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Gaussian bounds and collisions of variable speed random walks on lattices with power law conductances

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Abstract

We consider a weighted lattice $\mathbb{Z}^d$ with conductance $\mu_e = |e|^{-\alpha}$. We show that the heat kernel of a variable speed random walk on it satisfies a two-sided Gaussian bound by using an intrinsic metric. We also show that when $d = 2$ and $\alpha \in (-1, 0)$, two independent random walks on such weighted lattice will collide infinitely many times while they are transient.
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Keywords: Random walks; Heat kernel; Gaussian bound; Collisions; Intrinsic metric

1. Introduction

In [22], Hebisch and Saloff-Coste proved that when a group has polynomial volume growth of order D, the heat kernel of a constant speed random walk on the group satisfies a two-sided Gaussian estimate, i.e.,

$$c_1 t^{-D/2} \exp\left(-c_2 \frac{\rho(x, y)^2}{t}\right) \le p_t(x, y) \le c_3 t^{-D/2} \exp\left(-c_4 \frac{\rho(x, y)^2}{t}\right)$$

where $\rho(x, y)$ is the respective graph distance. Delmotte [17] gave equivalence of Gaussian bounds, parabolic Harnack inequalities, and the combination of volume regularity and Poincaré inequality. Later, there are many papers, such as [1–3,6,10,29], showing that Gaussian bounds hold for lattice $\mathbb{Z}^d$ with different random conductances. In this paper, we consider a deterministic

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weighted lattice which does not satisfy Poincaré inequalities for all (sufficiently large) balls or volume doubling property. We show that the two-side Gaussian bound still holds for a variable random walk on the weighted lattice, but with a metric which is not comparable to the Euclidean metric.

Let $\alpha \in \mathbb{R}$. For $x, y \in \mathbb{Z}^d$ with $|x-y|_1 = 1$, we set $\mu_{xy} = (|x|_\infty \vee |y|_\infty)^{-\alpha}$ for the conductance of (x, y). For convenience, we set $\mu_{xy} = 0$ if x and y are not nearest neighbor. Write $\mu_x = \sum_y \mu_{xy}$ and $\nu_x = (|x|_\infty \vee 1)^\alpha$ for each $x \in \mathbb{Z}^d$. Let $X = \{X_t : t \geq 0\}$ be a continuous time random walk on the lattice $\mathbb{Z}^d$ with generator

$$\mathscr{L} f(x) = \frac{1}{\nu_x} \sum_{y \in \mathbb{Z}^d} (f(y) - f(x)) \mu_{xy}.$$

Then X is a variable speed random walk waiting for an exponentially distributed time with mean $\frac{\nu_x}{\mu_x} \asymp |x|_\infty^{2\alpha}$ before jumping. The transition density of X with respect to ν is denoted by

$$p_t(x, y) = \frac{\mathbb{P}_x(X_t = y)}{\nu_y}.$$

To show the Gaussian bounds hold, we introduce a metric ρ of $\mathbb{Z}^d$. We call $x_0 \cdots x_m$ a path if $|x_{i+1} - x_i|_1 = 1$ for each $i < m$. Let $\rho(x, x) = 0$ for $x \in \mathbb{Z}^d$, and for $x, y \in \mathbb{Z}^d$ with $y \neq x$ set

$$\rho(x, y) = \min \left\{ \sum_{i=0}^m \nu_{z_i} : z_0 z_1 \cdots z_m \text{ is a path with } z_0 = x \text{ and } z_m = y \right\}.$$

Then there exists a constant $C = C(\alpha, d)$, such that

$$\frac{1}{\nu_x} \sum_{y \sim x} \rho(x, y)^2 \mu_{xy} \leq C \quad \text{for all } x. \tag{1.1}$$

Metrics satisfying (1.1) are called intrinsic metrics, see [20,31]. One may expect that analogues of diffusion processes on manifolds hold using the intrinsic metrics for random walks on graphs. For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$, write $B_\rho(x, r) = \{y \in \mathbb{Z}^d : \rho(x, y) \leq r\}$ for a ρ-ball. We extend ν to a measure on $\mathbb{Z}^d$ and set

$$V_\rho(x, r) = \nu(B_\rho(x, r)).$$

Theorem 1.1. *Let $\alpha > -1$. Let $x, y \in \mathbb{Z}^d$ and $t > 0$. If $t < (\nu_x \vee \nu_y)\rho(x, y)$, then*

$$p_t(x, y) \leq c_1 (\nu_x \nu_y)^{-1/2} \exp\left(-\frac{c_2 \rho(x, y)}{\nu_x \vee \nu_y} \left(1 \vee \log\left(\frac{(\nu_x \vee \nu_y)\rho(x, y)}{t} \right) \right) \right). \tag{1.2}$$

If $t \geq (\nu_x \vee \nu_y)\rho(x, y)$, then

$$p_t(x, y) \leq \frac{c_3}{\sqrt{V_\rho(x, t^{1/2}) V_\rho(y, t^{1/2})}} \exp\left(-c_4 \frac{\rho(x, y)^2}{t} \right) \tag{1.3}$$

and

$$p_t(x, y) \geq \frac{c_5}{\sqrt{V_\rho(x, t^{1/2}) V_\rho(y, t^{1/2})}} \exp\left(-c_6 \frac{\rho(x, y)^2}{t} \right). \tag{1.4}$$

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Remark 1.2. (1) In Lemmas 2.2 and 2.4, we give the bounds of $\rho(x, y)$ and $V_\rho(x, t^{1/2})$, respectively.
(2) Note that if $\alpha < -1$ then $\sup_{x,y} \rho(x, y) < \infty$. The behavior at the critical point $\alpha = -1$ is much complex. For example, one may consider the random conductance model with Cauchy tails separately, see [8,4]. In addition, one can see [27, p. 139, Example B.1] for a similar result that diffusion processes on Euclidean space with radial weights satisfy Gaussian estimates over a range of α.

Next, we consider the collision problem of random walks on these weighted lattices. As usual, we say that two walks X and X' collide infinitely often if there exists a sequence of (random) times $\{t_i : i \geq 1\}$ with $\lim_i t_i = \infty$ such that $X_{t_i} = X'_{t_i}$ almost surely for each i. In [28], Pólya first studied whether two independent simple random walks on $\mathbb{Z}^d$ collide infinitely often. He reduced it to the problem of a single walker returning to his starting point. Later Jain and Pruitt in [24] showed the Hausdorff dimension of the intersection of two independent stable processes, and Shieh in [30] gave a sufficient condition for infinite collisions of Lévy processes in $\mathbb{R}$. However, if the walks are not on a homogeneous space, the problem will be complicated. Recently in [23], Hutchcroft and Peres used the Mass-Transport Principle to prove that a recurrent reversible random rooted graph has the infinite collision property. Examples that two recurrent random walks will never meet, were shown in [9,7,25]. Here, we give another example that two transient random walks will collide infinitely often.

Theorem 1.3. *Let $\alpha > -1$. Let X' be an independent copy of X.*

(1) *Process X is recurrent if and only if $\alpha \geq d - 2$.*
(2) *If $d \leq 2$, then X and X' collide infinitely often.*
(3) *If $d \geq 3$, then X and X' collide finitely often.*

Remark 1.4. It is much interesting that X is not recurrent while X and X' collide infinitely often when $d = 2$ and $\alpha \in (-1, 0)$. Similarly, when $d \geq 3$ and $\alpha \geq d - 2$, X is recurrent while X and X' collide finitely often.

In Section 2, we obtain some geometric properties of the weighted lattice $\mathbb{Z}^d$. In Section 3, we obtain an upper bound on $p_T(w, w)$ by using the approach of Barlow and Chen [5], which in turn is based on [26,2]. In Section 4, we obtain the lower bounds of near diagonal transition probability by using the result of Delmotte [17] and a chain argument. In Section 5, we give the proof of Theorem 1.1. Section 6 deals with the proof of Theorem 1.3 by the two-sided Gaussian bounds obtained in Section 5.

Throughout this paper, we use the notation c, c' etc. to denote fixed positive constants which may vary on each appearance, and c_i to denote positive constants which are fixed in each argument. If we need to refer to constant c_1 of Lemma 2.1 elsewhere we will use the notation $c_{2.1.1}$. For any two functions f and g, we say $f \asymp g$ if there exists $c_i(\alpha, d) > 0$ such that $c_1 f \leq g \leq c_2 f$. For brevity, we write $|\cdot|_p$ for the L^p-norm of the Euclidean space $\mathbb{R}^d$, while $|\cdot|$ instead of $|\cdot|_\infty$ for the L^∞-norm. Write $B(x, r) = \{y \in \mathbb{Z}^d : |y - x| \leq r\}$ for an L^∞-ball.

2. Some geometric properties

Fix $\alpha > -1$ henceforth. In this section, we shall estimate the metric $\rho(x, y)$ and the volume $V_\rho(x, r)$, and give functional inequalities. Let us begin with the volume of a path.

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Lemma 2.1. *Let $z_0 \cdots z_n$ be a path with* $\max\{|z_0|, |z_n|, |z_0 - z_n|_1\} \geq n \geq 1$. *Then*

$$c_1 n(|z_0| \vee |z_n|)^\alpha \leq \sum_{i=0}^{n} \nu_{z_i} \leq c_2 n(|z_0| \vee |z_n|)^\alpha. \tag{2.1}$$

Proof. Without loss of generality, we may assume that $|z_0| \geq |z_n|$ in the following. (Otherwise, relabel z_{n-k} with z_k for all k.) Then

$$|z_0 - z_n|_1 \leq d|z_0 - z_n| \leq d|z_0| + d|z_n| \leq 2d|z_0|.$$

Using the condition $\max\{|z_0|, |z_n|, |z_0 - z_n|_1\} \geq n \geq 1$, we get

$$|z_0| \geq \tfrac{n}{2d} \vee 1. \tag{2.2}$$

Since $z_0 \cdots z_n$ is a path, we have $|z_i - z_0| \leq i$ for each i. So, $\nu_{z_i} = (|z_i| \vee 1)^\alpha$ takes value between $(|z_0| + i)^\alpha$ and $((|z_0| - i) \vee 1)^\alpha$. Hence $\nu_{z_i} \geq c|z_0|^\alpha$ for $i \leq \frac{n}{4d}$, which implies

$$\sum_{i=0}^{n} \nu_{z_i} \geq \sum_{i \leq n/(4d)} \nu_{z_i} \geq c\lceil n/4d \rceil \nu_{z_0} \geq c' n|z_0|^\alpha = c' n(|z_0| \vee |z_n|)^\alpha. \tag{2.3}$$

We have proved the lower bound in (2.1). For the upper bound, we consider two cases.

