_D(t,x,y)$ on $|x-y|\geq 1$ separately for the case $\beta=\infty$ and the case $\beta\in(1,\infty)$.

Proposition 3.5. *Suppose that $T > 0, a \in (0, 4^{-1}T^{-1/\alpha}]$, and $\beta = \infty$. Then, there exist constants $c_i = c_i(a, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$, such that for any x, y in a same component of D with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$, $|x-y| \geq 1$, and $t \leq T$ we have*

$$p_D(t, x, y) \geq c_1 \left(\frac{t}{T|x-y|} \right)^{c_2|x-y|}.$$

Proof. We fix $T > 0$ and $a \in (0, 4^{-1}T^{-1/\alpha}]$, and we let $R_1 := |x-y| \geq 1$. By our assumption for D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 R_1$ and $\delta_D(l(u)) \geq \lambda_2 a t^{1/\alpha}$ for every $u \in [0, |l|]$. We define k as the integer satisfying $(4 \leq)4\lambda_1 R_1 \leq k < 4\lambda_1 R_1 + 1 < 5\lambda_1 R_1$ and $r_t := 2^{-1}\lambda_2 a t^{1/\alpha} \leq 8^{-1}$. Let $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, with $i = 0, 1, 2, \ldots, k$. Then, $\delta_D(x_i) > 2r_t$ and $B_i = B(x_i, r_t) \subset B(x_i, 2r_t) \subset D$, with $i = 0, 1, 2, \ldots, k$.

Since $4\lambda_1 R_1 \leq k$, for each $y_i \in B_i$ we have

$$\begin{aligned} |y_i - y_{i+1}| &\leq |y_i - x_i| + |x_i - x_{i+1}| + |x_{i+1} - y_{i+1}| \\ &\leq \frac{1}{8} + \frac{|l|}{k} + \frac{1}{8} < \frac{\lambda_1 R_1}{4\lambda_1 R_1} + \frac{1}{4} \leq \frac{1}{2}. \end{aligned} \tag{3.4}$$

Moreover, $\delta_D(y_i) \geq \delta_D(x_i) - |y_i - x_i| > r_t > r_{t/k}$.

Thus, by Proposition 3.4 and (3.4), there are constants $c_i = c_i(a, T, \lambda_2) > 0$, $i = 1, 2$, such that for $(y_i, y_{i+1}) \in B_i \times B_{i+1}$ we have

$$p_D(t/k, y_i, y_{i+1}) \geq c_1 \left((t/k)^{-d/\alpha} \wedge \frac{t/k}{|y_i - y_{i+1}|^{d+\alpha}} \right) \geq c_2\, t/(Tk). \tag{3.5}$$

Observe that $4\lambda_1 R_1 \leq k < 2(k-1) < 8\lambda_1 R_1$ and $r_t \geq T^{1/\alpha} r_{t/(Tk)}$. Thus, from (3.5) we obtain

$$\begin{aligned} p_D(t, x, y) &\geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k, x, y_1) \cdots p_D(t/k, y_{k-1}, y) dy_{k-1} \cdots dy_1 \\ &\geq (c_2 t(Tk)^{-1})^k \Pi_{i=1}^{k-1} |B_i| \geq c_3 (c_4 t(Tk)^{-1})^{c_5 k} \geq c_6 (c_7 t(TR_1)^{-1})^{c_8 R_1} \\ &\geq c_9 (t(TR_1)^{-1})^{c_{10} R_1}. \quad \square \end{aligned}$$

Proposition 3.6. *Suppose that $T > 0, a \in (0, 4^{-1}T^{-1/\alpha}]$, and $\beta \in (1, \infty)$. Then, there exist constants $c_i = c_i(a, \beta, T, \lambda_1, \lambda_2) > 0$, $i = 1, 2$ such that for any x, y in a same component of D with $\delta_D(x) \wedge \delta_D(y) \geq at^{1/\alpha}$, $|x-y| \geq 1$, and $t \leq T$ we have*

$$p_D(t, x, y) \geq c_1 t \exp\left\{ -c_2 \left(|x-y| \left(\log \frac{T|x-y|}{t} \right)^{\frac{\beta-1}{\beta}} \wedge (|x-y|)^\beta \right) \right\}.$$

Proof. We fix $T > 0$ and $a \in (0, 4^{-1}T^{-1/\alpha}]$, and we let $R_1 := |x-y|$. If either $1 \leq R_1 \leq 2$ or $R_1(\log(TR_1/t))^{(\beta-1)/\beta} \geq (R_1)^\beta$, the proposition holds by virtue of Proposition 3.4. Thus, for the remainder of this proof we assume that $R_1 > 2$ and $R_1(\log(TR_1/t))^{(\beta-1)/\beta} < (R_1)^\beta$, which is equivalent to $R_1 \exp\{-(R_1)^\beta\} < t/T$.

