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A unified approach to self-normalized block sampling

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Abstract

The inference procedure for the mean of a stationary time series is usually quite different under various model assumptions because the partial sum process behaves differently depending on whether the time series is short or long-range dependent, or whether it has a light or heavy-tailed marginal distribution. In the current paper, we develop an asymptotic theory for the self-normalized block sampling, and prove that the corresponding block sampling method can provide a unified inference approach for the aforementioned different situations in the sense that it does not require the *a priori* estimation of auxiliary parameters. Monte Carlo simulations are presented to illustrate its finite-sample performance. The R function implementing the method is available from the authors.

MSC: primary 62G09; secondary 60G18

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1. Introduction

Given samples $X_1, \ldots, X_n$ from a stationary process $\{X_i\}_{i\in\mathbb{Z}}$ with mean $\mu = E(X_0)$, the sample average $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$ serves as a natural estimator for the population mean μ. To conduct statistical inference on the mean μ such as hypothesis testing or the construction of

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confidence intervals, one needs an asymptotic theory on the sample average for dependent data. The development of such a theory has been an active area of research. Consider first the classical case, where by assuming certain short-range dependence conditions, one obtains the usual central limit theorem, that is,

$$n^{1/2}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2), \tag{1}$$

where $\xrightarrow{d}$ denotes the convergence in distribution, and σ^2 is the long-run variance which typically is the sum of autocovariances of all orders. The short-range dependence conditions mentioned above include, but are not limited to, the m-dependence condition of Hoeffding and Robbins [34], the strong mixing condition of Rosenblatt [64] and its variants, and the p-stability condition based on functional dependence measures of Wu [84]; see also [36,57,49,13,87] and references therein. Once one has (1), an asymptotic $100(1-\alpha)\%$ confidence interval of μ can be constructed as

$$[\bar{X}_n - n^{-1/2}\sigma q_{1-\alpha/2},\ \bar{X}_n + n^{-1/2}\sigma q_{1-\alpha/2}] \tag{2}$$

where $q_{1-\alpha/2}$ is the $(1-\alpha/2)$-th quantile of the standard normal distribution. However, the implementation of (2) requires the estimation of a nuisance parameter σ, which can itself be a challenging problem and often relies on techniques including tapering and thresholding to achieve consistency; see for example [53,25,59,90] among others.

If the process $(X_i)_{i\in\mathbb{Z}}$ is heavy-tailed (distributional tail behaving like $x^{-\alpha}$ with $\alpha \in (1, 2)$) so that the variance is infinite, one typically has

$$n^{1-1/\alpha}\ell(n)^{-1}(\bar{X}_n - \mu) \xrightarrow{d} S_\alpha(\sigma, \beta, 0), \tag{3}$$

where $\ell(n)$ is a slowly varying function satisfying $\lim_{n\to\infty} \ell(an)/\ell(n) = 1$ for any $a > 0$, and $S_\alpha(\sigma, \beta, 0)$ is the centered α-stable random variable with scale parameter $\sigma > 0$ and skewness parameter $\beta \in [-1, 1]$. We refer the reader to the monographs by Samorodnitsky and Taqqu [65], Nolan [54] and Resnick [62] for an introduction. See also [2] for examples of heavy tails from finance, signal processing, networks, etc. Here the use of (3) for constructing confidence interval as in (2) becomes more difficult due to additional unknown parameters σ, α and β, as well as the unknown $\ell(n)$.

There has been a considerable amount of research focusing on the situation where the short-range dependence condition fails, and processes with *long-range dependence* (also called "long memory" or "strong dependence") has attracted a lot of attention in various fields including econometrics, finance, hydrology and telecommunication among others; see for example [48,20,45,8]. We also refer the reader to the monographs by Doukhan et al. [23], Giraitis et al. [26] and Beran et al. [10] for an introduction. For long-range dependent processes, it may be established that

$$n^{1-H}\ell(n)^{-1}(\bar{X}_n - \mu) \xrightarrow{d} Y, \tag{4}$$

where $H \in (1/2, 1)$ is the Hurst index (or the long memory index), $\ell(n)$ is a slowly varying function, and Y is typically a random variable which can be expressed by a multiple Wiener–Itô integral and is not necessarily Gaussian. The large sample theory of the form (4) has been studied by Davydov [17], Taqqu [75], Dobrushin and Major [21], Avram and Taqqu [4], Ho and Hsing [32], Wu [85] and Bai and Taqqu [6] among others. Therefore, the asymptotic behavior of the sample average and thus the inference procedure can become very different for long-range

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dependent processes, and the convergence rate in (4) depends critically on the Hurst index H which characterizes the dependence strength. Hence, in order to apply (4) for inference, unlike the case with short-range dependence and light tail, one needs to estimate in addition the Hurst index H and possibly the slowly varying function $\ell(n)$, which can be quite nontrivial. Furthermore, the distribution of a non-Gaussian Y (which also depends on H) has not been numerically evaluated in general. For the special case of the Rosenblatt distribution where it is evaluated, see [82].

There has recently been a surge of attention in using some random normalizers to avoid, or reduce the number of nuisance parameters that need to be estimated for statistical inference. For example, McElroy and Politis [50] considered using the sample standard deviation as the normalizer for inference on the mean of heavy-tailed linear processes that satisfy the strong mixing condition; see also [63] for the use of a similar normalizer for independent observations. Lobato [46], Shao [68], Zhou and Shao [91] and Huang et al. [35] used a normalization of the type

$$D_n = \left\{ n^{-1} \sum_{k=1}^{n} \left(\sum_{i=1}^{k} X_i - \frac{k}{n} \sum_{i=1}^{n} X_i \right)^2 \right\}^{1/2} \tag{5}$$

for finite-variance short-range dependent time series. Fan [24] used the normalizer D_n for long-range dependent time series with finite variances. Results have also been obtained by McElroy and Politis [52] using a lag-window normalizer instead of D_n in (5). McElroy and Politis [51], moreover, considered the following non-centered stochastic volatility model $X_i = \mu + \sigma_i Z_i$, $i \geq 1$, where $\{\sigma_i\}$ and $\{Z_i\}$ are independent, $\{\sigma_i\}$ is i.i.d. heavy-tailed and $\{Z_i\}$ is a Gaussian process. They proposed to use a random normalizer involving two terms that account for heavy-tailedness and long memory respectively. The term in their normalizer which accounts for long memory requires the choice of an additional tuning parameter. Therefore, it seems that the specific form of the normalization depends critically on the particular time series that is being considered, and different normalizers have been used in the literature to account for the heavy-tail and/or long-range dependent characteristics of the time series.

The current paper aims to provide a unified inference procedure by adopting the normalizer D_n in (5) and developing an asymptotic theory using self-normalized block sums. As observed by Shao [69], self-normalization itself is not able to fully avoid the problem of estimating the nuisance parameters, as the asymptotic distribution at least depends on the unknown Hurst index H for long-range dependent processes. In order to provide a unified approach that does not rely on the estimation of any nuisance parameter to determine the strength of dependence or heavy-tailedness, certain nonparametric techniques such as the block sampling[2] must be utilized to obtain the asymptotic quantiles. However, this requires developing an asymptotic theory on the self-normalized block sums for a general class of processes. This task may be nontrivial if we want it to include processes with long-range dependence and/or heavy-tails. Block sampling has been mainly studied in the literature in the *non-self-normalized* setting, where the normalizer converges in probability to a nonzero constant, thus simplifying the proof; see for example [29] for nonlinear transforms of Gaussian processes, Nordman and Lahiri [55] for linear processes, and Zhang et al. [89] for nonlinear transforms of linear processes. Jach et al. [37] applied block sampling to the model $X_i = \mu + \sigma_i Z_i$, $i \geq 1$, considered by McElroy and Politis [51]

[2] The following terms are used interchangeably in the literature: block sampling, subsampling, sampling window method.

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but with Z_i replaced by $g(Z_i)$ where g is a possibly nonlinear function with Hermite rank one. For more information on block sampling, see [71,43]. Betken and Wendler [11] recently obtained interesting results in the context of long-range dependence. They are briefly discussed in Section 3.2 (see (58)).

The current paper considers *self-normalized* block sums using D_n in (5) as normalizer. As observed by Fan [24], the development of an asymptotic theory in this case can be very nontrivial even for Gaussian processes. Developing a rigorous proof is stated as an open problem. The goal of this paper is to develop such a proof for nonlinear functions of Gaussian processes with either short or long-range dependence, and including heavy-tails.

The remaining of the paper is organized as follows. Section 2 introduces the self-normalized block sampling (SNBS) method, whose asymptotic theory is established in Section 3. Section 4 contains examples. Monte Carlo simulations are carried out in Section 5 to examine the finite-sample performance of the method.

2. Self-normalized block sampling

Let $X_1, \ldots, X_n$ be observations from a stationary process $(X_i)_{i\in\mathbb{Z}}$ with mean $\mu = E(X_0)$, and denote by $S_{j,k} = \sum_{i=j}^{k} X_i$, $j \le k$, its partial sums from j to k. Of particular interest is $S_{1,n} = \sum_{i=1}^{n} X_i$. We propose using the self-normalized quantity

$$T_n^* = \frac{S_{1,n} - n\mu}{D_n} \tag{6}$$

for making statistical inference on the mean μ, where D_n, defined in (5), can now be written

$$D_n = \left\{ n^{-1} \sum_{k=1}^{n} \left(S_{1,k} - \frac{k}{n} S_{1,n} \right)^2 \right\}^{1/2}. \tag{7}$$

In order to make inference on μ, we need to know the distribution $P(T_n^* \le x)$.

A first idea is to use the asymptotic distribution of (6). This would require knowing the weak limit of the normalized partial sum process, namely,

$$\{n^{-H}\ell(n)^{-1}(S_{\lfloor nt\rfloor} - n\mu),\ 0 \le t \le 1\} \Rightarrow \{Y(t),\ 0 \le t \le 1\}, \tag{8}$$

where $t \in [0,1]$, $\lfloor nt\rfloor$ denotes the largest integer not exceeding nt, and $\Rightarrow$ denotes weak convergence in Skorokhod space with suitable topology. By Lamperti [44], if (8) holds, then the process $Y(t)$ is self-similar with stationary increments, with Hurst index[3] $0 < H < 1$(H-sssi), and with $\ell(\cdot)$ a slowly varying function. Recall that a process $Y(t)$ is said to be self-similar with Hurst index H if $\{Y(ct),\ t \ge 0\}$ has the same finite-dimensional distributions as $\{c^H Y(t),\ t \ge 0\}$, for any $c > 0$.

The most important example of (8) is when $(X_i)_{i\in\mathbb{Z}}$ is short-range dependent and admits finite variance, in which case one expects

$$\{n^{-1/2}(S_{\lfloor nt\rfloor} - n\mu),\ 0 \le t \le 1\} \Rightarrow \{\sigma B(t),\ 0 \le t \le 1\}, \tag{9}$$

where $B(\cdot)$ is the standard Brownian motion, and $\sigma^2 > 0$ is the long-run variance; see for example, the invariance principle of Herrndorf [31] under strong mixing, and also the strong

[3] We exclude the degenerate case $H = 1$.

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invariance principle of Wu [86]. When $\{X_i\}$ is short-range dependent but has infinite variance with distributional tail regularly varying of order $-\alpha$ where $\alpha \in (1, 2)$, one has typically

$$\{n^{-1/\alpha}\ell(n)^{-1}(S_{\lfloor nt\rfloor} - n\mu),\ 0 \le t \le 1\} \Rightarrow \{L_{\alpha,\sigma,\beta}(t),\ 0 \le t \le 1\}, \tag{10}$$

where $L_{\alpha,\sigma,\beta}(t)$ is a centered α-stable Lévy process with scale parameter $\sigma > 0$ and skewness parameter $\beta \in [-1, 1]$. See, for example, [73,5,80,81,9] for the specification of the corresponding Skorohod topology.

Under long-range dependence, the limit in (8) can be quite complicated. A typical class of convergence in this case is

$$\{n^{-H}\ell(n)^{-1}(S_{\lfloor nt\rfloor} - n\mu),\ 0 \le t \le 1\} \Rightarrow \{cZ_{m,H}(t),\ 0 \le t \le 1\}, \tag{11}$$

where $1/2 < H < 1$, $Z_{m,H}(\cdot)$ is the mth order Hermite process which can be expressed by a multiple Wiener–Itô integral (see, e.g., [21,76]), and c is a constant depending on H, m and $\ell(n)$. A Hermite process $Z_{m,H}(\cdot)$ with $m \ge 2$ is non-Gaussian, and when $m = 1$ it is the Gaussian process called fractional Brownian motion, also denoted by $B_H(\cdot)$. One can also consider the anti-persistent case $H < 1/2$, where the limit can be more complicated than $Z_{m,H}(\cdot)$ (see [47]).

Applying the *same* normalization $n^{-H}\ell(n)^{-1}$ to both the numerator and denominator of T_n^* in (6), one can establish as in [46], via (8) and the Continuous Mapping Theorem that as $n \to \infty$,

$$T_n^* = \frac{n^{-H}\ell(n)^{-1}(S_{1,n} - n\mu)}{n^{-H}\ell(n)^{-1}\left\{n^{-1}\sum_{k=1}^{n}(S_{1,k} - \frac{k}{n}S_{1,n})^2\right\}^{1/2}} \xrightarrow{d} T := \frac{Y(1)}{D}, \tag{12}$$

with

$$D = \left[\int_0^1 \{Y(s) - sY(1)\}^2 ds\right]^{1/2}. \tag{13}$$

Note that $D > 0$ almost surely. Indeed, if $P(D = 0) > 0$, then with positive probability $Y(s) = sY(1)$, which has locally bounded variation. This cannot happen by Theorem 3.3 of Vervaat [83], since we assume $H < 1$.