Case I: $|z_0| \geq |z_n| \vee n$. Directly calculate

$$\sum_{i=0}^{n} \nu_{z_i} \leq \sum_{i=0}^{n} ((|z_0| + i)^\alpha + ((|z_0| - i) \vee 1)^\alpha) \leq 2 \sum_{i=|z_0|-n}^{i=|z_0|+n} (i \vee 1)^\alpha$$
$$\leq c_1 \int_{|z_0|-n}^{|z_0|+n} x^\alpha dx = \frac{c_1}{1+\alpha}((|z_0| + n)^{\alpha+1} - (|z_0| - n)^{\alpha+1}).$$

Since $\lim_{t \to 0+}((1+t)^{\alpha+1} - (1-t)^{\alpha+1})t^{-1} = 2(\alpha+1)$, we obtain

$$\sup_{t \in (0,1]} |((1+t)^{\alpha+1} - (1-t)^{\alpha+1})t^{-1}| \leq c_2.$$

Substituting $t = \frac{n}{|z_0|} \leq 1$ into the above inequality gives

$$\sum_{i=0}^{n} \nu_{z_i} \leq \frac{c_1}{1+\alpha}((|z_0| + n)^{\alpha+1} - (|z_0| - n)^{\alpha+1}) \leq \frac{c_1 c_2}{1+\alpha} n|z_0|^\alpha = cn(|z_0| \vee |z_n|)^\alpha.$$

Case II: $|z_n| \leq |z_0| < n$ and $|z_0 - z_n|_1 = n$. Then $z_0 \cdots z_n$ is an L^1-geodesic path, which implies $\{z_0, \ldots, z_n\} \subset B(0, n)$ and $|\{i : z_i \in B(0, r)\}| \leq (2d+1)r$ for each r. Write

$$k = \lceil \log_2 n \rceil, \qquad T_0 = B(0,1) \quad \text{and} \quad T_l = B(0, 2^l) - B(0, 2^{l-1}) \quad \text{for } l \geq 1.$$

Then

$$\sum_{i=0}^{n} \nu_{z_i} = \sum_{l=0}^{k} \sum_{i : z_i \in T_l} \nu_{z_i} \leq c \sum_{l=0}^{k} 2^{l\alpha} |\{i : z_i \in T_l\}| \leq c \sum_{l=0}^{k} 2^{\alpha l} |\{i : z_i \in B(0, 2^l)\}|$$
$$\leq c \sum_{l=0}^{k} 2^{\alpha l}(2d+1) \cdot 2^l = c' \sum_{l=0}^{k} 2^{(\alpha+1)l} \leq c'' 2^{(1+\alpha)(k-1)} \leq c'' n^{1+\alpha}.$$

Since $\frac{n}{2d} \leq |z_0| < n$, we still have $\sum_{i=0}^{n} \nu_{z_i} \leq c_2 n(|z_0| \vee |z_n|)^\alpha$ and complete the proof. □

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For $x \in \mathbb{Z}^d$ and $r \in \mathbb{R}^+$, we set

$$\rho_x(r) = (|x| \vee r)^\alpha r. \tag{2.4}$$

Then $\rho_x(\cdot)$ is strictly increasing and

$$\left(\frac{r}{s}\right)^{c_1} \le \frac{\rho_x(r)}{\rho_x(s)} \le \left(\frac{r}{s}\right)^{c_2}, \quad \forall\, r \ge s > 0. \tag{2.5}$$

A simple calculation gives, if $x, y \in \mathbb{Z}^d$ and $r \ge \kappa|x-y|$, then there exists $C = C(\alpha, \kappa) > 0$ such that

$$C^{-1}\rho_y(r) \le \rho_x(r) \le C\rho_y(r). \tag{2.6}$$

Set $\rho_x^{-1}(r) = (|x| \vee r^{1/(1+\alpha)})^{-\alpha} r$, which is the inverse function of ρ_x. Then $\rho_x^{-1}(\cdot)$ also satisfies (2.5) and (2.6).

Lemma 2.2. *Let $x, y \in \mathbb{Z}^d$ with $x \ne y$. Let γ be an L^1-geodesic path from x to y. Then*

$$\left\{\rho(x,y), \sum_{u\in V(\gamma)} \nu_u, \sum_{(u,v)\in E(\gamma)} \mu_{uv}^{-1}\right\} \subset [c_1\rho_x(|x-y|), c_1^{-1}\rho_x(|x-y|)].$$

Proof. By (2.6), we have $\rho_x(|x-y|) \asymp \rho_y(|x-y|)$. So, we may assume $|x| \ge |y|$ without loss generality. (Otherwise, exchange y with x.) Hence $|x| \ge \frac{1}{2}(|x|+|y|) \ge \frac{1}{2}|x-y|$, which implies

$$\rho_x(|x-y|) \asymp |x-y| \cdot |x|^\alpha. \tag{2.7}$$

Let $z_0 z_1 \cdots z_m$ be a ρ-geodesic path with $z_0 = x$ and $z_m = y$, then $m \ge |x-y|$, and so by Lemma 2.1,

$$\rho(x,y) = \sum_{k=0}^{m} \nu_{z_k} \ge \sum_{k=0}^{\lceil |x-y|/2\rceil} \nu_{z_k} \ge c\lceil |x-y|/2\rceil |x|^\alpha.$$

Since γ is a path from x to y, we have $\sum_{u\in V(\gamma)} \nu_u \ge \rho(x,y)$. Furthermore, by Lemma 2.1

$$\rho(x,y) \le \sum_{u\in V(\gamma)} \nu_u \le c|x-y|_1(|x| \vee |y|)^\alpha \le 2dc|x-y|\,|x|^\alpha.$$

Since $\mu_{uv}^{-1} \asymp (|u| \vee 1)^\alpha = \nu_u$ whenever $u \sim v$, we also have

$$\sum_{(u,v)\in E(\gamma)} \nu_{uv}^{-1} \asymp \sum_{u\in V(\gamma)} \nu_u.$$

Combining these inequalities together, we prove the lemma. □

Since $\rho_x(r)$ is increasing in r, Lemma 2.2 immediately implies Corollary 2.3 as follows. Recall that $B_\rho(x,r)$ is a ρ-ball. One can compare it with an L^∞-ball.

Corollary 2.3. *For any $x \in \mathbb{Z}^d$ and $r > 0$,*

$$B(x, \rho_x^{-1}(c_1 r)) \subset B_\rho(x,r) \subset B(x, \rho_x^{-1}(c_2 r)).$$

Recall that $V_\rho(x,r)$ is the volume of $B_\rho(x,r)$. Set $V(x,r) = \nu(B(x,r))$, similarly.

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Lemma 2.4. *Let $x \in \mathbb{Z}^d$ and $r > 0$.*

(1) $V(x, r) \asymp r^d(|x| \vee r)^\alpha \quad$ *if* $r \geq 1$.

(2) $V_\rho(x, r) \asymp V(x, \rho_x^{-1}(r)) \asymp \begin{cases} \nu_x & \text{if } r < \nu_x; \\ r^d|x|^{-(d-1)\alpha} & \text{if } \nu_x \leq r \leq |x|^{1+\alpha}; \\ r^{(d+\alpha)/(1+\alpha)} & \text{if } r > |x|^{1+\alpha}. \end{cases}$

Proof. (1) Let x_1 be the first coordinate of x. Set

$$\Lambda = \{s = (s_1, \ldots, s_d) \in B(x, r) : s_1 = x_1\}.$$

Write $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{Z}^d$. By Lemma 2.1, for each $s \in \Lambda$ we have

$$\sum_{l=-r}^{r} \nu_{s+le_1} \asymp r(|s - re_1| \vee |s + re_1|)^\alpha \asymp r(|s| \vee r)^\alpha \asymp r(|x| \vee r)^\alpha.$$

Hence,

$$V(x, r) = \sum_{s\in\Lambda} \sum_{l=-r}^{r} \nu_{s+le_1} \asymp |\Lambda| \cdot r(|x| \vee r)^\alpha \asymp r^d(|x| \vee r)^\alpha. \tag{2.8}$$

(2) Using (2.8) and Corollary 2.3, we get the desired result. □

Lemma 2.5. *Let $w \in \mathbb{Z}^d$ and $R \geq 1$. Then for any $x \in B(w, R)$ and $r \in [1, R]$,*

$$V(w, R) \leq c_1 \left(\frac{R}{r}\right)^{c_1} V(x, r). \tag{2.9}$$

Especially, $V(w, R) \leq c_1 R^{c_1} \nu_x$.

Proof. It follows directly from Lemma 2.4(1). □

Remark 2.6. If a measure m on $\mathbb{Z}^d$ satisfies $m(B(w, 2R)) \leq cm(B(w, R))$ for all $w \in \mathbb{Z}^d$ and $R \geq 1$ then we say that $(\mathbb{Z}^d, m)$ satisfies the volume doubling property. Hence $(\mathbb{Z}^d, \nu)$ satisfies the volume doubling property as shown in Lemma 2.5. However, $(\mathbb{Z}^d, \mu)$ does not satisfy it since that $\lim_{w\to\infty} \frac{\mu(B(w,|w|/2))}{\mu(B(w,|w|))} = 0$ when $\alpha > d$.

In [32] Virág, extending the early result of [18], showed that any convex subset A of the lattice $\mathbb{Z}^d$ satisfies the Poincaré inequality:

$$\min_a \sum_{u\in A} (f(u) - a)^2 \leq C_d D(A)^2 \sum_{u,v\in A, u\sim v} (f(u) - f(v))^2 \quad \text{for all } f : A \to \mathbb{R},$$

where $D(A) = \max\{|u - v| : u, v \in A\}$ is the diameter of A and C_d is a constant which depends on d. In the following, we shall apply their technique to our weighted lattices.

Lemma 2.7. *Let $x \in \mathbb{Z}^d$, $r > 0$. Then for any function f on $B(x, r)$,*

$$\min_a \sum_{u\in B(x,r)} (f(u) - a)^2 \nu_u \leq c_1[\rho_x(r)]^2 \sum_{u,v\in B(x,r)} (f(u) - f(v))^2 \mu_{uv}. \tag{2.10}$$

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Proof. If $r \in (0, 1)$ then inequality (2.10) is true since both sides are zero. So, let $r \geq 1$.