Let $k \geq 2$ be a positive integer such that

$$1 < R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta} \leq k < R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta} + 1 < 2R_1 \left(\log \frac{TR_1}{t} \right)^{-1/\beta}. \tag{3.6}$$

By our assumption for D, there is a length parameterized curve $l \subset D$ connecting x and y such that the total length $|l|$ of l is less than or equal to $\lambda_1 R_1$ and $\delta_D(l(u)) \geq \lambda_2 a t^{1/\alpha}$ for every $u \in [0, |l|]$. We define $r_t := (2^{-1}\lambda_2 a t^{1/\alpha}) \wedge ((6\lambda_1)^{-1}(\log(TR_1/t))^{1/\beta})$. Then, by (3.6) and the assumption $((\log(TR_1/t))^{1/\beta}) \vee 2 < R_1$ we have

$$\left(\frac{\lambda_2}{2} a T^{1/\alpha}\left(\frac{t}{TR_1}\right)^{1/\alpha}\right) \wedge \left(\frac{(2\log 2)^{1/\beta}}{6\lambda_1}\left(\frac{t}{TR_1}\right)^{1/\beta}\right)$$
$$\leq r_t \leq \frac{1}{6\lambda_1}\left(\log\frac{TR_1}{t}\right)^{1/\beta} < \frac{R_1}{3\lambda_1 k}. \tag{3.7}$$

We define $x_i := l(i|l|/k)$ and $B_i := B(x_i, r_t)$, with $i = 0, \dots, k$. Then, $\delta_D(y_i) \geq 2^{-1}\lambda_2 a t^{1/\alpha} > 2^{-1}\lambda_2 a (t/k)^{1/\alpha}$ for every $y_i \in B_i$. Note that from (3.7) we obtain

$$|y_i - y_{i+1}| \leq |x_i - x_{i+1}| + 2r_t \leq \left(\lambda_1 + \frac{2}{3\lambda_1}\right)\frac{R_1}{k}. \tag{3.8}$$

Thus, using Proposition 3.4 along with (3.6) and (3.8) we obtain

$$\begin{aligned}
p_D(t/k, y_i, y_{i+1}) &\geq c_1\left((t/k)^{-d/\alpha} \wedge \frac{t}{k} j(|y_i - y_{i+1}|)\right)\\
&\geq c_2\left(1 \wedge \left(\frac{t}{k}(R_1/k)^{-d-\alpha} e^{-c_3(R_1/k)^\beta}\right)\right)\\
&\geq c_4 \frac{t}{TR_1}\left(\frac{k}{R_1}\right)^{d+\alpha-1} e^{-c_3(R_1/k)^\beta}\\
&\geq c_4 \frac{t}{TR_1}\left(\log\frac{TR_1}{t}\right)^{-\frac{d+\alpha-1}{\beta}}\left(\frac{t}{TR_1}\right)^{c_3} \geq c_4\left(\frac{t}{TR_1}\right)^{c_5}.
\end{aligned} \tag{3.9}$$

Since the lower bound of r_t in (3.7) yields $r_t \geq c_6(t/(TR_1))^{(\alpha\wedge\beta)^{-1}}$, by using (3.9) and the semigroup property we conclude that

$$\begin{aligned}
p_D(t, x, y) &\geq \int_{B_1} \cdots \int_{B_{k-1}} p_D(t/k, x, y_1) \cdots p_D(t/k, y_{k-1}, y) dy_1 \cdots dy_{k-1}\\
&\geq c_7 \exp\{-c_8 k \log(TR_1/t)\}\\
&\geq c_7 \exp\left\{-c_8\left(R_1 \log\left(\frac{TR_1}{t}\right)^{-1/\beta} + 1\right)\log\frac{TR_1}{t}\right\}\\
&\geq c_7 \exp\left\{-c_9\left(R_1 \log\left(\frac{TR_1}{t}\right)^{1-1/\beta}\right)\right\}. \quad \square
\end{aligned}$$

Proof of the lower bound in (1.10). The proof for the two cases $\beta \in [0, 1]$ and $\beta \in (1, \infty]$ with $|x - y| < 1$ follows from Proposition 3.4 with $D = \mathbb{R}^d$. The proof for the remaining cases follows from Propositions 3.5 and 3.6 with $D = \mathbb{R}^d$. $\square$

4. Upper bound estimates

In this section, we derive the upper bound estimate for $p_D(t, x, y)$ as stated in Theorem 1.2. We first introduce a lemma that appears in [13]. The proof of the next lemma is identical to that of [13, Lemma 3.1], so we omit the proof.

Lemma 4.1. *Suppose that U_1, U_3, E are open subsets of $\mathbb{R}^d$, with $U_1, U_3 \subset E$ and $dist(U_1, U_3) > 0$. Let $U_2 := E\setminus(U_1 \cup U_3)$. If $x \in U_1$ and $y \in U_3$, then for every $t > 0$ we have*

$$\begin{aligned} p_E(t,x,y) &\le \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p_E(s,z,y) \\ &\quad + \int_0^t \mathbb{P}_x\left(\tau_{U_1} > s\right)\mathbb{P}_y\left(\tau_E > t-s\right) ds \cdot \sup_{u\in U_1, z\in U_3} J(u,z) \qquad (4.1)\\ &\le \mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \cdot \sup_{s<t, z\in U_2} p(s,z,y) \\ &\quad + \left(t \wedge \mathbb{E}_x[\tau_{U_1}]\right) \cdot \sup_{u\in U_1, z\in U_3} J(u,z). \qquad (4.2)\end{aligned}$$

For the remainder of this section we assume that $\eta \in (\alpha/2, 1]$, $T > 0$, and D is a $C^{1,\eta}$ open set with characteristics (R, Λ). Without loss of generality, we assume that $\Lambda > 1$ and $R < 10^{-1}$. Recall that b_1 is the constant in Theorem 2.6. We let

$$a_T = a_{T,R} := 2^{-1} b_1 R T^{-1/\alpha} < (200)^{-1} T^{-1/\alpha},$$

and for $x \in D$ we use z_x to denote a point on ∂D such that $|z_x - x| = \delta_D(x)$.

We first obtain the upper bound for the survival probability. Recall that Ψ is defined in (1.11).