In particular, in the short-range dependent case (9), one gets

$$T_n^* \xrightarrow{d} \frac{B(1)}{\left[\int_0^1 \{B(s) - sB(1)\}^2 ds\right]^{1/2}},$$

where the limit does not depend on any nuisance parameter. However, this nice property no longer holds in the other cases (10) and (11), since $Y(t)$ in either case involves additional parameters. Therefore, except for short-range dependent light-tailed processes, self-normalization itself is usually not able to fully avoid the problem of estimating the nuisance parameters, and we shall follow here Hall et al. [29] and consider a block sampling approach. See also Chapter 5

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of Politis et al. [60]. Let

$$T^*_{i,b_n} = \frac{S_{i,i+b_n-1} - b_n\mu}{\sqrt{b_n^{-1}\sum_{k=i}^{i+b_n-1}(S_{i,k} - b_n^{-1}(k-i+1)S_{i,i+b_n-1})^2}} =: \frac{S_{i,i+b_n-1} - b_n\mu}{D_{i,b_n}}, \quad 1 \le i \le n - b_n + 1, \tag{14}$$

which is the block version of T^*_n in (6) for the subsample $X_i, \ldots, X_{i+b_n-1}$, where b_n denotes the block size. Observe that there is a considerable overlap between successive blocks, since as i increases to $i+1$, the subsample becomes $X_{i+1}, \ldots, X_{i+b_n}$, and thus includes many of the same observations.

We consider using the empirical distribution function

$$\widehat{F}^*_{n,b_n}(x) = \frac{1}{n-b_n+1}\sum_{i=1}^{n-b_n+1} \mathrm{I}(T^*_{i,b_n} \le x), \tag{15}$$

where $\mathrm{I}(\cdot)$ is the indicator function, to approximate the distribution $P(T^*_n \le x)$ of T^*_n in (6). In practice, the mean μ in (14) is unknown and we shall replace it by the average $\bar{X}_n$ of the whole sample, which turns (14) into

$$T_{i,b_n} = \frac{S_{i,i+b_n-1} - b_n\bar{X}_n}{\sqrt{b_n^{-1}\sum_{k=i}^{i+b_n-1}(S_{i,k} - b_n^{-1}(k-i+1)S_{i,i+b_n-1})^2}}, \tag{16}$$

whose empirical distribution function is given by

$$\widehat{F}_{n,b_n}(x) = \frac{1}{n-b_n+1}\sum_{i=1}^{n-b_n+1} \mathrm{I}(T_{i,b_n} \le x). \tag{17}$$

The asterisk in T^*_{i,b_n} indicates that the centering involves the unknown population mean μ, in contrast to T_{i,b_n}, where the centering involves instead the sample average $\bar{X}_n$. We call the above inference procedure involving using $\widehat{F}_{n,b_n}(x)$ in (17) to approximate the distribution of T^*_n in (6), the *self-normalized block sampling (SNBS)* method. One can then construct confidence intervals or test hypotheses for the unknown population mean μ. For instance, to construct a one-sided $100(1-\alpha)\%$ confidence interval for μ, one gets first the α-th quantile q_α of the empirical distribution $\widehat{F}_{n,b_n}(x)$ in (17). Since

$$1 - \alpha \approx P(T^*_n \ge q_\alpha) = P\left(\frac{S_{1,n} - n\mu}{D_n} \ge q_\alpha\right) = P\left(\mu \le \bar{X}_n - q_\alpha D_n/n\right),$$

where D_n is defined in (7), then the $100(1-\alpha)\%$ confidence interval is constructed as

$$\left(-\infty,\ \bar{X}_n - q_\alpha D_n/n\right]. \tag{18}$$

The idea of using block sampling to approximate distributions of self-normalized quantities is not new, and it has been applied by Fan [24] and McElroy and Politis [52] to long-range dependent processes with finite variances. However, the aforementioned papers did not provide

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a full theoretical justification for their inference procedure based on block sampling, and as commented by Fan [24] such a task can be very nontrivial even for Gaussian processes and has been stated as an open problem. In addition, the aforementioned papers only considered the situation with finite variances, and therefore it has not been known whether one could unify the inference procedure for processes with long-range dependence and/or heavy-tails.

Recently, Jach et al. [37] considered this problem in the setting of stochastic volatility models where the error term can be nicely decomposed into two independent factors, with one being a function of long-range dependent Gaussian processes while the other being i.i.d. heavy-tailed.[4] But in their paper, the nonlinear function is restricted to have Hermite rank one and the choice of slowly varying functions is also greatly limited as neither $\log n$ nor $\log\log n$ are allowed. In addition, their random normalizer is specifically tailored to the aforementioned stochastic volatility model, and involves two different terms to account for the long-range dependent and heavy-tailed characteristics of the time series. Furthermore, the term in their normalizer that accounts for long-range dependence also requires the choice of an additional tuning parameter as in the estimation of the long-run variance for short-range dependent processes. We also mention that the proof of Jach et al. [37], which relies on the θ-weak dependence, does not seem to be applicable in the current setting, since using our random normalizer D_n in the denominator makes the self-normalized quantity a non-Lipschitz function of the data.

The current paper proposes to consider the use of (17) to provide a unified inference procedure without the estimation of a nuisance parameter for a wide class of processes, where the limit of the partial sum process can be a Brownian motion, an α-stable Lévy process, a Hermite process or other processes. In Section 3, we develop an asymptotic theory for the self-normalized block sums and establish the theoretical consistency of the aforementioned method, namely,

$$|\widehat{F}_{n,b_n}(x) - P(T_n^* \le x)| \to 0 \tag{19}$$

in probability as $n \to \infty$.

3. Asymptotic theory

We establish the asymptotic consistency of self-normalized block sampling for the following two classes of stationary processes: (a) nonlinear transforms of Gaussian stationary processes (called Gaussian subordination), and (b) those satisfying strong mixing conditions. The first allows for long-range dependence and non-central limits, while the second involves short-range dependent processes. Both classes allow for heavy-tails with infinite variance.

Let $D[0,1]$ be the space of càdlàg (right continuous with left limits) functions defined on $[0,1]$, endowed with Skorokhod's M_2 topology. The M_2 topology is weaker than the other topologies proposed by Skorokhod [72], in particular, weaker than the most commonly used J_1 topology. A sequence of function $x_n(t) \in D[0,1]$ converges to $x(t) \in D[0,1]$ in M_2 topology as $n \to \infty$, if and only if $\lim_n \sup_{t_1\le t\le t_2} x_n(t) = \sup_{t_1\le t\le t_2} x(t)$ and $\lim_n \inf_{t_1\le t\le t_2} x_n(t) = \inf_{t_1\le t\le t_2} x(t)$ for any t_1, t_2 at continuity points of $x(t)$ (see Statement 2.2.10 of Skorokhod [72]).

We consider the M_2 topology instead of J_1 since there are known examples in the heavy tailed case where convergence fails under J_1 but holds under M_2 (see [5,81,9]). To apply the continuous mapping argument, we need the following lemma.

[4] As noted in Section 4, we can recover the consistency result of Jach et al. [37] by replacing our normalization D_n by the one found in that paper.

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Lemma 3.1. *Integration on* $[0, 1]$ *is a continuous functional for* $D[0, 1]$ *under the* M_2 *topology.*

Proof. Suppose that $x_n(t) \to x(t)$ in the M_2 topology. For any partition $\mathcal{T} = \{0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1\}$, define $m_{i,n} = \inf_{t_{i-1} \le t \le t_i} x_n(t)$, $M_{i,n} = \sup_{t_{i-1} \le t \le t_i} x_n(t)$, $m_i = \inf_{t_{i-1} \le t \le t_i} x(t)$ and $M_i = \sup_{t_{i-1} \le t \le t_i} x(t)$, $i = 1, \ldots, k$. Note that

$$\sum_{i=1}^{k} m_{i,n}(t_i - t_{i-1}) \le \int_0^1 x_n(t)dt \le \sum_{i=1}^{k} M_{i,n}(t_i - t_{i-1}),$$
$$\sum_{i=1}^{k} m_i(t_i - t_{i-1}) \le \int_0^1 x(t)dt \le \sum_{i=1}^{k} M_i(t_i - t_{i-1}). \tag{20}$$

The function $x(t)$ is Riemann integrable since, as an element in $D[0, 1]$, it is a.e. continuous and bounded on $[0, 1]$. Riemann integrability implies that for any $\epsilon > 0$, one can choose a partition $\mathcal{T}$ so that

$$0 \le \sum_{i=1}^{k} M_i(t_i - t_{i-1}) - \sum_{i=1}^{k} m_i(t_i - t_{i-1}) < \epsilon. \tag{21}$$

Modify the partition, if necessary, so that all the t_i's are at continuity points of $x(t)$, without changing (21). This is possible since $x(t)$ has at most countable discontinuity points and is bounded. By the characterization of convergence in $D[0, 1]$ with M_2 topology, we have

$$\lim_n \sum_{i=1}^{k} m_{i,n}(t_i - t_{i-1}) = \sum_{i=1}^{k} m_i(t_i - t_{i-1}),$$
$$\lim_n \sum_{i=1}^{k} M_{i,n}(t_i - t_{i-1}) = \sum_{i=1}^{k} M_i(t_i - t_{i-1}). \tag{22}$$

Combining (20)–(22) concludes that $\limsup_n |\int_0^1 x_n(t)dt - \int_0^1 x(t)dt| \le \epsilon$. □

3.1. Results in the Gaussian subordination case

Let

$$\{\mathbf{Z}_i = (Z_{i,1}, \ldots, Z_{i,J}),\ i \in \mathbb{Z}\} \tag{23}$$

be an $\mathbb{R}^J$-valued Gaussian stationary process satisfying $\mathbb{E}Z_{i,j} = 0$ for any i, j. Define

$$\mathbf{Z}_p^q = \left(\mathbf{Z}_p, \ldots, \mathbf{Z}_q\right). \tag{24}$$

We shall view $\mathbf{Z}_p^q$ as a vector of dimension $J \times (q - p + 1)$ involving observations from time p to time q. The covariance matrix of $\mathbf{Z}_1^m$ will be written for convenience as a four-dimensional array involving i_1, i_2, j_2, j_2:

$$\Sigma_m = \left(\gamma_{j_1,j_2}(i_2 - i_1) := \mathbb{E}Z_{i_1,j_1}Z_{i_2,j_2}\right)_{1 \le i_1, i_2 \le m, 1 \le j_1, j_2 \le J}. \tag{25}$$

We assume throughout that Σ_m is non-singular for every $m \in \mathbb{Z}_+$. The cross-block covariance matrix between $\mathbf{Z}_1^m$ and $\mathbf{Z}_{k+1}^{k+m}$ is

$$\Sigma_{k,m} = \left(\gamma_{j_1,j_2}(i_2 + k - i_1) := \mathbb{E}Z_{i_1,j_1}Z_{i_2+k,j_2}\right)_{1 \le i_1, i_2 \le m, 1 \le j_1, j_2 \le J}. \tag{26}$$

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Let $\rho(\cdot,\cdot)$ denote the canonical correlation (maximum correlation coefficient) between $L^2(\Omega)$ random vectors $\mathbf{U} = (U_1, \ldots, U_p)$ and $\mathbf{V} = (V_1, \ldots, V_q)$. Let $\langle\cdot,\cdot\rangle$ denote the inner product in an Euclidean space of a suitable dimension. Then

$$\rho(\mathbf{U}, \mathbf{V}) = \sup_{\mathbf{x}\in\mathbb{R}^p, \mathbf{y}\in\mathbb{R}^q} \left|\mathrm{Corr}\Big(\langle \mathbf{x}, \mathbf{U}\rangle, \langle \mathbf{y}, \mathbf{V}\rangle\Big)\right|. \tag{27}$$

Let $\rho_{k,m}$ be the between-block canonical correlation:

$$\rho_{k,m} = \rho\left(\mathbf{Z}_1^m, \mathbf{Z}_{k+1}^{k+m}\right). \tag{28}$$

We now introduce the assumptions for the self-normalized block sampling procedure. $\{X_i\}$ is the stationary process (time series) we observe.

A1. $X_i = G(\mathbf{Z}_i, \ldots, \mathbf{Z}_{i-l}) = G(\mathbf{Z}_{i-l}^i)$ with mean $\mu = \mathbb{E}X_i$, where $\{\mathbf{Z}_i\}$ is a vector-valued stationary Gaussian process as in (23), and l is a fixed non-negative integer.

A2. We have weak convergence in $D[0, 1]$ endowed with the M_2 topology for the partial sum:

$$\left\{\frac{1}{n^H \ell(n)}(S_{\lfloor nt\rfloor} - n\mu),\ 0 \le t \le 1\right\} \Rightarrow \{Y(t),\ 0 \le t \le 1\},$$

for some nonzero H-sssi process $Y(t)$, where $0 < H < 1$ and $\ell(\cdot)$ is a slowly varying function.

A3. As $n \to \infty$, the block size $b_n \to \infty$, $b_n = o(n)$, and satisfies

$$\sum_{k=0}^{n} \rho_{k,l+b_n} = o(n), \tag{29}$$

where $\rho_{k,m}$ is the between-block canonical correlation defined in (28).