By [32, Proposition 2], for each $u, v \in \mathbb{Z}^d$ we can choose a path γ_{uv} such that, (1) γ_{uv} is an L^1-geodesic path from u to v; (2) each site in γ_{uv} has L^∞-distance less than 1 from the Euclidean line $\overline{uv}$. For $u, y \in \mathbb{Z}^d$, write

$$\Lambda_{uy} = \{s + z : s \in \gamma_{y,2y-u}, |z| \leq 4, z \in \mathbb{Z}^d\}.$$

By the construction, we have

$$1_{\{y\in\gamma_{uv}\}} \leq 1_{\{v\in\Lambda_{uy}\}} + 1_{\{u\in\Lambda_{vy}\}} \quad \text{for all } u, v, y.$$

By Lemma 2.2,

$$\sum_{v\in\Lambda_{uy}} \nu_v \leq \sum_{s\in\gamma_{y,2y-u}} \sum_{z\in\mathbb{Z}^d,|z|\leq 4} \nu_{s+z} \leq c \sum_{s\in\gamma_{y,2y-u}} \nu_s \leq c'\rho_y(|y-u|).$$

So, if $u, y \in B(x, r)$, we can use (2.6) and get

$$\sum_{v\in\Lambda_{u,y}} \nu_v \leq c\rho_y(2r) \leq c'\rho_x(r).$$

By Lemma 2.2, if $u, v \in B(x, r)$ then

$$\sum_{(y,z)\in E(\gamma_{uv})} \mu_{yz}^{-1} \leq c\rho_u(|u-v|) \leq c\rho_x(r).$$

Therefore, writing $B = B(x, r)$ and $\overline{f} = \sum_{u\in B} f(u)\nu_u$,

$$\begin{aligned}
\sum_{u\in B}(f(u)-\overline{f})^2\nu_u &\leq \frac{1}{\nu(B)} \sum_{u,v\in B}(f(u)-f(v))^2\nu_u\nu_v \\
&= \frac{1}{\nu(B)} \sum_{u,v\in B}\left(\sum_{(y,z)\in E(\gamma_{uv})}(f(y)-f(z))\right)^2 \nu_u\nu_v \\
&\leq \frac{1}{\nu(B)} \sum_{u,v\in B}\left(\sum_{(y,z)\in E(\gamma_{uv})}(f(y)-f(z))^2\mu_{yz}\right)\left(\sum_{(y,z)\in E(\gamma_{uv})}\mu_{yz}^{-1}\right)\nu_u\nu_v \\
&\leq \frac{c\rho_x(r)}{\nu(B)} \sum_{u,v\in B}\sum_{(y,z)\in E(\gamma_{uv})}(f(y)-f(z))^2\mu_{yz}\nu_u\nu_v \\
&\leq \frac{c\rho_x(r)}{\nu(B)} \sum_{y,z\in B}(f(y)-f(z))^2\mu_{yz}\sum_{u,v\in B}1_{\{y\in\gamma_{u,v}\}}\nu_u\nu_v \\
&\leq \frac{c\rho_x(r)}{\nu(B)} \sum_{y,z\in B}(f(y)-f(z))^2\mu_{yz}\left(\sum_{u\in B}\nu_u\sum_{v\in\Lambda_{u,y}}\nu_v + \sum_{v\in B}\nu_v\sum_{u\in\Lambda_{v,y}}\nu_u\right) \\
&\leq c'[\rho_x(r)]^2 \sum_{y,z\in B}(f(y)-f(z))^2\mu_{yz},
\end{aligned}$$

where the second inequality is by the Cauchy–Schwarz inequality. □

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Lemma 2.8. *Let $w \in \mathbb{Z}^d$, $R \geq 1$ and $r \in (0, \rho_w(R)]$. Let $g : B(w,R) \to \mathbb{R}^+$ with $\sum_{x\in B(w,R)} g(x)\nu_x \leq 1$. Then*

$$\sum_{x,y\in B(w,R)} (g(x)-g(y))^2\mu_{xy} \geq c_1 r^{-2}\left(\sum_{x\in B(w,R)} g(x)^2\nu_x - \frac{c_2}{V(w,R)}\left(\frac{\rho_w(R)}{r}\right)^{c_3}\right). \tag{2.11}$$

Proof. By (2.5) and (2.6), for each $x \in B(w,R)$, we have $\rho_x(R) \asymp \rho_w(R)$ and

$$\frac{\rho_x^{-1}(r)}{R} = \frac{\rho_x^{-1}(r)}{\rho_x^{-1}(\rho_x(R))} \geq c'\left(\frac{r}{\rho_x(R)}\right)^c \geq c''\left(\frac{r}{\rho_w(R)}\right)^c.$$

Write $\widehat{r} = \min\{\rho_x^{-1}(r) : x \in B(w,R)\} \wedge R$. Then

$$\frac{\widehat{r}}{R} \geq c'\left(\frac{r}{\rho_w(R)}\right)^c.$$

Using Lemma 2.5, we have

$$\frac{V(w,R)}{V(x,\widehat{r}/2)} \leq c\left(\frac{R}{\widehat{r}/2}\right)^c \leq c_1\left(\frac{\rho_w(R)}{r}\right)^{c_2}. \tag{2.12}$$

Choose $B_i = B(x_i, r_i)$, $i \leq N$ such that $B(w,R) = \cup_{i=1}^N B_i$ and $r_i \in [\frac{1}{2}\widehat{r}, \widehat{r}]$ for each i, and

$$|\{i : x \in B_i\}| \leq c_3 \quad \text{for each } x \in B(w,R). \tag{2.13}$$

Use Lemma 2.7,

$$\begin{aligned}
\sum_{x,y\in\mathcal{B}} (g(x)-g(y))^2\mu_{xy} &\geq c_3^{-1}\sum_{i=1}^N \sum_{x,y\in B_i} (g(x)-g(y))^2\mu_{xy} \\
&\geq c_3^{-1}c_{2.7.1}^{-1}\sum_{i=1}^N [\rho_{x_i}(\widehat{r})]^{-2}\sum_{x\in B_i}(g(x)-\overline{g}_i)^2\nu_x \\
&\geq c_3^{-1}c_{2.7.1}^{-1}\sum_{i=1}^N r^{-2}\left(\sum_{x\in B_i} g(x)^2\nu_x - \frac{\left(\sum_{x\in B_i} g(x)\nu_x\right)^2}{V(x_i,\widehat{r}/2)}\right) \\
&\geq c_3^{-1}c_{2.7.1}^{-1}\, r^{-2}\left(\sum_{x\in\mathcal{B}} g(x)^2\nu_x - \frac{c_1}{V(w,R)}\left(\frac{\rho_w(R)}{r}\right)^{c_2}\sum_{i=1}^N\left(\sum_{x\in B_i} g(x)\nu_x\right)^2\right),
\end{aligned}$$

where $\mathcal{B} = B(w,R)$ and $\overline{g}_i$ is the mean of g on B_i. Using (2.13), we get

$$\sum_{i=1}^N \sum_{x\in B_i} g(x)\nu_x \leq c\sum_{x\in\mathcal{B}} g(x)\nu_x \leq c.$$

Combining these inequalities with $\sum_i a_i^2 \leq (\sum_i a_i)^2$ for all $a_i \geq 0$, we complete the proof. □

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Remark 2.9. One cannot expect to improve Lemma 2.8 to the whole space such as

$$\sum_{x,y\in\mathbb{Z}^d}(g(x)-g(y))^2\mu_{xy}\geq c_1r^{-2}\sum_{x\in\mathbb{Z}^d}g(x)^2\nu_x-\frac{c_2r^{-2}}{V(w,R)}\left(\frac{\rho_w(R)}{r}\right)^{c_3} \tag{2.14}$$

for all $r\in(0,\rho_w(R)]$, and $g:\mathbb{Z}^d\to\mathbb{R}^+$ with $\sum_{x\in\mathbb{Z}^d}g(x)\nu_x\leq 1$.

To see this, let us fix $\alpha\in(-1,0)$ and $d\geq 2$. Choose $R\geq 1$ and $w\in\mathbb{Z}^d$ with $|w|=R^{-\alpha^{-1}}$. Then $\rho_w(R)=1$ and

$$V(w,R)\asymp R^d(|w|\vee R)^\alpha=R^{d-1}\to\infty. \tag{2.15}$$

Hence if $r=1$ and $R\to\infty$, then the second term in the right side of (2.14) tends to zero. On the other hand, let $s\geq 1$, and take

$$g(x)=A(s-|x|)1_{B(0,s)}(x),\quad x\in\mathbb{Z}^d,$$

where A is the constant which satisfies $\sum_x g(x)\nu_x=1$. Then

$$\sum_{x,y\in\mathbb{Z}^d}(g(x)-g(y))^2\mu_{xy}\leq A^2\sum_{x,y\in B(0,s)}\mu_{xy}\leq cA^2s^{d-\alpha},$$

and

$$\sum_{x\in\mathbb{Z}^d}g(x)^2\nu_x\geq\frac{A^2s^2}{4}\sum_{x\in B(0,s/2)}\nu_x\geq cA^2s^{2+d+\alpha}.$$

So, as s goes to infinity,

$$\sum_{x,y\in\mathbb{Z}^d}(g(x)-g(y))^2\mu_{xy}\ll\sum_{x\in\mathbb{Z}^d}g(x)^2\nu_x. \tag{2.16}$$

By (2.16) and the statement after (2.15), the inequality (2.14) fails.

3. On-diagonal upper bound estimates

Fix $w\in\mathbb{Z}^d$, $R\geq 1$ and $T=\rho_w(R)^2$. In this section, our aim is to give an upper bound of $p_T(w,w)$. Since ball $B(w,R)$ satisfies the inequality (2.11) while the whole space does not satisfy (2.14), we turn to the random walk X with reflection at $\partial_i B(w,R)$ as Barlow and Chen [5], obtain upper bounds of the corresponding heat kernels first, and then bring these bounds back to the original process.

Write $\mathcal{B}=B(w,R)$ for short. Let Y be the continuous time random walk on $\mathcal{B}$ with generator

$$\mathscr{L}_{\mathcal{B}}f(x)=\frac{1}{\nu_x}\sum_{y\in\mathcal{B}}(f(y)-f(x))\mu_{xy}.$$

For $x\in\mathbb{Z}^d$ and $r>0$, set

$$\tau_{x,r}=\inf\{t\geq 0:X_t\notin B(x,r)\}. \tag{3.1}$$

If Y and X start at the same vertex in $B(w,R-1)$, then we can couple Y and X on the same probability space such that

$$Y_s=X_s\quad\text{for } 0\leq s\leq\tau_{w,R-1}. \tag{3.2}$$

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We use $\mathbb{P}_x$ for both X and Y. Denote the heat kernel of Y by

$$q_t(x, y) = \frac{\mathbb{P}_x(Y_t = y)}{\nu_y}.$$

Proposition 3.1. *For $u \in \mathcal{B}$ and $t \in (0, T]$,*

$$q_t(u, u) \le \frac{c_1}{V(w, R)} \left(\frac{T}{t}\right)^{c_2}. \tag{3.3}$$

Especially, $q_T(w, w) \le \frac{c_1}{V(w,R)}$.