Lemma 4.2. *There exists a positive constant $c = c(\beta, R, \Lambda, \eta, \rho, T)$ such that for any $(t, x) \in (0, T] \times D$ we have $\mathbb{P}_x(\tau_D > t) \le c\Psi(t, x)$.*

Proof. By the definition of Ψ, we need to prove the lemma only for $\delta_D(x) \le a_T t^{1/\alpha}/8$. Let $U := D \cap B(z_x, a_T t^{1/\alpha})$. Since $\mathbb{P}_x(\tau_D > t) \le \mathbb{P}_x(\tau_U > t) + \mathbb{P}_x(X_{\tau_U} \in D)$, by Chebyshev's inequality, Lemma 2.3, and (2.11) we have $\mathbb{P}_x(\tau_D > t) \le t^{-1}\mathbb{E}_x[\tau_U] + c_1(a_T t^{1/\alpha})^{-\alpha}\mathbb{E}_x[\tau_U] \le c_2\delta_D(x)^{\alpha/2}/\sqrt{t} \le c_3\Psi(t, x)$. $\square$

Next, we use (4.2) to obtain the intermediate upper bound in which one boundary decay appears.

Proposition 4.3. *For any $a \le a_T$ and $\beta \in [0, \infty]$, there exists a positive constant $c = c(\beta, R, \Lambda, T, \eta, \rho, a)$ such that for every $(t, x, y) \in (0, T] \times D \times D$ with $|x - y| \ge 12at^{1/\alpha}\mathbf{1}_{\beta\in[0,1]} + 2 \cdot \mathbf{1}_{\beta\in(1,\infty)} + 2(1 + 2at^{1/\alpha}) \cdot \mathbf{1}_{\beta=\infty}$ we have*

$$p_D(t,x,y) \le c\Psi(t,x) \cdot \begin{cases} h_{C_1\wedge\gamma_1,\gamma_1,T}(t, |x-y|/3) & \text{if } \beta \in [0,\infty), \\ h_{C_1,\gamma_1,T}(t, |x-y|/2) & \text{if } \beta = \infty, \end{cases}$$

where C_1 is the constant in Theorem 1.1 and γ_1 is the constant in (1.2).

Proof. By virtue of Theorem 1.1 and the fact that $r \to h_{a,\gamma,T}(t, r)$ is decreasing, the theorem holds for $\delta_D(x) \ge at^{1/\alpha}/2$.

We now fix $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) < at^{1/\alpha}/2$ and $|x-y| \ge 12at^{1/\alpha}\mathbf{1}_{\beta\in[0,1]} + 2 \cdot \mathbf{1}_{\beta\in(1,\infty)} + 2(1 + 2at^{1/\alpha}) \cdot \mathbf{1}_{\beta=\infty}$, and we define $r_t := at^{1/\alpha}$. Let $U_1 := B(z_x, r_t) \cap D$, $U_3 := \{z \in D : |z - x| > |x - y|/2\}$, and $U_2 := D\setminus(U_1 \cup U_3)$. Then, $x \in U_1$ and $y \in U_3$. For $z \in U_2$, $|x - y|/2 \le |x - y| - |x - z| \le |z - y|$. Thus, by virtue of Theorem 1.1, we have

$$\begin{aligned}\sup_{s<t, z\in U_2} p(s,z,y) &\le c_0 \sup_{s<t, |z-y|>|x-y|/2} h_{C_1,\gamma_1,T}(s, |z-y|) \\ &\le c_1\left(1 \vee (6a)^{-d-\alpha}\right) h_{C_1,\gamma_1,T}(t, |x-y|/2).\end{aligned}$$

In fact, if $\beta \in (1, \infty]$, we have $|z - y| \geq |x - y|/2 > 1$ and so $h_{C_1,\gamma_1,T}(s, |z - y|)$ is increasing in s. Also, if $\beta \in [0, 1]$, we have $|z - y| \geq |x - y|/2 \geq 6at^{1/\alpha}$ and $sr^{-\alpha-d}e^{-\gamma r^\beta}$ is increasing in s. Thus, combining these observations with the fact $r \to h_{C_1,\gamma_1,T}(t, r)$ is decreasing, the second inequality above holds.

Moreover, from Lemma 2.3 and (2.11) in Theorem 2.6 we obtain

$$\mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right) \leq \mathbb{P}_x\left(Y_{\tau_{U_1}} \in D\right) \leq c_2 t^{-1}\mathbb{E}_x[\tau_{U_1}] \leq c_3\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}. \tag{4.3}$$

Hence, the first part of (4.2) in Lemma 4.1 is bounded as follows:

$$\mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right)\left(\sup_{s<t, z\in U_2} p(s, z, y)\right) \leq c_4\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} h_{C_1,\gamma_1,T}(t, |x - y|/2). \tag{4.4}$$

If $\beta \in [0, \infty)$, since $|x - y| \geq 12at^{1/\alpha}$ we have for $u \in U_1$ and $z \in U_3$ that

$$|u - z| \geq |z - x| - |x - z_x| - |u - z_x| > |x - y|/2 - 2at^{1/\alpha} \geq |x - y|/3. \tag{4.5}$$

Then, from (1.2)–(1.4), (1.6) and (2.11) we obtain

$$\begin{aligned}\mathbb{E}_x[\tau_{U_1}]\Big(\sup_{u\in U_1, z\in U_3} J(u, z)\Big) &\leq c_5\sqrt{t}\delta_D(x)^{\alpha/2}\frac{e^{-\gamma_1(|x-y|/3)^\beta}}{|x - y|^{d+\alpha}}\\ &\leq c_6\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} h_{\gamma_1,\gamma_1,T}(t, |x - y|/3).\end{aligned} \tag{4.6}$$