Remark 3.1. The data-generating specification in A1 allows us to get a variety of limits in A2, covering short-range dependence, long-range dependence, and heavy tails. When the covariance function of $X(n)$ is absolutely summable (short-range dependence), one typically gets in A2 convergence to Brownian motion (see, e.g., [14,33,16]). When the covariance of $X(n)$ is regularly varying of order between -1 and 0 (long-range dependence), one may get in A2 convergence to the Hermite-type processes (see, e.g., [75,21,76,3]).

Moreover, as shown in [74] in the case $J = 1$, when $G(\cdot)$ is chosen such that $X(n)$ is short-range dependent and heavy-tailed, so that $X(n)$ has infinite variance but finite mean, one can obtain in A2, convergence to an infinite-variance α-stable Lévy process; if $X(n)$ is long-range dependent and heavy-tailed, then the limit may be a finite-variance Hermite process, even though $X(n)$ may have infinite variance. All these situations are allowed under Assumptions A1–A3.

For sufficient conditions for Assumption A3 to hold, see Proposition 3.1 and Section 3.2.

Since the denominators in (12) are nonzero almost surely, Assumption A2, Lemma 3.1 and the Continuous Mapping Theorem imply the following (see [40, Corollary 4.5]):

Lemma 3.2. *T^*_{i,b_n} in (14) converges in distribution to T in (12).*

The following result allows us to relate the correlation of nonlinear functions to the correlation of linear functions.

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Lemma 3.3. *Let $(\mathbf{Z_i})_{i\in\mathbb{Z}}$ be a centered $\mathbb{R}^J$-valued Gaussian stationary process as in (23), and let $\mathbf{Z}_p^q$ be defined as in (24). Let $\mathcal{F}_{Jm}$ be the set of all functions F on $\mathbb{R}^{Jm}$ satisfying $\mathbb{E}F(\mathbf{Z}_1^m)^2 < \infty$. Then for $k \geq m$, one has*

$$\sup_{F,G\in\mathcal{F}_{Jm}} \left|\mathrm{Corr}\big(F(\mathbf{Z}_1^m), G(\mathbf{Z}_{k+1}^{k+m})\big)\right| = \rho\left(\mathbf{Z}_1^m, \mathbf{Z}_{k+1}^{k+m}\right) = \rho_{k,m}. \tag{30}$$

Proof. The equality is the well-known Gaussian maximal correlation equality. See, e.g., Theorem 1 of Kolmogorov and Rozanov [41] or Theorem 10.11 of Janson [38]. □

Our goal is to show that (19) holds, namely, $\widehat{F}_{n,b_n}$ is a consistent estimator of $P(T_n^* \leq x)$. This will be a consequence of the following theorem.

Theorem 3.1. *Assume that Assumptions* A1–A3 *hold. Let $F(x)$ be the CDF (cumulative distribution function) of T in (12), and let $\widehat{F}_{n,b_n}(x)$ be as in (17). As $n \to \infty$, we have*

$$\widehat{F}_{n,b_n}(x) \xrightarrow{p} F(x), \quad x \in C(F), \tag{31}$$

where $C(F)$ denotes the set of continuity points of $F(x)$. If $F(x)$ is continuous, then (31) can be strengthened to

$$\sup_x \left|\widehat{F}_{n,b_n}(x) - F(x)\right| \to 0 \quad \textit{in probability}. \tag{32}$$

Proof. *Step* 1. Let $\widehat{F}^*_{n,b_n}(x)$ be as in (15). To prove (31), we first show that

$$\widehat{F}^*_{n,b_n}(x) \xrightarrow{p} F(x), \quad x \in C(F), \tag{33}$$

where we have replaced $\widehat{F}_{n,b_n}(x)$ by $\widehat{F}^*_{n,b_n}(x)$. A bias–variance decomposition yields:

$$\begin{aligned}
\mathbb{E}\left(\left[\widehat{F}^*_{n,b_n}(x) - F(x)\right]^2\right) &= [\mathbb{E}\widehat{F}^*_{n,b_n}(x)]^2 - \mathbb{E}[2F(x)\widehat{F}^*_{n,b_n}(x)] \\
&\quad + F(x)^2 + \mathbb{E}[\widehat{F}^*_{n,b_n}(x)^2] - [\mathbb{E}\widehat{F}^*_{n,b_n}(x)]^2 \\
&= \left[\mathbb{E}\widehat{F}^*_{n,b_n}(x) - F(x)\right]^2 + \left[\mathbb{E}[\widehat{F}^*_{n,b_n}(x)^2] - [\mathbb{E}\widehat{F}^*_{n,b_n}(x)]^2\right] \\
&= \left[P(T^*_{i,b_n} \leq x) - P(T \leq x)\right]^2 + \mathrm{Var}\big[\widehat{F}^*_{n,b_n}(x)\big].
\end{aligned}$$

By Lemma 3.2, the squared bias $[P(T^*_{i,b_n} \leq x) - P(T \leq x)]^2$ converges to zero for $x \in C(F)$ as $b_n \to \infty$. We thus need to show that $\mathrm{Var}[\widehat{F}^*_{n,b_n}(x)] \to 0$. By the stationarity of $\{X_i\}$, which implies the stationarity of $\{T^*_{i,b_n}\}$ viewed as a process indexed by i, one has

$$\begin{aligned}
\mathrm{Var}[\widehat{F}^*_{n,b_n}(x)] &= \mathrm{Var}\left[\frac{1}{n-b_n+1}\sum_{i=1}^{n-b_n+1} \mathrm{I}\{T^*_{i,b_n} \leq x\}\right] \\
&= \frac{1}{(n-b_n+1)^2}\sum_{i,j=1}^{n-b_n+1} \mathrm{Cov}\left[\mathrm{I}\{T^*_{i,b_n} \leq x\}, \mathrm{I}\{T^*_{j,b_n} \leq x\}\right] \\
&\leq \frac{2}{n-b_n+1}\sum_{k=0}^{n} \left|\mathrm{Cov}\left[\mathrm{I}\{T^*_{1,b_n} \leq x\}, \mathrm{I}\{T^*_{k+1,b_n} \leq x\}\right]\right|,
\end{aligned} \tag{34}$$

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since for any covariance function $\gamma(\cdot)$ of a stationary sequence, we have

$$\sum_{i,j=1}^{p} |\gamma(i-j)| \le \sum_{|k|<p} (p-|k|)|\gamma(k)| \le 2p \sum_{k=0}^{p} |\gamma(k)|.$$

In view of Assumption A1, X_i depends on $\mathbf{Z}_i, \ldots, \mathbf{Z}_{i-l}$. By (14), T^*_{i,b_n} is a function of $X_i, \ldots, X_{i+b_n-1}$. Hence T^*_{1,b_n} depends not only on $\mathbf{Z}_1, \ldots, \mathbf{Z}_{b_n}$, but also on $\mathbf{Z}_{1-l}, \ldots, \mathbf{Z}_0$, and T^*_{k+1,b_n} depends on $\mathbf{Z}_{k+1-l}, \ldots, \mathbf{Z}_{k+b_n}$. We shall now apply Lemma 3.3 with the same k and $m = l + b_n$. Then when $k \ge l + b_n$, one has

$$\left|\mathrm{Cov}[\mathrm{I}\{T^*_{1,b_n} \le x\}, \mathrm{I}\{T^*_{k+1,b_n} \le x\}]\right| \le \frac{1}{4}\left|\mathrm{Corr}[\mathrm{I}\{T^*_{1,b_n} \le x\}, \mathrm{I}\{T^*_{k+1,b_n} \le x\}]\right| \le \frac{1}{4}\rho_{k,b_n+l} \tag{35}$$

where we have used the following fact[5]: if $0 \le X \le 1$, then $\mathrm{Var}[X] \le 1/4$. We have

$$\mathrm{Var}[\widehat{F}^*_{n,b_n}(x)] \le \frac{1}{2(n-b_n+1)} \sum_{k=0}^{n} \rho_{k,b_n+l}, \tag{36}$$

which converges to zero because of Assumption A3. Hence $\widehat{F}^*_{n,b_n}(x) \xrightarrow{p} F(x)$ for $x \in C(F)$. Step 1 of the proof is now complete.

Step 2. We now show that

$$\widehat{F}_{n,b_n}(x) \xrightarrow{p} F(x) \quad \text{for } x \in C(F),$$

that is, we go from (33) to (31). To do so, we follow the proof of Theorem 11.3.1 of Politis et al. [60], and express (17) as

$$\widehat{F}_{n,b_n}(x) = \frac{1}{n-b_n+1} \sum_{i=1}^{n-b_n+1} \mathrm{I}\{T^*_{i,b_n} \le x + b_n(\bar{X}_n - \mu)/D_{i,b_n}\}, \tag{37}$$

where D_{i,b_n} is as in (14). The goal is to show that $b_n(\bar{X}_n - \mu)/D_{i,b_n}$ is negligible. For $\epsilon > 0$, define

$$\begin{aligned} R_n(\epsilon) &= \frac{1}{n-b_n+1} \sum_{i=1}^{n-b_n+1} \mathrm{I}\{b_n(\bar{X}_n - \mu)/D_{i,b_n} \le \epsilon\} \\ &= \frac{1}{n-b_n+1} \sum_{i=1}^{n-b_n+1} \mathrm{I}\{(b_n^H \ell(b_n))^{-1} D_{i,b_n} \ge \epsilon^{-1} b_n(\bar{X}_n - \mu)(b_n^H \ell(b_n))^{-1}\}. \end{aligned} \tag{38}$$

Since $R_n(\epsilon)$ is an average of indicators, we have $R_n(\epsilon) \le 1$. Our goal is to show that $R_n(\epsilon) \xrightarrow{p} 1$. Note that as $n \to \infty$,

$$\frac{D_{i,b_n}}{b_n^H \ell(b_n)} = \frac{1}{b_n^H \ell(b_n)} \left(b_n^{-1} \sum_{k=i}^{i+b_n-1} \left(S_{i,k} - b_n^{-1}(k-i-1) S_{i,i+b_n-1} \right)^2 \right)^{1/2}$$

converges in distribution to D in (13) by Assumption A2 and continuous mapping. Moreover, since $b_n = o(n)$, $H < 1$ and $n(\bar{X}_n - \mu) n^{-H} \ell(n)^{-1}$ converges in distribution to $Y(1)$ by

[5] If $0 \le X \le 1$, then $\mu = \mathbb{E}X \in [0,1]$, $\mathbb{E}X^2 \le \mu$ and $\mathrm{Var}[X] \le \mu - \mu^2$ is maximized at $\mu = 1/2$, so that $\mathrm{Var}[X] \le 1/4$ (for more general results, see [19, Lemma 2.2]).

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Assumption A2, we have

$$b_n(\bar{X}_n - \mu)(b_n^H \ell(b_n))^{-1} = n(\bar{X}_n - \mu)n^{-H}\ell(n)^{-1}\,\frac{n^{H-1}\ell(n)}{b_n^{H-1}\ell(b_n)} \xrightarrow{p} 0.$$

Hence for any $\delta > 0$, with probability tending to 1 as $n \to \infty$, one has

$$1 \geq R_n(\epsilon) \geq \frac{1}{n - b_n + 1} \sum_{i=1}^{n-b_n+1} \mathrm{I}\{(b_n^H \ell(b_n))^{-1} D_{i,b_n} \geq \delta\epsilon^{-1}\}. \tag{39}$$

Since as T^*_{i,b_n} in Step 1, D_{i,b_n} is also a function of $X_i, \ldots, X_{i+b_n-1}$, we can follow a same argument as in Step 1, replacing T^*_{i,b_n} by $(b_n^H \ell(b_n))^{-1} D_{i,b_n}$ to obtain a similar result as in (33), namely that the empirical distribution of $(b_n^H \ell(b_n))^{-1} D_{i,b_n}$ converges in probability to that of D at all points of continuity of the distribution of D. Therefore

$$\frac{1}{n - b_n + 1} \sum_{i=1}^{n-b_n+1} \mathrm{I}\{(b_n^H \ell(b_n))^{-1} D_{i,b_n} \geq \delta\epsilon^{-1}\} \xrightarrow{p} P(D \geq \delta\epsilon^{-1}) \tag{40}$$

for $\delta\epsilon^{-1}$ at continuity point of the CDF of D. Since $P(D > 0) = 1$, we can choose δ small enough to make $P(D \geq \delta\epsilon^{-1})$ as close to 1 as desired. In view of (39) and (40), we conclude that as $n \to \infty$,

$$R_n(\epsilon) \xrightarrow{p} 1 \tag{41}$$

for any $\epsilon > 0$. Now notice that each summand in the sum (37) satisfies

$$\begin{aligned}
&\mathrm{I}\{T^*_{i,b_n} \leq x + b_n(\bar{X}_n - \mu)/D_{i,b_n}\} \\
&\quad = \left[\mathrm{I}\{T^*_{i,b_n} \leq x + b_n(\bar{X}_n - \mu)/D_{i,b_n}\}\right] \\
&\qquad \times \left[\mathrm{I}\{b_n(\bar{X}_n - \mu)/D_{i,b_n} \leq \epsilon\} + \mathrm{I}\{b_n(\bar{X}_n - \mu)/D_{i,b_n} > \epsilon\}\right] \\
&\quad \leq \mathrm{I}\{T^*_{i,b_n} \leq x + \epsilon\} + \mathrm{I}\{b_n(\bar{X}_n - \mu)/D_{i,b_n} > \epsilon\},
\end{aligned} \tag{42}$$

so that by plugging these inequalities in (37) and using (38), we get

$$\widehat{F}_{n,b_n}(x) \leq \widehat{F}^*_{n,b_n}(x + \epsilon) + 1 - R_n(\epsilon).$$

But by (41), $R_n(\epsilon) \xrightarrow{p} 1$. So for any $\gamma > 0$, one has

$$\widehat{F}_{n,b_n}(x) \leq \widehat{F}^*_{n,b_n}(x + \epsilon) + \gamma$$

with probability tending to 1 as $n \to \infty$. We can now use (33) to replace $\widehat{F}^*_{n,b_n}(x+\epsilon)$ by $F(x+\epsilon)$, so that for arbitrary $\gamma' > \gamma$, and for any $x + \epsilon \in C(F)$, one has $\widehat{F}_{n,b_n}(x) \leq F(x + \epsilon) + \gamma'$ with probability tending to 1 as $n \to \infty$. Now letting $\epsilon \downarrow 0$ through $x + \epsilon \in C(F)$ and using the continuity of $F(\cdot)$ at x, one gets with probability tending to 1 that

$$\widehat{F}_{n,b_n}(x) \leq F(x) + \gamma'', \quad x \in C(F), \tag{43}$$

for any $\gamma'' > \gamma'$.