Proof. Given Lemma 2.8, the proof is similar to [2, Proposition 3.1] and [5, Proposition 3.2], so we omit it. □

Lemma 3.2. *Let $x_1, x_2 \in \mathcal{B}$ with $|x_1 - x_2| \ge \frac{1}{16}R$. If $t \le c_1 T$ and $R \ge c_2$, then*

$$q_t(x_1, x_2) \le \tfrac{1}{4V(w,R)}. \tag{3.4}$$

Proof. Write $\eta = \max_{x\in\mathcal{B}} \nu_x$. By (2.5) and (2.6), we have

$$\frac{T^{1/2}}{\eta} = \frac{\rho_w(R)}{\max\limits_{x\in\mathcal{B}} \rho_x(1)} = \inf_{x\in\mathcal{B}} \left\{ \frac{\rho_w(R)}{\rho_x(R)} \cdot \frac{\rho_x(R)}{\rho_x(1)} \right\} \ge c_1 R^{c_1}. \tag{3.5}$$

Set $c_2 = 2^{|\alpha|+2}d$. Let $\widetilde{\nu}_x = \eta^{-1}\nu_x$, $\widetilde{\mu}_{xy} = \eta\mu_{xy}$ and $\widetilde{\rho}(x, y) = c_2^{-1}\eta^{-1}\rho(x, y)$ for $x, y \in \mathcal{B}$. Then

$$\begin{cases} \dfrac{1}{\widetilde{\nu}_x} \displaystyle\sum_{y\in\mathcal{B}} \widetilde{\rho}(x, y)^2 \widetilde{\mu}_{xy} \le 1; \\ \widetilde{\rho}(x, y) \le 1 \quad \text{whenever } x \sim y. \end{cases} \tag{3.6}$$

Hence $\widetilde{\rho}(\cdot, \cdot)$ is an adapted metric, which was introduced by Davies [15,16]. Let $Z_s = Y_{\eta^2 s}$, for $s \ge 0$. Then Z has the generator

$$\widetilde{\mathscr{L}}_{\mathcal{B}} f(x) = \frac{1}{\widetilde{\nu}_x} \sum_{y\in\mathcal{B}} (f(y) - f(x))\widetilde{\mu}_{xy}.$$

We state that there exists constant $c, c' > 0$ such that if $s \le c\eta^{-2}T$ and $R \ge c'$ then

$$\mathbb{P}_{x_1}(Z_s = x_2) \le \frac{\nu_{x_2}}{4V(w, R)}. \tag{3.7}$$

If this is true, then we have (3.4) and prove the lemma.

We now prove (3.7). Set $c_3 = c_{3.1.2} + c_1^{-1}c_{2.5.1}$. For each $i \in \{1, 2\}$, define

$$f_{x_i}(s) = \frac{V(w, R)}{c_{3.1.1}\nu_{x_i}} \left(\frac{\eta^2 s}{T}\right)^{c_3}, \quad s \ge 0.$$

Then by Proposition 3.1, for $s \le \eta^{-2}T$,

$$\mathbb{P}_{x_i}(Z_s = x_i) = \mathbb{P}_{x_i}(Y_{\eta^2 s} = x_i) = q_{\eta^2 s}(x_i, x_i)\nu_{x_i} \le \frac{1}{f_{x_i}(s)}. \tag{3.8}$$

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Given the on-diagonal upper bounds as in (3.8), we can use the 'two-point' method directly to obtain a Gaussian upper bound for $\mathbb{P}_{x_1}(Z_s = x_2)$. The 'two-point' method was initiated by Grigor'yan [21] (in the manifold case) and developed by [14,19] (for random walks on graphs). In our another paper [11], we also considered the 'two-point' method and used the maximum principle to improve the result of [19] by no longer requiring a lower bound on the reversible measure $(\widetilde{\nu}_x)$. So that, the method is potentially very useful to locally irregular graphs and random conductance models, see [5,10]. The metric $d_v(x, y)$ in [11] is just $\widetilde{\rho}(x, y)$ and one can easily check that $f_{x_i}(s)$ is $(1, 2)$–regular on $(0, T]$. (See [11] for the definition.) By (3.5) and Lemma 2.5, for $s \le \eta^{-2}T$,

$$\begin{aligned}\frac{f_{x_i}(s)}{s^{c_3}} &= \frac{V(w, R)}{c_{3.1.1}\nu_{x_i}} \cdot \left(\frac{\eta^2}{T}\right)^{c_3} \le \frac{c_{2.5.1}R^{c_{2.5.1}}}{c_{3.1.1}} \cdot (c_1 R^{c_1})^{-2c_3} \\ &= c' R^{c_{2.5.1}-2c_1c_3} \le c' R^{-c_{2.5.1}} \le c'.\end{aligned}$$

Therefore, by [11, Theorem 1.1] for $s \in \left(\widetilde{\rho}(x_1, x_2),\ \eta^{-2}T\right]$,

$$\mathbb{P}_{x_1}(Z_s = x_2) \le \frac{c_4(\widetilde{\nu}_{x_2}/\widetilde{\nu}_{x_1})^{1/2}}{\sqrt{f_{x_1}(c_5 s) f_{x_2}(c_5 s)}} \exp\left(-c_6 \frac{\widetilde{\rho}(x_1, x_2)^2}{s}\right) \tag{3.9}$$

$$= \frac{c_7 \nu_{x_2}}{V(w, R)} \left(\frac{T}{\eta^2 s}\right)^{c_3} \exp\left(-c_6 c_2^{-2} \frac{\rho(x_1, x_2)^2}{\eta^2 s}\right). \tag{3.10}$$

By Lemma 2.2 and the condition $|x_1 - x_2| \ge \frac{1}{16}R$, we have

$$\rho(x_1, x_2) \ge c_{2.2.1}\rho_{x_1}\left(\tfrac{1}{16}R\right) \ge c_8 \rho_w(R) = c_8 T^{1/2}. \tag{3.11}$$

Substituting (3.11) into (3.10) gives

$$\mathbb{P}_{x_1}(Z_s = x_2) \le \frac{c_7 \nu_{x_2}}{V(w, R)} \left(\frac{T}{\eta^2 s}\right)^{c_3} \exp\left(-c_6 c_2^{-2} c_8^2 \frac{T}{\eta^2 s}\right),$$

which implies (3.7) holds for each $s \in \left(\widetilde{\rho}(x_1, x_2),\ c_9\eta^{-2}T\right]$, provided $c_9 > 0$ is small enough.

On the other hand, by [11, Corollary 2.8] we have the 'long range' bounds, that is, if $s \le \widetilde{\rho}(x_1, x_2)$ then

$$\mathbb{P}_{x_1}(Z_s = x_2) \le c'(\widetilde{\nu}_{x_2}/\widetilde{\nu}_{x_1})^{1/2} e^{-c\widetilde{\rho}(x_1, x_2)}. \tag{3.12}$$

Using (3.5) and (3.11), we have

$$\widetilde{\rho}(x_1, x_2) = c_2^{-1}\eta^{-1}\rho(x_1, x_2) \ge c\eta^{-1}T^{1/2} \ge c' R^{c''}. \tag{3.13}$$

Combining these inequalities with Lemma 2.5,

$$\begin{aligned}\mathbb{P}_{x_1}(Z_s = x_2) &\le c(\widetilde{\nu}_{x_2}/\widetilde{\nu}_{x_1})^{1/2} e^{-c' R^{c''}} = \frac{c\nu_{x_2}}{V(w, R)} \cdot \frac{V(w, R)}{(\nu_{x_2}\nu_{x_1})^{1/2}} e^{-c' R^{c''}} \\ &\le \frac{c\nu_{x_2}}{V(w, R)} \cdot c_{2.5.1} R^{c_{2.5.1}} e^{-c' R^{c''}}.\end{aligned}$$

So, (3.7) holds again if $s \le \widetilde{\rho}(x_1, x_2)$ and $R \ge c$. □

Lemma 3.3. *Let $t \le c_1 T$ and $x \in B(w, \frac{7}{8}R)$. If $R \ge c_2$ then*

$$\mathbb{P}_x\left(Y_t \notin B\left(x, \tfrac{1}{16}R\right)\right) \le \tfrac{1}{4}.$$

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Proof. By Lemma 3.2, we get

$$\mathbb{P}_x\left(Y_t \notin B\left(x, \tfrac{1}{16}R\right)\right) = \sum_{y\in\mathcal{B}-B\left(x,\frac{1}{16}R\right)} q_t(x,y)\nu_y$$
$$\leq \sum_{y\in\mathcal{B}-B\left(x,\frac{1}{16}R\right)} \frac{\nu_y}{4V(w,R)} \leq \frac{1}{4}. \quad \square$$

Now we bring these bounds of the reflection process back to the original process. Note that X and Y agree until time $\tau_{w,R-1}$.

Lemma 3.4. *If $R \geq c_1$ then for $x \in B(w, \frac{5}{8}R)$,*

$$\mathbb{P}_x(\tau_{x,R/8} < c_2 T) \leq \tfrac{1}{2}.$$

Proof. Given Lemma 3.3, the proof is similar to [5, Lemma 4.1], so we omit it. $\square$

Proposition 3.5. *Let $w \in \mathbb{Z}^d$, $R > 0$ and $T = \rho_w(R)^2$. Then*

$$\mathbb{P}_w(X_T = w) \leq \frac{c_1\nu_w}{V(w,R)}.$$

Proof. If $R < (c_{3.4.1} \vee c_{3.2.2})$ then by Lemma 2.5,

$$\frac{\nu_w}{V(w,R)} \geq c'R^{-c} \geq c'(c_{3.4.1} \vee c_{3.2.2})^{-c} \geq c_1^{-1}\mathbb{P}_w(X_T = w).$$

So, let $R \geq (c_{3.4.1} \vee c_{3.2.2})$. Given Lemma 3.4, similar to the inequality (4.6) of Barlow and Chen [5] we obtain

$$p_{c_2T}(w,w) \leq q_{c_2T}(w,w) + \sup_{0<s\leq c_2T}\max_{y\in A} q_s(y,w),$$

where $c_2 = c_{3.4.1} \wedge c_{3.2.1} \wedge 1$ and $A = B(w, 5R/8) - B(w, 5R/8 - 1)$. By Proposition 3.1 and Lemma 3.2,

$$p_T(w,w) \leq p_{c_2T}(w,w) \leq \frac{c_3}{V(w,R)}. \quad \square$$

4. Near diagonal lower bound estimates

In this section, we shall prove the following lower bounds for the near diagonal transition probabilities. Recall $\tau_{x,r}$ from (3.1). Fix $\delta \in (0, 1/2)$. We will use the notation K_i to denote constants which depend on δ, α and d, while $c_i = c_i(\alpha, d)$ as before.

Theorem 4.1. *Let $w \in \mathbb{R}^d$ and $R \geq 1$. For $x_1, x_2 \in B(w,R)$ and $t \in [\delta\rho_w(R)^2,\ 2\rho_w(R)^2]$,*

$$\mathbb{P}_{x_1}(X_t = x_2,\ \tau_{w,c_1R} > t) \geq K_2\frac{\nu_{x_2}}{V(w,R)}. \tag{4.1}$$

Since $(\mathbb{Z}^d, \mu)$ does not satisfy the volume doubling property, we cannot obtain the lower bound by a general approach. Let us begin with the easy case: balls far from the origin.