If $\beta = \infty$, since $|u - z| \geq |x - y|/2 - 2at^{1/\alpha} \geq 1$ we have $J(u, z) = 0$ on $U_1 \times U_3$. Hence, by applying (4.4) and (4.6) to (4.2) for the case $\beta \in [0, \infty)$ and applying (4.4) to (4.2) for the case $\beta = \infty$, we reach the conclusion. $\square$

For notational convenience, we denote by X the process Y in the case $\beta = 0$, and we let $J^X(x, y) := \kappa(x, y)|x - y|^{-d-\alpha}$ be its jumping kernel. By Meyer's construction (e.g., see [15, Section 4.1]), when $\beta \in (0, \infty]$ the process Y can be constructed from X by removing jumps of size greater than 1 with suitable rate. Let $p_D^X(t, x, y)$ be the transition density function of X on D. For $\beta \in (0, \infty]$, we define

$$\mathcal{J}(x) := \int_{\mathbb{R}^d} \kappa(x, y)|x - y|^{-(d+\alpha)}\left(1 - \psi_1(|x - y|)^{-1}\right) dy,$$

where $\psi_1(|x - y|)$ is defined in (1.2). Then, $\|\mathcal{J}\|_\infty \leq c_1\int_{|z|\geq 1}|z|^{-(d+\alpha)}dz < \infty$. By [1, Lemma 3.6] we have

$$p_D(t, x, y) \leq e^{T\|\mathcal{J}\|_\infty} p_D^X(t, x, y) \quad \text{for any } (t, x, y) \in (0, T] \times D \times D. \tag{4.7}$$

(4.7) and the upper bound of $p_D^X(t, x, y)$, which is given next, imply the upper bound of $p_D(t, x, y)$ for $|x - y| < M$ for some $M > 0$.

Proposition 4.4. *There exists a positive constant $c = c(R, \Lambda, \eta, \rho, T)$ such that for any $(t, x, y) \in (0, T] \times D \times D$ we have*

$$p_D^X(t, x, y) \leq c\,\Psi(t, x)\,\Psi(t, y)\left(t^{-d/\alpha} \wedge t|x - y|^{-\alpha-d}\right).$$

Proof. The semigroup property, Theorem 1.1 (for $\beta = 0$), and Lemma 4.2 yield

$$\begin{aligned} p_D^X(t/2, x, y) &\le \left(\sup_{z,w \in D} p_D^X(t/4, z, w) \right) \int_D p_D^X(t/4, x, z) dz \\ &\le c_1 t^{-d/\alpha} \mathbb{P}_x(\tau_D > t/4) \le c_2 t^{-d/\alpha} \Psi(t, x). \end{aligned}$$

Thus, by Proposition 4.3 and Theorem 1.1 (for $\beta = 0$), we obtain

$$p_D^X(t/2, x, y) \le c_3 \Psi(t, x) \cdot \left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d} \right) \le c_4 \Psi(t, x) p^X(t/2, x, y).$$

Combining this with Theorem 1.1 (for $\beta = 0$), we conclude that

$$\begin{aligned} p_D^X(t, x, y) &= \int_D p_D^X(t/2, x, z) \cdot p_D^X(t/2, z, y) dz \\ &\le c_3^2 \Psi(t, x) \Psi(t, y) \int_{\mathbb{R}^d} p^X(t/2, x, z) p^X(t/2, z, y) dz \\ &= c_4^2 \Psi(t, x) \Psi(t, y) p^X(t, x, y) \\ &\le c_5 \Psi(t, x) \Psi(t, y) \left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d} \right). \quad \square \end{aligned}$$

Combining Propositions 4.3 and 4.4, we have the following proposition.

Proposition 4.5. *There exists a positive constant $c = c(\beta, R, \Lambda, T, \eta, \rho)$ such that for every $(t, x, y) \in (0, T] \times D \times D$ we have*

$$p_D(t, x, y) \le c \Psi(t, x) \cdot \begin{cases} h_{C_1 \wedge \gamma_1, \gamma_1, T}(t, |x-y|/3) & \text{if } \beta \in [0, \infty), \\ h_{C_1, \gamma_1, T}(t, |x-y|/2) & \text{if } \beta = \infty, \end{cases}$$

where C_1 is the constant in Theorem 1.1 and γ_1 is the constant in (1.2).

Next, we provide the upper bound estimates for $p_D(t, x, y)$ in the case $\beta \in (0, \infty]$.

Proof of Theorem 1.2(1). We let $r_t := a_T t^{1/\alpha}$. By Proposition 4.5 and the symmetry of $p_D(t, x, y)$, we may assume that $\delta_D(x) \vee \delta_D(y) < r_t$.

If $\beta = \infty$ and $6 < |x-y| \le 6(1 \vee C_1^{-1})$, then by (4.7) and Proposition 4.4 we have

$$p_D(t, x, y) \le c_1 \Psi(t, x) \Psi(t, y)(t/T) \le c_1 \Psi(t, x) \Psi(t, y)(t/T)^{C_1|x-y|/6}.$$

If $\beta \in [0, \infty)$ and $|x-y| \le 6(1 \vee C_1^{-1})$ or $\beta = \infty$ and $|x-y| \le 6$, by (4.7) and Proposition 4.4, we have

$$p_D(t, x, y) \le e^{T\|\mathcal{J}\|_\infty} p_D^X(t, x, y) \le c_2 \Psi(t, x) \Psi(t, y) \left(t^{-d/\alpha} \wedge t|x-y|^{-\alpha-d} \right).$$

Thus, the theorem holds for $|x-y| \le 6(1 \vee C_1^{-1})$.