A similar argument, which replaces (42) by

$$\mathrm{I}\{T_{i,b_n} \leq x\} \geq \mathrm{I}\{T^*_{i,b_n} \leq x - \epsilon\} - \mathrm{I}\{b_n(\bar{X}_n - \mu)/D_{i,b_n} < -\epsilon\},$$

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will show that for any $\gamma'' > 0$, with probability tending to 1,

$$\widehat{F}_{n,b_n}(x) \geq F(x) - \gamma'', \quad x \in C(F). \tag{44}$$

Combining (43) and (44), one gets

$$P(|\widehat{F}_{n,b_n}(x) - F(x)| \leq \gamma'') \to 1$$

as $n \to \infty$, and thus (31) holds.

Step 3. We now show (32). If $F(x)$ is continuous, then by the already established (31), we have $\widehat{F}_{n,b_n}(x) \to F(x)$ in probability for any $x \in \mathbb{R}$. Let n_i be an arbitrary subsequence, one can then choose a further subsequence of n_i, still denoted as n_i, so that $\widehat{F}_{n_i}(x) \to F(x)$ almost surely for all rational x by a diagonal subsequence argument. Then by Lemma A9.2 (ii) of Gut [28], $\sup_{x\in\mathbb{R}} |\widehat{F}_{n_i}(x) - F(x)| \to 0$ almost surely, and therefore $\sup_{x\in\mathbb{R}} |\widehat{F}_n(x) - F(x)| \to 0$ in probability. Hence (32) is proved. □

Consistency (19) is a simple corollary of Theorem 3.1.

Corollary 3.1. *Assume that Assumptions* A1–A3 *hold. Then as* $n \to \infty$,

$$|\widehat{F}_{n,b_n}(x) - P(T_n^* \leq x)| \to 0 \quad \textit{in probability} \tag{45}$$

for $x \in C(F)$. *If* $F(x)$ *is continuous, then the preceding convergence can be strengthened to*

$$\sup_{x\in\mathbb{R}} |\widehat{F}_{n,b_n}(x) - P(T_n^* \leq x)| \to 0 \quad \textit{in probability}. \tag{46}$$

Proof. The first result (45) follows directly from the triangle inequality

$$|\widehat{F}_{n,b_n}(x) - P(T_n^* \leq x)| \leq |\widehat{F}_{n,b_n}(x) - F(x)| + |P(T_n^* \leq x) - F(x)|,$$

where $x \in C(F)$ and $F(x) = P(T \leq x)$, by combining Theorem 3.2 or 3.1 with (12). For the second result (46), one uses also the fact that (12) implies $\sup_{x\in\mathbb{R}} |P(T_n^* \leq x) - F(x)| \to 0$ as $n \to \infty$ if $F(x)$ is continuous (see again Lemma A9.2 (ii) of Gut [28]). □

Bai and Taqqu [7] recently proved the following proposition, showing that the bound (29) holds for a large class of models with long-range dependence. Thus, for these models, one has the freedom to choose any $b_n = o(n)$, irrespective of the long-range dependence parameter H.

Proposition 3.1 (*[7, Theorems 2.2 and 2.3]*)**.** *Consider the case* $J = 1$. *Suppose that the spectral density of the underlying Gaussian* $\{Z_i\}$ *is given by*

$$f(\lambda) = f_H(\lambda) f_0(\lambda),$$

where $f_H(\lambda) = |1-e^{i\lambda}|^{-2H+1}$, $1/2 < H < 1$, *and* $f_0(\lambda)$ *is a spectral density which corresponds to a covariance function (or Fourier coefficient)* $\gamma_0(n) = \int_{-\pi}^{\pi} f_0(\lambda) e^{in\lambda} d\lambda$. *Assume that the following hold:*

(a) *There exists* $c_0 > 0$ *such that* $f_0(\lambda) \geq c_0$ *for all* $\lambda \in (-\pi, \pi]$;
(b) $\sum_{n=-\infty}^{\infty} |\gamma_0(n)| < \infty$;
(c) $\gamma_0(n) = o(n^{-1})$.

Then the condition (29) *in Assumption* A3 *holds if* $b_n = o(n)$. *The result extends to the case where the underlying Gaussian* $\{\mathbf{Z}_i\}$ *is* J*-dimensional with independent components.*

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In Proposition 3.1, $f_H(\lambda)$ is the spectral density of a FARIMA$(0, d, 0)$ sequence with $d = H - 1/2$, and $f_0(\lambda)$ is the spectral density of a sequence with short-range dependence.

Under the assumptions in Proposition 3.1, the spectral density $f(\lambda)$ cannot have a slowly varying factor which diverges to infinity or converges to zero at $\lambda = 0$, because $f_0(\lambda)$ is bounded away from infinity and zero. For $H \in (1/2, 1)$, the FARIMA(p, d, q) model with $d = H - 1/2$ and the fractional Gaussian noise model satisfy the assumptions of Proposition 3.1. See Examples 2.1 and 2.2 of Bai and Taqqu [7].

We thus have the following result which we formulate for simplicity in the univariate case $J = 1$.

Corollary 3.2. *Assume that Assumptions* A1–A2 *hold with* $J = 1$*, and the underlying Gaussian* $\{Z_i\}$ *satisfies the assumptions in Proposition* 3.1*. If* $b_n \to \infty$ *and* $b_n = o(n)$*, then the conclusions of Theorem* 3.1 *and Corollary* 3.1 *hold.*

3.2. Further analysis of Assumption A3

In this section, we discuss the critical Assumption A3, which involves the covariance structure of the underlying Gaussian $\{\mathbf{Z}_i\}$. In particular, we shall give the general bound (49) for the canonical correlation $\rho_{k,m}$ in (28), and discuss how it relates to Assumption A3. As noted in Proposition 3.1, however, this bound, in the long memory case, can be improved substantially so as to provide more flexibility on the choice of the block size b_n.

To state this general bound, define

$$M_\gamma(k) = \max_{n>k} \max_{1\le j_1, j_2\le J} |\gamma_{j_1,j_2}(n)|, \tag{47}$$

and

$$\lambda_m = \text{ the } \textit{minimum} \text{ eigenvalue of } \Sigma_m. \tag{48}$$

Note that $\lambda_m > 0$ since Σ_m is assumed to be positive definite.

Lemma 3.4. *Let* $\rho_{k,m}$ *be as in* (28)*,* $M_\gamma(k)$ *be as in* (47) *and* λ_m *be as in* (48)*. We have the bound*

$$\rho_{k,m} \le \min\left\{ Jm \frac{M(k-m)}{\lambda_m}, 1 \right\}. \tag{49}$$

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be (column) vectors in $\mathbb{R}^{Jm}$. Note that each $\mathbf{Z}_1^m = (\mathbf{Z}_1, \ldots, \mathbf{Z}_m)$ and $\mathbf{Z}_{k+1}^{k+m} = (\mathbf{Z}_{k+1}, \ldots, \mathbf{Z}_{k+m})$ are Jm-dimensional Gaussian vectors translated by k units in the time index. Therefore by (27),

$$\rho_{k,m} = \rho\left(\mathbf{Z}_1^m, \mathbf{Z}_{k+1}^{k+m}\right) = \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^{Jm}} \frac{\mathbb{E}\left[\langle \mathbf{x}, \mathbf{Z}_1^m\rangle \langle \mathbf{y}, \mathbf{Z}_{k+1}^{k+m}\rangle\right]}{\left(\mathrm{Var}[\langle \mathbf{x}, \mathbf{Z}_1^m\rangle]\right)^{1/2}\left(\mathrm{Var}[\langle \mathbf{y}, \mathbf{Z}_{k+1}^{k+m}\rangle]\right)^{1/2}}$$

$$= \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^{Jm}} \frac{\mathbf{x}^T \Sigma_{k,m} \mathbf{y}}{\sqrt{\mathbf{x}^T \Sigma_m \mathbf{x}}\sqrt{\mathbf{y}^T \Sigma_m \mathbf{y}}}, \tag{50}$$

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where Σ_m is as in (25), $\Sigma_{k,m}$ is as in (26). By relations 6.58(a) and 6.62(a) in [66], one has

$$\rho_{k,m} = \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^{Jm}} \frac{\left|\mathbf{x}^T \Sigma_{k,m}\mathbf{y}\right|}{\sqrt{\mathbf{x}^T \Sigma_m \mathbf{x}}\sqrt{\mathbf{y}^T \Sigma_m \mathbf{y}}} \le \sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^{Jm}} \frac{1}{\lambda_m} \frac{\left|\mathbf{x}^T \Sigma_{k,m}\mathbf{y}\right|}{\|\mathbf{x}\|\,\|\mathbf{y}\|} \le \frac{1}{\lambda_m}\sigma_{k,m}, \tag{51}$$

where λ_m is the smallest eigenvalue of Σ_m, and $\sigma_{k,m}$ is the maximum singular value[6] of $\Sigma_{k,m}$. By Seber [66, 4.66(b) and 4.67(b)], $\sigma_{k,m}$ is bounded by the linear size of the matrix $\Sigma_{k,m}$ times the maximum absolute value of all the elements of the matrix. Since the matrix $\Sigma_{k,m}$ has linear size Jm, we have

$$\begin{aligned}\sigma_{k,m} &\le Jm \max_{1\le i_1,i_2\le m} \max_{1\le j_1,j_2\le J} |\gamma_{j_1,j_2}(i_2+k-i_1)| \\ &\le Jm \max_{n>k-m} \max_{1\le j_1,j_2\le J} |\gamma_{j_1,j_2}(n)| = JmM_\gamma(k-m).\end{aligned}$$

The bound (49) is then obtained by noting that $\rho_{k,m} \le 1$ in view of (30). □

Example 3.1. Consider the important scalar case $J = 1$, where $\mathbf{Z}_i = Z_i$. Denote the covariance function of $\{Z_i\}$ by $\gamma(n)$ and its spectral density by $f(\omega)$. In this case, it is known that Σ_m is non-singular for any m if $\lim_{n\to\infty}\gamma(n) = 0$ (see Proposition 5.1.1 of Brockwell and Davis [15]), and that the minimum eigenvalue λ_m satisfies

$$\lambda_m \ge 2\pi\ \operatorname{ess\,inf}_\omega f(\omega), \quad \text{and} \quad \lim_{m\to\infty}\lambda_m = 2\pi\ \operatorname{ess\,inf}_\omega f(\omega), \tag{52}$$

where "ess inf" denotes the essential infimum with respect to Lebesgue measure on $[-\pi,\pi)$ (see [27, Chapter 5.2]). If $J = 1$, $M_\gamma(k)$ also reduces to

$$M_\gamma(k) = \max_{n>k}|\gamma(n)|. \tag{53}$$

Remark 3.2. Consider the vector case but suppose that $\{Z_{i,1}\},\ldots,,\ldots,\{Z_{i,J}\}$ are mutually independent, i.e., $\gamma_{j_1,j_2}(n) = \gamma_{j_1,j_2}(n)\mathrm{I}\{j_1 = j_2\}$. Let $\Gamma_{m,j} = \big(\gamma_{j,j}(i_1-i_2)\big)_{1\le i_1,i_2\le m}$. In this case, we have a block-diagonal $\Sigma_m = \operatorname{diag}(\Gamma_{m,1},\ldots,\Gamma_{m,J})$. Let $\Gamma_{k,m,j} = \big(\gamma_{j,j}(i_2+k-i_1)\big)_{1\le i_1,i_2\le m}$. We also have a block-diagonal $\Sigma_{k,m} = \operatorname{diag}(\Gamma_{k,m,1},\ldots,\Gamma_{k,m,J})$. Let $\rho_{k,m,j}$ be the between-block canonical correlation $\rho(\mathbf{Z}^m_{1,j},\mathbf{Z}^m_{k,j})$ in component j, $j = 1,\ldots,J$. The block-diagonal structure implies that

$$\rho_{k,m} = \max\{\rho_{k,m,j},\ j = 1,\ldots,J\}.$$

Proposition 3.2. *Assumption* A3 *holds if* $b_n = o(n)$ *and*

$$\sum_{k=0}^{n}\min\left\{\frac{b_n}{\lambda_{b_n+l}}M_\gamma(k),\,1\right\} = o(n). \tag{54}$$

Proof. In view of Lemma 3.4, we have

$$\sum_{k=0}^{n}\rho_{k,b_n+l} \le (b_n+l) + \sum_{k=b_n+l}^{n}\min\left\{Jb_n\frac{M(k-b_n-l)}{\lambda_{b_n+l}},\,1\right\} = o(n)$$

since $b_n = o(n)$. Hence Assumption A3 holds. □

[6] Note that $\Sigma_{k,m}$ is not a symmetric matrix. The square of its singular values are the eigenvalues of $\Sigma^T_{k,m}\Sigma_{k,m}$, which is symmetric and non-negative definite.