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Lemma 4.2. *Let $w \in \mathbb{Z}^d$ and $R \geq 1$ with $|w| \geq 32R$. Then for any $x_1, x_2 \in B(w, R)$ and $t \in [\delta\rho_w(R)^2, 2\rho_w(R)^2]$,*

$$\mathbb{P}_{x_1}(X_t = x_2,\ \tau_{w,8R} > t) \geq K_1 R^{-d}. \tag{4.2}$$

Proof. Applying Lemma 3.4 on $B(w, 8R)$, we can choose constant $c_1 \in (0, 1/2)$ which satisfies

$$\mathbb{P}_x(\tau_{x,R} > c_1\rho_w(R)^2) \geq \tfrac{1}{2}, \quad \text{for all } x \in B(w, R). \tag{4.3}$$

Since $|w| \geq 32R$, we have $\rho_w(R) = R|w|^\alpha$ and $B(w, 16R) \subset B(w, |w|/2)$. So, for each $x, y \in B(w, 16R)$ with $x \sim y$,

$$\nu_x \in [4dc_2^{-1}|w|^\alpha, c_2|w|^\alpha] \quad \text{and} \quad \mu_{xy} \in [c_2^{-1}|w|^{-\alpha}, c_2|w|^{-\alpha}].$$

For $x, y \in B(w, 16R)$, set

$$\widetilde{\nu}_x = c_2|w|^{-\alpha}\nu_x \quad \text{and} \quad \widetilde{\mu}_{xy} = \begin{cases} c_2^{-1}|w|^\alpha\mu_{xy}, & \text{if } x \neq y; \\ \widetilde{\nu}_x - c_2^{-1}|w|^\alpha \displaystyle\sum_{z\in B(w,16R)\setminus\{x\}} \mu_{xz}, & \text{if } x = y. \end{cases}$$

Then $\widetilde{\nu}_x, \widetilde{\mu}_{xy} \in [c_3, c_3^{-1}]$ for all $x \in B(w, 16R)$ and $y \in B(w, 16R) \cap B(x, 1)$. Let Z be the (constant speed) continuous time random walk on $B(w, 16R)$ with generator

$$\widetilde{\mathscr{L}} f(u) = \frac{1}{\widetilde{\nu}_u} \sum_{v\in B(w,16R)} (f(v) - f(u))\widetilde{\mu}_{uv}.$$

Then Z and X can be coupled in the same probability space such that

$$Z_s = X_{c_2^{-2}|w|^{2\alpha}s}, \quad \text{for all } s < \sigma = c_2^2|w|^{-2\alpha}\tau_{w,8R},$$

where $\sigma := \inf\{s \geq 0 : Z_s \notin B(w, 8R)\}$.

Fix $x_1, x_2 \in B(w, R)$, and let $u(s, y) = \mathbb{P}_{x_1}(Z_s = y, \sigma > s)/\widetilde{\nu}_y$ for each $y \in B(w, 16R)$ and $s \geq 0$. Then u is a positive solution of the heat equation $\frac{\partial u}{\partial s} = \widetilde{\mathscr{L}} u$ on $(0, \infty) \times B(w, 4R)$. One can easily check that the weighted graph $(B(w, 16R), \widetilde{\mu}, \widetilde{\nu})$ satisfies the volume regularity $DV(c)$, the Poincaré inequality $P(c')$ and the local ellipticity property $\Delta(c'')$, where

$$DV(c): \sum_{y\in B(x,2r)\cap B(w,16R)} \widetilde{\nu}_y \leq c \sum_{y\in B(x,r)\cap B(w,16R)} \widetilde{\nu}_y \quad \text{for } x \in B(w, 16R) \text{ and } r \geq 1;$$

$P(c')$: for all $B(x, r) \subset B(w, 16R)$ and all $f : B(w, 16R) \to \mathbb{R}$,

$$\min_a \sum_{u\in B(x,r)} (f(u) - a)^2\widetilde{\nu}_u \leq c'r^2 \sum_{u,v\in B(x,r)} (f(u) - f(v))^2\widetilde{\mu}_{uv};$$

$\Delta(c'')$: $\widetilde{\mu}_{uv} \geq c''\widetilde{\nu}_u$ for all $u, v \in B(w, 16R)$ with $\widetilde{\mu}_{uv} \neq 0$.

So, $u(s, y)$ satisfies the Harnack inequality on $B(w, 4R)$, see [17, Theorem 1.7] (or [5, Theorem 7.2 and Corollary 6.4]). Therefore,

$$\max_{\left[\frac{1}{2}s_0, s_0\right]\times B(w,2R)} u \leq K_1^{-1} \min_{[\delta c_2^2R^2, 2c_2^2R^2]\times B(w,2R)} u,$$

where $s_0 = c_1\delta c_2^2R^2$. Furthermore, for any $s \in [\delta c_2^2R^2, 2c_2^2R^2]$,

$$\mathbb{P}_{x_1}(Z_s = x_2, \sigma > s) \geq K_1 \left(\sum_{z\in B(w,2R)} \widetilde{\nu}_z\right)^{-1} \sum_{z\in B(w,2R)} \mathbb{P}_{x_1}(Z_{s_0} = z, \sigma > s_0)$$

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$$\geq K_1 c_3 |B(w,2R)|^{-1} \mathbb{P}_{x_1}(Z_{s_0} \in B(w,2R), \sigma > s_0)$$
$$\geq K_1 c_3 (5R)^{-d} \mathbb{P}_{x_1}(\inf\{h : Z_h \notin B(x_1,R)\} > s_0). \tag{4.4}$$

Since $X_t = Z_{c_2^2|w|^{-2\alpha}t}$ for all $t < \tau_{w,8R}$, inequality (4.4) can be rewritten as

$$\mathbb{P}_{x_1}(X_t = x_2, \tau_{w,8R} > t) \geq K_2 R^{-d} \mathbb{P}_{x_1}(\tau_{x_1,R} > \delta c_1 R^2 |w|^{2\alpha}),$$
$$t \in [\delta R^2 |w|^{2\alpha}, 2R^2 |w|^{2\alpha}].$$

Using (4.3), we finish the proof. □

Lemma 4.3. *For any $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$ and $x \in B(0,R)$,*

$$\mathbb{P}_x(|X_t| > K_1 R,\ \tau_{0,c_2 R} > t) \geq \tfrac{1}{4}.$$

Proof. By Proposition 3.5, for any $x, y \in \mathbb{Z}$ and $t > 0$,

$$p_t(x,y) \leq (p_t(x,x)p_t(y,y))^{1/2} \leq c_1 (V(x, \rho_x^{-1}(t^{1/2}))V(y, \rho_y^{-1}(t^{1/2})))^{-1/2}.$$

So, by Lemma 2.4 if $x, y \in B(0,R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$, then

$$p_t(x,y) \leq K_1 t^{-(d+\alpha)/(2+2\alpha)} \leq K_2 R^{-d-\alpha}.$$

Fix $x \in B(0,R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$. By Lemma 2.4 again, for each $\varepsilon \in (0,1)$,

$$\mathbb{P}_x(|X_t| \leq \varepsilon R) = \sum_{y \in B(0,\varepsilon R)} p_t(x,y) v_y \leq V(0,\varepsilon R) \cdot K_2 R^{-(d+\alpha)}$$
$$\leq c_2 (\varepsilon R)^{d+\alpha} \cdot K_2 R^{-(d+\alpha)} = K_2 c_2 \varepsilon^{d+\alpha}.$$

Hence there exists $\varepsilon_0 = \varepsilon_0(\delta, \alpha, d) > 0$ such that

$$\mathbb{P}_x(|X_t| \leq \varepsilon_0 R) \leq \tfrac{1}{4}. \tag{4.5}$$

On the other hand, applying Lemma 3.4 gives

$$\mathbb{P}_x(\tau_{0,(c+1)R} < t) \leq \mathbb{P}_x(\tau_{x,cR} < R^{2+2\alpha}) \leq \tfrac{1}{2}. \tag{4.6}$$

Combining (4.6) with (4.5), we finish the proof. □

Lemma 4.4. *Let $R \geq 1$. Let $x_1, x_2 \in B(0,R) \setminus B(0,\delta R)$ and $t \in [\delta R^{2+2\alpha}, R^{2+2\alpha}]$. Then*

$$\mathbb{P}_{x_1}(X_t = x_2,\ \tau_{0,c_0 R} > t) \geq K_1 R^{-d}. \tag{4.7}$$

Proof. Write $\mathbb{T} = B(0,R) \setminus B(0,\delta R)$. Then $\rho_w(R) \subset [K_1^{-1} R^{1+\alpha}, K_1 R^{1+\alpha}]$ and $B(w,8R) \subset B(0,10R)$ for all $w \in \mathbb{T}$. We now consider two cases.

Case 1. ($d = 1$ and $x_1 x_2 > 0$) or $d \geq 2$. The lower bound will be proved by a standard chaining argument. Let $w_i \in \mathbb{T}$, $i = 0, \ldots, K_2$ be such that $w_0 = x_1, w_{K_2} = x_2$ and $|w_{i+1} - w_i| \leq \frac{\delta}{64} R$ for each i. Using Lemma 4.2 on $B(w_{i+1}, \frac{\delta}{32} R)$ gives

$$\mathbb{P}_u(X_{t/K_2} = v, \tau_{0,10R} > t) \geq K_3 R^{-d} \quad \text{for } u \in B\left(w_i, \tfrac{\delta}{64} R\right),\ v \in B\left(w_{i+1}, \tfrac{\delta}{64} R\right).$$

So,

$$\mathbb{P}_{x_1}(X_t = x_2,\ \tau_{0,10R} > t) \geq \mathbb{P}_{x_1}\left(X_{it/K_2} \in B\left(w_i, \tfrac{\delta}{64} R\right),\ 1 \leq i \leq K_2 - 1, X_t = x_2\right)$$

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$$\geq \left(\left(\tfrac{\delta}{64}R\right)^d \cdot K_3 R^{-d}\right)^{K_2-1} \cdot K_3 R^{-d} = K_4 R^{-d}.$$

Case 2. $d = 1$ and $x_1 x_2 < 0$. Then $\mathbb{T} = ([-R, -\delta R] \cup [\delta R, R]) \cap \mathbb{Z}$. By symmetric, we may assume further that $x_1 \in [\delta R, R]$ and $x_2 \in [-R, -\delta R]$ without lose of generality. By Lemma 4.3, there exist $c_0 = c_0(\alpha, d) > 10$ and $K_5 = K_5(\delta, \alpha, d) \in (0, 1)$ such that

$$\mathbb{P}_x(|X_{t/3}| > K_5 R,\ \tau_{0,c_0 R} > t/3) \geq \tfrac{1}{4}, \quad x \in B(0, R). \tag{4.8}$$

For conciseness, we write $\widehat{\mathbb{P}}$ for the measure of the process X killed on exiting $B(0, c_0 R)$. Let $\delta' = \delta'(\delta, \alpha, d) \in (0, \frac{1}{3}\delta)$ be a small constant, whose value will be taken later. Set $x_* = \lfloor 2\delta' R \rfloor$. By the result of the first case, we have

$$\inf_{s\in[\delta' R^{2+2\alpha}, R^{2+2\alpha}]} \ \inf_{x,y\in B(0,R)\setminus B(0,\delta' R) \text{ with } xy>0} \widehat{\mathbb{P}}_x(X_s = y) \geq K_6 R^{-1}.$$