For the remainder of the proof, we assume that $\delta_D(x) \vee \delta_D(y) < r_t$ and $|x-y| > 6(1 \vee C_1^{-1})$. For any x with $\delta_D(x) < r_t$, let $z_x \in \partial D$ such that $\delta_D(x) = |z_x - x|$. Let $U_1 := B(z_x, r_t) \cap D$, $U_3 := \{z \in D : |z-x| > |x-y|/2\}$, and $U_2 := D \backslash (U_1 \cup U_3)$. Note that $x \in U_1$ and $y \in U_3$ and $|x-y|/2 \le |z-y|$ for $z \in U_2$. Thus, by Proposition 4.5

we have

$$\begin{aligned}
&\sup_{s<t, z\in U_2} p_D(s,z,y) \\
&\le \sup_{s<t, z\in U_2} c_4 \frac{\delta_D(y)^{\alpha/2}}{\sqrt{s}} \cdot \big(h_{C_1\wedge\gamma_1,\gamma_1,T}(s,|z-y|/3)\cdot \mathbf{1}_{\beta\in[0,\infty)} \\
&\quad + h_{C_1,\gamma_1,T}(s,|z-y|/2)\cdot \mathbf{1}_{\beta=\infty}\big) \\
&\le c_4\, \delta_D(y)^{\alpha/2} \sup_{s<t, |x-y|/2\le|z-y|} \frac{1}{\sqrt{s}} \cdot \big(h_{C_1\wedge\gamma_1,\gamma_1,T}(s,|z-y|/3)\cdot \mathbf{1}_{\beta\in[0,\infty)} \\
&\quad + h_{C_1,\gamma_1,T}(s,|z-y|/2)\cdot \mathbf{1}_{\beta=\infty}\big) \\
&\le c_5\, \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \cdot \big(h_{C_1\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6)\cdot \mathbf{1}_{\beta\in[0,\infty)} \\
&\quad + h_{C_1,\gamma_1,T}(t,|x-y|/4)\cdot \mathbf{1}_{\beta=\infty}\big).
\end{aligned} \tag{4.8}$$

The last inequality is clear for $\beta \in [0,\infty)$ by definition of $h_{a,\gamma,T}$, and for $\beta = \infty$ we used the fact that $s \to s^{-1/2}(s/Tr)^{ar}$ is increasing if $ar \ge 1$. Hence, from (4.3) and (4.8) we obtain

$$\begin{aligned}
&\mathbb{P}_x\left(Y_{\tau_{U_1}} \in U_2\right)\Big(\sup_{s<t, z\in U_2} p_D(s,z,y)\Big) \\
&\le c_6 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \cdot \begin{cases} h_{C_1\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta\in[0,\infty), \\ h_{C_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta=\infty. \end{cases}
\end{aligned} \tag{4.9}$$

However, by Lemma 4.2 we have

$$\begin{aligned}
\int_0^t \mathbb{P}_x(\tau_{U_1}>s)\mathbb{P}_y(\tau_D>t-s)ds &\le \int_0^t \mathbb{P}_x(\tau_D>s)\mathbb{P}_y(\tau_D>t-s)ds \\
&\le c_7\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\int_0^t s^{-1/2}(t-s)^{-1/2}ds \\
&\le c_8\, t\, \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}.
\end{aligned} \tag{4.10}$$

For $\beta \in [0,\infty)$, we have $|u-z| \ge |x-y|/3$ for $(u,z) \in U_1 \times U_3$ as in (4.5). Thus, from (1.2)–(1.4) and (4.10) we obtain

$$\begin{aligned}
&\int_0^t \mathbb{P}_x\left(\tau_{U_1}>s\right)\mathbb{P}_y\left(\tau_D>t-s\right)ds \cdot \Big(\sup_{u\in U_1, z\in U_3} J(u,z)\Big) \\
&\le c_9 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\, t\, \frac{e^{-\gamma_1(|x-y|/3)^\beta}}{|x-y|^{d+\alpha}} \\
&\le c_9 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} h_{\gamma_1,\gamma_1,T}(t,|x-y|/3).
\end{aligned} \tag{4.11}$$

If $\beta = \infty$, since $|u-z| > 1$, $J(u,z) = 0$ on $U_1 \times U_3$. Therefore, by applying (4.9) and (4.11) in (4.1) of Lemma 4.1 for $\beta \in [0,\infty)$ and applying (4.9) for $\beta = \infty$, we prove the theorem for $|x-y| > 6(1 \vee C_1^{-1})$ and $\delta_D(x) \vee \delta_D(y) < r_t$. □

5. Lower bound estimates

We proved the preliminary lower bound estimates in Section 3. In this section, combining these results with the key estimate in (2.12), we give the full lower bound estimate for $p_D(t,x,y)$ with the boundary decay terms. We first introduce the next lemma.

Lemma 5.1. *Suppose that E_1, E_2, E are open subsets of $\mathbb{R}^d$ with $E_1, E_2 \subset E$ and* $\mathrm{dist}(E_1, E_2) > 0$. *If $x \in E_1$ and $y \in E_2$, then for all $t > 0$ we have*

$$p_E(t,x,y) \geq t\ \mathbb{P}_x(\tau_{E_1} > t)\ \mathbb{P}_y(\tau_{E_2} > t) \inf_{(u,w)\in E_1\times E_2} J(u,w).$$

Proof. See the proof of [11, Lemma 3.3]. □

For the remainder of this section we assume that $\eta \in (\alpha/2, 1]$, $T > 0$, and D is a $C^{1,\eta}$ open set with characteristics (R, Λ). Without loss of the generality, we assume that $\Lambda > 4$ and $R < 10^{-1}$. We let

$$\widehat{a}_T = a_{T,R} := 2^{-5} R T^{-1/\alpha} < 2^{-5} 10^{-1} T^{-1/\alpha},$$

and for $x \in D$ we use z_x to denote a point on ∂D such that $|z_x - x| = \delta_D(x)$.