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Implications of Proposition 3.2.

We discuss here the implications of Condition (54) in various specific situations. This discussion is restricted to the case $J = 1$ which is of most interest. This discussion can be easily extended to the case of independent components via the observation made in Remark 3.2. Let $c, C > 0$ be generic constants whose value can change from expression to expression. The notation $a \asymp b$ means $cb \le a \le Cb$ for some $0 < c < C$. Assume throughout that the covariance $\gamma(n) \to 0$ and $b_n = o(n)$ as $n \to \infty$. We distinguish two cases: $\operatorname{ess\,inf}_\omega f(\omega) > 0$ and $\operatorname{ess\,inf}_\omega f(\omega) = 0$.

1. Assume first $\operatorname{ess\,inf}_\omega f(\omega) > 0$.

In view of (52), the minimum eigenvalue λ_m is bounded below away from zero, and hence Condition (54) holds if

$$b_n \sum_{k=0}^{n} M_\gamma(k) = o(n), \tag{55}$$

where $M_\gamma(k)$ is expressed as (53). Consider the case $\sum_{k=0}^{\infty} M_\gamma(k) < \infty$, which implies the typical *short-range dependence* condition: $\sum_{k=1}^{\infty} |\gamma(k)| < \sum_{k=0}^{\infty} M_\gamma(k) < \infty$. Then (55) reduces to $b_n = o(n)$. We get in particular:

Corollary 3.3. *Suppose that* $\operatorname{ess\,inf}_\omega f(\omega) > 0$, *and* $|\gamma(n)| \le d_n$, *where* d_n *is non-increasing and summable (typically,* $d_n = cn^{-\beta}$ *for some constant* $c > 0$ *and* $\beta > 1$*). If* $b_n = o(n)$, *then Assumption* A3 *holds.*

Proof. $|\gamma(k)| \le d_k$ implies $M_\gamma(k) \le d_k$, and hence $\sum_{k=0}^{\infty} M_\gamma(k) < \infty$. □

Consider now the situation relevant to *long-range dependence*:

$$\gamma(k) = k^{2H-2} L(k), \quad 1/2 < H < 1, \tag{56}$$

where $L(k)$ is a slowly varying function at infinity. By Theorem 1.5.3 of Bingham et al. [12], Condition (56) implies that $M_\gamma(k) \sim k^{2H-2} L(k)$, which entails that $\sum_{k=0}^{n} M_\gamma(k) \le cn^{2H-1} L(n)$. Thus (55) holds if

$$b_n = o(n^{2-2H} L(n)^{-1}). \tag{57}$$

So, the larger the H, the smaller the block size b_n.

Corollary 3.4. *Suppose that* $\operatorname{ess\,inf}_\omega f(\omega) > 0$, *and* $|\gamma(n)| \le n^{2H-2} L(n)$, *where* $1/2 < H < 1$ *and* L *is slowly varying. If* $b_n = o(n^{2-2H} L(n)^{-1})$, *then Assumption* A3 *holds.*

The case $|\gamma(k)| \le k^{2H-2} L(k)$ also encompasses the seasonal long memory situations (see, e.g., [30]), where $\gamma(k)$ oscillates within a power-law envelope.

In the long-range dependent case, Betken and Wendler [11] obtained recently a bound for $\rho_{k,m}$ in (28) using a result of Adenstedt [1] under some additional assumptions. Their bound allows (29) to hold under the block size condition

$$b_n = o(n^{3/2-H-\epsilon}) \tag{58}$$

with arbitrarily small $\epsilon > 0$. The condition (58) is better than (57) for each H, and $b_n = O(n^{1/2})$ is always allowed.

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We have also seen that if the model satisfies the assumptions of Proposition 3.1, one can choose

$$b_n = o(n),$$

irrespective of the value of $H \in (1/2, 1)$.

2. Assume now $\operatorname{ess\,inf}_\omega f(\omega) = 0$.

As mentioned in (52), the smallest covariance eigenvalue λ_m converges to $\operatorname{ess\,inf}_\omega f(\omega) = 0$ as $m \to \infty$. The rate of convergence has been investigated by a number of authors. See, e.g., [39,61,67,77,56]. It involves the order of the zeros of $f(\omega)$. We say $f(\omega)$ has a *zero of order* $\nu > 0$ at $\omega = \omega_0$ if $f(\omega) \asymp |\omega - \omega_0|^\nu$. Roughly speaking, the rate at which λ_m converges to zero follows the highest order of the zeros of $f(\omega)$, and the rate of convergence to zero cannot be faster than exponential:

$$\lambda_m \geq e^{-cm} \tag{59}$$

for some $c > 0$ (see [61,77]). Let us focus on the situation where $f(\omega)$ has a finite number of zeros of polynomial orders. Specifically, suppose that $f(\omega)$ has zeros of order $\nu_1, \ldots, \nu_p$ at p distinct points $\omega_1, \ldots, \omega_p$, and $f(\omega)$ stays positive outside arbitrary neighborhoods of $\omega_1, \ldots, \omega_p$. Then by Theorem 2.2 of Novosel'tsev and Simonenko [56], one has $\lambda_m \asymp m^{-\nu}$ where

$$\nu = \max(\nu_1, \ldots, \nu_p).$$

Therefore,

$$\lambda_{b_n+l} \asymp (b_n + l)^{-\nu} \asymp b_n^{-\nu}$$

and since $M_\gamma(k)$ is non-increasing, we have

$$\begin{aligned} &\sum_{k=0}^{n} \min\left\{\frac{b_n}{\lambda_{b_n+l}} M_\gamma(k), 1\right\} \\ &\leq \sum_{k=0}^{p_n} 1 + C b_n^{1+\nu} \sum_{k=p_n+1}^{n} M_\gamma(k) \leq C\left(p_n + n b_n^{1+\nu} M_\gamma(p_n)\right). \end{aligned} \tag{60}$$

To satisfy (54), we need the last expression in (60) to be of order $o(n)$. This will be so if as $n \to \infty$, $p_n = o(n)$, and

$$b_n = o\left([M_\gamma(p_n)]^{-1/(1+\nu)}\right). \tag{61}$$

To get the weakest restriction on b_n, let in addition p_n grow fast enough so that $n/p_n = o(n^\delta)$ for any $\delta > 0$ (e.g., choose $n/p_n \asymp \log n$). We have the following two typical cases:

- $M_\gamma(k) = O(e^{-k})$ decays exponentially. In this case, $[M_\gamma(p_n)]^{-1/(1+\nu)} = O(e^{p_n/(1+\nu)})$, so the condition (61) is certainly satisfied when $b_n = o(n)$. Hence Assumption A3 holds with $b_n = o(n)$;
- $M_\gamma(k) = O(k^{-\beta})$, $\beta > 0$. In this case, (54) holds when

$$b_n = o(n^{\beta/(1+\nu)-\epsilon}) \tag{62}$$

for arbitrarily small $\epsilon > 0$. So the worst case is when β is close to 0 and ν is large.

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A nice example involving both ν and β is when $Z(n)$ is *anti-persistent* (also called negative memory), e.g., the fractional Gaussian noise (the increments of fractional Brownian motion) with $H < 1/2$, and FARIMA(p, d, q) with $d = H - 1/2$ so that $-1/2 < d < 0$. In this case, we have $\beta = 2 - 2H$ and $\nu = 1 - 2H$ in (62), and hence (54) holds with $b_n = o(n^{1-\epsilon})$. Therefore:

Corollary 3.5. *Suppose that* $\{Z_n\}$ *is fractional Gaussian noise with* $H < 1/2$ *or* FARIMA(p, d, q) *with* $-1/2 < d < 0$. *If* $b_n = o(n^{1-\epsilon})$ *for* $\epsilon > 0$ *arbitrarily small, then Assumption* A3 *holds.*

Remark 3.3. We also mention that in [89] which studies non-self-normalized block sampling for sample mean, the condition $b_n = o(n^{1-\epsilon})$ for arbitrarily small $\epsilon > 0$ is shown to suffice for consistency. The framework in their paper assumes $\{X_i\}$ to be a univariate nonlinear transform of linear *non-Gaussian* processes. But it is not clear how to adapt their proof to a setting involving the self-normalization considered here.

3.3. Strong mixing case

Given a stationary process $\{X_i\}$, let $\mathcal{F}_a^b$ be the σ-field generated by $X_a, \ldots, X_b$, where $-\infty \le a \le b \le +\infty$. Recall that the strong mixing (or α-mixing) coefficient is defined as

$$\alpha(k) = \sup \left\{ |P(A)P(B) - P(A \cap B)|,\ A \in \mathcal{F}_{-\infty}^0, B \in \mathcal{F}_k^\infty \right\}. \tag{63}$$

Note that $0 \le \alpha(k) \le 1$. The process $\{X_i\}$ is said to be *strong mixing* if

$$\lim_{k \to +\infty} \alpha(k) = 0.$$

We refer the reader to Bradley [13] for more details. We shall use the following inequality which can be found in Lemma A.0.2 of Politis et al. [60].

Lemma 3.5. *If* $U \in \mathcal{F}_{-\infty}^0$ *and* $V \in \mathcal{F}_k^\infty$, *and* $0 \le U, V \le 1$ *almost surely, then*

$$|\mathrm{Cov}(U, V)| \le \alpha(k) \le 1.$$

We shall assume:

B1. $\{X_i\}$ is a strong mixing stationary process with mean $\mu = \mathbb{E}X_i$.

B2. We have the weak convergence in $D[0, 1]$ endowed with M_2 topology of the partial sum:

$$\left\{ \frac{1}{n^H \ell(n)} (S_{\lfloor nt \rfloor} - n\mu),\ 0 \le t \le 1 \right\} \Rightarrow \{Y(t),\ 0 \le t \le 1\},$$

for some nonzero H-sssi process $Y(t)$, where $0 < H < 1$ and $\ell(\cdot)$ is a slowly varying function.

B3. The block size $b_n \to \infty$ and $b_n = o(n)$ as $n \to \infty$.

The following theorem establishes the consistency of the self-normalized block sampling under the strong mixing framework.

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Theorem 3.2. *The conclusions of Theorem 3.1 and of Corollary 3.1 hold under Assumptions* B1–B3 .

Proof. The structure of the proof and many details are similar to those of Theorem 3.1. We only highlight the key differences. See also [60] or [71].

In Step 1, we again need to show (33). The term $[P(T^*_{i,b_n} \le x) - P(T \le x)]^2 \to 0$ as before. We need to establish $\mathrm{Var}[\widehat{F}^*_{n,b_n}(x)] \to 0$. We still have the bound (34).

In view of Lemma 3.5, one has that,

$$\left|\mathrm{Cov}[\mathrm{I}\{T^*_{1,b_n} \le x\}, \mathrm{I}\{T^*_{k+1,b_n} \le x\}]\right| \le \begin{cases} 1 & \text{if } k < b_n, \\ \alpha(k - b_n + 1), & \text{if } k \ge b_n; \end{cases}$$

where $\alpha(\cdot)$ is the mixing coefficient in (63). Hence from (34), we have

$$\begin{aligned} \mathrm{Var}[\widehat{F}^*_{n,b_n}(x)] &\le \frac{2}{n - b_n + 1}\left(\sum_{k=0}^{b_n-1}\left|\mathrm{Cov}\big[\mathrm{I}\{T^*_{1,b_n} \le x\}, \mathrm{I}\{T^*_{k+1,b_n} \le x\}\big]\right|\right. \\ &\quad \left. + \sum_{k=b_n}^{n}\left|\mathrm{Cov}\big[\mathrm{I}\{T^*_{1,b_n} \le x\}, \mathrm{I}\{T^*_{k+1,b_n} \le x\}\big]\right|\right) \\ &\le \frac{2}{(n - b_n + 1)}\left[b_n + \sum_{k=b_n}^{n}\alpha(k - b_n + 1)\right] \\ &= \frac{2b_n}{(n - b_n + 1)} + \frac{2}{(n - b_n + 1)}\sum_{k=1}^{n-b_n+1}\alpha(k), \end{aligned} \tag{64}$$

which converges to zero as $n \to \infty$, because $b_n = o(n)$ by Assumption B3, and $\alpha(k) \to 0$ as $k \to \infty$ by Assumption B1 and by applying a Cesàro summation. Hence (33) is proved.

Step 2 and 3 proceed exactly as the proof of Theorem 3.1. The argument in the proof of Corollary 3.1 shows that the conclusion of that corollary continues to hold under Assumptions B1–B3. □

Remark 3.4. In view of Shao [68], the self-normalized block sampling method considered in this paper may be extended to more general statistics beyond the sample mean. There are two aspects to consider, self-normalization and block sampling. For the *self-normalization* aspect to work, the general statistics needs to be approximately linear, namely, it admits a functional Taylor expansion in the sense of (2) in [68]. In this case, Assumption A2 or B2 needs to be replaced by a modified version of Assumption 1 of Shao [68]. Furthermore, the remainder term in the aforementioned functional Taylor expansion has to satisfy a negligibility condition (see Assumption 2 of Shao [68] or Assumption II of Shao [70]). Validating these conditions for particular statistics (e.g., sample quantiles) and particular models (e.g., the Gaussian subordination model in Assumption A1) may be considered in future work. The *block sampling* aspect is likely to continue to be valid, since as shown in the proofs of Theorems 3.1 and 3.2, the key is to have a bound on the between-block correlation, as the one in Proposition 3.1 in the long-memory Gaussian subordination framework, or as in Lemma 3.5 in the strong mixing framework.