So,

$$\widehat{\mathbb{P}}_{x_1}(X_{t/3} \in (\delta' R, 2\delta' R)) \geq \tfrac{1}{2} K_6 \delta' \quad \text{and} \quad \inf_{s\in[t/3,t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2) \geq K_6 R^{-1}.$$

For $x \in \mathbb{Z}$, we define $\sigma_x = \inf\{t \geq 0 : X_t = x\}$, the hitting time of vertex x. By the strong Markov property,

$$\begin{aligned}
\mathbb{P}_{x_1}(X_t = x_2,\ \tau_{0,c_0 R} > t) &\geq \widehat{\mathbb{P}}_{x_1}\left(\sigma_{x_*} < \tfrac{t}{3}, \sigma_{-x_*} < \tfrac{2t}{3}, X_t = x_2\right) \\
&\geq \widehat{\mathbb{P}}_{x_1}\left(\sigma_{x_*} < \tfrac{t}{3}\right) \widehat{\mathbb{P}}_{x_*}\left(\sigma_{-x_*} < \tfrac{t}{3}\right) \inf_{s\in[t/3,t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2) \\
&\geq \widehat{\mathbb{P}}_{x_1}(X_{t/3} \in (\delta' R, 2\delta' R)) \widehat{\mathbb{P}}_{x_*}\left(\sigma_{-x_*} < \tfrac{t}{3}\right) \inf_{s\in[t/3,t]} \widehat{\mathbb{P}}_{-x_*}(X_s = x_2) \\
&\geq \tfrac{1}{2} K_6 \delta' \cdot \widehat{\mathbb{P}}_{x_*}\left(\sigma_{-x_*} < \tfrac{t}{3}\right) \cdot K_6 R^{-1}.
\end{aligned} \tag{4.9}$$

So, we need a lower bound of $\widehat{\mathbb{P}}_{x_*}(\sigma_{-x_*} < \frac{t}{3})$. By Lemma 2.2, for any $x \in \mathbb{N}$ and $r, s \geq 2|x|$,

$$\mathbb{P}_x(\sigma_{x-r} > \sigma_{x+s}) = \frac{\sum_{i=x-r}^{x-1} \mu_{i,i+1}^{-1}}{\sum_{i=x-r}^{x+s-1} \mu_{i,i+1}^{-1}} \leq \frac{c_{2.2.1}^{-1} \rho_x(r)}{c_{2.2.1} \rho_x(r+s-1)} = c_{2.2.1}^{-2} \left(\frac{r}{r+s-1}\right)^{1+\alpha}.$$

So, there exists $c_1 = c_1(\alpha, d) \in \{1, 2, 3, \ldots\}$ such that

$$\mathbb{P}_x(\sigma_{-x} > \sigma_{c_1 x}) \leq \tfrac{1}{8}, \quad \text{for all } x \in \mathbb{N}. \tag{4.10}$$

We now take $\frac{1}{3}\delta c_1^{-1} K_5$ as the value of δ'. Then $x_* = \lfloor \frac{2}{3}\delta c_1^{-1} K_5 R \rfloor \in B(0, R)$. By (4.8),

$$\mathbb{P}_{x_*}\left(\sigma_{-x_*} \wedge \sigma_{c_1 x_*} < \tfrac{1}{3}t,\ \tau_{0,c_0 R} > \tfrac{1}{3}t\right) \geq \mathbb{P}_{x_*}(|X_{t/3}| > K_5 R,\ \tau_{0,c_0 R} > t/3) \geq \tfrac{1}{4}. \tag{4.11}$$

Combining (4.11) with (4.10), we get

$$\widehat{\mathbb{P}}_{x_*}\left(\sigma_{-x_*} < \tfrac{t}{3}\right) \geq \tfrac{1}{8}.$$

Substituting the above inequality into (4.9), we show that (4.7) still holds. □

Proof of Theorem 4.1. If $|w| \geq 32R$, then one can take $c_1 = 8$ in (4.1) and the problem is reduced to Lemma 4.2. So, let $R > |w|/32$ in the following. Then

$$\rho_w(R) \in \left[c_1 R^{1+\alpha}, \tfrac{1}{2} c_2 R^{1+\alpha}\right] \quad \text{and} \quad B(w, R) \subset B(0, 40R).$$

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Fix $t \in [\delta\rho_w(R)^2, 2\rho_w(R)^2]$. Then $t \in [\delta c_1 R^{2+2\alpha}, c_2 R^{2+2\alpha}]$. By Lemma 4.3, for any $x \in B(0, 40R)$,

$$\mathbb{P}_x(|X_{t/3}| > K_1 R,\ \tau_{0,c_3 R} > t/3) \geq \tfrac{1}{4}. \tag{4.12}$$

Let $c_4 = c_3 \vee c_2^{1/(2+2\alpha)}$. Then by Lemma 4.4,

$$\mathbb{P}_x(X_{t/3} = y, \tau_{0,c_5 R} > t/3) \geq K_2 R^{-d} \quad \text{for } x, y \in B(0, c_4 R) \setminus B(0, K_1 R). \tag{4.13}$$

Write $\mathbb{T} = B(0, c_3 R) \setminus B(0, K_1 R)$, and use $\widehat{\mathbb{P}}$ to denote the measure of the process X killed on exiting $B(0, c_5 R)$. Then we can rewrite (4.12) and (4.13) as

$$\widehat{\mathbb{P}}_x(X_{t/3} \in \mathbb{T}) \geq \tfrac{1}{4} \quad \text{for } x \in B(0, 40R)$$

and

$$\widehat{\mathbb{P}}_x(X_{t/3} = y) \geq K_2 R^{-d} \quad \text{for } x, y \in \mathbb{T}.$$

Therefore, for any $x_1, x_2 \in B(w, R) \subset B(0, 40R)$,

$$\begin{aligned}
\mathbb{P}_{x_1}(X_t = x_2, \tau_{0,c_5 R} > t) &\geq \sum_{x,y\in\mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x,\ X_{2t/3} = y,\ X_t = x_2) \\
&= \sum_{x,y\in\mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x)\widehat{\mathbb{P}}_x(X_{t/3} = y)\widehat{\mathbb{P}}_y(X_{t/3} = x_2) \\
&\geq K_2 R^{-d} \sum_{x,y\in\mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x)\widehat{\mathbb{P}}_y(X_{t/3} = x_2) \\
&= K_2 R^{-d} \sum_{x,y\in\mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x)\widehat{\mathbb{P}}_{x_2}(X_{t/3} = y)\frac{\nu_{x_2}}{\nu_y} \\
&\geq K_2 R^{-d} \frac{\nu_{x_2}}{\max_{y\in\mathbb{T}} \nu_y} \sum_{x,y\in\mathbb{T}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} = x)\widehat{\mathbb{P}}_{x_2}(X_{t/3} = y) \\
&\geq K_2 R^{-d} \frac{\nu_{x_2}}{K_3 R^{\alpha}} \widehat{\mathbb{P}}_{x_1}(X_{t/3} \in \mathbb{T})\widehat{\mathbb{P}}_{x_2}(X_{t/3} \in \mathbb{T}) \\
&\geq \frac{K_2}{16K_3} \frac{\nu_{x_2}}{R^{d+\alpha}}.
\end{aligned} \tag{4.14}$$

Substituting $V(w, R) \geq cR^{d+\alpha}$ and $\tau_{0,c_5 R} \leq \tau_{w,cR}$ into (4.14), we complete the proof. □

5. Proof of Theorem 1.1

Lemma 5.1. *There exists constant $c_1 > 0$ such that for any $x, y \in \mathbb{Z}^d$,*

$$(\nu_x \vee \nu_y)|\log \nu_x - \log \nu_y|^3 \leq c_1 \rho(x, y).$$

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Proof. Let $|x| > |y| \geq 1$. Directly calculate

$$\begin{aligned}\frac{(\nu_x \vee \nu_y)}{(|x| \vee |y|)^\alpha} \cdot \frac{|\log \nu_x - \log \nu_y|^3}{|x-y|} &\leq \frac{|x|^\alpha \vee |y|^\alpha}{|x|^\alpha} \cdot \frac{|\log(|x|^\alpha) - \log(|y|^\alpha)|^3}{|x|-|y|} \\ &= \left(\tfrac{|y|}{|x|}\right)^{\alpha\wedge 0} \cdot |\alpha|^3 \frac{\log^3(|x|/|y|)}{(|x|/|y|-1)|y|} \\ &\leq |\alpha|^3 \sup_{t>1}\left\{t^{(-\alpha)\vee 0} \cdot \frac{\log^3 t}{t-1}\right\}.\end{aligned}$$

Since $\alpha > -1$, the supremum of the right side is finite and hence if $|x| > |y| \geq 1$ then

$$(\nu_x \vee \nu_y)|\log \nu_x - \log \nu_y|^3 \leq c(|x| \vee |y|)^\alpha |x-y| \leq c'\rho(x,y).$$

The proof of the rest case is the same and so we omit the details. □

Proof of Theorem 1.1. We obtain the Gaussian upper bounds by the same way as Lemma 3.2. Write $\eta = \nu_x \vee \nu_y$ for short. Set $\widetilde{\nu}_u = \eta^{-1}\nu_v$ and $\widetilde{\mu}_{uv} = \eta\mu_{uv}$ for each $u, v \in \mathbb{Z}^d$. Denote $\widetilde{\rho} : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$ by

$$\widetilde{\rho}(u,v) = \left((2^{|\alpha|+2}d)^{-1} \cdot \eta^{-1}\rho(u,v)\right) \wedge |u-v|_1.$$

Then $\widetilde{\rho}(\cdot,\cdot)$ is an adapted metric of $\mathbb{Z}^d$, that is, for all $u \in \mathbb{Z}^d$,

$$\begin{cases} \dfrac{1}{\widetilde{\nu}_u}\displaystyle\sum_{v\in\mathbb{Z}^d} \widetilde{\rho}(u,v)^2\widetilde{\mu}_{uv} \leq 1; \\ \widetilde{\rho}(u,v) \leq 1 \quad \text{whenever } v \sim u. \end{cases}$$

By Lemma 2.2,

$$\eta^{-1}\rho(x,y) \leq (\nu_x \vee \nu_y)^{-1} \cdot c(|x| \vee |y|)^\alpha |x-y| \leq c|x-y|_1.$$

So,

$$c_1^{-1}\eta^{-1}\rho(x,y) \leq \widetilde{\rho}(x,y) \leq c_1\eta^{-1}\rho(x,y). \tag{5.1}$$

Set $Z_s = X_{\eta^2 s}$ for $s \geq 0$. Then Z has the generator

$$\widetilde{\mathscr{L}} f(u) = \frac{1}{\widetilde{\nu}_u}\sum_{v\in\mathbb{Z}^d}(f(v)-f(u))\widetilde{\mu}_{uv}.$$

By Proposition 3.5 for each $z \in \{x, y\}$,

$$\mathbb{P}_z(Z_s = z) = \mathbb{P}_z(X_{\eta^2 s} = z) \leq \frac{c_2\nu_z}{V(z, \rho_z^{-1}(\eta s^{1/2}))} := \frac{1}{f_z(s)}.$$