The next two lemmas are crucial to obtain the lower bound on the survival probability where x is near the boundary of D.

Lemma 5.2. *For any $a \leq \widehat{a}_T$, there exists a positive constant $c = c(a, \beta, R, \Lambda, T, \eta, \rho)$ such that for every $t < T$ and $x \in D$ with $\delta_D(x) < at^{1/\alpha}$ we have*

$$\mathbb{P}_x(\tau_{B(z_x, 10at^{1/\alpha})\cap D} > t/3) \geq c\ \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.$$

Proof. Without loss of generality, we assume that $z_x = 0$. Consider a coordinate system $CS := CS_0$ such that $B(0,R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS : y_d > \phi(\widetilde{y})\}$, where ϕ is a $C^{1,\eta}$ function such that $\phi(0) = 0$, $\nabla\phi(0) = (0,\dots,0)$, $\|\nabla\phi\|_\infty \leq \Lambda$, and $|\nabla\phi(\widetilde{y}) - \nabla\phi(\widetilde{w})| \leq \Lambda|\widetilde{y} - \widetilde{w}|^\eta$.

Let $\psi(\widetilde{y}) = 2\Lambda|\widetilde{y}|$ and $V := \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS : y_d > \psi(\widetilde{y})\}$. Then, since $\psi(\widetilde{y}) \geq 2\Lambda|\widetilde{y}|^{\eta+1}$, the mean value theorem yields $\{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS : y_d > \psi(\widetilde{y})\} \subset B(0,R) \cap D$.

Let $U_1 := B(0, 2at^{1/\alpha}) \cap D$, $U_2 := B(0, 10at^{1/\alpha}) \cap D$, and

$$W := \{y = (\widetilde{y}, y_d) \in B(0, 8at^{1/\alpha}) \setminus B(0, 2at^{1/\alpha}) \text{ in } CS : y_d > \psi(\widetilde{y})\}.$$

Since $\Lambda|\widetilde{w}| = \psi(\widetilde{w})/2 < w_d/2$ for $w \in W$, we have

$$\delta_D(w) > (1+\Lambda)^{-1}(w_d - \phi(\widetilde{w})) > (1+\Lambda)^{-1}(w_d - \Lambda|\widetilde{w}|) > 2^{-1}(1+\Lambda)^{-1} w_d. \quad (5.1)$$

Moreover, since $|\widetilde{w}| \leq (2\Lambda)^{-1}|w| \leq \Lambda^{-1} 4at^{1/\alpha} \leq at^{1/\alpha}$ for $w \in W$, we have

$$w_d^2 = |w|^2 - |\widetilde{w}|^2 \geq (2at^{1/\alpha})^2 - (at^{1/\alpha})^2 = 3(at^{1/\alpha})^2 \quad \text{for } w \in W. \quad (5.2)$$

Combining (5.1) and (5.2), we obtain $\delta_D(w) > (1+\Lambda)^{-1} at^{1/\alpha}$. Thus, $B(w, r_1 at^{1/\alpha}) \subset U_2$ for $w \in W$, where $r_1 := (1+\Lambda)^{-1}$. Hence, by virtue of the strong Markov property, Lemma 3.1, and (2.12), we have

$$\mathbb{P}_x(\tau_{U_2} > t/3) \geq \mathbb{P}_x(\tau_{U_2} > t/3, Y_{\tau_{U_1}} \in W) = \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{U_2} > t/3) : Y_{\tau_{U_1}} \in W]$$

$$\geq \mathbb{E}_x[\mathbb{P}_{Y_{\tau_{U_1}}}(\tau_{B(Y_{\tau_{U_1}}, r_1 a t^{1/\alpha})} > t/3) : Y_{\tau_{U_1}} \in W]$$
$$\geq \left(\inf_{z\in\mathbb{R}^d} \mathbb{P}_z(\tau_{B(z, r_1 a t^{1/\alpha})} > t/3)\right) \mathbb{P}_x(Y_{\tau_{U_1}} \in W) \geq c_1\, \mathbb{P}_x(Y_{\tau_{U_1}} \in W)$$
$$\geq c_2 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}. \quad \square$$

We introduce the following definition for the subsequent lemma.

Definition 5.3. Let $0 < \kappa \leq 1/2$. We say that an open set D is κ-fat if there is $R_1 > 0$ such that for all $x \in \overline{D}$ and all $r \in (0, R_1]$ there is a ball $B(A_r(x), \kappa r) \subset D \cap B(x, r)$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D.

It is clear that a $C^{1,\eta}$ open set D with characteristics (R, Λ) is always a κ-fat set whose characteristics (R_1, κ) depend only on R, Λ, and d. Hereinafter, without loss of generality, we assume that $R \leq R_1$ (by choosing R smaller if necessary) and that $A_r(x)$ is always the point $A_r(x) \in D$ in Definition 5.3 for D. Recall that Ψ is defined in (1.11).