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4. Examples

The first two examples of models concern Assumptions A1–A3. They both involve a phase transition.

Example 4.1. Suppose that

$$X_i = G(Z_i) = Z_i^2,$$

where $\{Z_i\}$ is a standardized stationary Gaussian process with covariance $\gamma(n) = n^{2d-1}L(n)$, with $d \in (0, 1/2)$, and $L(n)$ is a positive slowly varying function. Then Assumption A1 is satisfied. Moreover, by Taqqu [75] in the case $d < 1/4$ and Breuer and Major [14] and Chambers and Slud [16] in the case $d > 1/4$, Assumption A2 holds with the following dichotomy:

$$\begin{cases} H = 1/2,\ \ell(n) = 1,\ Y(t) = \sigma B(t) & \text{if } d < 1/4; \\ H = 2d,\ \ell(n) = L(n),\ Y(t) = c_H Z_{2,H}(t) & \text{if } d > 1/4, \end{cases}$$

where $\sigma^2 = \sum_n \mathrm{Cov}[X(n), X(0)]$, c_H is a positive constant, $B(t)$ is the standard Brownian motion and $Z_{2,H}$ is the standard Rosenblatt process (second-order Hermite process). Assume in addition that the assumptions for $\{Z_i\}$ in Proposition 3.1 hold. Then one can choose a block size $b_n = o(n)$ to satisfy Assumption A3. Hence Theorem 3.1 and Corollary 3.1 hold. Without the additional assumptions in Proposition 3.1, Assumption A3 is guaranteed at least by the choice $b_n = o(n^{1-2d}L(n)^{-1})$ in view of (57).

Example 4.2. Let F_α be the cdf of t_α distribution with $1 < \alpha < 2$, so that it has finite mean but infinite variance. Let Φ be the cdf of a standard normal. Suppose that

$$X_i = F_\alpha^{-1}(\Phi(Z_i)),$$

where $\{Z_i\}$ is a standardized stationary Gaussian process with covariance $\gamma(n) = n^{2d-1}L(n)$, $d \in (0, 1/2)$, and $L(n)$ is a positive slowly varying function. The marginal distribution of $\{X_i\}$ is a t_α. Then Assumption A1 is satisfied. By Sly and Heyde [74], Assumption A2 holds with the following dichotomy (for $0 < d < 1/2$, $1 < \alpha < 2$):

$$\begin{cases} H = 1/\alpha,\ \ell(n) = 1,\ Y(t) = c_1 L_\alpha(t) & \text{if } d + 1/2 < 1/\alpha; \\ H = d + 1/2,\ \ell(n) = L(n),\ Y(t) = c_2 B_H(t) & \text{if } d + 1/2 > 1/\alpha, \end{cases}$$

where c_1 and c_2 are positive constants, $L_\alpha(t)$ is a symmetric α-stable Lévy process, and $B_H(t)$ is a standard fractional Brownian motion. Assume in addition that the assumptions for $\{Z_i\}$ in Proposition 3.1 hold. This will be the case if $\{Z_i\}$ is fractional Gaussian noise or FARIMA(p, d, q). Then $b_n = o(n)$ implies (29). Hence Theorem 3.1 and Corollary 3.1 hold. Without the additional assumptions in Proposition 3.1, Assumption A3 is guaranteed at least by the choice $b_n = o(n^{1-2d}L(n)^{-1})$ in view of (57).

Example 4.3. Consider the following long-memory stochastic duration (LMSD) model (for modeling inter-trade duration, see [18]):

$$X_i = \xi_i \exp(Z_i),$$

where $\{\xi_i\}$ are i.i.d. positive random variables satisfying $P(\xi_i > x) \sim Ax^{-\alpha}$ as $x \to \infty$, $A > 0$, $\alpha \in (1, 2)$, Z_i is a Gaussian linear process $Z_i = \sum_{j=1}^{\infty} j^{d-1} l(j)\epsilon_{i-j}$ with $d \in (0, 1/2)$, $l(j)$ a positive and slowly varying function, $\{\epsilon_i\}$ i.i.d. centered Gaussian, and $\{\epsilon_i\}$ is independent of

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$\{\xi_i\}$. Note that $\mu = \mathbb{E}X_i > 0$. The model has the interesting feature that although $\mathbb{E}X_i^2 = \infty$, it has the following finite covariance for $h \neq 0$, namely,

$$\mathrm{Cov}[X_i, X_{i+h}] = \mathrm{Cov}[\exp(Z_0), \exp(Z_h)]\mu_\xi^2 \sim ch^{2d-1}l^2(h),$$

as $h \to \infty$, where $\mu_\xi = \mathbb{E}\xi_i$, and we have used the fact that the exponential function has Hermite rank 1 (see [75]). To satisfy Assumption A1, one can rewrite the model as

$$X_i = g(Z_i')\exp(Z_i),$$

where $\{Z_i'\}$ are i.i.d. standard Gaussian with g chosen such that $g(Z_i')$ is equal in distribution to ξ_i. This makes the model satisfy Assumption A1 with $J = 2$, $l = 0$, $\mathbf{Z}_i = (Z_i', Z_i)$ and $G(x_1, x_2) = g(x_1)\exp(x_2)$. By (4.100) and (4.101) of Beran et al. [10], Assumption A2 holds with the following dichotomy:

$$\begin{cases} H = 1/\alpha,\ \ell(n) = 1,\ Y(t) = c_\alpha L_{\alpha,1,1}(t) & \text{if } d + 1/2 < 1/\alpha; \\ H = d + 1/2,\ \ell(n) = l^2(n),\ Y(t) = c_d B_H(t) & \text{if } d + 1/2 > 1/\alpha, \end{cases}$$

where c_α, c_d are positive constants, $L_{\alpha,1,1}(t)$ is an α-stable Lévy process with skewness $\beta = 1$ (see (10)), and $B_H(t)$ is the standard fractional Brownian motion. If in addition, the assumptions for $\{Z_i\}$ in Proposition 3.1 hold, then Assumption A3 is satisfied if $b_n = o(n)$. Hence Theorem 3.1 and Corollary 3.1 hold. Without the additional assumptions in Proposition 3.1, Assumption A3 is at least satisfied if $b_n = o(n^{1-2d}l(n)^{-2})$ (see (57) and Remark 3.2).

Remark 4.1. Consider the non-centered stochastic volatility model $X_i = \sigma_i g(Z_i) + \mu$ in [37], where σ_i and $g(Z_i)$ are independent, σ_i is i.i.d. with heavy tails and $\{Z_i\}$ is Gaussian with long-range dependence and g has Hermite rank one. This model can be similarly embedded into Assumption A1. However, as far as we know, the functional convergence[7] needed in Assumption A2 has not been established (only the marginal convergence was established in [37]). Assumption A2 for this model is, nevertheless, expected to hold in view of its similarity[8] to the model treated in [42, Theorem 4.1] (see also Theorem 4.19 of Beran et al. [10]). Checking Assumption A2 in details is outside the scope of the current paper. Assumption A3 is dealt with as in Example 4.3.

Nevertheless, the consistency of the self-normalized block sampling in [37] can be shown to hold under our A1 and A3 framework. This is done by adopting the normalization of Jach et al. [37], with A2 replaced by marginal convergence involving partial sums and sample covariances,[9] and to ensure A3, by assuming $b_n = o(n)$ and that $\{Z_i\}$ is a long-range dependent sequence satisfying the assumptions of Proposition 3.1.

We now give two examples with *strong mixing*. The first involves a nonlinear time series and the second involves heavy tails.

Example 4.4. Suppose that

$$X_i = \rho|X_{i-1}| + \epsilon_i, \quad 0 < \rho < 1, \tag{65}$$

[7] The weak convergence assumed in Assumption A2 allowed us to take advantage of Lemma 3.1 in order to establish Lemma 3.2.

[8] Both Jach et al. [37] and Kulik and Soulier [42] treated stochastic volatility models of the form $X_i = L_i H_i$ (for limit theorems it does not matter whether a level is added or not), where L_i has finite variance and is long-range dependent, while H_i has infinite variance and is i.i.d. The difference between the two papers is that in [37] L_i is centered and H_i is not, while in [42] H_i is centered and L_i is not.

[9] More precisely, convergence in distribution of a 3-dimensional vector specified in Theorem 3 of Jach et al. [37].

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where ϵ_i's are i.i.d. standard Gaussian. Thus $\{X_i\}$ follows a threshold autoregressive model [78]. The Markov process $\{X_i\}$ is strong mixing because it is ergodic[10] (see [58, Theorem 2.1], or [22, p.103]), and hence Condition B1 holds. The conditions of Theorem 3(ii) of Wu [84] are satisfied[11] and therefore Condition B2 holds with $H = 1/2$, $\ell(n) = 1$ and $Y(t) = \sigma B(t)$, where $\sigma^2 = \sum_n \gamma(n) > 0$ and $B(t)$ is standard Brownian motion. Condition B3 holds for any block size $b_n = o(n)$. Therefore, Theorem 3.2 holds.

In the following example, both Assumptions A1–A3 and B1–B3 hold.

Example 4.5. Consider the MA(1) model

$$X_i = \epsilon_i + a\epsilon_{i-1},$$

where $a \geq 0$ and $\{\epsilon_i\}$ are i.i.d. Assume that $\mathbb{E}\epsilon_i = 0$, $\mathbb{E}\epsilon_i^2 = \infty$, and ϵ_i is in the domain of attraction of a stable distribution with an index $\alpha \in (1, 2)$. Let $b_n = o(n)$. By choosing appropriate transforms, we can express ϵ_i as function of Gaussian. Therefore Assumption A1 holds. Assumption B1 holds because $\{X_i\}$ is 2-dependent. By Theorem 2' of Avram and Taqqu [5], Assumption A2 and B2 hold with $H = 1/\alpha$, some slowly varying function $\ell(n)$, and $Y(t)$ is an α-stable Lévy process. Also A3 holds with any $b_n = o(n)$ since $\rho_{k,m} = 0$ when $k \geq m + 2$. Therefore, both assumptions A1–A3 and B1–B3 hold in this case.

5. Monte Carlo simulations

We shall carry out here Monte Carlo simulations to examine the finite-sample performance of the self-normalized block sampling (SNBS) method and make a comparison with the recent result of Zhang et al. [89]. Instead of resorting to self-normalization, the method of Zhang et al. [89] exploits the regularly varying property of the asymptotic variance to avoid the problem of estimating the nuisance Hurst index. We first consider the case with Gaussian subordination. For this, let

$$X_i = K(Z_i), \quad Z_i = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}, \ i = 1, \ldots, n, \tag{66}$$

where $K(\cdot)$ is a possibly nonlinear transformation and $\{\epsilon_k\}$ are i.i.d. standard normal random variables.[12] We consider the following configurations for (66):

(a) $K(x) = x$ and $a_j = (1 + j)^{d-1}$, $j \geq 0$;
(b) $K(x) = x^2$ and $a_j = (1 + j)^{d-1}$, $j \geq 0$;
(c) $K(x) = \Phi_t^{-1}[\Phi_N\{(\sum_{j=0}^{\infty} a_j^2)^{-1/2}x\}]$ and $a_j = (1 + j)^{d-1}$, $j \geq 0$,

where Φ_N is the CDF of the standard normal and Φ_t is the CDF of the Student's t-distribution with degree of freedom 1.5, whose tail probability decays like $|x|^{-3/2}$ as $|x| \to \infty$ so that it has infinite variance but finite mean.

[10] That is, the Markov chain is irreducible aperiodic and positive recurrent (see [79]).

[11] In the terminology of Wu [84], $R(x, \epsilon) = \rho|x| + \epsilon$, $L_\epsilon = \rho$, $\delta_p(n) = O(n^r)$ for some $0 < r < 1$, so that $\sum_{n=0}^{\infty} n\delta_p(n) < \infty$, implying Theorem 3(ii).

[12] To generate the process, we use the approximation $Z_i \approx \sum_{j=0}^{\lfloor n^{3/2} \rfloor - 1} a_j \epsilon_{i-j}$ in our simulation, and the fast Fourier transform (FFT) as mentioned in [88] is implemented to facilitate the computation. Note that the cutoff $n^{3/2}$ is much greater than the sample size n.