By Lemma 2.5 and the inequality (2.5), for each $s \geq (\log \nu_x - \log \nu_y)^2$ we have

$$\begin{aligned} f_z(s) &\leq \frac{V(z, \rho_z^{-1}(\eta e^{s^{1/2}}))}{c_2\nu_z} \leq c(\rho_z^{-1}(\eta e^{s^{1/2}}))^c \leq c'\left(\frac{\eta e^{s^{1/2}}}{\rho_z(1)}\right)^{c'} \\ &\leq c'\left(\frac{\nu_x \vee \nu_y}{\nu_x \wedge \nu_y} e^{s^{1/2}}\right)^{c'} \leq c' e^{2c's^{1/2}}. \end{aligned}$$

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Therefore, similar to (3.9) we can apply [11, Theorem 5.1] and get

$$\mathbb{P}_x(Z_s = y) \le \frac{c_3(\nu_y/\nu_x)^{1/2}}{\sqrt{f_x(s/c_3)f_y(s/c_3)}} \exp\left(-\frac{\widetilde{\rho}(x,y)^2}{c_3 s}\right)$$
$$\text{for all } s \ge |c_3 \log(\nu_x/\nu_y)|^3 \vee \widetilde{\rho}(x,y).$$

By the inequality (5.1) and Lemma 5.1,

$$|c_3 \log(\nu_x/\nu_y)|^3 \le c_4 \widetilde{\rho}(x,y).$$

So, for each $t \ge c_4 c_1 \eta \rho(x,y)$, we have $\eta^{-2}t \ge |c_3 \log(\nu_x/\nu_y)|^3 \vee \widetilde{\rho}(x,y)$ and

$$\mathbb{P}_x(X_t = y) = \mathbb{P}_x(Z_{\eta^{-2}t} = y) \le \frac{c_5(\nu_y/\nu_x)^{1/2}}{\sqrt{f_x(\eta^{-2}t/c_5)f_y(\eta^{-2}t/c_5)}} \exp\left(\frac{\widetilde{\rho}(x,y)^2}{c_5\eta^{-2}t}\right)$$
$$\le \frac{c\nu_y}{\sqrt{V(x,\rho_x^{-1}(t^{1/2}))V(y,\rho_y^{-1}(t^{1/2}))}} \exp\left(-c'\frac{\rho(x,y)^2}{t}\right). \tag{5.2}$$

Further, by Lemma 2.4

$$p_t(x,y) \le \frac{c}{\sqrt{V_\rho(x,t^{1/2})V_\rho(y,t^{1/2})}} \exp\left(-c'\frac{\rho(x,y)^2}{t}\right), \quad t \ge c_4 c_1 \eta \rho(x,y). \tag{5.3}$$

On the other hand, by [11, Corollary 2.8], if $s \le c_4 c_1^2 \widetilde{\rho}(x,y)$ then

$$\mathbb{P}_x(Z_s = y) \le c(\widetilde{\nu}_y/\widetilde{\nu}_x)^{1/2} \exp\Big(-c'\widetilde{\rho}(x,y)\big(1 \vee \log(\widetilde{\rho}(x,y)/s)\big)\Big).$$

Hence for each $t \le c_4 c_1 \eta \rho(x,y)$,

$$\mathbb{P}_x(X_t = y) = \mathbb{P}_x(Z_{\eta^{-2}t} = y) \le c(\widetilde{\nu}_y/\widetilde{\nu}_x)^{1/2} \exp\Big(-c'\widetilde{\rho}(x,y)\big(1 \vee \log\big(\eta^2\widetilde{\rho}(x,y)/t\big)\big)\Big)$$
$$\le c(\nu_y/\nu_x)^{1/2} \exp\Big(-c''\eta^{-1}\rho(x,y)\big(1 \vee \log(\eta\rho(x,y)/t)\big)\Big). \tag{5.4}$$

Combining (5.4) with (5.3), we conclude that both (1.2) and (1.3) are true.

The Gaussian lower bound is proved by the standard chaining argument. If $t \ge \rho(x,y)^2$, then there exists $c_6 > 1$ such that $t \ge c_6^{-2}\rho_x(|x-y|)^2$. Applying Theorem 4.1 on $B(x, \rho_x^{-1}(c_6 t^{1/2}))$, we get

$$p_t(x,y) \ge \frac{c}{V(x,\rho_x^{-1}(c_6 t^{1/2}))} \ge \frac{c'}{V_\rho(x,t^{1/2})}. \tag{5.5}$$

So, let $(\nu_x \vee \nu_y)\rho(x,y) \le t \le \rho(x,y)^2$. Fix an L_1-geodesic path γ from x to y. By Lemma 2.2, there exists $c_7 > 1$ such that

$$\nu(\gamma) \le c_7 \rho(x,y).$$

Set $r = t/\rho(x,y)$, then

$$\rho(x,y) \ge r \ge \nu_x \vee \nu_y = \max_{z\in\gamma} \nu_z.$$

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Hence there exists a sequence of vertices $y = z_0, z_1 \cdots, z_m = x$ on the path γ, such that

$$m \le 2c_7\rho(x,y)/r = 2c_7\frac{\rho(x,y)^2}{t} \quad \text{and} \quad r \le \rho(z_{i-1},z_i) \le 2r \quad \text{for } i \le m.$$

As a result,

$$|z_{i-1}-z_{i-2}| \le c_8'\rho^{-1}_{z_{i-1}}(\rho(z_{i-1},z_{i-2})) \le c_8'\rho^{-1}_{z_{i-1}}(2\rho(z_{i-1},z_i)) \le (c_8-1)|z_i-z_{i-1}|.$$

Write $r_i = |z_i - z_{i-1}|$, $F_i = B(z_i, r_i)$ and $F_i^* = B(z_i, c_8 r_i)$ for $i \le m$. Then

$$F_{i-1} \cup F_i \subset F_i^*.$$

Set $s = (4c_7)^{-1}r^2$. Then $s \asymp \rho(z_i, z_{i-1})^2 \asymp \rho_{z_i}(r_i)^2$. As (5.5), we have

$$p_s(y',x') \ge \frac{c_9}{\nu(F_i^*)} \quad \text{for } y' \in F_{i-1},\ x' \in F_i. \tag{5.6}$$

By Lemma 2.5, for $y' \in F_{i-1}$,

$$\mathbb{P}_{y'}(X_s \in F_i) \ge c_9\frac{\nu(F_i)}{\nu(F_i^*)} \ge c_{10}.$$

Note that

$$t - ms = t - m\cdot(4c_7)^{-1}r^2 \ge t - (2c_7\rho(x,y)/r)\cdot(4c_7)^{-1}r\cdot(t/\rho(x,y)) = \tfrac{t}{2}.$$

So, as (5.5) we can get

$$p_{t-ms}(x,y') \ge \frac{c_{11}}{V_\rho(x,t^{1/2})} \quad \text{for } y' \in F_m. \tag{5.7}$$

Therefore,

$$\begin{aligned} p_t(x,y) = p_t(y,x) &\ge \mu_x^{-1}\mathbb{P}_y(X_{is} \in F_i,\ 1 \le i \le m,\ X_t = x) \\ &\ge c_{10}^m \min_{y' \in F_m} \mathbb{P}_{y'}(X_{t-ms} = x)\mu_x^{-1} \\ &= c_{10}^m \min_{y' \in F_m} p_{t-ms}(x,y') \ge c_{10}^m \frac{c_{11}}{V_\rho(x,t^{1/2})} \\ &\ge \frac{c_{11}}{V_\rho(x,t^{1/2})}\exp\{-c_{10}'m\} \ge \frac{c_{11}}{V_\rho(x,t^{1/2})}\exp\left\{-2c_7c_{10}'\frac{\rho(x,y)^2}{t}\right\}, \end{aligned}$$

which implies (1.4). We have completed the proof of Theorem 1.1. □

6. Proof of Theorem 1.3

Proof of Theorem 1.3. (1) By Theorem 1.1 and Lemma 2.4, if $\alpha < d-2$ then

$$\int_1^\infty p_t(0,0)dt \le c\int_1^\infty t^{-(d+\alpha)/(2+2\alpha)}dt = \frac{2+2\alpha}{d-2-\alpha}c < \infty.$$

Hence if $\alpha < d-2$ then X is transient. Similarly, if $\alpha \ge d-2$ then $\int_1^\infty p_t(0,0)dt = \infty$ and so X is recurrent.

(2) Let X' be an independent copy of X. We use $\mathbb{P}_{x,x'}$ for the probability measure of the processes X and X' which start from x and x' respectively.

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If $d = 1$ then

$$\int_1^\infty \mathbb{P}_{0,0}(X_t = X'_t = 0)dt = \int_1^\infty \mathbb{P}_0(X_t = 0)\mathbb{P}_0(X'_t = 0)dt$$
$$= \int_1^\infty p_t(0,0)^2 dt \geq c\int_1^\infty t^{-2(1+\alpha)/(2+2\alpha)}dt = \infty.$$

So, (X, X') is recurrent, which implies X and X' collide at the origin infinitely often.

Let $d = 2$. Fix $\lambda = \lceil 100c_{4.1.1}\rceil > 100$. For $k \geq 1$, we set

$$t_k = \lambda^{2k(1+\alpha)},$$
$$\mathbb{T}_k = B(0, 2\lambda^k) - B(0, \lambda^k),$$
$$\theta_k = \inf\{t \geq 0 : |X_t| \geq \lambda^{k+1}\}, \qquad \theta'_k = \inf\{t \geq 0 : |X'_t| \geq \lambda^{k+1}\}$$

and

$$H_k = \int_0^{\theta_k \wedge \theta'_k \wedge 2t_k} 1_{\{X_t = X'_t \in \mathbb{T}_k\}} dt.$$

So, if $H_k > 0$ then there exists at least one collision of X and X' before their breaking out of $B(0, \lambda^{k+1})$. We shall use the second moment method to estimate the probability of the event $\{H_k > 0\}$ as the approach of [12,13]. Fix $x, y \in B(0, \lambda^k)$. Then

$$\mathbf{E}_{x,y}(H_k) = \int_0^{2t_k} \mathbb{P}_{x,y}(X_t = X'_t \in \mathbb{T}_k, \theta_k > t, \theta'_k > t)dt$$
$$\geq \int_{t_k}^{2t_k} \sum_{u\in\mathbb{T}_k} \mathbb{P}_{x,y}(X_t = u, X'_t = u, \theta_k > t, \theta'_k > t)dt$$
$$= \int_{t_k}^{2t_k} \sum_{u\in\mathbb{T}_k} \mathbb{P}_x(X_t = u, \theta_k > t)\mathbb{P}_y(X'_t = u, \theta'_k > t)dt. \tag{6.1}$$