Lemma 5.4. *For any $\beta \in [0, \infty]$, there exists a positive constant $c = c(\beta, R, \Lambda, T, \eta, \rho) > 0$ such that, for every $t < T$ and $x \in D$, we can find x_1 with $\delta_D(x_1) \geq 2^{-1}\kappa \widehat{a}_T t^{1/\alpha}$ and $|x_1 - x| \leq 6\widehat{a}_T t^{1/\alpha}$ such that*

$$\int_{B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})} p_D(t/3, x, z)dz \geq c\,\Psi(t, x).$$

Proof. For $\delta_D(x) < 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$, let $x_1 = A_{6\widehat{a}_T t^{1/\alpha}}(z_x)$. Let $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$ and $B_{z_x} := B(z_x, 5\kappa\widehat{a}_T t^{1/\alpha}) \cap D$ so that $B_{x_1} \cap B_{z_x} = \emptyset$. By Lemmas 5.1, 5.2 and 3.1,

$$\begin{aligned}\int_{B_{x_1}} p_D(t/3, x, z)dz &\geq \frac{t}{3}\int_{B_{x_1}} \mathbb{P}_x(\tau_{B_{z_x}} > t/3)\; \mathbb{P}_z(\tau_{B_{x_1}} > t/3) \cdot \inf_{(u,w)\in B_{z_x}\times B_{x_1}} J(u, w)dz \\ &\geq \frac{t}{3}\; \mathbb{P}_x(\tau_{B_{z_x}} > t/3) \cdot c_1 \int_{B_{x_1}} dz \cdot c_2 \frac{1}{(t^{1/\alpha})^{d+\alpha}} \\ &= c_3 \mathbb{P}_x(\tau_{B_{z_x}} > t/3) \geq c_4\; \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.\end{aligned}$$

For $\delta_D(x) \geq 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$, let $x_1 = x$ and $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$. By Lemma 3.1, there exists a constant $c_5 = c_5(\alpha, \beta, R, T, d, L_3) > 0$ such that

$$\int_{B_{x_1}} p_D(t/3, x, z)dz \geq \int_{B_{x_1}} p_{B_{x_1}}(t/3, x, z)dz = \mathbb{P}_x(\tau_{B_{x_1}} > t/3) > c_5.$$

This proves the lemma. $\square$

We are now ready to give the proof of the lower bound estimates for $p_D(t, x, y)$. Recall our assumption that $\eta \in (\alpha/2, 1]$ and D is a $C^{1,\eta}$ open set. For the cases $\beta \in (1, \infty)$ with $|x - y| \geq 1$ and $\beta = \infty$ with $|x - y| > 4/5$, we assume in addition that the path distance in each connected component of D is comparable to the Euclidean distance with characteristic λ_1. Note that combining this assumption with $C^{1,\eta}$ assumption entails that D satisfies the assumption made before Proposition 3.5.

Proof of Theorems 1.2(2) and 1.2(3). By Lemma 5.4, for any $x, y \in D$, there exist $x_1, y_1 \in D$ such that $\delta_D(x_1) \wedge \delta_D(y_1) \geq 2^{-1}\kappa\widehat{a}_T t^{1/\alpha}$ and $|x_1 - x| \vee |y_1 - y| \leq 6\widehat{a}_T t^{1/\alpha}$, and there exists a constant $c_1 = c_1(\eta, \rho, \beta, R, \Lambda, T) > 0$ independent of x, y such that

$$\int_{B_{x_1}} p_D(t/3, x, z)dz \int_{B_{y_1}} p_D(t/3, y, z)dz \geq c_1 \Psi(t, x)\Psi(t, y), \tag{5.3}$$

where $B_{x_1} := B(x_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$ and $B_{y_1} := B(y_1, (\kappa/4)\widehat{a}_T t^{1/\alpha})$. Thus, by the semigroup property we have

$$\begin{aligned} p_D(t, x, y) &= \int_D \int_D p_D(t/3, x, u)p_D(t/3, u, w)p_D(t/3, w, y)dudw \\ &\geq \int_{B_{x_1}} p_D(t/3, x, u)du \\ &\quad \times \int_{B_{y_1}} p_D(t/3, y, w)dw \left(\inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3, u, w) \right) \\ &\geq c_1 \Psi(t, x)\Psi(t, y) \inf_{(u,w)\in B_{x_1}\times B_{y_1}} p_D(t/3, u, w). \end{aligned} \tag{5.4}$$

We now carefully calculate the lower bounds of $p_D(t/3, u, w)$ on $B_{x_1} \times B_{y_1}$. Since $|x - x_1| \vee |y - y_1| \leq 6\widehat{a}_T t^{1/\alpha}$, for $u \in B_{x_1}$ and $w \in B_{y_1}$ we have

$$\begin{aligned} |x - y| - 20^{-1} &\leq |x - y| - (12 + (\kappa/2))\widehat{a}_T t^{1/\alpha} \\ &\leq |u - w| \leq |x - y| + (12 + (\kappa/2))\widehat{a}_T t^{1/\alpha} \leq |x - y| + 20^{-1} \end{aligned} \tag{5.5}$$

and $\delta_D(u) \wedge \delta_D(w) \geq (\kappa/4)\widehat{a}_T t^{1/\alpha}$.