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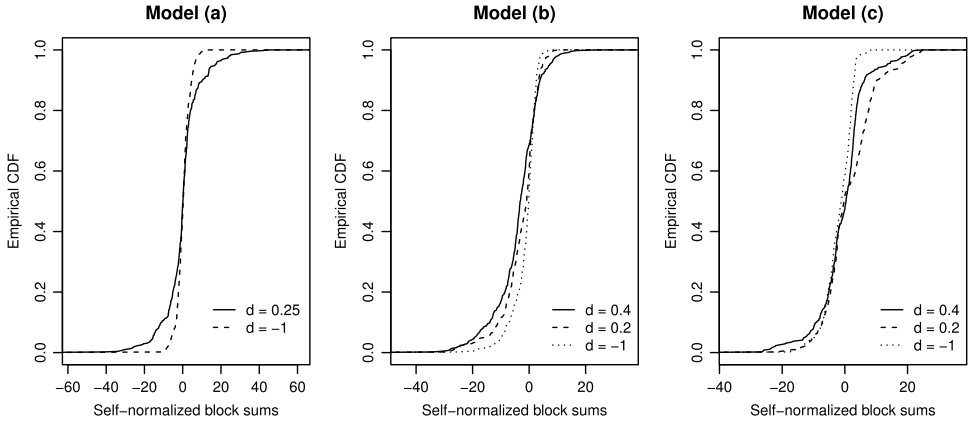

Fig. 1. Examples of realized $\hat{F}_{n,b_n}$ for models (a)–(c) with $n = 500$, $c = 1$ and different choices of d. The x-axis represents the self-normalized block sums, which have been appropriately centered and scaled.

Case (a) represents the Gaussian linear process which has been extensively used in the literature for modeling time series data. It has long-range dependence if $0 < d < 1/2$. We let $d \in \{0.25, -1\}$. The choice $d = 0.25$ corresponds to long-range dependence (LRD) and the choice $d = -1$ corresponds to short-range dependence (SRD).

Case (b) involves an additional nonlinear transformation and now $\{X_i\}$ is LRD if $0.25 < d < 0.5$. We let $d \in \{0.4, 0.2, -1\}$. When $d = 0.4$, both $\{Z_i\}$ and $\{X_i\}$ have LRD (the limit for $\{X_i\}$ is the Rosenblatt process); when $d = 0.2$, $\{Z_i\}$ has LRD and $\{X_i\}$ has SRD (the limit for $\{X_i\}$ is Brownian motion); when $d = -1$, both $\{Z_i\}$ and $\{X_i\}$ have SRD (the limit for $\{X_i\}$ is Brownian motion). See for example [85,89].

Case (c) corresponds to a process $\{X_i\}$ with marginal distribution t with 1.5 degrees of freedom and hence with infinite variance. We let $d \in \{0.4, 0.2, -1\}$. When $d = 0.4$ and $d = 0.2$, both $\{Z_i\}$ and $\{X_i\}$ have LRD (the limit for $\{X_i\}$ is the fractional Brownian motion); when $d = -1$, both $\{Z_i\}$ and $\{X_i\}$ have SRD (the limit for $\{X_i\}$ is symmetric $(3/2)$-stable Lévy motion). See [74] for the boundary between SRD and LRD in the heavy tail case. We also consider the situation with a non-constant slowly varying function, where we let $a_j = (1 + j)^{d-1} \log(1 + j)$, $j \geq 0$, and denote the corresponding cases by (a*), (b*) and (c*), respectively.

We consider the problem of constructing the lower and upper one-sided confidence interval where the nominal level is taken as 90%; see also [55,89] for similar performance assessment of this type. Following Zhang et al. [89], we use throughout the block sizes $b_n = \lfloor cn^{0.5} \rfloor$, $c \in \{0.5, 1, 2\}$. This does not necessarily represent the optimal choice of b_n, but provides us with a spectrum of reasonable block sizes in our finite-sample simulations. For each realization we compute the self-normalized block sums and its empirical distribution function $\hat{F}_{n,b_n}$ as in (17). Examples of realized $\hat{F}_{n,b_n}$ can be found in Fig. 1 for models (a)–(c) with different choices of d. Let q_α ($\alpha = 10\%$) be the 10%-quantile of $\hat{F}_{n,b_n}$, then the lower 90% one-sided confidence interval can be constructed as

$$\left(-\infty,\ \bar{X}_n - n^{-1} \left\{ n^{-1} \sum_{k=1}^{n} \left(S_{1,k} - \frac{k}{n} S_{1,n} \right)^2 \right\}^{1/2} q_\alpha \right].$$

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Table 1
Empirical coverage probabilities of lower and upper (paired in parentheses) one-sided 90% confidence intervals with different combinations of the index d, sample size n and block size $b_n = \lfloor cn^{0.5} \rfloor$ when $a_j = (1+j)^{d-1}$, $j \geq 0$.

d	c	$n = 100$		$n = 500$	
		SNBS	ZHWW2013	SNBS	ZHWW2013
Model (a)					
0.25	0.5	(88.3, 91.1)	(86.8, 90.3)	(92.2, 92.0)	(92.0, 91.5)
	1	(86.1, 86.6)	(85.7, 85.3)	(89.6, 91.2)	(89.3, 91.3)
	2	(82.3, 83.7)	(81.0, 82.2)	(87.5, 87.5)	(87.4, 87.2)
−1	0.5	(93.5, 94.2)	(93.0, 92.9)	(93.2, 93.1)	(92.9, 93.0)
	1	(89.5, 90.7)	(89.0, 90.2)	(91.4, 92.1)	(91.1, 91.7)
	2	(87.1, 86.3)	(86.9, 85.6)	(90.0, 89.0)	(89.9, 89.5)
Model (b)					
0.4	0.5	(90.3, 95.7)	(89.2, 95.2)	(93.2, 96.2)	(92.9, 95.6)
	1	(84.7, 93.6)	(83.8, 92.7)	(88.2, 94.8)	(88.4, 94.9)
	2	(75.9, 91.8)	(75.3, 91.4)	(84.3, 92.8)	(84.0, 92.9)
0.2	0.5	(94.6, 95.8)	(94.0, 94.8)	(95.7, 96.0)	(95.8, 95.6)
	1	(88.8, 93.6)	(88.2, 93.3)	(93.8, 93.6)	(93.7, 93.9)
	2	(81.4, 91.5)	(80.3, 90.8)	(89.4, 92.0)	(89.3, 91.9)
−1	0.5	(97.6, 86.3)	(97.5, 85.5)	(97.0, 86.0)	(97.0, 86.1)
	1	(94.1, 84.2)	(93.5, 83.3)	(94.5, 86.5)	(94.3, 86.5)
	2	(87.2, 84.0)	(86.7, 83.6)	(91.3, 86.6)	(91.2, 86.7)
Model (c)					
0.4	0.5	(74.8, 84.4)	(72.5, 82.9)	(82.2, 78.0)	(81.8, 77.1)
	1	(78.0, 76.9)	(76.5, 75.8)	(77.7, 79.3)	(76.9, 78.9)
	2	(75.5, 73.4)	(74.8, 72.2)	(74.6, 78.6)	(73.8, 78.4)
0.2	0.5	(78.8, 81.4)	(76.7, 79.0)	(80.8, 79.9)	(80.0, 79.6)
	1	(77.0, 80.6)	(75.9, 79.6)	(79.1, 80.8)	(78.7, 80.0)
	2	(77.9, 74.8)	(76.6, 74.1)	(81.1, 77.3)	(80.9, 76.3)
−1	0.5	(82.3, 83.7)	(80.9, 82.2)	(83.6, 85.3)	(83.3, 84.2)
	1	(84.1, 80.0)	(83.2, 79.4)	(81.6, 86.0)	(80.6, 85.5)
	2	(87.4, 71.2)	(86.2, 70.3)	(82.0, 82.9)	(81.7, 82.8)

Similarly, if $q_{1-\alpha}$ $(1-\alpha = 90\%)$ denotes the 90%-quantile of $\hat{F}_{n,b_n}$, then the corresponding upper 90% one-sided confidence interval is

$$\left[\bar{X}_n - n^{-1} \left\{ n^{-1} \sum_{k=1}^{n} (S_{1,k} - \frac{k}{n} S_{1,n})^2 \right\}^{1/2} q_{1-\alpha}, \ +\infty \right).$$

See (18) for details.

In Tables 1 and 2, we report the empirical coverage probabilities of the constructed confidence intervals based on 5000 realizations for each scenario.[13] For example, Table 1 displays the following results of simulation. If $d = 0.25$, $c = 0.5$ and $n = 100$, then the self-normalized block sampling (SNBS) simulation yielded the following: the lower 90% confidence interval included

[13] When evaluating the empirical coverage probability of the constructed confidence interval, we use the averaged mean of 1000 realizations as an approximation to the true mean.

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Table 2
Empirical coverage probabilities of lower and upper (paired in parentheses) one-sided 90% confidence intervals with different combinations of the index d, sample size n and block size $b_n = \lfloor cn^{0.5} \rfloor$ when $a_j = (1+j)^{d-1}\log(1+j)$, $j \geq 0$.

d	c	$n = 100$		$n = 500$	
		SNBS	ZHWW2013	SNBS	ZHWW2013
Model (a*)					
0.25	0.5	(87.8, 87.9)	(86.4, 86.5)	(91.9, 91.6)	(92.0, 91.6)
	1	(84.0, 84.2)	(82.7, 83.0)	(90.4, 89.4)	(90.1, 88.9)
	2	(78.0, 79.2)	(76.8, 78.6)	(84.7, 85.1)	(84.4, 84.8)
−1	0.5	(93.7, 93.6)	(93.1, 92.5)	(94.0, 94.4)	(93.6, 94.4)
	1	(90.9, 89.8)	(90.1, 88.6)	(93.2, 91.9)	(92.8, 92.0)
	2	(86.4, 86.4)	(85.9, 85.4)	(90.6, 90.3)	(90.2, 90.1)
Model (b*)					
0.4	0.5	(84.7, 95.1)	(83.3, 94.3)	(90.3, 98.0)	(90.4, 97.7)
	1	(80.4, 92.3)	(79.4, 91.8)	(86.2, 96.0)	(86.4, 96.0)
	2	(71.7, 90.2)	(70.5, 89.6)	(79.5, 93.6)	(79.6, 93.9)
0.2	0.5	(89.3, 96.8)	(88.7, 96.3)	(94.3, 97.7)	(94.4, 97.5)
	1	(83.6, 93.9)	(83.0, 93.3)	(90.8, 96.7)	(91.1, 96.7)
	2	(77.7, 91.2)	(76.8, 90.4)	(85.1, 95.6)	(85.1, 95.2)
−1	0.5	(98.3, 86.3)	(97.9, 85.6)	(97.1, 87.0)	(97.1, 87.1)
	1	(93.1, 85.2)	(92.8, 84.6)	(95.1, 85.6)	(94.7, 85.7)
	2	(88.6, 84.1)	(87.9, 83.5)	(92.2, 85.9)	(92.1, 85.5)
Model (c*)					
0.4	0.5	(86.3, 85.8)	(84.6, 83.7)	(92.6, 88.0)	(92.3, 88.0)
	1	(83.4, 77.7)	(82.3, 76.1)	(87.3, 83.3)	(87.7, 83.5)
	2	(74.9, 75.2)	(73.2, 74.0)	(81.1, 81.7)	(81.1, 81.1)
0.2	0.5	(83.0, 85.2)	(80.4, 83.3)	(86.6, 84.6)	(86.8, 84.9)
	1	(80.4, 80.5)	(79.5, 78.7)	(84.4, 81.7)	(84.5, 80.7)
	2	(77.9, 73.5)	(76.9, 72.9)	(80.4, 78.8)	(80.9, 78.3)
−1	0.5	(83.9, 83.1)	(82.3, 81.7)	(88.5, 84.0)	(87.5, 83.0)
	1	(80.6, 83.1)	(80.0, 81.7)	(86.8, 83.4)	(85.9, 82.8)
	2	(83.2, 76.7)	(82.2, 75.8)	(85.8, 82.3)	(85.4, 81.5)

the unknown mean μ, 88.3% of the times and the upper 90% confidence interval included the unknown mean μ, 91.1% of the times. We also report the results of the subsampling method of Zhang et al. [89] for a comparison in the column ZHWW2013. Note that the method of Zhang et al. [89] does not take advantage of the technique of self-normalization and therefore it requires an additional bandwidth to utilize the regularly varying property of the asymptotic variance.[14]

It can be seen from Tables 1 and 2 that the method proposed in this paper performs reasonably well, as most of the empirical coverage probabilities are reasonably close to their nominal level of 90%, except for situations with heavy tails where deviations under small sample sizes are expected. However, the results seem to improve as the sample size increases from $n = 100$ to $n = 500$ and the performance is comparable to the method of Zhang et al. [89].[15] Note that the

[14] In Tables 1 and 2, we let the second bandwidth be $l_n = \lfloor n^{0.9} \rfloor$ when using the method of Zhang et al. [89]. Many other choices are possible. We also used $l_n = \lfloor 0.5n^{0.9} \rfloor$ and obtained similar results.

[15] The theoretical assumptions in [89] do not allow for infinite variance.

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Table 3
Empirical coverage probabilities of lower and upper (paired in parentheses) one-sided 90% confidence intervals with the TAR model (65) for different combinations of sample size n and block size $b_n = \lfloor cn^{0.5} \rfloor$.

c	$n = 100$		$n = 500$	
	SNBS	ZHWW2013	SNBS	ZHWW2013
0.5	(92.1, 94.3)	(91.7, 93.7)	(93.2, 89.6)	(93.0, 89.4)
1	(90.0, 88.9)	(88.8, 88.8)	(91.0, 88.0)	(91.3, 88.5)
2	(86.9, 84.7)	(86.1, 84.0)	(89.9, 87.2)	(90.1, 87.3)

choice of sample size $n = 100$ is considered to be challengingly small for inference of long-range dependent processes. Because of self-normalization, our method has the advantage over the one by Zhang et al. [89] in not requiring the choice of a second bandwidth.

Finally, consider the strong mixing Example 4.4, where $X_i = \rho|X_{i-1}| + \epsilon_i$, following the threshold autoregressive model [78]. The ϵ_i's are i.i.d. Gaussian. The results for $\rho = 0.5$ are summarized in Table 3. Observe that the method works quite well in this case as well.