Note that $t_k = 2^{-2-2\alpha}\rho_0(2\lambda^k)^2$ and $\lambda^{k+1} = \lceil 100c_{4.1.1}\rceil\lambda^k$. Employing Theorem 4.1 on $B(0, 2\lambda^k)$, we get for each $u, v \in B(0, 2\lambda^k)$ and $t \in [t_k, 2t_k]$,

$$\mathbb{P}_u(X_t = v, \theta_k > t) \geq \frac{c\nu_v}{V(0, 2\lambda^k)}.$$

By Lemma 2.4, for $v \in \mathbb{T}_k$,

$$\frac{\nu_v}{V(0, 2\lambda^k)} \geq c\frac{|v|^\alpha}{(2\lambda^k)^{2+\alpha}} \geq c'\lambda^{-2k}.$$

Hence $\mathbb{P}_u(X_t = v, \theta_k > t) \geq c\lambda^{-2k}$ for each $u \in \{x, y\}$, $v \in \mathbb{T}_k$ and $t \in [t_k, 2t_k]$. Therefore, inequality (6.1) becomes

$$\mathbf{E}_{x,y}(H_k) \geq (c\lambda^{-2k})^2 \cdot |\mathbb{T}_k| \cdot (2t_k - t_k) \geq c^2\lambda^{-4k} \cdot c'(\lambda^k)^2 \cdot \lambda^{2k(1+\alpha)} = c''\lambda^{2k\alpha}. \tag{6.2}$$

On the other hand, for any $u \in \mathbb{T}_k$,

$$\mathbf{E}_{u,u}(H_k) \leq \int_0^{2t_k} \sum_{w\in\mathbb{T}_k} [\mathbb{P}_u(X_t = w)]^2 dt$$
$$\leq \frac{\max_{w\in\mathbb{T}_k} \nu_w}{\nu_u} \int_0^{2t_k} \sum_{w\in\mathbb{T}_k} \mathbb{P}_u(X_t = w)\mathbb{P}_w(X_t = u)dt$$

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$$\begin{aligned}
&\le c\int_0^{2t_k} \mathbb{P}_u(X_{2t}=u)dt \le c\nu_u^2 + c\int_{\nu_u^2}^{2t_k} \mathbb{P}_u(X_{2t}=u)dt\\
&\le c\nu_u^2 + \int_{\nu_u^2}^{2t_k} \frac{c'\nu_u}{V_\rho(u,t^{1/2})}dt\\
&\le c\nu_u^2 + c''\nu_u\int_{\nu_u^2}^{2t_k} |u|^\alpha t^{-1}dt,
\end{aligned}$$

where the last second inequality is by (1.3), while the last by Lemma 2.4. Hence

$$\begin{aligned}
\mathbf{E}_{u,u}(H_k) &\le c\nu_u^2 + c\nu_u|u|^\alpha(\log(2t_k) - \log(\nu_u^2))\\
&\le c'\lambda^{2k\alpha} + c'\lambda^{2k\alpha}\cdot(\log(\lambda^{2k(1+\alpha)}) - \log(\lambda^{2k\alpha})) \le c''k\lambda^{2k\alpha}. \qquad (6.3)
\end{aligned}$$

By the strong Markov property,

$$\begin{aligned}
\mathbf{E}_{x,y}(H_k^2) &= 2\mathbf{E}_{x,y}\left(\int_0^{\theta_k\wedge\theta_k'\wedge 2t_k} 1_{\{X_t=X_t'\in\mathbb{T}_k\}}dt\int_t^{\theta_k\wedge\theta_k'\wedge 2t_k} 1_{\{X_s=X_s'\in\mathbb{T}_k\}}ds\right)\\
&\le 2\mathbf{E}_{x,y}\left(\int_0^{\theta_k\wedge\theta_k'\wedge 2t_k} 1_{\{X_t=X_t'\in\mathbb{T}_k\}}\mathbf{E}_{X_t,X_t'}(H_k)dt\right)\\
&\le 2\sup_{u\in\mathbb{T}_k}\mathbf{E}_{u,u}(H_k)\mathbf{E}_{x,y}(H_k).
\end{aligned}$$

So, by (6.3), (6.2) and the Cauchy–Schwarz inequality,

$$\mathbb{P}_{x,y}(H_k>0) \ge \frac{[\mathbf{E}_{x,y}(H_k)]^2}{\mathbf{E}_{x,y}(H_k^2)} \ge \frac{\mathbf{E}_{x,y}(H_k)}{2\sup_{u\in\mathbb{T}_k}\mathbf{E}_{u,u}(H_k)} \ge \frac{c}{k}.$$

Therefore, when X and X' start from $x, y\in B(0,\lambda^k)$ respectively, the probability that they will collide before their breaking out $B(0,\lambda^{k+1})$, is not less than $\frac{c}{k}$. Note that $\sum_k \frac{1}{k}=\infty$. Using the second Borel–Cantelli Lemma as [13, Theorem 1.1], we prove that X and X' collide infinitely often when $d=2$.

(3) Let $d\ge 3$. For $k\ge 0$, set

$$\mathbb{T}_k = B(0,2^{k+1}) - B(0,2^k) \quad\text{and}\quad Z_k = \int_0^\infty 1_{\{X_t=X_t'\in\mathbb{T}_k\}}dt.$$

Then

$$\begin{aligned}
\mathbf{E}_{0,0}(Z_k) &= \sum_{u\in\mathbb{T}_k}\int_0^\infty [\mathbb{P}_0(X_t=u)]^2dt\\
&= \sum_{u\in\mathbb{T}_k}\int_{t_k}^\infty [\mathbb{P}_0(X_t=u)]^2dt + \sum_{u\in\mathbb{T}_k}\int_{s_k}^{t_k}[\mathbb{P}_0(X_t=u)]^2dt\\
&\quad + \sum_{u\in\mathbb{T}_k}\int_0^{s_k}[\mathbb{P}_0(X_t=u)]^2dt\\
&= I_1+I_2+I_3,
\end{aligned}$$

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where $s_k = (1 \vee 2^{k\alpha})2^{k(1+\alpha)}$ and $t_k = 2^{k(2+2\alpha)}$. We shall deal with the three sums separately. Since $t_k \geq c\rho(0,u)^2$ for $u \in \mathbb{T}_k$, we can use Theorem 1.1 and Lemma 2.4, and get

$$\begin{aligned} I_1 &\leq \sum_{u\in\mathbb{T}_k} \int_{t_k}^{\infty} \frac{c\nu_u^2}{V_\rho(0,t^{1/2})V_\rho(u,t^{1/2})}dt \\ &\leq |\mathbb{T}_k| \cdot \max_{u\in\mathbb{T}_k} \nu_u^2 \cdot \int_{t_k}^{\infty} c' t^{-(d+\alpha)/(1+\alpha)}dt \\ &\leq 2^{dk} \cdot c'' 2^{2k\alpha} \cdot c'''(2^{2k(1+\alpha)})^{1-(d+\alpha)/(1+\alpha)} \\ &= c2^{k(2+2\alpha-d)}. \end{aligned}$$

Next, since $(1 \vee \nu_u)\rho(0,u) \geq cs_k$ and $t_k^{1/2} \leq c'|u|^{1+\alpha}$ for $u \in \mathbb{T}_k$, using Theorem 1.1 and Lemma 2.4 again gives

$$\begin{aligned} I_2 &\leq \sum_{u\in\mathbb{T}_k} \int_{s_k}^{t_k} \frac{c\nu_u^2}{V_\rho(0,t^{1/2})V_\rho(u,t^{1/2})} \exp\left(-\frac{\rho(0,u)^2}{ct}\right) dt \\ &\leq \sum_{u\in\mathbb{T}_k} \int_{s_k}^{t_k} \frac{c'\nu_u^2}{t^{(d+\alpha)/(2+2\alpha)} \cdot t^{d/2}|u|^{-(d-1)\alpha}} \exp\left(-\frac{2^{2k(1+\alpha)}}{c't}\right) dt \\ &\leq |\mathbb{T}_k| \cdot \max_{u\in\mathbb{T}_k}\{\nu_u^2|u|^{(d-1)\alpha}\} \cdot \int_0^{\infty} c' t^{-(d+\alpha)/(2+2\alpha)-d/2} \exp\left(-\frac{2^{2k(1+\alpha)}}{c't}\right) dt \\ &\leq 2^{dk} \cdot c'' 2^{(d+1)k\alpha} \cdot (2^{2k(1+\alpha)})^{1-(d+\alpha)/(2+2\alpha)-d/2} \cdot c''' \int_0^{\infty} x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx \\ &= 2^{k(2+2\alpha-d)} \cdot c''c''' \int_0^{\infty} x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx. \end{aligned}$$

Since $d \geq 3$, we have $\int_0^{\infty} x^{(d+\alpha)/(2+2\alpha)+d/2-2} e^{-x} dx < \infty$ and so,

$$I_2 \leq c2^{k(2+2\alpha-d)}.$$

For the remaining term, applying Theorem 1.1 we still have

$$\begin{aligned} I_3 &\leq \sum_{u\in\mathbb{T}_k} \int_0^{s_k} (\nu_u/\nu_0) \exp\left(-c(\nu_0 \vee \nu_u)^{-1}\rho(0,u)\Big(1 \vee \log\big((\nu_0 \vee \nu_u)\rho(0,u)/t\big)\Big)\right) dt \\ &\leq |\mathbb{T}_k| \cdot s_k \cdot \max_{u\in\mathbb{T}_k} \nu_u \exp\big(-c(\nu_0 \vee \nu_u)^{-1}\rho(0,u)\big) \\ &\leq 2^{dk} \cdot (1 \vee 2^{k\alpha})2^{k(1+\alpha)} \cdot 2^{k\alpha} \exp\left(-c'(1 \vee 2^{k\alpha})^{-1} \cdot 2^{k(1+\alpha)}\right) \\ &= 2^{ck} e^{-c'2^{k(1+\alpha\wedge 0)}}. \end{aligned}$$

Since $\alpha > -1$, we have $I_3 \leq c2^{k(2+2\alpha-d)}$ again. Therefore,

$$\mathbf{E}_{0,0}(Z_k) \leq c2^{k(2+2\alpha-d)}. \tag{6.4}$$

On the other hand, once X and X' collide at some vertex u and some time t, then with at least e^{-2} probability they will stick together during time $[t, t+\nu_u/\mu_u)$, which implies

$$\mathbf{E}_{0,0}(Z_k | Z_k > 0) \geq c \min_{u\in\mathbb{T}_k} \nu_u/\mu_u \geq c'2^{2k\alpha}.$$

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So, for each $k \geq 0$,

$$\mathbb{P}_{0,0}(Z_k > 0) = \frac{\mathbf{E}_{0,0}(Z_k)}{\mathbf{E}_{0,0}(Z_k | Z_k > 0)} \leq c2^{-(d-2)k}.$$

Therefore,

$$\sum_k \mathbb{P}_{0,0}(Z_k > 0) \leq c \sum_k 2^{-(d-2)k} < \infty.$$

By the Borel–Cantelli Lemma, we completed the proof of (3). □

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