If $\beta \in [0, 1]$, then by considering the cases $|x - y| \leq 15\widehat{a}_T t^{1/\alpha}$ and $|x - y| > 15\widehat{a}_T t^{1/\alpha}$ separately using Proposition 3.4 and (5.5) we obtain

$$\begin{aligned} p_D(t/3, u, w) &\geq c_2 \left(t^{-d/\alpha} \wedge t|u - w|^{-d-\alpha} e^{-\gamma_2|u-w|^\beta} \right) \\ &\geq c_3 \left(t^{-d/\alpha} \wedge t|x - y|^{-d-\alpha} e^{-\gamma_2|x-y|^\beta} \right). \end{aligned}$$

If $\beta \in (1, \infty]$ and $|x - y| \leq 4/5$, then (5.5) yields $|u - w| \leq |x - y| + 20^{-1} < 1$. Thus, by considering the cases $|x-y| \leq 15\widehat{a}_T t^{1/\alpha}$ and $|x-y| > 15\widehat{a}_T t^{1/\alpha}$ separately using Proposition 3.4 and (5.5), we have $p_D(t/3, u, w) \geq c_4 \left(t^{-d/\alpha} \wedge (t|x - y|^{-d-\alpha})\right)$.

If $\beta \in (1, \infty]$ and $4/5 \leq |x - y|$, then (5.5) yields $|u - w| \asymp |x - y|$.

We now consider $p_D(t/3, u, w)$ in each of the remaining cases.

(1) If $\beta \in (1, \infty)$ and $4/5 \leq |x - y| < 2$, then $|u - w| \asymp 1$. Thus, by Proposition 3.4, we have $p_D(t/3, u, w) \geq c_5 t$.

(2) If $\beta = \infty$ and $4/5 \leq |x - y| < 2$, then by Propositions 3.4 and 3.5 we have

$$p_D(t/3, u, w) \geq c_6 \frac{4t}{5T|x - y|} \geq c_6 \left(\frac{4t}{5T|x - y|} \right)^{5|x-y|/4}.$$

(3) If $\beta \in (1, \infty)$ and $2 \le |x-y|$, then $1 < |u-w|$ and from Proposition 3.6 and (5.5) we obtain

$$\begin{aligned} p_D(t/3, u, w) &\ge c_7 t \exp\left\{ -c_8 \left(|u-w| \left(\log \frac{T|u-w|}{t} \right)^{\frac{\beta-1}{\beta}} \wedge |u-w|^\beta \right) \right\} \\ &\ge c_7 t \exp\left\{ -c_8 \left((5|x-y|/4) \left(\log \left(\frac{T(|x-y|+20^{-1})}{t} \right) \right)^{\frac{\beta-1}{\beta}} \right.\right. \\ &\qquad \left.\left. \wedge\ (5|x-y|/4)^\beta \right) \right\} \\ &\ge c_7 t \exp\left\{ -c_9 \left(|x-y| \left(\log \frac{T|x-y|}{t} \right)^{\frac{\beta-1}{\beta}} \wedge |x-y|^\beta \right) \right\}. \end{aligned}$$

The last inequality comes from the inequality $\log r \le \log(r+b) \le 2\log r$ for $r \ge 2 \vee b$.

(4) If $\beta = \infty$ and $2 \le |x-y|$, then $1 < |u-w|$ and from Proposition 3.5 and (5.5) we have

$$\begin{aligned} p_D(t/3, u, w) &\ge c_{11} \left(\frac{t}{T|u-w|} \right)^{c_{10}|u-w|} \ge c_{11} \left(\frac{t}{T(|x-y|+20^{-1})} \right)^{c_{12}|x-y|} \\ &\ge c_{11} \left(\frac{t}{T|x-y|} \right)^{2c_{12}|x-y|} \ge c_{11} \left(\frac{4t}{5T|x-y|} \right)^{2c_{12}|x-y|}. \end{aligned}$$

The second last inequality holds by virtue of the inequality $r^2 \ge r+b$ for $r \ge 2 \vee b$.

Hence, combining (5.4) with the above observations on the lower bound of $p_D(t/3, u, w)$, we have proved Theorem 1.2(2) and 1.2(3). □

Proof of Theorem 1.2(4). Let $D(x)$ and $D(y)$ be connected components containing x and y, respectively. By definition of a $C^{1,\eta}$ open set, the distance between x and y is at least R. Using Lemma 5.4, we find that $x_1 \in D(x)$ and $y_1 \in D(y)$. We then define B_{x_1} and B_{y_1} in the same way as when beginning the proof of Theorem 1.2(2) and 1.2(3) so that (5.3) holds and for any $u \in B_{x_1}$ and $w \in B_{y_1}$ we have $3R/4 \le 3|x-y|/4 \le |u-w| \le 5|x-y|/4$. By Proposition 3.4, for every $u \in B_{x_1}$ and $w \in B_{y_1}$ we have

$$p_D(t/3, u, w) \ge c_1 \frac{t}{|u-w|^{d+\alpha}} e^{-\gamma_2 |u-w|^\beta} \ge c_2 \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma_2 (5|x-y|/4)^\beta}.$$

Therefore,

$$\begin{aligned} p_D(t, x, y) &\ge \int_{B_{x_1}} \int_{B_{y_1}} p_D(t/3, x, w) p_D(t/3, u, w) p_D(t/3, w, y) dw dv \\ &\ge \int_{B_{x_1}} p_D(t/3, x, u) du \int_{B_{y_1}} p_D(t/3, y, w) dw \cdot \inf_{(u,w) \in B_{x_1} \times B_{y_1}} p_D(t/3, u, w) \\ &\ge c_3 \Psi(t, x) \Psi(t, y) \cdot \frac{t}{|x-y|^{d+\alpha}} e^{-\gamma_2 (5|x-y|/4)^\beta}. \quad \square \end{aligned}$$

Proof of Theorem 1.2(5). Note that, since D is bounded and connected, the estimate for $p_D(t, x, y)$ at small time is the same as that obtained for a symmetric stable process in [10].

Thus, the remainder of the proof of Theorem 1.2(5) using the estimate for $p_D(t, x, y)$ at small time is routine (see [10]) and we omit it here.

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