The `R` function implementing the method is available from the authors.

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References

[1] R. Adenstedt, On large-sample estimation for the mean of a stationary random sequence, Ann. Statist. (1974) 1095–1107.
[2] R. Adler, R. Feldman, M.S. Taqqu (Eds.), A Practical Guide to Heavy Tails: Statistical Techniques and Applications, Birkhäuser, 1998.
[3] M.A. Arcones, Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors, Ann. Probab. (1994) 2242–2274.
[4] F. Avram, M.S. Taqqu, Noncentral limit theorems and Appell polynomials, Ann. Probab. 15 (1987) 767–775.
[5] F. Avram, M.S. Taqqu, Weak convergence of sums of moving averages in the α-stable domain of attraction, Ann. Probab. (1992) 483–503.
[6] S. Bai, M.S. Taqqu, Generalized Hermite processes, discrete chaos and limit theorems, Stochastic Process. Appl. 124 (2014) 1710–1739.
[7] S. Bai, M.S. Taqqu, On the validity of resampling methods under long memory, 2015. ArXiv Preprint arXiv:1512.00819.
[8] R. Baillie, Long memory processes and fractional integration in econometrics, J. Econometrics 73 (1996) 5–59.
[9] B. Basrak, D. Krizmanić, J. Segers, A functional limit theorem for dependent sequences with infinite variance stable limits, Ann. Probab. 40 (2012) 2008–2033.
[10] J. Beran, Y. Feng, S. Ghosh, R. Kulik, Long-Memory Processes, Springer, 2013.
[11] A. Betken, M. Wendler, Subsampling for general statistics under long range dependence, 2015. ArXiv Preprint arXiv:1509.05720.
[12] N. Bingham, C. Goldie, J. Teugels, Regular Variation, in: Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 1989.
[13] R. Bradley, Introduction to Strong Mixing Conditions, Kendrick Press, Utah, 2007.
[14] P. Breuer, P. Major, Central limit theorems for non-linear functionals of Gaussian fields, J. Multivariate Anal. 13 (1983) 425–441.
[15] P. Brockwell, R. Davis, Time Series: Theory and Methods, Springer, 1991.

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[16] D. Chambers, E. Slud, Central limit theorems for nonlinear functionals of stationary Gaussian processes, Probab. Theory Related Fields 80 (1989) 323–346.
[17] Y. Davydov, The invariance principle for stationary processes, Theory Probab. Appl. 15 (1970) 487–498.
[18] R. Deo, M. Hsieh, C.M. Hurvich, Long memory in intertrade durations, counts and realized volatility of NYSE stocks, J. Statist. Plann. Inference 140 (2010) 3715–3733.
[19] S. Dharmadhikari, K. Joag-Dev, Upper bounds for the variances of certain random variables, in: Communications in Statistics-Theory and Methods, Vol. 18, 1989, pp. 3235–3247.
[20] Z. Ding, C. Granger, R. Engle, A long memory property of stock market returns and a new model, J. Empir. Finance 1 (1993) 83–106.
[21] R.L. Dobrushin, P. Major, Non-central limit theorems for non-linear functional of Gaussian fields, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 50 (1979) 27–52.
[22] P. Doukhan, Mixing, Springer, 1994.
[23] P. Doukhan, G. Oppenheim, M.S. Taqqu (Eds.), Theory and Applications of Long-Range Dependence, Birkhäuser, 2003.
[24] Z. Fan, Statistical issues and developments in time series analysis and educational measurement (Ph.D. thesis), University of Illinois at Urbana-Champaign, 2010.
[25] J. Flegal, G. Jones, Batch means and spectral variance estimators in Markov chain Monte Carlo, Ann. Statist. 38 (2010) 1034–1070.
[26] L. Giraitis, H. Koul, D. Surgailis, Large Sample Inference for Long Memory Processes, World Scientific Publishing Company Incorporated, 2012.
[27] U. Grenander, G. Szegö, Toeplitz Forms and their Applications, Vol. 321, Univ of California Press, 1958.
[28] A. Gut, Probability: A Graduate Course, Springer Science & Business Media, 2006.
[29] P. Hall, B.-Y. Jing, S.N. Lahiri, On the sampling window method for long-range dependent data, Statist. Sinica 8 (1998) 1189–1204.
[30] M. Haye, M. Viano, Limit theorems under seasonal long-memory, in: Theory and Applications of Long-Range Dependence, 2003, pp. 102–110.
[31] N. Herrndorf, A functional central limit theorem for weakly dependent sequences of random variables, Ann. Probab. (1984) 141–153.
[32] H.-C. Ho, T. Hsing, Limit theorems for functionals of moving averages, Ann. Probab. 25 (1997) 1636–1669.
[33] H. Ho, T. Sun, A central limit theorem for non-instantaneous filters of a stationary Gaussian process, J. Multivariate Anal. 22 (1987) 144–155.
[34] W. Hoeffding, H. Robbins, The central limit theorem for dependent random variables, Duke Math. J. 15 (1948) 773–780.
[35] Y. Huang, S. Volgushev, X. Shao, On self-normalization for censored dependent data, J. Time Series Anal. 36 (2015) 109–124.
[36] I.A. Ibragimov, I.V. Linnik, Independent and Stationary Sequences of Random Variables, Wolters-Noordhoff, Groningen, 1971.
[37] A. Jach, T. McElroy, D.N. Politis, Subsampling inference for the mean of heavy-tailed long-memory time series, J. Time Series Anal. 33 (2012) 96–111.
[38] S. Janson, Gaussian Hilbert Spaces, Vol. 129, Cambridge University Press, 1997.
[39] M. Kac, W. Murdock, G. Szego, On the eigen-values of certain hermitian forms, J. Ration. Mech. Anal. 2 (1953) 767–802.
[40] O. Kallenberg, Foundations of Modern Probability, Springer Science & Business Media, 2006.
[41] A. Kolmogorov, Y. Rozanov, On strong mixing conditions for stationary Gaussian processes, Theory Probab. Appl. 5 (1960) 204–208.
[42] R. Kulik, P. Soulier, Limit theorems for long-memory stochastic volatility models with infinite variance: partial sums and sample covariances, Adv. Appl. Probab. 44 (2012) 1113–1141.
[43] S. Lahiri, Resampling Methods for Dependent Data, Springer Science & Business Media, 2003.
[44] J. Lamperti, Semi-stable stochastic processes, Trans. Amer. Math. Soc. 104 (1962) 62–78.
[45] W. Leland, M.S. Taqqu, W. Willinger, D. Wilson, On the self-similar nature of ethernet traffic (extended version), IEEE/ACM Trans. Netw. 2 (1994) 1–15.
[46] I.N. Lobato, Testing that a dependent process is uncorrelated, J. Amer. Statist. Assoc. 96 (2001) 1066–1076.
[47] P. Major, Limit theorems for non-linear functionals of Gaussian sequences, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 57 (1981) 129–158.
[48] B. Mandelbrot, J. Wallis, Noah, Joseph, and operational hydrology, Water Resour. Res. 4 (1968) 909–918.
[49] M. Maxwell, M. Woodroofe, Central limit theorems for additive functionals of Markov chains, Ann. Probab. 28 (2000) 713–724.

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[50] T. McElroy, D.N. Politis, Robust inference for the mean in the presence of serial correlation and heavy-tailed distributions, Econometric Theory 18 (2002) 1019–1039.
[51] T. McElroy, D.N. Politis, Self-normalization for heavy-tailed time series with long memory, Statist. Sinica 17 (2007) 199–220.
[52] T. McElroy, D.N. Politis, Distribution theory for the studentized mean for long, short, and negative memory time series, J. Econometrics 177 (2013) 60–74.
[53] W.K. Newey, K.D. West, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, Econometrica 55 (1987) 703–708.
[54] J.P. Nolan, Stable Distributions—Models for Heavy Tailed Data, Birkhauser, Boston, 2015, in preparation, Chapter 1 online at: http://academic2.American.edu/˜jpnolan.
[55] D.J. Nordman, S.N. Lahiri, Validity of the sampling window method for long-range dependent linear processes, Econometric Theory 21 (2005) 1087–1111.
[56] A. Novosel'tsev, I. Simonenko, Asymptotic dependence of the extreme eigenvalues of truncated Toeplitz matrices on the rate at which the symbol attains its extremum, St. Petersburg Math. J. 16 (2005) 713–718.
[57] M. Peligrad, On the asymptotic normality of sequences of weak dependent random variables, J. Theoret. Probab. 9 (1996) 703–715.
[58] J.D. Petruccelli, S.W. Woolford, A threshold AR(1) model, J. Appl. Probab. (1984) 270–286.
[59] D.N. Politis, Higher-order accurate, positive semi-definite estimation of large-sample covariance and spectral density matrices, Econometric Theory 27 (2011) 703–744.
[60] D. Politis, J. Romano, M. Wolf, Subsampling, in: Springer Series in Statistics, Springer, New York, 1999.
[61] M. Pourahmadi, Remarks on extreme eigenvalues of Toeplitz matrices, Int. J. Math. Math. Sci. 11 (1988) 23–26.
[62] S. Resnick, Heavy-Tail Phenomena: Probabilistic and Statistical Modeling, Springer Science & Business Media, 2007.
[63] J.P. Romano, M. Wolf, Subsampling inference for the mean in the heavy-tailed case, Metrika 50 (1999) 55–69.
[64] M. Rosenblatt, A central limit theorem and a strong mixing condition, Proc. Natl. Acad. Sci. USA 42 (1956) 43–47.
[65] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Vol. 1, CRC Press, 1994.
[66] G. Seber, A Matrix Handbook for Statisticians, Vol. 15, John Wiley & Sons, 2008.
[67] S. Serra, On the extreme eigenvalues of Hermitian (block) Toeplitz matrices, in: Linear Algebra and its Applications, Vol. 270, 1998, pp. 109–129.
[68] X. Shao, A self-normalized approach to confidence interval construction in time series, J. R. Stat. Soc. Ser. B Stat. Methodol 72 (2010) 343–366.
[69] X. Shao, A simple test of changes in mean in the possible presence of long-range dependence, J. Time Series Anal. 32 (2011) 598–606.
[70] X. Shao, Self-normalization for time series: a review of recent developments, J. Amer. Statist. Assoc. 110 (512) (2015) 1797–1817.
[71] M. Sherman, E. Carlstein, Replicate histograms, J. Amer. Statist. Assoc. 91 (1996) 566–576.
[72] A. Skorokhod, Limit theorems for stochastic processes, Theory Probab. Appl. 1 (1956) 261–290.
[73] A. Skorokhod, Limit theorems for stochastic processes with independent increments, Theory Probab. Appl. 2 (1957) 138–171.
[74] A. Sly, C. Heyde, Nonstandard limit theorem for infinite variance functionals, Ann. Probab. 36 (2008) 796–805.
[75] M.S. Taqqu, Weak convergence to fractional Brownian motion and to the Rosenblatt process, Z. Wahrscheinlichkeitstheor. Verwandte Geb. 31 (1975) 287–302.
[76] M.S. Taqqu, Convergence of integrated processes of arbitrary Hermite rank, Probab. Theory Related Fields 50 (1979) 53–83.
[77] P. Tilli, Universal bounds on the convergence rate of extreme Toeplitz eigenvalues, in: Linear Algebra and its Applications, Vol. 366, 2003, pp. 403–416.
[78] H. Tong, Non-linear Time Series: A Dynamic System Approach, Oxford University Press, Oxford, 1990.
[79] R.L. Tweedie, Sufficient conditions for ergodicity and recurrence of Markov chains on a general state space, Stochastic Process. Appl. 3 (1975) 385–403.
[80] M. Tyran-Kamińska, Convergence to Lévy stable processes under some weak dependence conditions, Stochastic Process. Appl. 120 (2010) 1629–1650.
[81] M. Tyran-Kamińska, Functional limit theorems for linear processes in the domain of attraction of stable laws, Statist. Probab. Lett. 80 (2010) 975–981.
[82] M.S. Veillette, M.S. Taqqu, Properties and numerical evaluation of the Rosenblatt distribution, Bernoulli 19 (2013) 982–1005.
[83] W. Vervaat, Sample path properties of self-similar processes with stationary increments, Ann. Probab. (1985) 1–27.

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[84] W.B. Wu, Nonlinear system theory: another look at dependence, Proc. Natl. Acad. Sci. USA 102 (2005) 14150–14154.
[85] W.B. Wu, Unit root testing for functionals of linear processes, Econometric Theory 22 (2006) 1–14.
[86] W.B. Wu, Strong invariance principles for dependent random variables, Ann. Probab. 35 (2007) 2294–2320.
[87] W.B. Wu, Asymptotic theory for stationary processes, Stat. Interface 4 (2011) 207–226.
[88] W.B. Wu, G. Michailidis, D. Zhang, Simulating sample paths of linear fractional stable motion, IEEE Trans. Inform. Theory 50 (2011) 1086–1096.
[89] T. Zhang, H.-C. Ho, M. Wendler, W.B. Wu, Block sampling under strong dependence, Stochastic Process. Appl. 123 (2013) 2323–2339.
[90] T. Zhang, W.B. Wu, Inference of time-varying regression models, Ann. Statist. 40 (2012) 1376–1402.
[91] Z. Zhou, X. Shao, Inference for linear models with dependent errors, J. R. Stat. Soc. Ser. B Stat. Methodol 75 (2013) 323–343.

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